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Revenue and Efficiency Effects of Resale in First-Price Auctions*

Isa Hafalir† and Vijay Krishna‡

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Abstract

We study first-price auctions in a model with asymmetric, independent private values. Asymmetries lead to inefficient allocations thereby creating a motive for resale after the auction is over. In our model, resale takes place via monopoly pricing—the winner of the auction makes a take-it-or-leave-it offer to the loser. Our goal is to compare equilibria of the first-price auction without resale (FPA) with those of the first-price auction with resale (FPAR). For the three major families of distributions for which equilibria of the FPA are available in closed form, we show that resale possibilities increase the revenue of the original seller. We also show by example that, somewhat paradoxically, resale may actually decrease efficiency.

1 Introduction

This paper studies how resale possibilities affect the performance of first-price auctions in terms of revenue and efficiency. Resale has an important role to play only when the equilibrium allocation of the auction is inefficient so that unrealized gains from trade remain. One important source of inefficiency in auctions is the presence of ex ante asymmetries among bidders. For instance, one of the bidders may be inherently strong relative to the other in the sense that his values are stochastically higher. Because the two bidders draw values from different distributions, their equilibrium bidding strategies are also different. As a result, it may be that the person who wins the auction is not the one with the higher realized value.

It is commonly argued that resale possibilities are detrimental to the original seller. This intuition comes from the fact that resale is thought to dilute the market power of a seller. For instance, a price discriminating monopolist would be hurt by

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resale (see, for instance, Wilson 1993, p. 11). This is because resale imposes additional constraints on the set of final allocations that are available to the monopolist. The same is true in Myerson's (1981) analysis of optimal auctions.

It is also argued that resale markets are good for efficiency because they allow buyers to reallocate the object after the auction is over. While this is certainly true in an ex post sense, this argument does not take into account that if buyers anticipate that resale may take place, this will affect their bidding strategies and the resulting allocations.

The purpose of this paper is to examine the validity of these arguments in the context of first-price auctions. Here we study a simple model of resale with two buyers who bid in a first-price auction to obtain a single indivisible object. We postulate an environment with *asymmetric* independent private values. In the benchmark model of a first-price auction (FPA), there is no resale and the allocation of the auction is final. In the model of a first-price auction with resale (FPAR), the winner of the auction may, if he so wishes, sell the object to the other buyer. The resale transaction is assumed to take place via a take-it-or-leave-it offer from the winner to the loser. In effect, the winner acts as a monopolist during the resale transaction.

In an earlier paper, Hafalir and Krishna (2008), we showed that the first-price auction with resale has an equilibrium in which the bidders use monotone bidding strategies and moreover, that the equilibrium is unique in this class. The equilibrium has a key characteristic: even though the bidders are asymmetric, in equilibrium their bid distributions are identical. This symmetry property proves to be very convenient in characterizing the equilibrium and evaluating its performance.

Finding equilibria of first-price auctions without resale, however, is known to be a thorny problem. An equilibrium exists and is typically unique but is characterized by a pair of linked differential equations which can be analytically solved only rarely. To the best of our knowledge, analytic expressions for the bidding strategies in asymmetric first-price auctions are available for only a few classes of distributions:

1. The class $\mathcal{P}$, identified by Plum (1992), consists of distributions F_1 and F_2 which are both proportional to x^a for some fixed $a > 0$ but have differing supports.

2. The class $\mathcal{C}_1$, identified by Cheng (2006), consists of distribution F_1 and F_2 which are proportional to $x^{a_1}$and x^{a_2}, respectively. It is assumed that $a_1 > a_2$ and that the ratio of the supports is a particular constant depending on a_1 and a_2.

3. The class $\mathcal{C}_2$, also identified by Cheng (2007), consists of distribution F_1 and F_2 in which F_1 is a (translated) power distribution and F_2 is a truncated exponential distribution. This class is different from the first two in that (i) the lower ends of the supports of the distributions are not the same; and (ii) one of them has a mass point.

Our main finding below is that in all known cases where the bidding strategies can be calculated in the first-price auction without resale, resale *increases* revenue—the FPAR is revenue superior to the FPA. Thus contrary to the intuition derived

from the analysis of optimal auctions (or optimal nonlinear pricing), resale actually improves the revenue of the seller for a fixed *suboptimal* mechanism—in this case, the first-price auction. We conjecture that resale possibilities enhance the revenue of the original seller—that is, the FPAR is revenue superior to the FPA in general.

We also show a somewhat paradoxical result. The presence of resale may actually *decrease* social surplus. An auction with resale may lead to an allocation that is so inefficient that the gains from post-auction trade are unable to recover these losses relative to a situation without resale. Again this is contrary to the intuition that resale promotes efficiency.[1]

Related Literature Asymmetric first-price auctions were already studied by Vickrey (1961). He studied environments in which bidder 1's value, say v_1, was commonly known while the other bidder's value was uniformly distributed. In that case, there is no pure strategy equilibrium. Vickrey (1961) showed that there was a mixed strategy equilibrium in which only bidder 1 randomizes.

Asymmetric first-price auctions without resale are known to have pure strategy equilibria under quite general conditions (see, for instance, Athey, 2001). Moreover, the equilibrium is typically unique (see, for instance, Maskin and Riley, 2003). Closed-form expressions for bidding strategies and revenue comparisons are rare, however. Griesmer, et al. (1967) derive closed-form equilibrium bidding strategies in a first-price auction in which bidders draw values from uniform distributions, but over different supports. Plum (1992) extends this to the class of distribution pairs $\mathcal{P}$ in which the two value distributions are of the form x^a, again over different supports. Cheng (2006 and 2007) identifies two classes of distribution pairs $\mathcal{C}_1$ and $\mathcal{C}_2$ and shows that in both these classes, the bidding strategies in a first-price auction are linear. Working with distributions from $\mathcal{P}$, Cantillon (2007) shows how asymmetry affects revenue in first-price auctions. In this paper, we will also consider distributions from $\mathcal{P}$, $\mathcal{C}_1$ and $\mathcal{C}_2$.

In the absence of general analytic results, some researchers have resorted to numerical methods (Marshall, et al. 1994).

There is a small but growing literature on auctions with resale. Gupta and Lebrun (1999) consider resale possibilities but assume that at the end of the auction both *values* are announced. In contrast, in our model, the auctioneer knows only the bids and not the values and so these cannot be made public. Haile (2003) considers resale possibilities in a *symmetric* model. At the time of bidding, however, buyers have only noisy information regarding their true values, which are revealed only after the auction. While resale can never improve the outcome of Myerson's (1981) optimal auction, Zheng (2002) identifies conditions under which the seller can achieve the same profits even with resale. Garratt and Tröger (2006) consider resale in Vickrey's asymmetric auction in which one of the bidder's value is commonly known to be 0.

[1] Apart from revenue and efficiency considerations, there may be other reasons to restrict resale. For instance, in the auction of 3G spectrum licenses in the UK, the government had a specific objective of attracting new firms to the industry. Certain licenses were reserved for new entrants and could not be resold.

Garratt and Tröger (2006) show that there is a unique mixed strategy equilibrium in the first-price auction in which the revenue is positive.

In an earlier paper (Hafalir and Krishna, 2008), discussed below, we have compared the revenue from the first-price auction with resale to the second-price auction, showing that the former is superior for all regular distributions.

2 Preliminaries

There are two risk-neutral buyers, labelled 1 and 2, who seek to buy a single object. Buyers' values X_1 and X_2 are private and independently distributed. Buyer i's value for the object, X_i, is distributed according to the cumulative distribution function F_i with support $[0, \omega_i]$. It is assumed that each F_i admits a continuous density $f_i \equiv F_i'$ and that this density is positive on $(0, \omega_i)$.

We suppose throughout that bidder 1 is "strong" (and bidder 2 is "weak") in the sense of first-order stochastic dominance: for all x, $F_1(x) \leq F_2(x)$.

We assume that both F_i are *regular* as defined by Myerson (1981); that is, the functions

$$x - \frac{1 - F_i(x)}{f_i(x)} \tag{1}$$

are increasing.

We consider two environments.

First-price auctions without resale (FPA) In the first, the object is sold via a first-price auction and there is no further trade. In this case, there is typically a unique equilibrium with strictly increasing bidding strategies β_i. The inverse bid functions ϕ_i satisfy the following system of differential equations: for $j = 1, 2$ and $i \neq j$,

$$\frac{d}{db} \ln F_j\left(\phi_j(b)\right) = \frac{1}{\phi_i(b) - b} \tag{2}$$

See, for instance, Maskin and Riley (2000).

First-price auctions with resale (FPAR) In the second, the object is again sold via a first-price auction but now the winner of auction may sell the object to the loser by making a take-it-or-leave-it offer. We suppose that at the end of the auction, the losing bid is not announced.[2] In this case, there is a unique monotonic equilibrium in which the inverse bid functions ϕ_j, $j = 1, 2$, satisfy the system:

$$\frac{d}{db} \ln F_j\left(\phi_j(b)\right) = \frac{1}{p(b) - b} \tag{3}$$

where

$$p(b) = \arg\max_p \left[F_1(\phi_1(b)) - F_1(p)\right] p + F_1(p)\, \phi_2(b) \tag{4}$$

[2] It can be shown that if the losing bid is announced, then there does not exist a monotonic equilibrium (see Krishna, 2002). For other specifications of the resale stage, see Hafalir and Krishna (2008).

is the *pricing function* that determines the monopoly price set in equilibrium by bidder 2 when he wins with a bid of b. An implication of this is that for all bids b that occur in equilibrium

$$F_1(\phi_1(b)) = F_2(\phi_2(b)) \tag{5}$$

that is, the bid distributions are identical. It can be shown that if we define a distribution F by

$$F(p) = F_2\left(p - \frac{F(p) - F_1(p)}{f_1(p)}\right) \tag{6}$$

then the revenue to the original seller is just

$$R^{FPAR} = \int_0^{\omega_1} (1 - F(p))^2 \, dp \tag{7}$$

See Hafalir and Krishna (2008) for details.

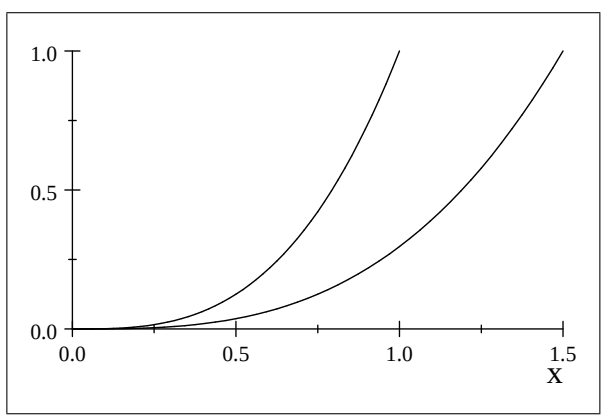
3 Asymmetric First-Price Auctions

When bidders are asymmetric, closed-form solutions for the equilibrium bidding strategies in a first-price auction (FPA) are difficult to obtain. To the best of our knowledge, there are only three classes of distribution-pairs for which equilibria in the FPA are explicitly known.

1. Plum (1992) derives the bidding strategies in a first-price auction when the distributions of values belong to the class $\mathcal{P}$ consisting of F_1 and F_2 such that:

   $$F_1(x) = \left(\frac{x}{\omega_1}\right)^a \text{ and } F_2(x) = \left(\frac{x}{\omega_2}\right)^a$$

   where $a > 0$ and $\omega_1 > \omega_2$. An example for the case $a = 3$, $\omega_1 = \frac{3}{2}$ and $\omega_2 = 1$ is depicted below.

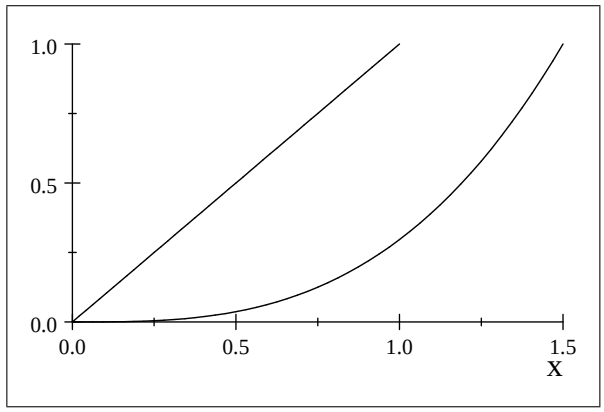


Distributions from Plum's class

2. Cheng (2006) derives the bidding strategies in a first-price auction when the distributions of values belong to the class $\mathcal{C}_1$ consisting of F_1 and F_2 such that:

$$F_1(x) = \left(\frac{x}{\omega_1}\right)^{a_1} \text{ and } F_2(x) = \left(\frac{x}{\omega_2}\right)^{a_2}$$

where $a_1 > a_2 > 0$ and $\omega_2 = \frac{a_2}{a_2+1}\frac{a_1+1}{a_1}\omega_1$. An example for the case $a_1 = 3$, $a_2 = 1$, $\omega_1 = \frac{3}{2}$ and $\omega_2 = 1$ is depicted below.

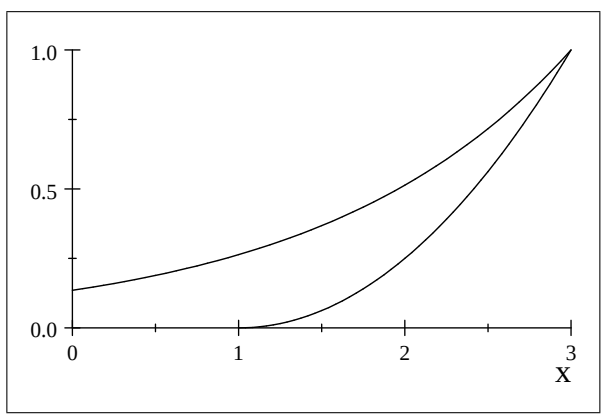


Distributions from Cheng's class 1

3. Cheng (2007) also derives the bidding strategies in a first-price auction when the distributions of values belong to the class $\mathcal{C}_2$ consisting of F_1 and F_2 such that:

$$\begin{aligned} F_1(x) &= \left(\frac{x-1}{a}\right)^a \text{ over } [1, a+1] \\ F_2(x) &= \exp\left(\frac{a}{a+1}x - a\right) \text{ over } [0, a+1] \end{aligned}$$

where $a > 0$.[3] An example for the case $a = 2$ is depicted below.



Distributions from Cheng's class 2

[3] This is actually a sub-class of a two-parameter class of distributions studied by Cheng (2007).

4 Distribution Class $\mathcal{P}$

4.1 Equilibrium without resale

For this class, Plum (1992) finds that the equilibrium strategies $\beta_i^N : [0, \omega_i] \to \mathbb{R}$ are:

$$\beta_1^N(x) = \frac{\left(1 + kx^{a+1}\right)^{\frac{a}{a+1}} - 1}{kx^a} \text{ and } \beta_2^N(x) = \frac{1 - \left(1 - kx^{a+1}\right)^{\frac{a}{a+1}}}{kx^a} \tag{8}$$

where

$$k \equiv \frac{1}{\omega_2^{a+1}} - \frac{1}{\omega_1^{a+1}} > 0$$

The maximum amount bid by either bidder is

$$\overline{b}^N = \omega_1 \omega_2 \frac{\omega_1^a - \omega_2^a}{\omega_1^{a+1} - \omega_2^{a+1}}$$

The inverse bid functions $\phi_i^N : [0, \overline{b}^N] \to [0, \omega_i]$, however, cannot be written in closed form. Nevertheless, if we define $A(x) = \left(1 + kx^{a+1}\right)^{\frac{1}{a+1}}$ and $B(x) = \left(1 - kx^{a+1}\right)^{\frac{1}{a+1}}$, then it can be easily verified that

$$\beta_1^N(x) = \beta_2^N\left(\frac{x}{A(x)}\right) \text{ and } \beta_2^N(x) = \beta_1^N\left(\frac{x}{B(x)}\right)$$

If we write $y = \phi_2^N\left(\beta_1^N(x)\right)$ as the value for 2 such that he bids the same as 1 does when the latter's value is x, then

$$b = \beta_1^N(x) = \beta_2^N\left(\frac{x}{A(x)}\right) = \beta_2^N(y)$$

Since β_1^N and β_2^N are both increasing, we obtain the identity

$$y = \frac{x}{A(x)} \tag{9}$$

4.2 Equilibrium with resale

We now turn to first-price auctions with resale. From (5) we have that the inverse bidding strategies with resale satisfy

$$F_1(\phi_1(b)) = \left(\frac{\phi_1(b)}{\omega_1}\right)^a = \left(\frac{\phi_2(b)}{\omega_2}\right)^a = F_2(\phi_2(b))$$

so that

$$\phi_2(b) = \frac{\omega_2}{\omega_1}\phi_1(b)$$

Next we determine the pricing function $p(b)$ making use of (4):

$$p(b) = \arg\max \frac{1}{\omega_1^a}\left[\phi_1(b)^a - p^a\right]\left[p - \frac{\omega_2}{\omega_1}\phi_1(b)\right]$$

If we let $\phi_1(b) = x$, then the regularity of F_1 (see (1)) guarantees that $p(b)$ is the unique solution to the first-order condition

$$ap^{a-1}\frac{\omega_2}{\omega_1}x + x^a - (a+1)\,p^a = 0$$

Notice that if we substitute $p = cx$ for some c, then the first-order condition becomes

$$x^a\left[\frac{\omega_2}{\omega_1}ac^{a-1} + 1 - (a+1)\,c^a\right] = 0$$

The left-hand side of the equation above is decreasing in c. When $c = 1$ it is negative. When $c = \sqrt{\frac{\omega_2}{\omega_1}}$, the bracketed expression on the left-hand side is

$$a\left(\frac{\omega_2}{\omega_1}\right)^{\frac{a+1}{2}} + 1 - (a+1)\left(\frac{\omega_2}{\omega_1}\right)^{\frac{a}{2}}$$

and we claim that this is positive. To see this, note that the function $at^{\frac{a+1}{2}} + 1 - (a+1)\,t^{\frac{a}{2}}$ is minimized at $t = 1$ and its value there is 0.

Thus we have verified that there exists a c satisfying $\sqrt{\frac{\omega_2}{\omega_1}} < c < 1$ such that the monopoly pricing function is

$$p(b) = c\phi_1(b) \tag{10}$$

where $\frac{\omega_2}{\omega_1}ac^{a-1} + 1 - (a+1)\,c^a = 0$.

To find the inverse bidding strategies in the FPAR consider the differential equation (3), which now becomes

$$\frac{a\phi_1'(b)}{\phi_1(b)} = \frac{1}{c\phi_1(b) - b}$$

The solution to this is linear:

$$\phi_1^R(b) = \frac{a+1}{ac}b \tag{11}$$

and so we have

$$\phi_2^R(b) = \frac{a+1}{ac}\frac{\omega_2}{\omega_1}b \tag{12}$$

and as a result, $p(b) = \frac{a+1}{a}b$.

Thus equilibrium bidding strategies in the first-price auction with resale are

$$\beta_1^R(x) = \frac{ac}{a+1}x \text{ and } \beta_2^R(x) = \frac{\omega_1}{\omega_2}\frac{ac}{a+1}x$$

The maximum amount bid is $\overline{b}^R = \frac{ac}{a+1}\omega_1$.

4.3 Revenue comparison

The distribution of revenues—the highest bid—in the FPA is:

$$\begin{aligned} L^N(b) &= F_1\left(\phi_1^N(b)\right) F_2\left(\phi_2^N(b)\right) \\ &= \left(\frac{\phi_1^N(b)\,\phi_2^N(b)}{\omega_1\omega_2}\right)^a \end{aligned}$$

and similarly, the distribution of revenues in the FPAR is

$$L^R(b) = \left(\frac{\phi_1^R(b)\,\phi_2^R(b)}{\omega_1\omega_2}\right)^a$$

We will argue that for all b, $L^N(b) > L^R(b)$ which will imply that the revenues in the FPAR stochastic dominate the revenues in the FPA. This is equivalent to showing that for all b,

$$\frac{\left(\phi_1^N(b)\,\phi_2^N(b)\right)^{\frac{1}{2}}}{b} > \frac{\left(\phi_1^R(b)\,\phi_2^R(b)\right)^{\frac{1}{2}}}{b}$$

Making use of (9), (11) and (12), we obtain

$$\begin{aligned} \frac{\left(\phi_1^N(b)\,\phi_2^N(b)\right)^{\frac{1}{2}}}{b} &= \frac{\phi_1^N(b)}{\left(A\left(\phi_1^N(b)\right)\right)^{\frac{1}{2}}}\frac{k\left(\phi_1^N(b)\right)^a}{A(\phi_1^N(b))^a - 1} \\ \frac{\left(\phi_1^R(b)\,\phi_2^R(b)\right)^{\frac{1}{2}}}{b} &= \frac{a+1}{ac}\left(\frac{\omega_2}{\omega_1}\right)^{\frac{1}{2}} \end{aligned}$$

Since $\sqrt{\frac{\omega_2}{\omega_1}} < c$,

$$\frac{a+1}{a} > \frac{a+1}{ac}\left(\frac{\omega_2}{\omega_1}\right)^{\frac{1}{2}}$$

Therefore, it suffices to show that

$$\frac{1}{\left(A\left(\phi_1^N(b)\right)\right)^{\frac{1}{2}}}\frac{k\left(\phi_1^N(b)\right)^{a+1}}{A(\phi_1^N(b))^a - 1} \geq \frac{a+1}{a}$$

for all b.

It is convenient to write $z = A\left(\phi_1^N(b)\right)$ in above inequality. We claim that the function,

$$H(z) = \frac{z^{a+1} - 1}{z^{a+\frac{1}{2}} - z^{\frac{1}{2}}}$$

which is defined for $z \geq 1$, is bounded below by $\frac{a+1}{a}$. We show this by establishing that $H(1) = \frac{a+1}{a}$ and that H is increasing.

Lemma 1 $H(1) = \frac{a+1}{a}$

Proof. At $z = 1$, we have a $\frac{0}{0}$ indeterminacy, but we can use L'Hopital's rule to conclude

$$\lim_{z\to 1} \frac{z^{a+1}-1}{z^{a+\frac{1}{2}} - z^{\frac{1}{2}}} = \lim_{z\to 1} \frac{(a+1)\, z^{a}}{\left(a+\frac{1}{2}\right) z^{a-\frac{1}{2}} - \frac{1}{2} z^{-\frac{1}{2}}} = \frac{a+1}{a}$$

■

Lemma 2 *$H'(z) > 0$ for $z > 1$.*

Proof. Note that

$$H'(z) = \frac{z^{2a+1} - 1 - \left(z^{a+1} - z^{a}\right)(2a+1)}{2z^{\frac{3}{2}}\left(z^{a}-1\right)^{2}}$$

We want to show that for all $z > 1$,

$$\gamma(z) \equiv \left(z^{2a+1} - 1\right) - \left(z^{a+1} - z^{a}\right)(2a+1) > 0$$

Note that $\gamma(1) = 0$ and

$$\gamma'(z) = (2a+1)\left(z^{2a} - (a+1)\, z^{a} - a z^{a-1}\right)$$

Again $\gamma'(1) = 0$ and

$$\begin{aligned} \gamma''(z) &= (2a+1)\left(2a z^{2a-1} - a(a+1)\, z^{a-1} - a(a-1)\, z^{a-2}\right) \\ &= (2a+1)\, a z^{a-2}\left(2z^{a+1} - (a+1)\, z - (a-1)\right) \end{aligned}$$

Now note that the function $\psi(z) = 2z^{a+1} - (a+1)\, z - (a-1) > 0$ for all $z > 1$ since $\psi(1) = 0$ and $\psi'(z) = 2(a+1)\, z^{a} - (a+1) > 0$ for all $z > 1$. Thus we have argued that $\gamma''(z) > 0$. Now the fact that $\gamma'(1) = 0$ implies that for all $z > 1$, $\gamma'(z) > 0$. Finally, the fact that $\gamma(1) = 0$ now implies that for all $z > 1$, $\gamma(z) > 0$. ■

Thus we have shown that when F_1 and F_2 belong to the class studied by Plum (1992), then the FPAR results in a higher revenue than the FPA.[4]

Proposition 1 *When the value distributions belong to the class $\mathcal{P}$, the revenue from a first-price auction with resale is greater than that from a first-price auction without resale.*

[4] This proof shows that $\phi_1^R(b)\,\phi_2^R(b) < \phi_1^N(b)\,\phi_1^N(b)$ for all $b \in \left[0, \min\{\bar{b}^R, \bar{b}^N\}\right]$ which in turn implies that $\bar{b}^R > \bar{b}^N$ (as otherwise we should have $\phi_1^N(\bar{b}^R)\phi_1^N(\bar{b}^R) > \omega_1\omega_2$, which is not possible). This fact can be also shown by using direct arguments.

5 Distribution Class $\mathcal{C}_1$

5.1 Equilibrium without resale

For this class, Cheng (2006) finds that the equilibrium strategies in the FPA are in fact linear:

$$\beta_1^N(x) = \frac{a_2}{a_2+1}x \text{ and } \beta_2^N(x) = \frac{a_1}{a_1+1}x$$

which gives the maximum bid of

$$\overline{b}^N = \frac{a_2}{a_2+1}\omega_1 = \frac{a_1}{a_1+1}\omega_2$$

The resulting inverse bid functions are

$$\phi_1^N(b) = \frac{a_2+1}{a_2}b \text{ and } \phi_2^N(b) = \frac{a_1+1}{a_1}b$$

5.2 Equilibrium with resale

For Cheng's class $\mathcal{C}_1$, equilibrium strategies in the FPAR can only be determined implicitly.

Nevertheless, because of (7), it is sufficient to determine the distribution F and because of (6) this can be done without an explicit expression for the equilibrium strategies. For this class, condition (6) that determines the distribution F becomes

$$T(F(p)) \equiv \frac{a_2(a_1+1)}{(a_2+1)}F(p)^{\frac{1}{a_2}}\left(\frac{p}{\omega_1}\right)^{a_1-1} + F(p) - (a_1+1)\left(\frac{p}{\omega_1}\right)^{a_1} = 0 \qquad (13)$$

5.3 Revenue comparison

The distribution of revenues in the FPA is:

$$L^N(b) = \left(\frac{a_2+1}{a_2}\frac{b}{\omega_1}\right)^{a_1}\left(\frac{a_1+1}{a_1}\frac{b}{\omega_2}\right)^{a_2} = \left(\frac{b}{\overline{b}^N}\right)^{a_1+a_2}$$

The revenue from the FPA is then given by

$$R^{FPA} = \overline{b}^N - \int_0^{\overline{b}^N}\left(\frac{b}{\overline{b}^N}\right)^{a_1+a_2}db = \frac{a_1+a_2}{a_1+a_2+1}\overline{b}^N = \frac{a_1+a_2}{a_1+a_2+1}\frac{a_2}{a_2+1}\omega_1$$

Consider the distribution function

$$G(p) = \left(\frac{p}{\omega_1}\frac{(a_1+a_2)(a_2+1)}{a_2(2+a_2+a_1)}\right)^{\frac{a_1+a_2}{2}}$$

It is routine to confirm that

$$\int_0^{\omega_1\frac{a_2(2+a_2+a_1)}{(a_1+a_2)(a_2+1)}}(1-G(p))^2\,dp = \frac{a_1+a_2}{a_1+a_2+1}\frac{a_2}{a_2+1}\omega_1 = R^{FPA}$$

In other words, G is a distribution such that a symmetric first-price auction in which both bidders draw values from G is *revenue equivalent* to an asymmetric first-price auction in which the bidders draw values from F_1 and F_2, respectively.

We will show that F determined in (13) *stochastically dominates* G; that is, for all $p > 0$, $F(p) < G(p)$. First, note that T, defined in (13), is an increasing function. Therefore it suffices to show that $T(G(p)) > 0$.

Lemma 3 $T(G(p)) > 0$.

Proof.

$$\begin{aligned} T(G(p)) &= \frac{a_2(a_1+1)}{(a_2+1)}\left(\frac{(a_2+1)(a_1+a_2)}{a_2(2+a_2+a_1)}\frac{p}{\omega_1}\right)^{\frac{a_1+a_2}{2a_2}}\left(\frac{p}{\omega_1}\right)^{a_1-1} \\ &\quad + \left(\frac{(a_2+1)(a_1+a_2)}{a_2(2+a_2+a_1)}\frac{p}{\omega_1}\right)^{\frac{a_1+a_2}{2}} - (a_1+1)\left(\frac{p}{\omega_1}\right)^{a_1} \\ &= \left(\frac{p}{\omega_1}\right)^{\frac{a_1+a_2}{2}}\left((a_1+1)\frac{a_2}{a_2+1}\left(\frac{(a_2+1)(a_1+a_2)}{a_2(2+a_2+a_1)}\right)^{\frac{a_1+a_2}{2a_2}}\left(\frac{p}{\omega_1}\right)^{\frac{a_1-a_2}{2a_2}+\frac{a_1-a_2}{2}}\right. \\ &\quad \left. + \left(\frac{(a_2+1)(a_1+a_2)}{a_2(2+a_2+a_1)}\right)^{\frac{a_1+a_2}{2}} - (a_1+1)\left(\frac{p}{\omega_1}\right)^{\frac{a_1-a_2}{2}}\right) \end{aligned}$$

Let $\frac{p}{\omega_1} = r$, $\frac{(a_2+1)(a_1+a_2)}{a_2(2+a_2+a_1)} = m$, since $\left(\frac{p}{\omega_1}\right)^{\frac{a_1+a_2}{2}} > 0$, it suffices to show that

$$D(r) = m^{\frac{a_1+a_2}{2}} - (a_1+1)\, r^{\frac{a_1-a_2}{2}}\left(1 - \frac{a_2}{a_2+1} m^{\frac{a_1+a_2}{2a_2}} r^{\frac{a_1-a_2}{2a_2}}\right) > 0$$

Note that the function $r^{\frac{a_1-a_2}{2}}\left(1 - \frac{a_2}{a_2+1} m^{\frac{a_1+a_2}{2a_2}} r^{\frac{a_1-a_2}{2a_2}}\right)$ is maximized at

$$r = m^{-\frac{2a_2}{a_1-a_2}\frac{a_1+a_2}{2a_2}}$$

Therefore the minimum value of D is given by

$$\begin{aligned} \underline{D} &= m^{\frac{a_1+a_2}{2}} - (a_1+1)\, m^{-\frac{a_1+a_2}{2}}\left(1 - \frac{a_2}{a_2+1}\right) \\ &= m^{-\frac{a_1+a_2}{2}}\left(m^{a_1+a_2} - \frac{a_1+1}{a_2+1}\right) \end{aligned}$$

Therefore, it suffices to show that

$$\left(\frac{(a_2+1)(a_1+a_2)}{a_2(2+a_2+a_1)}\right)^{a_1+a_2} - \frac{a_1+1}{a_2+1} > 0$$

or equivalently that

$$\left(1 + \frac{a_1-a_2}{a_2(2+a_2+a_1)}\right)^{a_1+a_2} > 1 + \frac{a_1-a_2}{a_2+1}$$

and this follows from the fact that[5]

$$\left(1+\frac{a_1-a_2}{a_2\left(2+a_2+a_1\right)}\right)^{a_1+a_2}>1+\left(a_1+a_2\right)\frac{a_1-a_2}{a_2\left(2+a_2+a_1\right)}>1+\frac{a_1-a_2}{a_2+1}$$

■

Thus we have shown that FPAR gives more revenue than FPA for Cheng's class $\mathcal{C}_1$.

Proposition 2 *When the value distributions belong to the class $\mathcal{C}_1$, the revenue from a first-price auction with resale is greater than that from a first-price auction without resale.*

6 Distribution Class $\mathcal{C}_2$

6.1 Equilibrium without resale

For this class Cheng (2007) finds that the equilibrium strategies in the FPA are both affine.

$$\begin{aligned}\beta_1^N(x) &= x-1 \text{ for } x\in[1,a+1]\\ \beta_2^N(x) &= \frac{a}{a+1}x \text{ for } x\in[0,a+1]\end{aligned}$$

with a maximum bid of

$$\bar{b}^N=a$$

The resulting inverse bid functions are

$$\phi_1^N(b)=b+1 \text{ and } \phi_2^N(b)=\frac{a+1}{a}b$$

Let us find a symmetric auction (with value distribution G and equilibrium inverse bid function ϕ) with the same distribution of winning bids (selling prices) as the given asymmetric first-price auction (FPA). This requires that

$$G\left(\phi(b)\right)^2=F_1\left(\phi_1^N(b)\right)F_2\left(\phi_2^N(b)\right)$$

for all $b\in[0,a]$, by taking the logarithm and derivative of both sides we obtain

$$\frac{d}{db}\ln G\left(\phi(b)\right)=\frac{1}{2}\frac{d}{db}\ln F_1\left(\phi_1^N(b)\right)+\frac{1}{2}\frac{d}{db}\ln F_2\left(\phi_2^N(b)\right)$$

From the necessary conditions for an FPA, this is the same as

$$\begin{aligned}\frac{1}{\phi(b)-b} &= \frac{1}{2}\frac{1}{\phi_2^N(b)-b}+\frac{1}{2}\frac{1}{\phi_1^N(b)-b}\\ &= \frac{a+b}{2b}\end{aligned}$$

[5] Since $(1+x)^k>1+kx$ for $x>0$. This can be seen by noting the derivative of the former function is always greater than the latter.

and so the inverse bidding strategy in the equivalent auction is

$$\phi(b) = b + \frac{2b}{a+b}$$

for all $b \in [0, a]$.

Solving the differential equation

$$\frac{d}{db} \ln G(\phi(b)) = \frac{a+b}{2b}$$

result in

$$G(\phi(b)) = \left(\frac{b}{a}\right)^{\frac{a}{2}} e^{\frac{b-a}{2}}$$

or

$$G(x) = \left(\frac{\beta(x)}{a}\right)^{\frac{a}{2}} e^{\frac{\beta(x)-a}{2}} \tag{14}$$

where $\beta(\cdot)$ is the inverse of $\phi(\cdot)$ and given by

$$\beta(x) = \frac{1}{2}x - \frac{1}{2}a + \frac{1}{2}\sqrt{4a - 4x + 2ax + a^2 + x^2 + 4} - 1$$

for all $x \in [0, a+1]$.

6.2 Equilibrium with resale

Once again, because of (7), it is sufficient to determine the distribution of resale prices F and because of (6) this can be done without an explicit expression for the equilibrium strategies.

We know that the symmetric auction (with value distribution F) which would give the same bid distribution as *first-price auction with resale* satisfies

$$F(x) = F_2\left(x - \frac{F(x) - F_1(x)}{f_1(x)}\right) \tag{15}$$

for all $x \in [\underline{x}, a+1]$ where $\underline{x} > 1$ satisfies

$$\underline{x} - \frac{e^{-a} - F_1(\underline{x})}{f_1(\underline{x})} = 0$$

6.3 Revenue comparison

We will show that the revenue with resale is higher by showing that showing that F, which is implicitly defined by (15), first-order stochastically dominates G, which is defined by (14).

Below we consider $a > 1$. When $a \in (0, 1]$, revenue superiority of FPAR is obvious. This follows from the observation that maximum bid in FPA is $a \leq 1$, whereas the minimum bid in FPAR is $\underline{x} > 1$.

We have

$$F(x) = \exp\left(\frac{a}{a+1}\left(x - \frac{F(x) - \left(\frac{x-1}{a}\right)^a}{\left(\frac{x-1}{a}\right)^{a-1}}\right) - a\right)$$

and so

$$\ln F(x) - x + \frac{1}{a+1} + a + \frac{a}{a+1}\frac{F(x)}{\left(\frac{x-1}{a}\right)^{a-1}} = 0$$

Since both $\ln F(x)$ and $\frac{a}{a+1}\frac{F(x)}{\left(\frac{x-1}{a}\right)^{a-1}}$ are increasing in x, for G to stochastically dominate F, it is enough to show that

$$\ln G(x) - x + \frac{1}{a+1} + a + \frac{a}{a+1}\frac{G(x)}{\left(\frac{x-1}{a}\right)^{a-1}} \geq 0$$

which is equivalent to

$$\ln\left(\left(\frac{\beta(x)}{a}\right)^{\frac{a}{2}} e^{\frac{\beta(x)-a}{2}}\right) - x + \frac{1}{a+1} + a + \frac{a}{a+1}\frac{\left(\frac{\beta(x)}{a}\right)^{\frac{a}{2}} e^{\frac{\beta(x)-a}{2}}}{\left(\frac{x-1}{a}\right)^{a-1}} \geq 0$$

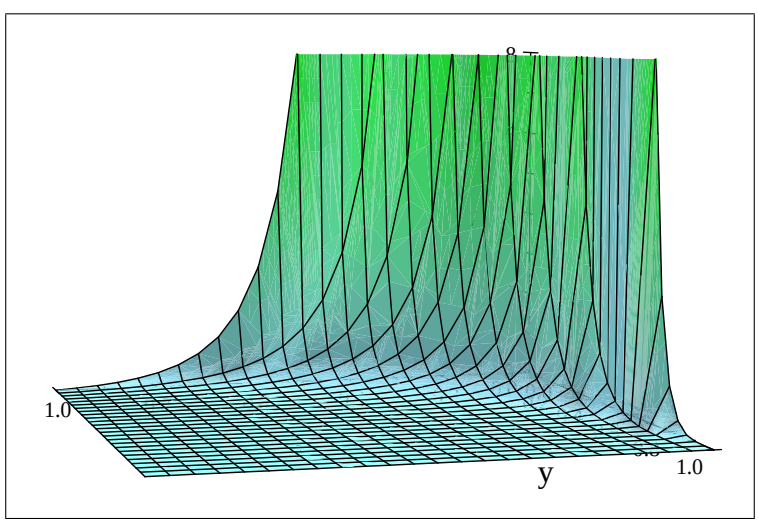
for all $x \in [\underline{x}, a+1]$ and $a > 1$. By change of variables $z = \frac{x-1}{a}$and $c = \frac{1}{a}$, this is equivalent to showing that

$$\phi(z,c) \equiv \ln\left((\gamma(z)c)^{\frac{1}{2c}} e^{\frac{\gamma(z)-\frac{1}{c}}{2}}\right) - \left(\frac{z}{c}+1\right) + \frac{c}{c+1} + \frac{1}{c} + \frac{1}{c+1}\frac{(\gamma(z)c)^{\frac{1}{2c}} e^{\frac{\gamma(z)-\frac{1}{c}}{2}}}{z^{\frac{1}{c}-1}} \geq 0$$

for all $z \in (0,1]$ and $c \in (0,1)$ where

$$\gamma(z) = \frac{1}{2}\sqrt{\frac{4}{c} + \frac{2}{c}\left(\frac{z}{c}+1\right) + \left(\frac{z}{c}+1\right)^2 - 4\frac{z}{c} + \frac{1}{c^2}} - \frac{1}{2c} + \frac{z}{2c} - \frac{1}{2}$$

This inequality $\phi(z,c) \geq 0$ for all $z \in (0,1]$ and $c \in (0,1)$ is hard to verify analytically but can be verified by numerical methods. For instance, plotting $\phi(z,c)$ shows that it is non-negative, as in the graph below:



Proposition 3 *When the value distributions belong to the class $\mathcal{C}_2$, the revenue from a first-price auction with resale is greater than that from a first-price auction without resale.*

7 Resale and efficiency

In this section, we examine the effects of resale as they pertain to efficiency. One line of thought, loosely associated with the "Chicago School," suggests that if the allocation from an auction is inefficient, then resale markets will reallocate in a way so as to ensure full efficiency. For resale markets to be fully efficient requires at the very least, as is now well understood, the absence of market power and the absence of incomplete information.

In the model we have formulated resale does not result in fully efficient outcomes. This is because it also takes place under incomplete information—the seller is unsure of the precise value of the buyer—and exercises his monopoly power. With positive probability, an inefficient allocation remains inefficient even after the resale stage is over.

But does the presence of resale markets *enhance* efficiency? In other words, is the total social surplus higher with resale than without, even if neither reaches full efficiency levels? Ex post, of course, resale can only help to increase social surplus—with positive probability the object is transferred to the buyer with the higher value. Its ex ante effects, on the other hand, are not so clear since the possibility of resale affects bidding behavior and hence also how the object is allocated by the auction.

Surplus without resale For a particular realization of values (x_1, x_2), bidder 2 wins if and only if (neglecting ties, since they occur with zero probability):

$$\beta_2^N(x_2) > \beta_1^N(x_1)$$

or equivalently, when

$$x_2 > \phi_2^N(\beta_1^N(x_1)) \equiv Q^N(x_1) \tag{16}$$

Bidder 2 with value $Q^N(x_1)$ would bid the same amount as bidder 1 with value x_1.

The ex ante expected *social surplus* in the first-price auction without resale is

$$S^{FPA} = \int_0^{\omega_1} \left(\int_0^{Q^N(x_1)} x_1 dF_2(x_2) + \int_{Q^N(x_1)}^{\omega_2} x_2 dF_2(x_2) \right) dF_1(x_1) \tag{17}$$

For fixed x_1, the first integral in the parentheses is the surplus when bidder 1 wins and the second is the surplus when bidder 2 wins.

Surplus with resale In the model with resale, the results of the auction do not, of course, represent the final allocation. Again, for a particular realization of the values (x_1, x_2), bidder 2 wins if and only if:

$$\beta_2^R(x_2) > \beta_1^R(x_1)$$

or equivalently, when

$$x_2 > \phi_2^R(\beta_1^R(x_1)) \equiv Q^R(x_1) \tag{18}$$

Suppose that 2 wins the auction when his value is x_2. In equilibrium, he will set a monopoly price of $p\left(\beta_2^R(x_2)\right)$. Bidder 1 will accept this offer if $p\left(\beta_2^R(x_2)\right) < x_1$, or equivalently if

$$x_2 < \min \phi_2^R\left(p^{-1}(x_1)\right) \equiv P(x_1) \tag{19}$$

Otherwise, the offer will be refused and the object will remain with bidder 2. Bidder 2 with value $P(x_1)$ would offer to sell the object for a price equal to x_1.

The ex ante expected *social surplus* in the first-price auction with resale is

$$S^{FPAR} = \int_0^{\omega_1} \left(\int_0^{Q^R(x_1)} x_1 dF_2(x_2) + \int_{Q^R(x_1)}^{P(x_1)} x_1 dF_2(x_2) + \int_{P(x_1)}^{\omega_2} x_2 dF_2(x_2) \right) dF_1(x_1)$$

For fixed x_1, the first integral in the parentheses is the surplus when bidder 1 wins and so there is no resale. The second integral is the surplus when 2 wins and sets a price low enough so that 1 accepts the offer and there is resale. The third integral is the surplus when 2 wins but sets a price so high that 1 rejects the offer.

The expression above can be simplified to

$$S^{FPAR} = \int_0^{\omega_1} \left(\int_0^{P(x_1)} x_1 dF_2(x_2) + \int_{P(x_1)}^{\omega_2} x_2 dF_2(x_2) \right) dF_1(x_1) \tag{20}$$

7.1 Surplus comparison for class $\mathcal{P}$

Consider, once again, distributions in the class $\mathcal{P}$, identified by Plum (1992), that is

$$F_1(x) = \left(\frac{x}{\omega_1}\right)^a \text{ and } F_2(x) = x^a$$

for some $\omega_1 > 1$. Without loss of generality, we have set $\omega_2 = 1$.

We now proceed to compare the ex ante surplus from an FPA against the ex ante surplus from an FPAR.[6]

Surplus without resale The equilibrium strategies in the FPA are determined as in (8):

$$\beta_1^N(x) = \frac{\left(1 + kx^{a+1}\right)^{\frac{a}{a+1}} - 1}{kx^a} \text{ and } \beta_2^N(x) = \frac{1 - \left(1 - kx^{a+1}\right)^{\frac{a}{a+1}}}{kx^a}$$

where

$$k \equiv 1 - \frac{1}{\omega_1^{a+1}} > 0$$

[6] We are grateful to a referee for generously providing the calculations given below, both generalizing and simplifying our earlier arguments.

and it is easily verified that the function Q^N defined in (16) is

$$Q^N(x_1) = \frac{x_1}{\left(1 + kx_1^{a+1}\right)^{\frac{1}{a+1}}}$$

Now using these expressions in (17) we have

$$\begin{aligned} S^{FPA} &= \int_0^{\omega_1} \left(\int_0^{Q^N(x_1)} x_1 dF_2(x_2) + \int_{Q^N(x_1)}^1 x_2 dF_2(x_2) \right) dF_1(x_1) \\ &= \int_0^{\omega_1} \left(x_1 \frac{x_1^a}{\left(1 + kx_1^{a+1}\right)^{\frac{a}{a+1}}} + \int_{Q^N(x_1)}^1 a x_2^a dx_2 \right) dF_1(x_1) \\ &= \int_0^{\omega_1} \left(\frac{x_1^{a+1}}{\left(1 + kx_1^{a+1}\right)^{\frac{a}{a+1}}} + \frac{a}{a+1} \frac{1 - \omega_1^{-a-1} x_1^{a+1}}{1 + kx_1^{a+1}} \right) dF_1(x_1) \end{aligned}$$

Changing variables to $y = x_1/\omega_1$, we have

$$\frac{1}{\omega_1} S^{FPA} = \int_0^1 \left(\frac{y^{a+1}}{\left(\omega_1^{-a-1} + ky^{a+1}\right)^{\frac{a}{a+1}}} + \left(\frac{a}{a+1}\right) \frac{1 - y^{a+1}}{\omega_1 \left(1 + k\omega_1^{a+1} y^{a+1}\right)} \right) d(y^a)$$

Now as $\omega_1 \to \infty$, $k \to 1$ and so we have

$$\lim_{\omega_1 \to \infty} \frac{1}{\omega_1} S^{FPA} = \int_0^1 y d(y^a) = \frac{a}{a+1}$$

Surplus with resale The function P, defined in (19) is

$$P(x_1) = \min\left\{ \frac{x_1}{c\omega_1}, 1 \right\}$$

where $\frac{1}{\omega_1} a c^{a-1} + 1 - (a+1) c^a = 0$.

Using (20), the expected surplus in the first-price auction with resale (FPAR) is

$$\begin{aligned} S^{FPAR} &= \int_0^{\omega_1} \left(\int_0^{P(x_1)} x_1 dF_2(x_2) + \int_{P(x_1)}^{\omega_2} x_2 dF_2(x_2) \right) dF_1(x_1) \\ &= \int_0^{\omega_1} \left(x_1 P(x_1)^a + \frac{a}{a+1} \left(1 - P(x_1)^{a+1}\right) \right) dF_1(x_1) \end{aligned}$$

Changing variables to $y = x_1/\omega_1$, let us define (with some abuse of notation)

$$P(y) = \min\left\{ \frac{y}{c}, 1 \right\}$$

and so

$$\frac{1}{\omega_1} S^{FPAR} = \int_0^1 \left(yP(y)^a + \frac{a}{a+1} \left(\frac{1 - P(y)^{a+1}}{\omega_1} \right) \right) d(y^a)$$

Now as $\omega_1 \to \infty$, $c \to \gamma = (a+1)^{-\frac{1}{a}}$ and so we have

$$\begin{aligned}\lim_{\omega_1 \to \infty} \frac{1}{\omega_1} S^{FPAR} &= \int_0^1 yP(y)^a \, d(y^a) \\ &= \frac{a}{a+1}\left(1 - \frac{(a+1)^{-\frac{a+1}{a}}}{a+2}\right)\end{aligned}$$

Surplus comparison The limit of the ratio of the surpluses from the two mechanisms is

$$\lim_{\omega_1 \to \infty} \frac{S^{FPAR}}{S^{FPA}} = 1 - \frac{(a+1)^{-\frac{a+1}{a}}}{a+2} < 1$$

and thus when ω_1 is large, a somewhat surprising phenomenon arises—the social surplus under resale may be *smaller* than the social surplus without resale. Of course, resale always increases efficiency ex post—given any allocation resulting from the auction, resale can only improve social surplus. But the presence of resale causes speculative bidding by the weak bidder. As a result, the allocation at the end of the auction is so inefficient that even post-auction resale is unable to compensate enough. Thus resale may decrease ex ante efficiency! As shown above, this phenomenon occurs when the asymmetry is large because that is when the benefits to speculative bidding are also large.

For this class of distributions, the degree of asymmetry can be measured by one parameter alone, ω_1. When the degree of asymmetry is small, that is, $\omega_1 - \omega_2 = \omega_1 - 1$ is small, it is indeed the case that resale improves efficiency—the expected social surplus under resale is higher than the expected social surplus without resale. For the case when $a = 1$, the surplus from the FPAR is smaller for all $\omega_1 > \omega_1^* \simeq 1.95$.

These features can be seen in Figure 3 that depicts the allocations resulting from FPA and FPAR for a special case when $a = 1$ and $\omega_1 = 2$. The dotted line has slope 1 and thus determines the efficient allocations—bidder 1 is allocated the object when the realized values (x_1, x_2) lie below and to the right of the dotted line so that $x_1 > x_2$. The allocations resulting from a first-price auction without resale (FPA) are determined by the curve $Q^N = \phi_2^N \beta_1^N$ (as in (16)) and bidder 2 wins if $x_2 > Q^N(x_1)$. The allocations resulting from a first-price auction with resale (FPAR) are determined in two stages. First, the allocation at the end of the auction stage is determined by $Q^R = \phi_2^R \beta_1^R$ (as defined in (18)), but this, of course, is not the final allocation. Second, resale activity changes the allocation at the end of the auction to that determined by P (defined in (19)). This takes into account situations in which bidder 2 wins but then successfully sells the object to bidder 1. For instance, if the realized vector of values, (x_1, x_2), were in between P and Q^R, bidder 2 would win the auction but then successfully resell the object to bidder 1.

As is apparent in the figure, resale is beneficial ex post—the surplus from P is greater than the surplus from Q^R. But the curves Q^N and P intersect and so the resulting surpluses cannot be unequivocally ranked. Indeed, the ex ante surplus from a first-price auction with resale may be lower than that without resale—in the example this occurs when $\omega_1 > \omega_1^*$.

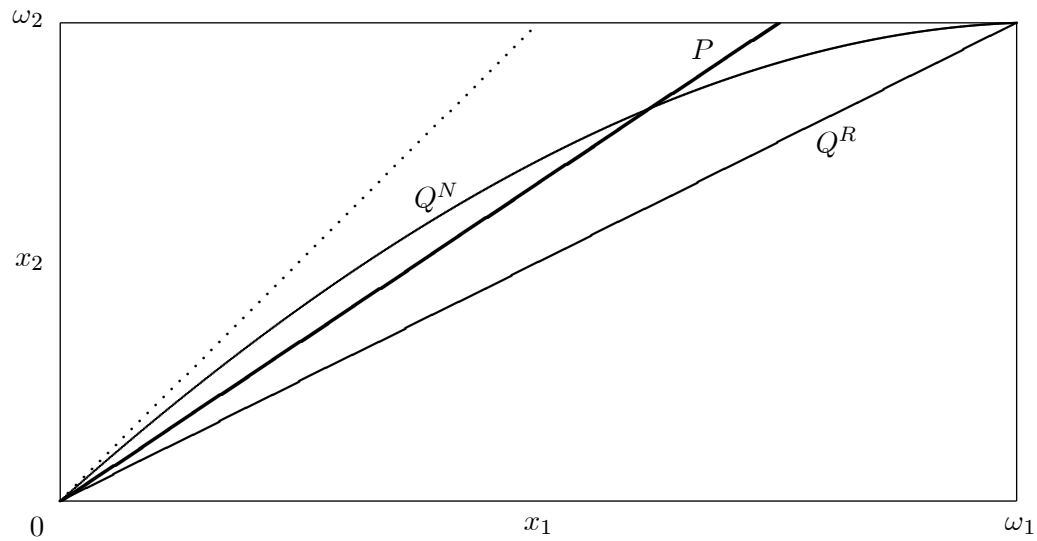


Figure 1: Resale may decrease efficiency

8 Conclusion

We have compared the performance of first-price auctions with and without resale. This comparison has been carried out for the three main classes of distributions $\mathcal{P}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ for which the equilibrium of the first-price auction can be explicitly characterized. We have shown that in both cases, resale improves the revenue of the original seller. In an earlier paper, Hafalir and Krishna (2008), we have shown that the first-price auction with resale (FPAR) is revenue superior to the second-price auction (SPA) whenever the distributions are regular. Thus it is also true that for the class of distributions for which the SPA is revenue superior to the FPA (see Maskin and Riley, 2000 for examples), the FPAR is revenue superior to the FPA. Thus we are led to conjecture:

Conjecture 1 *For all regular distributions, the revenue from the first-price auction with resale is higher than that from the first-price auction, that is,*

$$R^{FPAR} > R^{FPA}$$

In addition to the fact that the conjecture has been shown to be true in all known examples, it is also supported by the following intuition. It is commonly understood that asymmetry among bidders is detrimental for the seller (see Cantillon (2007) for a precise statement of this property). Resale serves to decrease the degree of asymmetry between bidders. For the weak bidder, winning is more valuable with

resale than without resale—he derives some option value from being able to resell the object if he wins. Conversely, for the strong bidder, winning is less valuable—he also derives some option value from being able to buy the object later if he loses. Thus the auction with resale is "more symmetric" than without resale. This benefits the seller.

We also examined the efficiency properties of resale. While resale does not restore efficiency to a first-price auction, we showed that, in fact, it may decrease efficiency.

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