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2008

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Published In
American Economic Review, 98, 1, 87-112.

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Asymmetric Auctions with Resale

Isa Hafalir and Vijay Krishna*

Abstract

We study first- and second-price auctions with resale in a model with independent private values. With asymmetric bidders, the resulting inefficiencies create a motive for post-auction trade which, in our model, takes place via monopoly pricing—the winner makes a take-it-or-leave-it offer to the loser. We show (i) a first-price auction with resale has a unique monotonic equilibrium; and (ii) with resale, the expected revenue from a first-price auction exceeds that from a second-price auction. The inclusion of resale possibilities thus permits a general revenue ranking of the two auctions that is not available when these are excluded. (*JEL* D44)

In a first-price auction, asymmetries among bidders typically result in inefficient allocations—that is, the winner of the auction may not be the person who values the object the most. This inefficiency creates a motive for post-auction resale and when bidders take resale possibilities into account, their bidding behavior is affected as well. Standard models of such auctions, by and large, implicitly assume either that resale possibilities do not exist or that bidders do not take these into account when formulating bids.

There are at least two reasons why resale possibilities should be considered explicitly. The first one is positive. If, after the auction is over, bidders see that there are potential gains from trade, then they will naturally engage in such trade. And it seems unlikely that the seller can prevent bidders from engaging in post-auction trade, even if, for some reason, resale was deemed disadvantageous. In the auction of 3G spectrum licenses in the UK, post-auction trade was restricted by the government. The bidders, however, were easily able to circumvent these restrictions. TIW, a Canadian firm, bid successfully for the most valuable license "A" with a winning bid in excess of £4 billion. The telecommunications company, Hutchison, then acquired the license by buying TIW itself.[1] Similarly, after the auction, France Telecom, an unsuccessful bidder, acquired Orange, a successful bidder. British Telecom created a wholly-owned subsidiary that bid successfully in the auction. After the auction, this subsidiary

[1] Actually, Hutchison had acquired a small stake in TIW prior to the auction and in fact, provided funds for its bid. We are grateful to Kenneth Binmore and Tilman Börgers for providing us with details of the UK 3G auctions.

was floated on the stock market and sold. Thus, restrictions on the buying and selling of licenses were circumvented by the buying and selling of companies that owned the licenses. The actions of Hutchison and British Telecom prior to the auction suggest that bidders fully anticipated post-auction resale possibilities.

The second reason to consider auctions with resale is normative. There has been recent interest in the design of efficient auctions, especially in the context of privatization. If post-auction resale results in efficiency, however, then from the planner's perspective, an inefficient auction is just as good. Are the allocations from an inefficient auction followed by post-auction resale indeed efficient?

In this paper, we study the effects of post-auction resale in a simple model with two bidders whose private values are independently, but perhaps asymmetrically, distributed. Equilibrium allocations in first-price auctions are then inefficient and bidders have the incentive to engage in post-auction trade. In our basic model, resale takes place via monopoly pricing—the winner of the auction makes a take-it-or-leave-it offer to the loser.

We first show that every equilibrium has the feature that, despite the asymmetries, the distributions of bids of the two bidders are identical (Proposition 1). Symmetry would not be surprising if resale took place under complete information and so was always efficient. In that case, each bidder would bid as if his value were the maximum of the two values. In our model, the symmetry of bid distributions is striking because post-auction resale also takes place under incomplete information and so is not always efficient. Here it occurs as a result of some cost-benefit calculations at the margin. The symmetry of the bid distributions is key—we show in Theorem 1 how the it may be used to construct an equilibrium of the first-price auction with resale.

The possibility of resale also affects incentives in second-price auctions. It is no longer a dominant strategy to bid one's value. It is, nevertheless, a robust equilibrium—the strategies do not depend on the value distributions (Proposition 2). In this equilibrium, of course, the auction allocates efficiently and so there is no resale.

Our main result (Theorem 2) is that once resale possibilities are admitted, the expected revenue from a first-price auction exceeds that from a second-price auction. We thus obtain a *general* revenue ranking of the two auction formats. We require only that the value distributions be regular in the sense defined by Roger Myerson (1981), ensuring that the monopoly pricing problem at the resale stage is well behaved. We remind the reader that in the absence of resale, the two auctions cannot be unambiguously ranked (see the work of Eric Maskin and John Riley (2000)).

The proof of Theorem 2 uses a technique borrowed from the calculus of variations. To the best of our knowledge, the use of this technique is new to auction theory and it will,

perhaps, prove useful in other applications as well.

The results reported above concern a particular resale institution—monopoly pricing—in which the winner of the auction has all the bargaining power. In Section 5, we show, however, that this is inessential by considering more general resale mechanisms. In particular, we examine the monopsony mechanism in which the loser has all the bargaining power and then more generally, mechanisms in which bargaining power is shared, perhaps unequally. All of the results reported above extend to these alternative, and more general, resale institutions.

Related Literature Equilibrium analysis of asymmetric first-price auctions has posed many challenges. Some of the difficulties were already pointed out by William Vickrey (1961) in his pioneering paper. He constructed an example in which bidder 1's value, say a, was commonly known while the other's was uniformly distributed. In that case, there is an equilibrium of the first-price auction in which bidder 1 randomizes. Vickrey (1961) showed that for some values of a, the revenue from a first-price auction exceeded that from a second-price auction; for other values of a, the revenue ranking was reversed.

Since then, progress in the area has been sporadic at best. In asymmetric first-price auctions without resale, pure strategy equilibria exist under quite permissive conditions, as a consequence of general existence results (see, for instance, Philip Reny (1999), Susan Athey (2001) or Matthew Jackson and Jeroen Swinkels (2005)). There is, again under weak conditions, a unique equilibrium (see, for instance, the work of Eric Maskin and John Riley (2003) or that of Bernard Lebrun (2006)). But characterization results and revenue comparisons are few and far between. James Griesmer, et al. (1967) derive closed-form equilibrium bidding strategies in a first-price auction in which bidders draw values from uniform distributions, but over different supports. Michael Plum (1992) extends this to situations in which the two value distributions are of the form v^n, again over different supports. Harrison Cheng (2006) identifies a class of distribution pairs which lead to linear bidding strategies. For this class, he shows that the first-price auction is revenue superior to the second-price auction. For specific examples of distribution pairs, Estelle Cantillon (2007, forthcoming) shows how asymmetry affects revenue in first-price auctions. In the absence of general analytic results, some researchers have resorted to numerical methods (see, for instance, the work of Robert Marshall, et al. (1994)).

Maskin and Riley (2000) derive the most comprehensive characterization and revenue ranking results concerning first- and second-price auctions in the presence of asymmetries. They consider problems in which one of the bidders is unambiguously stronger than the other. Specifically, the distribution of one bidder (conditionally) stochastically dominates that of the other. Maskin and Riley (2000) are able to identify circumstances in which one

or the other auction is revenue superior. For instance, the second-price auction is revenue superior if the distribution of the weak bidder is obtained from that of the strong bidder by reassigning probability mass toward lower values. Gadi Fibich, Arieh Gavious and Aner Sela (2004) have shown that when the bidders are “nearly symmetric,” the difference in revenues is of a smaller magnitude than the difference, appropriately measured, in the underlying distributions. Thus, for small asymmetries, the auctions are nearly revenue equivalent.

Madhurima Gupta and Bernard Lebrun (1999) consider resale possibilities in a manner not unlike this paper. They assume, however, that at the end of the auction both *values* are announced. This means, of course, that resale is always efficient. But it is not clear how the auctioneer would come to know the values themselves. In contrast, in our model, the auctioneer knows only the bids and not the values. Philip Haile (2003) considers resale possibilities in a *symmetric* model. At the time of bidding, however, buyers have only noisy information regarding their true values, which are revealed to them only after the auction. There is a motive for resale because although the winner of the auction may have received the highest signal, he may not have the highest true value. No general revenue ranking obtains. Charles Zheng (2002) identifies conditions under which the outcomes of Myerson’s (1981) optimal auction can be achieved when resale is permitted.

The model of Rod Garratt and Thomas Tröger (2006) is closest to ours in spirit. The crucial difference is that they assume, as in Vickrey’s (1961) example, that the value of one of the bidders is commonly known, and moreover, is equal to zero. This bidder, thus, participates in the auction for purely “speculative” reasons—he has no use value for the object. He only benefits if he can resell the object to the other bidder.[2] In the efficient equilibrium of the second-price auction, the revenue is obviously zero. Garratt and Tröger (2006) show that there is a unique mixed strategy equilibrium in the first-price auction in which the revenue is positive. We allow for general continuous distributions and so their model may be viewed as a limiting case of ours.

1 Preliminaries

A single indivisible object is for sale. There are two risk-neutral buyers, labelled s (“strong”) and w (“weak”), with independently distributed private values, V_s and V_w. Buyer i’s value for the object, V_i, is distributed according to the cumulative distribution function F_i with support $[0, a_i]$. It is assumed that each F_i admits a continuous density $f_i \equiv F_i'$ and that this density is positive on $(0, a_i)$. We suppose that for all v, $F_s(v) \leq F_w(v)$, so that the

[2] In a supplement to their paper, Garratt and Tröger (2006) also consider situations in which there is one speculator and many identical other bidders.

distribution of values of the strong bidder (first-order) *stochastically dominates* that of the weak bidder.[3]

We assume that both F_i are *regular* in the sense of Myerson (1981) so that for $i = s, w$, the *virtual value*, defined as

$$v - \frac{1 - F_i(v)}{f_i(v)}$$

is a strictly increasing function of the actual value v. This ensures that the price at the resale stage is uniquely determined and is characterized by the first-order conditions for a maximum.[4]

In later sections we will need to consider conditional distributions of the form $F_i(v \mid V_i \leq a) = F_i(v) / F_i(a)$ with support $[0, a]$. The *virtual value* of the conditional distribution $F_i(v \mid V_i \leq a)$ is

$$v - \frac{F_i(a) - F_i(v)}{f_i(v)}$$

It is easily verified that if F_i is regular then the conditional distribution $F_i(\cdot \mid V_i \leq a)$ is also regular.

2 First-Price Auction with Resale

Our model of the first-price auction with resale (FPAR) is the following. The buyers first participate in a standard sealed-bid first-price auction. The winning bid is publicly announced. We assume—as is common in real-world auctions—that the losing bid is *not* announced.[5]

In the second stage, the winner of the auction—say j—may, if he wishes to, offer to sell the object to the other bidder $i \neq j$ at some price p. If the offer is accepted by i, a sale ensues. If the offer is rejected, the original owner j retains the object. Thus resale takes place via a take-it-or-leave-it offer by the winner of the auction.[6]

Note that if i loses the auction, then the announcement of the winning bid b_j carries no useful information—that is, whether or not i will accept an offer is independent of what he believes j's value to be. Thus the equilibrium would be unaffected if neither bid were

[3]The assumption that the two distributions are stochastically ranked is made for expositional ease only. In the supplemental material accompanying this paper, we show that all of our results remain true without this assumption.

[4]As shown by Jeremy Bulow and John Roberts (1989), the virtual value can be interpreted as the "marginal revenue" of a monopolist who faces a demand curve $1 - F_i(p)$.

[5]This assumption is discussed in more detail below in Remark 1.

[6]All bargaining power thus lies with the seller and from his perspective, this is, of course, the optimal resale mechanism. In Section 5 below, we show that our analysis extends to resale mechanisms in which all bargaining power lies with the buyer and then, more generally, to mechanisms in which it is shared.

announced.

As usual, we work backwards and first outline behavior in the resale stage.

2.1 Resale Stage

Suppose that the two bidders follow *continuous* and *strictly increasing* bidding strategies β_s and β_w with inverses ϕ_s and ϕ_w, respectively.[7]

Suppose that bidder j with value v_j wins the auction with a bid of b. As a result, he would infer that bidder i's value $V_i \leq \phi_i(b)$. If $v_j < \phi_i(b)$, then there are potential gains from trade and so bidder j will set a ("monopoly") price p that solves

$$\max_p [F_i(\phi_i(b)) - F_i(p)]p + F_i(p)v_j \tag{1}$$

The first term is j's expected payoff from the event $V_i \geq p$ in which bidder i accepts his offer. The second term is his payoff from the event $V_i < p$, in which case bidder i rejects it.

The first-order condition for j's maximization problem can be rewritten as

$$p - \frac{F_i(\phi_i(b)) - F_i(p)}{f_i(p)} = v_j \tag{2}$$

Since F_i is regular, the left-hand side is increasing and so (2) has a unique solution. Moreover, (2) is also sufficient for j's maximization problem. Thus there is a unique price $p_j(b, v_j)$ that maximizes j's payoff from resale. Clearly, $v_j < p_j(b, v_j) < \phi_i(b)$. It follows immediately from (2) that the optimal price $p_j(b, v_j)$ is an increasing function of both b and v_j.

Let $R_j(b, v_j)$ denote bidder j's optimal expected revenue from resale (which may or may not take place), that is, the value of (1). For future reference, note that as a result of the envelope theorem,

$$\frac{\partial}{\partial b} R_j(b, v_j) = f_i(\phi_i(b))\, \phi_i'(b)\, p_j(b, v_j) \tag{3}$$

If bidder j wins the auction with a bid of b and $v_j \geq \phi_i(b)$, then there are no potential gains from trade and so bidder j will not offer the object for sale.

2.2 Bidding Stage

We begin by deriving some necessary conditions that equilibrium bidding strategies must satisfy. At the time of the auction, both bidders anticipate that behavior in the resale stage

[7] It can be shown that all equilibria must have these properties. See the supplement to this paper.

will be as specified above.

2.2.1 Necessary Conditions

Suppose that, in equilibrium, each bidder i follows a *continuous* and *strictly increasing* bidding strategy $\beta_i : [0, a_i] \to \mathbb{R}$, so that $\beta_i(v_i)$ is the bid submitted by i when his value is v_i.

It may be verified that $\beta_s(0) = \beta_w(0) = 0$. If a bidder with a value of zero bids a positive amount, bidders with values close to zero would surely lose money (details may be found in the supplement). It is also easy to verify that $\beta_s(a_s) = \beta_w(a_w) \equiv \overline{b}$.

As above, let $\phi_i : [0, \overline{b}] \to [0, a_i]$ denote i's inverse bidding strategy in equilibrium, that is, $\phi_i = \beta_i^{-1}$. Fix a bid b and suppose that $\phi_j(b) < \phi_i(b)$. This means that if j wins with a bid of b, then there are potential gains from trade and so j will make an offer to i. If, on the other hand, i wins with bid of b, then there are no potential gains from trade and so i will not make an offer to j. Thus the bid b itself determines the direction of the resale transaction, that is, the identities of the seller and the buyer.

Suppose bidder i follows ϕ_i. Bidder j's expected payoff when his value is $v_j \equiv \phi_j(b)$ and he deviates by bidding a c close to b is

$$R_j(c, v_j) - F_i(\phi_i(c))\, c \tag{4}$$

where $R_j(c, v_j)$, defined above as the value of (1), is his expected revenue from resale. If j loses the auction, then $\phi_j(c) < \phi_i(c)$ implies that bidder i will not offer to resell to him and so his payoff is 0. Since it is optimal for j to bid b, the first-order condition for maximizing (4), by using (3), results in

$$\begin{aligned} 0 &= f_i(\phi_i(b))\, \phi_i'(b)\, p_j(b, v_j) \\ &\quad - f_i(\phi_i(b))\, \phi_i'(b)\, b - F_i(\phi_i(b)) \end{aligned}$$

where $p_j(b, v_j)$ is defined as the solution to (1). Since $v_j = \phi_j(b)$, writing $p(b) \equiv p_j(b, \phi_j(b))$, the first-order condition results in the differential equation

$$\frac{d}{db} \ln F_i(\phi_i(b)) = \frac{1}{p(b) - b} \tag{5}$$

Note that p depends on both ϕ_s and ϕ_w.

Now suppose bidder j follows an equilibrium strategy ϕ_j. Bidder i's expected payoff when

his value is $v_i \equiv \phi_i(b)$ and he deviates by bidding a c close to b is

$$(v_i - c)F_j(\phi_j(c)) + \int_{\phi_j(c)}^{a_j} \left[v_i - p_j(\beta_j(v_j), v_j)\right]_+ f_j(v_j)\, dv_j \tag{6}$$

where $[x]_+ = \max\{x, 0\}$. This is because if i wins the auction, he never resells to j and so his profit is simply $v_i - c$. The second term is i's expected payoff from buying the object from j. Since it is optimal for i to bid b, the first-order condition for maximizing (6) is

$$\begin{aligned} 0 &= [v_i - b]\, f_j(\phi_j(b))\phi_j'(b) - F_j(\phi_j(b)) \\ &\quad - \left[v_i - p_j\left(b, \phi_j(b)\right)\right] f_j(\phi_j(b))\phi_j'(b) \end{aligned}$$

Again writing $p_j(b, \phi_j(b)) = p(b)$, the first-order condition becomes

$$\frac{d}{db} \ln F_j\left(\phi_j(b)\right) = \frac{1}{p(b) - b} \tag{7}$$

which is the *same* as (5).

We have argued that if ϕ_s, ϕ_w are the equilibrium inverse bid functions in a first-price auction with resale, then they satisfy the same differential equation (5) or (7). This was derived using the first-order necessary conditions for local deviations to be unprofitable.[8] In the appendix we show that the differential equations are, in fact, also sufficient.

Recall that in any increasing equilibrium, the highest bids must be the same, say $\bar{b}$. Thus $F_s\left(\phi_s\left(\bar{b}\right)\right) = 1 = F_w\left(\phi_w\left(\bar{b}\right)\right)$. Since the boundary conditions for the two differential equations are the same, it now follows immediately that

Proposition 1 *If ϕ_s and ϕ_w are strictly increasing equilibrium inverse bidding strategies, then for all b,*

$$F_s(\phi_s(b)) = F_w(\phi_w(b))$$

that is, the bid distributions of the two bidders are identical.

Since $F_s \leq F_w$, the equality of the bid distributions also implies that $\phi_s \geq \phi_w$ or equivalently that

[8] We have argued that the differential equations hold at any b such that $\phi_j(b) < \phi_i(b)$. If b is such that $\phi_j(b) = \phi_i(b)$, then whoever wins at that bid will set a price $p(b) = \phi_i(b) = \phi_j(b)$ and the same arguments as given above show that the differential equations still hold.

Corollary 1 *The weak bidder bids more aggressively than the strong bidder; that is, for all* v, $\beta_w(v) \geq \beta_s(v)$.

2.2.2 Symmetrization

Proposition 1 identifies a remarkable property of first-price auctions with resale—even though the bidders are asymmetric, in equilibrium they bid in a way that the resulting bid distributions $F_i(\phi_i(\cdot))$ are the same. In this sense, resale *symmetrizes* the auction. Since this property plays an important role in what follows, it is worth exploring the underlying reasons.[9]

As a first step, consider a standard first-price auction *without resale* (FPA) and let φ_s and φ_w be the equilibrium inverse bidding strategies. Suppose bidder i with value $v_i = \varphi_i(b)$ raises his bid slightly from b to $b+\varepsilon$. This benefits bidder i only against the types of bidder j to whom i loses the auction by bidding b but wins by bidding $b+\varepsilon$. By doing this, bidder i gains approximately $v_i - b = \varphi_i(b) - b$ whenever $\varphi_j(b) < v_j < \varphi_j(b+\varepsilon)$. Writing the first-order condition for optimality yields the pair of differential equations: for $j = s, w$ and $i \neq j$,

$$\frac{d}{db} \ln F_j\left(\varphi_j(b)\right) = \frac{1}{\varphi_i(b) - b} \tag{8}$$

Notice that the right-hand side is the inverse of the marginal gain accruing to i from increasing his bid.[10]

Now consider a first-price auction *with resale* (FPAR) with equilibrium inverse bidding strategies ϕ_s and ϕ_w. Suppose that for all b, $\phi_w(b) < \phi_s(b)$. This means that in equilibrium if w wins with a bid of b, so that his value $v_w = \phi_w(b)$, then he will try to resell the object to bidder s since there are potential gains from trade. On the other hand, if s wins with a bid of b, he will not resell the object to bidder w since there are no gains from trade.

Suppose the weak bidder with value $v_w = \phi_w(b)$ raises his bid slightly from b to $b+\varepsilon$. As before, we look at how much w gains against strong bidder types such that $\phi_s(b) < v_s < \phi_s(b+\varepsilon)$. When he bids b, the weak bidder loses against these types of bidder s and since there is no resale, the weak bidder's payoff is 0. When he bids $b+\varepsilon$, however, he wins against these types of bidder s and is able to resell to them at a price of $p(b)$ for a gain of $p(b) - b$.

What about the strong bidder? Suppose bidder s with value $v_s = \phi_s(b)$ raises his bid slightly from b to $b+\varepsilon$ and again consider the benefit to s against those bidder w types such

[9]Gupta and Lebrun (1998) allude to this kind of symmetry in passing although the main thrust of their analysis concerns a different model—one in which values are announced at the end of the auction.

[10]A consequence of this is that in a first-price auction without resale, the distribution of bids of the strong bidder stochastically dominates that of the weak bidder (see Maskin and Riley, 2000). As a referee pointed out, without resale, the bid distributions are identical if and only if the value distributions are.

that $\phi_w(b) < v_w < \phi_w(b+\varepsilon)$. When he bids b, bidder s loses against these types of bidder w but is able to buy the object from them at a price of approximately $p(b)$. His payoff thus approximately equals $v_s - p(b)$. When he bids $b+\varepsilon$, he wins against these types of bidder w and so his payoff is $v_s - b$. The gain in payoff for s from increasing his bid from b to $b+\varepsilon$ is thus approximately equal to $(v_s - b) - (v_s - p(b)) = p(b) - b$, the same as w's gain!

In contrast to (8), the right-hand sides of (5) and (7) are identical.

The symmetrization effects of resale come from the fact that the *marginal gain* to both bidders from a higher bid is the same: $p(b) - b$. For the weak bidder (the "seller"), the marginal gain is just the profit from resale, that is, $p(b) - b$. For the strong bidder (the "buyer"), the marginal gain is the difference in the "retail price" $p(b)$ he pays when he loses the auction but buys from bidder w and the "wholesale price" b that he pays when he wins the auction and buys directly from the auctioneer.

The distributions of equilibrium bids in an asymmetric first-price auction with resale are thus observationally equivalent to the distribution of equilibrium bids in a symmetric first-price auction. In other words, given F_s and F_w, there exists a distribution F such that a first-price auction (FPA) in which both bidders draw values from F is equivalent, in terms of equilibrium bid distributions, to a first-price auction with resale (FPAR) in which bidders draw values from F_s and F_w, respectively. This also means that the two auctions are revenue equivalent.

We now show how F may be obtained from F_s and F_w without any knowledge of the equilibrium bidding strategies. Given distributions F_s and F_w, let F be such that for all p,

$$F(p) = F_w\left(p - \frac{F(p) - F_s(p)}{f_s(p)}\right) \tag{9}$$

Then F is a uniquely determined distribution function such that $F_s(p) \leq F(p) \leq F_w(p)$. Moreover, if $F_s(p) < F_w(p)$, then $F_s(p) < F(p) < F_w(p)$. These properties follow in a straightforward manner because the regularity of F_s guarantees that the conditional virtual value in the right-hand side of (9) is increasing.

The construction of F has a simple geometric interpretation, as depicted in Figure 1. The distribution F is such that it passes through the point b, which bisects the line segment ac. The length of the line segment ab is just $p - F_w^{-1}(F(p))$. And since bd/bc is $f_s(p)$, the slope of F_s at p, the length of bc is $[F(p) - F_s(p)]/f_s(p)$. Equation (9) requires that these be equal.

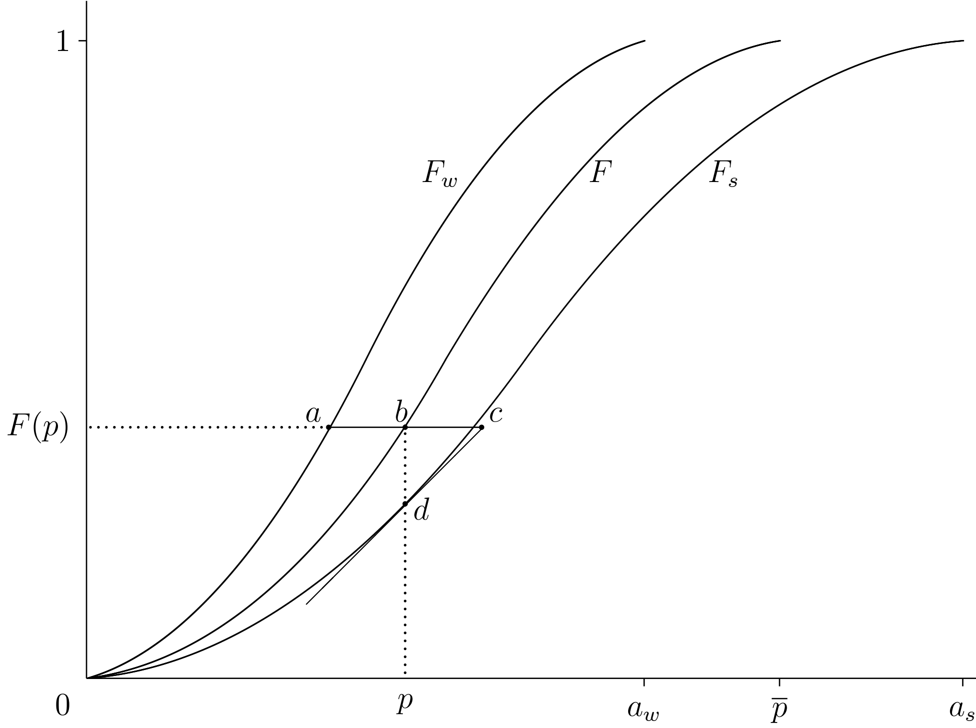

Figure 1: Construction of F

2.2.3 Equivalent Symmetric Auction

Now consider a *symmetric* first-price auction without resale in which there are two bidders and both draw values independently from the distribution function F on $[0, \overline{p}]$ as defined above in (9).

The equilibrium strategies in a symmetric auction can, of course, be derived explicitly and are given by

$$\beta(v) = \frac{1}{F(v)} \int_0^v y f(y)\, dy$$

Let $\overline{b} = \beta(\overline{p})$. Define the equilibrium inverse bid function for the symmetric auction as

$$\phi(b) \equiv \beta^{-1}(b) \tag{10}$$

so that the distribution of bids for each bidder is $F(\phi(b))$. A necessary condition for ϕ to be the equilibrium inverse bidding strategy in the symmetric auction is that

$$\frac{d}{db} \ln F(\phi(b)) = \frac{1}{\phi(b) - b} \tag{11}$$

2.3 Equilibrium with Resale

In this section, we establish that the first-price auction with resale has a pure strategy equilibrium in which each bidder follows a strictly increasing bidding strategy. The equilibrium is unique in the class of pure strategy equilibria with nondecreasing bidding strategies.

The proof that there is a strictly increasing equilibrium is constructive. Given regular distribution functions F_s and F_w, construct F as in (9). Consider a symmetric first-price auction in which each bidder draws values independently from F. In symmetric auctions, it is known that a symmetric equilibrium β exists and is strictly increasing. We will use the equilibrium β to construct equilibrium bidding strategies β_s and β_w for the asymmetric first-price auction with resale.

Theorem 1 *Suppose F_s and F_w are regular. Then there is an equilibrium in the first-price auction with resale in which the bidding strategies are strictly increasing.*

Proof. The proof is by construction.

Given F_s and F_w, let F be determined as in (9). Let ϕ, as defined above in (10), be the equilibrium inverse bidding strategy in the symmetric auction in which bidders draw values from F. Let $\bar{b}$ be the maximum bid in the symmetric auction and define inverse bidding strategies $\phi_s : \left[0, \bar{b}\right] \rightarrow [0, a_s]$ and $\phi_w : \left[0, \bar{b}\right] \rightarrow [0, a_w]$ in the asymmetric first-price with resale as follows:

$$F_s\left(\phi_s\left(b\right)\right) = F(\phi\left(b\right)) \tag{12}$$

$$F_w\left(\phi_w\left(b\right)\right) = F(\phi\left(b\right)) \tag{13}$$

Then using (11), we have that for $i = s, w$

$$\frac{d}{db} \ln F_i\left(\phi_i\left(b\right)\right) = \frac{1}{\phi\left(b\right) - b}$$

We claim that ϕ_s and ϕ_w are equilibrium inverse bidding strategies in the first-price auction with resale.

The definition of F in (9) implies that

$$F_w^{-1}\left(F\left(\phi\left(b\right)\right)\right) = \phi\left(b\right) - \frac{F\left(\phi\left(b\right)\right) - F_s\left(\phi\left(b\right)\right)}{f_s\left(\phi\left(b\right)\right)}$$

and since $F\left(\phi\left(b\right)\right) = F_w\left(\phi_w\left(b\right)\right) = F_s\left(\phi_s\left(b\right)\right)$,

$$\phi_w\left(b\right) = \phi\left(b\right) - \frac{F_s\left(\phi_s\left(b\right)\right) - F_s\left(\phi\left(b\right)\right)}{f_s\left(\phi\left(b\right)\right)}$$

which is precisely the first-order condition for

$$\max_p [F_s(\phi_s(b)) - F_s(p)]p + F_s(p)\phi_w(b)$$

Regularity implies that the first-order condition is both necessary and sufficient for a maximum. Since $p(b)$ was defined to be the solution to the maximization problem, we have that for all b,

$$p(b) = \phi(b)$$

Finally, note that $F_s(p(b)) < F_w(p(b))$ is equivalent to $\phi_w(b) < \phi_s(b)$. This is because (12) and (13) imply that $F_s(p(b)) < F(p(b))$, which is equivalent to $F_s(p(b)) < F_s(\phi_s(b))$ and so also to $p(b) < \phi_s(b)$. Similarly, $F(p(b)) < F_w(p(b))$ is equivalent to $F_w(\phi_w(b)) < F_w(p(b))$ and so also to $\phi_w(b) < p(b)$. Thus, $F_s(p(b)) < F_w(p(b))$ if and only if $\phi_w(b) < \phi_s(b)$.

We have thus argued that if ϕ_s and ϕ_w are determined by (12) and (13), then they satisfy the differential equations (5) and (7) where $p(b)$ is determined by the solution to (2) when $j = w$ and $v_w = \phi_w(b)$.

Thus as constructed, the functions ϕ_s and ϕ_w satisfy the conditions of Proposition 4 in the appendix and so constitute equilibrium inverse bidding strategies.

Remark 1 Theorem 1 relies on the assumption that at the end of the auction, the losing bid is not announced. If the losing bid is announced, the value of the losing bidder would be revealed in any strictly increasing equilibrium. This creates an incentive for a bidder to bid lower, so that if he were to lose, then the other bidder would think that his value is smaller than it actually is. This effect overwhelms the loss from not winning with a lower bid and it is known that no strictly increasing equilibrium exists (Krishna, 2002, Chapter 4). In fact, a stronger result holds: if the losing bid is announced, there is no nondecreasing equilibrium with (partial) pooling either.

Remark 2 It can be shown that the equilibrium constructed in Theorem 1 is, in fact, the only equilibrium in which bidders follow nondecreasing bidding strategies. This is established in the supplement to this paper.

Remark 3 It may be verified that the strong bidder's strategy does not constitute a strict best-response. In particular, the strong bidder is indifferent between his equilibrium bid and bidding 0 and buying (or attempting to buy) on the resale market.

2.4 An Example

It is useful to consider an explicit example to illustrate the various constructs.

Suppose that $F_s(v) = v/a_s$ over $[0, a_s]$ and $F_w(v) = v/a_w$ over $[0, a_w]$ where $a_s \geq a_w$; that is, the value distributions are both uniform, but over different supports.

It may be verified that with uniformly distributed values, the symmetrizing distribution, F, is also uniform. Specifically, $F(v) = 2v/(a_s + a_w)$ over support $\left[0, \frac{1}{2}(a_s + a_w)\right]$. The associated pricing function $p(b) = 2b$. The equilibrium inverse bidding strategies are: $\phi_s(b) = 4a_s b/(a_s + a_w)$ and $\phi_w(b) = 4a_w b/(a_s + a_w)$. The highest bid $\bar{b} = \frac{1}{4}(a_s + a_w)$.

Notice that if $3a_w < a_s$, then $\phi_w(b) < b$, or equivalently, $\beta_w(v) > v$; that is, bidder w bids more than his value in a first-price auction with resale. The reason, of course, is that he anticipates being able to resell the object to bidder s for a profit. Thus, bidder w's motives have a substantial "speculative" component. The model of Garratt and Tröger (2006), in which the weak bidder is known to have a value of 0, is an extreme instance of this. Since the weak bidder derives no value from object himself, he is driven purely by speculative motives.

3 Second-Price Auction with Resale

We now study properties of the second-price auction with resale. Our model is the same as that in previous sections except for the change in the auction format—that is, there is a second-price auction and then the winner, if he so wishes, can resell the object to the other bidder via a take-it-or-leave-it offer. There is one important difference, however. Under second-price rules, the winner of the auction inevitably knows the *losing* bid—after all this is the price he pays in the auction. Thus, unlike in a first-price auction, the winner can condition the price offered in the resale stage on the losing bid.[11] This, of course, considerably simplifies the inference problem faced by a winning bidder and puts the losing bidder in a weak position during resale.

Resale Stage Suppose bidder i follows a nondecreasing bidding strategy β_i in the auction. Suppose also that bidder j wins the auction and pays a price of b_i which is in the range of β_i; that is, i's bid. He then infers that bidder i's value is in the set $\beta_i^{-1}(b_i) = \{v : \beta_i(v) = b_i\}$. If $\beta_i^{-1}(b_i)$ is a singleton, say $\beta_i^{-1}(b_i) = \{v_i\}$, then it is optimal for j to offer the object to i only if $v_j < v_i$ and in that case, set a price $p = v_i$.

[11] Recall that a first-price auction with resale does not have a monotonic equilibrium if the losing bid is known to the winner. See Remark 1.

Bidding Stage With private values, a standard second-price auction—without the possibility of resale—has some important and well-known features. First, it is a *weakly dominant* strategy for each bidder to bid his true value. Second, the resulting equilibrium is, of course, *efficient*, even in an asymmetric environment. Third, there is a continuum of other (inefficient) equilibria (see Andreas Blume and Paul Heidhues (2004) for a complete classification).

Our first observation is that once there is the possibility of resale, it is *not* a weakly dominant strategy to bid one's value in a second-price auction. As the example below shows, if one of the bidders, say s, bids more than his value, the other bidder may gain by bidding less than his value. This is because a lower bid in the auction may lead to a lower resale price.

Example 1 *The values $V_s, V_w \in [0,1]$. Suppose that bidder s bids according to a continuous and strictly increasing strategy β_s such that $\beta_s(v) > v$, for all $v \in (0,1)$, and, if he wins, has beliefs $\mu_s(b_w, v) = 1$ if $v = b_w$ and 0 otherwise; that is, s believes that w is following the strategy $\beta_w(v_w) = v_w$.*

Suppose that bidder s has value $v_s \in (0,1)$ and bidder w's value v_w is such that $v_s < v_w < \beta_s(v_s)$. If bidder w bids v_w, then s will win the auction and will offer the object to w at price $p = v_w$. So bidder w's payoff from bidding his value is 0. If bidder w reduces his bid to a b_w such that $v_s < b_w < v_w$, then again bidder s will win the auction but now offer to sell the object to w at price $p = b_w$. By accepting this offer, bidder w can make a profit of $v_w - b_w$. Thus in this situation it is strictly better for bidder w to bid $b_w < v_w$ than to bid v_w.

Robust Equilibrium While not weakly dominant, if both bidders bid their values and the winner prices optimally, then this nevertheless results in an equilibrium of the second-price auction with resale. Of course, this results in an efficient allocation and so, in equilibrium, there is no resale.

Proposition 2 *There is an equilibrium of the second-price auction with resale in which both bidders bid their values.*

Proof. Consider the following strategies. In the auction, each bidder bids his value; that is, $\beta_i(v_i) = v_i$. After the auction, the winner i believes that j's value $V_j = b_j$ and offers to sell at a price $p_i = b_j$ if and only if $b_j > v_i$; the loser responds optimally to the price offer, if any.

Suppose bidder i follows the strategy outlined above.

Suppose bidder j deviates and bids $b < v_j$. If $v_i < b < v_j$, bidder j wins for a price of v_i and so there is no resale. So his payoff is $v_j - v_i$ which is the same as if he bid v_j. If $b < v_i < v_j$, then i wins and j's payoff is zero since again there is no offer of resale. So if $b < v_i < v_j$, j's payoff is zero if he bids v and $v_j - v_i$ if he bids v_j. Finally, if $b < v_j < v_i$, bidder j loses the auction and his payoff is 0 whether he bids b or v_j. Thus underbidding is not profitable.

Now suppose bidder j bids $b > v_j$. If $v_i > b$, then j's payoff is 0 since he loses and i will not resell to him. If $b > v_i > v_j$, then again his payoff is zero, because he will pay v_i for the object and then resell to i for V_i. If $b > v_j > v_i$, then it makes no difference whether he bids b or v_j. Thus overbidding is not profitable either.

We have thus argued it is a best response for bidder j to follow the strategy $\beta_j(v) = v$, also. The optimality of the proposed strategies in the resale stage is clear.

Remark 4 The "bid-your-value" strategies constitute not only an equilibrium but, in fact, one that is robust—that is, the proposed strategies constitute a perfect Bayesian for all distributions F_s, F_w of values that are strictly increasing and continuous. This is easily verified since the proof of Proposition 2 did not make use of the distributions. It can also be argued that the "bid-your-value" equilibrium is the unique robust equilibrium. This last result is proved in the supplement to this paper.[12]

4 Revenue Comparison

Recall that a *first-price auction with resale* (FPAR) in which bidders draw values from F_s and F_w, respectively, has the same bid distribution as a symmetric first-price auction in which both bidders draw values from F. This is because if $\phi(\cdot)$ is the equilibrium inverse bidding strategy in the auxiliary auction, then for all $b \in \left[0, \overline{b}\right]$, $F(\phi(b)) = F_j\left(\phi_j(b)\right)$.

Hence, in equilibrium, the expected revenue accruing to the auctioneer from a first-price auction with resale (FPAR) is

$$
\begin{aligned}
R^{FPAR}(F_s, F_w) &= R^{FPA}(F, F) \\
&= R^{SPA}(F, F) \\
&= \int_0^{\overline{p}} (1 - F(p))^2 \, dp
\end{aligned}
\tag{14}
$$

[12]A paper by Tilman Börgers and Timothy McQuade (2006) develops the correct notion of a "robust" equilibrium in multi-stage games such as ours.

where F is defined in (9) and $R^{SPA}(F, F)$ denotes the revenue from a symmetric second-price auction (SPA). The second equality is a consequence of the revenue equivalence principle. The third equality is a well-known formula for the expectation of the minimum of two independent random variables, both of which are distributed according to F.

In a *second-price auction with resale* (SPAR), the expected revenue from the efficient equilibrium is

$$\begin{aligned} &R^{SPAR}(F_s, F_w) \\ &= \int_0^{a_w} (1 - F_s(v))(1 - F_w(v))\, dv \end{aligned} \tag{15}$$

The right-hand side of the formula above is simply $E[\min\{V_s, V_w\}]$.

Example For the asymmetric uniform distributions as in Section 2.4, the expected revenue in a first-price auction with resale is

$$R^{FPAR} = \tfrac{1}{6}(a_s + a_w)$$

whereas the expected revenue in a second-price auction with resale is

$$R^{SPAR} = \frac{a_w(3a_s - a_w)}{6a_s}$$

The difference between the two

$$R^{FPAR} - R^{SPAR} = \frac{(a_s - a_w)^2}{6a_s}$$

and this is positive as long as $a_s > a_w$.

We now show that the revenue superiority of the first-price auction with resale over its second-price counterpart is general. Our main result is

Theorem 2 *The seller's revenue from a first-price auction with resale is at least as great as that from a second-price auction with resale.*

The proof of Theorem 2, which appears in the appendix, proceeds as follows. For a fixed distribution of the strong bidder, F_s, consider the difference in the revenues between the two auctions:

$$\Delta = R^{FPAR}(F_s, F_w) - R^{SPAR}(F_s, F_w)$$

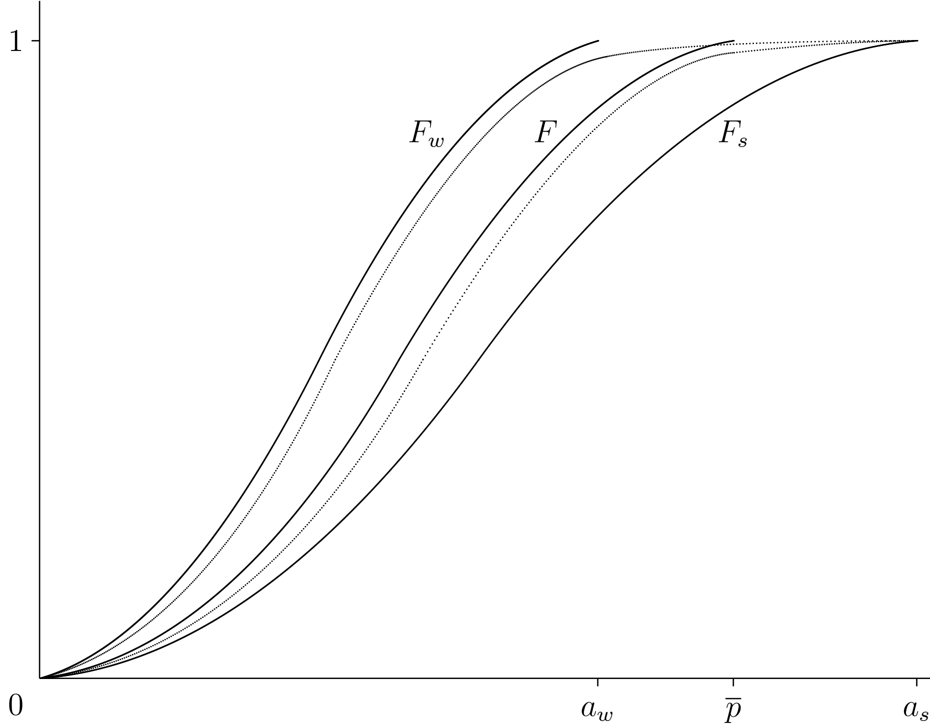

Figure 2: Perturbation in the Direction of Symmetry

When $F_w = F_s$, this difference is zero because in that case $F_w = F_s = F$ also and then (14) and (15) are identical, so that $\Delta = 0$ (This also follows from the revenue equivalence principle.) Now consider an ε-perturbation of F_w in the direction of F_s. As depicted in Figure 2, such a perturbation affects F also, bringing the situation closer to one with symmetric bidders. The proof shows that such a perturbation must *decrease* Δ as long as we are not in a symmetric situation. Since in the symmetric situation, the value of Δ is zero, it must be positive whenever F_w is not the same as F_s. Figure 3 is a schematic depiction of this claim.

The calculation of the derivative of Δ with respect to ε (the perturbation) is carried out using a simple technique from the calculus of variations (this is also discussed in the appendix) and uses the regularity condition on F_s.

5 Other Resale Mechanisms

In our analysis of asymmetric auctions with resale, we assumed that post-auction trade took place via a take-it-or-leave-it offer from the winner of the auction. This mechanism—henceforth referred to as the *monopoly* mechanism—is salient in that it places all bargaining power in the hands of the seller and so is, of course, optimal from his perspective. But

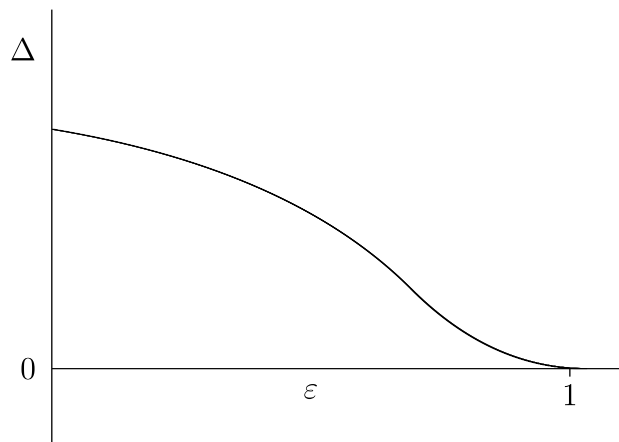

Figure 3: Revenue Difference

one may wonder whether our results are robust to changes in the way resale takes place. For instance, what if resale takes place via a take-it-or-leave-it offer from the loser of the auction—that is, via the *monopsony* mechanism? More generally, what if bargaining power is shared between the buyer and the seller, perhaps unequally?

In this section, we analyze the robustness of our results in this direction.

Recall that in our basic model, at the end of the auction, only the winning bid was announced. When resale takes place via the monopoly mechanism, this is the same as if no bid were announced. This is because information about the winner's bid is irrelevant to the other bidder—he faces a take-it-or-leave-it offer from the winner. In other resale mechanisms, say when the buyer makes a take-it-or leave-it offer, information regarding the winner's value is no longer irrelevant—for instance, if the winning bid revealed the winner's value, then the loser could extract all surplus from the winner during resale. It can be shown that if the winning bid is announced when resale takes place via monopsony, then there is no monotonic equilibrium.

In the extensions of the basic model that follow, we assume that *no bids are announced* at the end of the auction. Thus only the identity of the winner is commonly known.

We first observe that the symmetrization result of Section 3, Proposition 1, can be generalized to include a large class of resale mechanisms.[13]

[13]We developed this result following a suggestion of John Riley.

5.1 Symmetrization Redux

There is an almost unlimited variety of possible trading mechanisms for resale and a separate analysis of each one would be tedious. It is more fruitful, instead, to posit a mechanism in the abstract, and then to identify some common features that lead to symmetrization. Specifically, a trading mechanism may be thought of as consisting of two functions q and t that specify, for any pair of values v_i and v_j, (i) the probability $q(v_i, v_j)$ that i buys the object from j; and (ii) the transfer $t(v_i, v_j)$ from i to j if, in fact, a transaction takes place. We suppose that the mechanism is incentive compatible, that is, both parties want to report their true values, and also individually rational, that is, both parties wish to participate.[14]

The monopoly resale mechanism, considered above, can, of course, be accommodated in this framework. In that case, the probability of sale $q(v_i, v_j) = 1$, if $v_i \geq p\left(\beta_j(v_j)\right)$; that is, if i's value exceeds j's offer and $q(v_i, v_j) = 0$, otherwise. The transfer $t(v_i, v_j) = p\left(\beta_j(v_j)\right)$, is, naturally, just the monopoly price set by j.

Now consider the following situation: a first-price auction is conducted and then resale takes place via the mechanism (q, t). Suppose that there is an equilibrium of the two-stage game in which the bidding strategies $\left(\phi_i, \phi_j\right)$ are continuous and increasing. We know that in this equilibrium, both players will report their values truthfully at the resale stage.

Is it the case that for an *arbitrary* resale mechanism, the distribution of bids in equilibrium is identical? The answer is clearly no since if $(q, t) = (0, 0)$, that is, no trade/transfer ever takes place, then clearly symmetrization does not obtain.

Suppose, however, that the mechanism has the property that if the seller's value is the *lowest* possible and the buyer's value is the *highest* possible, then trade is sure to take place. We call this the *sure-trade* property.

Now consider a bid b such that $v_j = \phi_j(b) < \phi_i(b) = v_i$. If j wins with a bid of b then he knows that i's value is at most $\phi_i(b)$. Similarly, if i loses with a bid of b then he knows that j's value is at least $\phi_j(b)$. The sure-trade property requires that $q\left(\phi_i(b), \phi_j(b)\right) = 1$. In other words, if the values of the two parties are such that they bid the same amount b in the auction, then trade takes place with probability one. Of course, the monopoly mechanism has this property since if bidder j wins the object, then the price $p(b)$ he will set will be strictly less than $\phi_i(b)$ and so this offer will be accepted if i's value is $\phi_i(b)$.

We now argue that for any mechanism satisfying the sure-trade property, a necessary condition for equilibrium is that the bid distributions be symmetric.

To see this, suppose that the bid b is such that $\phi_j(b) < \phi_i(b)$.

[14] We have specified the resale mechanism in a direct form, that is, the outcome depends on the values only. Roger Myerson and Mark Satterthwaite (1983) show, via the so-called revelation principle, how any equilibrium of any trading mechanism can be formulated in this way.

Consider bidder j with a value $v_j = \phi_j(b)$. Consider what happens if bidder j were to deviate from equilibrium behavior by bidding $c \neq b$ during the auction but to report truthfully at the resale stage. Suppose further that c is close enough to b so that $\phi_j(c) < \phi_i(c)$ also. His payoff from doing so is

$$\int_0^{\phi_i(c)} \left[q(v_i, v_j)\, t(v_i, v_j) + (1 - q(v_i, v_j))\, v_j\right] f_i(v_i)\, dv_i$$
$$-F_i(\phi_i(c))\, c$$

(This is analogous to (4) in Section 2.2). Since such a deviation cannot be profitable, the expected payoff must be maximized at b and the first-order condition can be written as (all functions are evaluated at b)

$$\begin{aligned} 0 &= q\left(\phi_i, \phi_j\right) t\left(\phi_i, \phi_j\right) f_i(\phi_i)\, \phi_i' \\ &\quad + \left[1 - q\left(\phi_i, \phi_j\right)\right] v_j f_i(\phi_i)\, \phi_i' \\ &\quad - f_i(\phi_i)\, b\phi_i' - F_i(\phi_i) \\ &= t\left(\phi_i, \phi_j\right) f_i(\phi_i)\, \phi_i' \\ &\quad - f_i(\phi_i)\, b\phi_i' - F_i(\phi_i) \end{aligned}$$

since the sure-trade property ensures that $q\left(\phi_i, \phi_j\right) = 1$. Rearranging this, results in

$$\frac{d}{db} \ln F_i(\phi_i(b)) = \frac{1}{t\left(\phi_i(b), \phi_j(b)\right) - b}$$

which is a generalization of (5).

Now consider bidder i with a value $v_i = \phi_i(b)$. His payoff from bidding c and then behaving truthfully at the resale stage is

$$\begin{aligned} &(v_i - c)\, F_j\left(\phi_j(c)\right) \\ &+ \int_{\phi_j(c)}^{a_j} q(v_i, v_j)\left[v_i - t(v_i, v_j)\right] f_j(v_j)\, dv_j \end{aligned}$$

(This is analogous to (6) in Section 2.2). Again, since it is optimal for i to bid b, the

first-order condition results in (all functions are evaluated at b)

$$\begin{aligned} 0 &= (v_i - b) f_j(\phi_j) \phi_j' - bF_j(\phi_j) \\ &\quad -q(\phi_i, \phi_j) v_i f_j(\phi_j) \phi_j' \\ &\quad +q(\phi_i, \phi_j) t(\phi_i, \phi_j) f_j(\phi_j) \phi_j' \\ &= \left(t(\phi_i, \phi_j) - b\right) f_j(\phi_j) \phi_j' \\ &\quad -bF_j(\phi_j) \end{aligned}$$

by using the sure-trade property again. Rearranging this results in the same differential equation as above:

$$\frac{d}{db} \ln F_j\left(\phi_j(b)\right) = \frac{1}{t\left(\phi_i(b), \phi_j(b)\right) - b}$$

Thus the symmetrization result generalizes to all resale mechanisms with the sure-trade-at-the-margin property.

Proposition 3 *Suppose that resale takes place via a mechanism with the sure-trade property. If ϕ_s and ϕ_w are continuous and strictly increasing inverse bidding strategies associated with an equilibrium, then for all b,*

$$F_s(\phi_s(b)) = F_w(\phi_w(b))$$

that is, the bid distributions of the two bidders are identical.

Remark 5 Notice that the symmetry result above does not directly make use of the regularity assumption.

5.2 Revenue Comparisons

5.2.1 Monopsony

We have shown above that the symmetrization result, first derived in Proposition 1, in fact generalizes for all resale mechanisms with the sure-trade property. We have mentioned that the monopoly mechanism satisfies this assumption. But notice that the so-called monopsony mechanism also satisfies the sure-trade assumption. To see this, suppose bidder j wins the object with a bid of b and $\phi_j(b) < \phi_i(b)$. Under the monopsony mechanism the losing bidder i will make a take-it-or-leave-it offer to j and the price he will offer, say $r(b)$ will be strictly greater than $\phi_j(b)$. This offer is sure to be accepted if j's value is indeed $\phi_j(b)$. Thus if the two values are $\phi_i(b)$ and $\phi_j(b)$, then trade takes place for sure.

Proposition 3 now guarantees that the bid distributions in any increasing equilibrium are symmetric.

The analysis parallels that of the earlier sections except that the monopoly pricing function $p(b)$ is replaced by the monopsony pricing function $r(b)$, which is the solution to

$$\max_{r} \left[F_j(r) - F_j\left(\phi_j(b)\right)\right](\phi_i(b) - r) \tag{16}$$

(where $\phi_j(b) \leq \phi_i(b)$). For both the strong and the weak bidder, a pair of differential equation analogous to (5) and (7) characterizes the equilibrium bidding strategy—it is necessary only to replace $p(b)$ by $r(b)$. Of course, the monopsony price $r(b)$ typically differs from the monopoly price $p(b)$ and so the equilibrium bidding strategies when resale is via monopsony are different from the equilibrium bidding strategies when resale is via monopoly.

All of our other results also extend (details may be found in the supplement). Specifically,

1. As in (9), F_s and F_w uniquely determine a distribution G of monopsony resale prices:

$$G(r) = F_s\left(r - \frac{G(r) - F_w(r)}{f_w(r)}\right)$$

   The distribution G has a geometric interpretation similar to that of F in Figure 1. In general, $G \neq F$.

2. As in Theorem 1, there exists an equilibrium with strictly increasing bidding strategies.

3. As in Theorem 2, the revenue from a first-price auction with monopsony resale is at least as great as that from a second-price auction with resale.

5.2.2 Random Proposer Mechanism

Our results also extend to a class of mechanisms in which the bargaining power is shared, perhaps unequally. Specifically, we consider a mechanism in which with probability k, the seller makes a take-it-or-leave-it offer and with probability $1-k$, the buyer makes a take-it-or-leave-it offer. We refer to this as the *random proposer* mechanism.

The value of k determines the allocation of bargaining power between the seller and the buyer. When $k=1$, this reduces to the monopoly resale mechanism in which all bargaining power lies with the seller. When $k=0$, it reduces to the monopsony mechanism in which all bargaining power lies with the buyer.

The random proposer mechanism also satisfies the sure-trade property. If the values of the buyer and the seller are $\phi_i(b)$ and $\phi_j(b)$, respectively, then trade takes place for sure

regardless of whether the buyer or the seller makes a take-it-or-leave-it offer. So again, by Proposition 3, bid distributions are symmetric.

The analysis of first-price auctions when resale is via the random proposer mechanism also parallels the analysis when resale is via the monopoly mechanism—the monopoly price $p(b)$ now needs to be replaced by the expected price $kp(b) + (1-k)\,r(b)$.

Once again, in general, $p(b) \neq r(b)$ and so for $k \in (0,1)$, the expected price is distinct from both $p(b)$ and $r(b)$. Thus the equilibrium bidding strategies are now different from both those in the case of monopoly resale and those in the case of monopsony resale. This in turn implies that the pricing functions $p(b)$ and $r(b)$ in the case of a random proposer are also different from those resulting in the case of a pure monopoly or a pure monopsony.

It can be shown that the expected revenue from a first-price auction in which resale is via a random proposer is a $k : 1-k$ weighted average of the revenue when resale is via monopoly and when resale is via monopsony.

The following theorem extends our main result.

Theorem 3 *The seller's revenue from a first-price auction with resale via the random proposer mechanism is at least as great as that from a second-price auction.*

Theorem 3 naturally subsumes the two extreme cases of monopoly (when $k = 1$) and of monopsony (when $k = 0$). A proof may be found in the supplement to this paper.

6 Further Extensions

6.1 Interdependent values

We have assumed from the beginning that bidders' evaluation of the object was private. In a more general set up, however, there may be a "common value" component to the values. Specifically, suppose that each bidder receives a private signal x_i regarding the value and that values are interdependent. In other words, the value to bidder i is of the form $v_i(x_i, x_j)$ and is increasing in both signals. Further, suppose that the signals of the two bidders are independently distributed according to distributions F_i and F_j.

It can then be shown that in equilibrium, the bid distributions are symmetric in the same fashion as in Proposition 1. The proof is virtually the same.

The revenue ranking result of Theorem 2, however, does not generalize. When there is a substantial common value component, it may be that the second-price auction with resale is revenue superior to the first-price auction with resale.

6.2 More than two bidders

In this paper, we have restricted attention to the case of two bidders. Considering resale when there are three or more bidders poses additional conceptual and technical difficulties.[15] First, there are many reasonable ways to resell to more than one buyer. For instance, the winner of the auction could hold a second auction—perhaps of a different format—himself. Alternatively, he could post a fixed price and sell the object at random to all buyers who are willing to buy at that price. He could make price offers sequentially to different buyers.

Second, it can be shown that the symmetry result in Proposition 1 does not extend in general to the case of three or more bidders. In one special case, however, the analysis is tractable. Suppose there are two identical weak bidders and only one strong bidder. Resale takes place via a posted price. Then it can be shown that again symmetry obtains as in Proposition 1.

We hope to explore the case of three or more bidders in future work.

7 Conclusion

We have shown that a consideration of resale possibilities allows for a simpler characterization of equilibrium strategies in first-price auctions than available when resale is not admitted. In our model, equilibrium strategies can be explicitly computed in a relatively simple manner as in the proof of Theorem 1. Moreover, we obtain a general revenue ranking result between first- and second-price auctions that is not available in the standard model. Thus this appears to be one of those happy circumstances where complicating the model with a real-world feature—resale—actually simplifies the analysis.

A Appendix

Here we show that the pair of differential equations in (5) for $j = s, w$ are sufficient to characterize equilibrium. In other words, the solution to the differential equations results in equilibrium bidding strategies—no deviations are profitable.

Proposition 4 *The strictly increasing and onto functions* $\phi_s : [0, \overline{b}] \to [0, a_s]$ *and* $\phi_w : [0, \overline{b}] \to [0, a_w]$ *are equilibrium inverse bidding strategies for the first-price auction with*

[15] Zheng (2002) also finds that when there are three or more bidders, Myerson's optimal auction is robust to resale only under stringent conditions (see the paper by Tymofiy Mylovanov and Thomas Tröger, 2005).

resale if and only if for all $b \in \left[0, \bar{b}\right]$,

$$\begin{aligned}
\frac{d}{db} \ln F_s\left(\phi_s\left(b\right)\right) &= \frac{1}{p\left(b\right) - b} \\
\frac{d}{db} \ln F_w\left(\phi_w\left(b\right)\right) &= \frac{1}{p\left(b\right) - b}
\end{aligned}$$

where $p\left(b\right)$ *is the solution to*

$$\max_p [F_s(\phi_s(b)) - F_s(p)]p + F_s(p)\phi_w\left(b\right)$$

Proof. Note that the boundary conditions are determined by the condition that the ϕ_i be strictly increasing and onto.

The necessity of the differential equations has already been shown. It remains to show that these are sufficient.

Suppose bidder j follows the equilibrium inverse bidding strategy ϕ_j. We will argue that when bidder i has a value of v_i, he cannot do better than to bid b such that $\phi_i\left(b\right) = v_i$. We do this by showing that neither underbidding nor overbidding can be profitable.

Notice that the differential equations can be rewritten as: for $j = s, w$ and for all b,

$$(p\left(b\right) - b) f_j(\phi_j(b))\phi_j'(b) - F_j(\phi_j(b)) = 0 \tag{A1}$$

Case 1 (Underbidding): Suppose bidder i bids c such that $\phi_i\left(c\right) < v_i$.

Case 1a: $\phi_j\left(c\right) < \phi_i\left(c\right) < v_i$. If i wins the auction with a bid of c, then his payoff is simply $(v_i - c)$ since there are no benefits to reselling. If i loses, however, j will offer to sell the object to him for a price of $p(\beta_j\left(v_j\right))$ and so i's payoff is $\max\left\{v_i - p(\beta_j\left(v_j\right)), 0\right\}$. Thus i's expected payoff is

$$\Pi_i\left(c, v_i\right) = \left(v_i - c\right) F_j\left(\phi_j\left(c\right)\right) + \int_{\phi_j(c)}^{a_j} \left[v_i - p\left(\beta_j\left(v_j\right)\right)\right]_+ f_j\left(v_j\right) dv_j$$

where $\left[x\right]_+ = \max\left\{x, 0\right\}$. Differentiating with respect to c and using (A1), results in

$$\frac{\partial \Pi_i}{\partial c} = (p\left(c\right) - c) f_j(\phi_j(c))\phi_j'(c) - F_j(\phi_j(c)) = 0$$

Case 1b: $\phi_i\left(c\right) \leq \phi_j\left(c\right) < v_i$. If i wins the auction with a bid of c, then his payoff is simply $(v_i - c)$ since again there are no benefits to reselling. Similarly, if i loses, bidder j will not offer to sell the object to him since from j's perspective, there appear to be no benefits

to selling to i. Thus i's expected payoff is simply

$$\Pi_i(c, v_i) = (v_i - c) F_j\left(\phi_j(c)\right)$$

and so again by using (A1),

$$\begin{aligned} \frac{\partial \Pi_i}{\partial c} &= (v_i - c) f_j(\phi_j(c))\phi_j'(c) - F_j(\phi_j(c)) \\ &> (\phi_j(c) - c) f_j(\phi_j(c))\phi_j'(c) - F_j(\phi_j(c)) \\ &\geq (p(c) - c) f_j(\phi_j(c))\phi_j'(c) - F_j(\phi_j(c)) \\ &= 0 \end{aligned}$$

Case 1c: $\phi_i(c) < v_i \leq \phi_j(c)$. If i wins the auction with a bid of c, then he may resell it to bidder j since again there are potential gains from trade. His expected payoff from winning is

$$R_i(c, v_i) = \max[F_j(\phi_j(c)) - F_j(p)]p + F_j(p)v_i$$

If i loses, bidder j will not offer to sell the object to him since from j's perspective, there appear to be no gains from trade. Thus i's expected payoff from bidding c is

$$\Pi_i(c, v_i) = R_i(c, v_i) - F_j\left(\phi_j(c)\right) c$$

and using the envelope theorem, and the fact that $p_i(c, v_i) > p_i(c, \phi_i(c)) \equiv p(c)$,

$$\begin{aligned} \frac{\partial \Pi_i}{\partial c} &= (p_i(c, v_i) - c) f_j\left(\phi_j(c)\right) \phi_j'(c) - F_j\left(\phi_j(c)\right) \\ &> (p(c) - c) f_j\left(\phi_j(c)\right) \phi_j'(c) - F_j\left(\phi_j(c)\right) \\ &= 0 \end{aligned}$$

Case 2 (Overbidding): Suppose bidder i bids c such that $v_i < \phi_i(c)$.

Case 2a: $\phi_j(c) < v_i < \phi_i(c)$. If i wins the auction with a bid of c, then his payoff is simply $(v_i - c)$ since there is no benefit from reselling to j. On the other hand, if i loses, j will offer to sell the object to him for a price of $p(\beta_j(v_j))$ and so i's payoff if he loses is $\max\{v_i - p(\beta_j(v_j)), 0\}$. Thus i's expected payoff from bidding c is

$$\Pi_i(c, v_i) = (v_i - c) F_j\left(\phi_j(c)\right) + \int_{\phi_j(c)}^{a_j} \left[v_i - p\left(\beta_j(v_j)\right)\right]_+ f_j(v_j)\, dv_j$$

Differentiating with respect to c,

$$\begin{aligned}\frac{\partial \Pi_i}{\partial c} &= (v_i - c) f_j(\phi_j(c))\phi_j'(c) - F_j(\phi_j(c)) - [v_i - p(c)]_+ f_j\left(\phi_j(c)\right)\phi_j'(c) \\ &\leq (v_i - c) f_j(\phi_j(c))\phi_j'(c) - F_j(\phi_j(c)) - [v_i - p(c)] f_j\left(\phi_j(c)\right)\phi_j'(c) \\ &= (p(c) - c) f_j(\phi_j(c))\phi_j'(c) - F_j(\phi_j(c)) \\ &= 0\end{aligned}$$

since $[v_i - p(c)]_+ \geq [v_i - p(c)]$.

Case 2b: $v_i \leq \phi_j(c) < \phi_i(c)$. If i wins the auction with a bid of c, then he may resell it to bidder j since again there are potential gains from trade. If he loses, bidder j will offer to sell the object to him for a price of $p\left(\beta_j(v_j)\right)$ but this price will always exceed v_i and so i will refuse the offer. Thus i's expected payoff from bidding c is just

$$\Pi_i(c, v_i) = R_i(c, v_i) - F_j\left(\phi_j(c)\right) c$$

and again using the envelope theorem and the fact that $p_i(c, v_i) \leq \phi_j(c) < p_j\left(c, \phi_j(c)\right) \equiv p(c)$,

$$\begin{aligned}\frac{\partial \Pi_i}{\partial c} &= (p_i(c, v_i) - c) f_j\left(\phi_j(c)\right)\phi_j'(c) - F_j\left(\phi_j(c)\right) \\ &< (p(c) - c) f_j\left(\phi_j(c)\right)\phi_j'(c) - F_j\left(\phi_j(c)\right) \\ &= 0\end{aligned}$$

Case 2c: $v_i < \phi_i(c) \leq \phi_j(c)$. If i wins the auction with a bid of c, then he may resell it to bidder j since again there are potential gains from trade. His expected payoff from winning is the monopoly profit $R_i(c, v_i)$. If he loses, bidder j will not offer to sell the object to him since from j's perspective, there appear to be no gains from trade. Thus i's expected payoff from bidding c is again

$$\Pi_i(c, v_i) = R_i(c, v_i) - F_j\left(\phi_j(c)\right) c$$

and the argument is the same as in Case 2b, except that now $p_i(c, v_i) < p_i(c, \phi_i(c)) \equiv p(c)$.

We have thus argued that for all c such that $\phi_i(c) < v_i$, $\frac{\partial \Pi_i}{\partial c} \geq 0$ and for all c such that $\phi_i(c) > v_i$, $\frac{\partial \Pi_i}{\partial c} \leq 0$. Thus bidding a b such that $\phi_i(b) = v_i$ is a best response to ϕ_j.

B Appendix

This appendix contains the proof of Theorem 2 which ranks the first- and second-price auctions with resale in terms of expected revenue. Before proceeding with the proof, some preliminaries are in order.

Calculus of Variations In what follows, we will make use of a simple technique from the calculus of variations that is used to derive the *Euler equation.* (See, for instance, Section 3 in Kamien and Schwartz, 1981).

Consider the integral

$$\Delta = \int_0^a \Phi\left(p, M\left(p\right), m\left(p\right)\right) dp$$

where $M : [0, a] \to \mathbb{R}$ and $m(p) = M'(p)$. Suppose $Z(p) : [0, a] \to \mathbb{R}$ is a *variation* satisfying $Z(0) = Z(a) = 0$ and let $z(p) \equiv Z'(p)$. Define

$$\Delta(\varepsilon) = \int_0^a \Phi\left(p, M + \varepsilon Z, m + \varepsilon z\right) dp$$

to be the value of the integral when M is perturbed by εZ. Differentiating with respect to ε,

$$\Delta'(\varepsilon) = \int_0^a \left[\Phi_M Z + \Phi_m z\right] dp$$

where $\Phi_M \equiv \partial\Phi/\partial M$ and $\Phi_m \equiv \partial\Phi/\partial m$. Integrating by parts,

$$\begin{aligned} \int_0^a \Phi_m z dp &= \Phi_m Z\Big|_0^a - \int_0^a \frac{d}{dp}\left(\Phi_m\right) Z dp \\ &= -\int_0^a \frac{d}{dp}\left(\Phi_m\right) Z dp \end{aligned}$$

since $Z(0) = Z(a) = 0$.

Thus we obtain,

(B1)
$$\Delta'(\varepsilon) = \int_0^a \left[\Phi_M - \frac{d}{dp}\left(\Phi_m\right)\right] Z dp$$

where both Φ_M and $\frac{d}{dp}(\Phi_m)$ are evaluated at $(p, M + \varepsilon Z, m + \varepsilon z)$.

Notation In the proof below it is convenient to reformulate the problem in terms of the *decumulative* distribution functions[16]: for $p \in [0, a_s]$, $H_s(p) \equiv 1 - F_s(p)$, $H_w(p) \equiv 1 - F_w(p)$

[16] As usual, if $p > a_w$, $H_w(p) = 0$ and similarly, if $p > \overline{p}$, $H(p) = 0$.

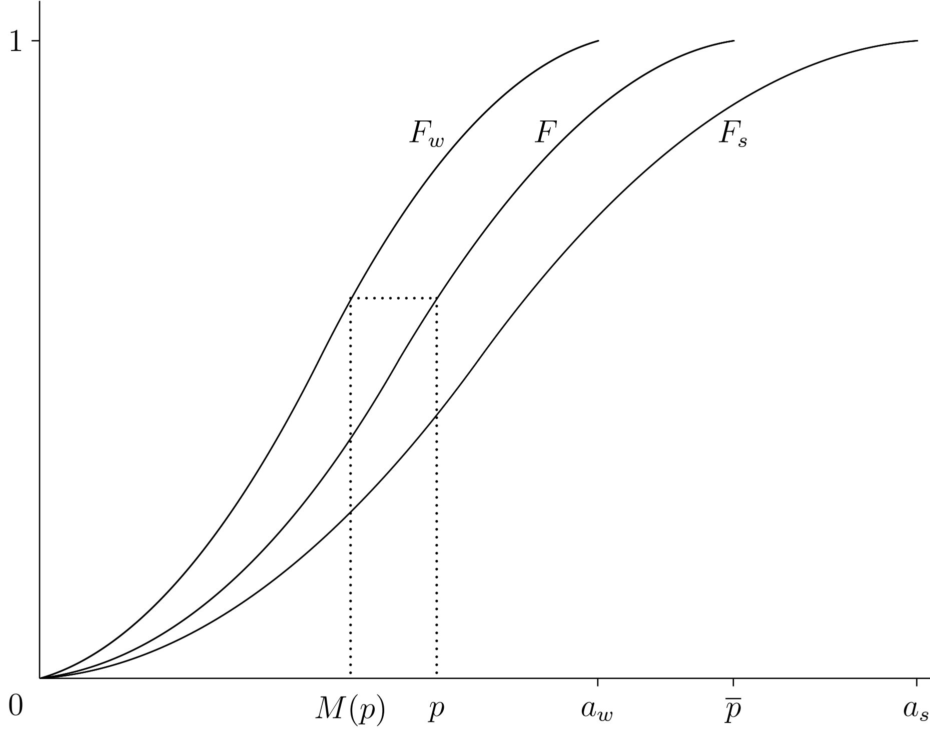

Figure 4: The Function M

and $H(p) \equiv 1 - F(p)$. Also, let $h_s(p) = H_s'(p)$, $h_w(p) = H_w'(p)$ and $h(p) = H'(p)$. Notice that in terms of decumulative functions, equation (9) in the body of the paper can be rewritten as:

$$H(p) = H_w\left(p - \frac{H(p) - H_s(p)}{h_s(p)}\right)$$

or equivalently,

$$H(p) = H_s(p) + \left[p - H_w^{-1}H(p)\right] h_s(p) \tag{B2}$$

The regularity assumption; that is, the monotonicity of virtual values $p + (H_s(p)/h_s(p))$, is equivalent to

$$2h_s(p)^2 - H_s(p)\, h_s'(p) > 0 \tag{B3}$$

Proof of Theorem 2. In terms of the decumulative functions, the difference between the revenue from a FPAR and the revenue from a SPAR can be written as

$$\Delta = \int_0^{a_s} H(p)^2\, dp - \int_0^{a_s} H_s(v)\, H_w(v)\, dv \tag{B4}$$

(See equations (14) and (15) in the body of the paper.)

Define the function $M : [0, a_s] \to [0, a_s]$ by

$$\begin{aligned} M(p) &= p - \frac{F(p) - F_s(p)}{f_s(p)} \\ &= p - \frac{H(p) - H_s(p)}{h_s(p)} \end{aligned}$$

to be the value of bidder w for which he would set a monopoly price of p. It follows from the definition of F (see (9) in the body of the paper) that for $p \in [0, \overline{p}]$, $M(p) = F_w^{-1} F(p)$ (see Figure 4). M is an increasing function satisfying $M(0) = 0$, $M(a_s) = a_s$ and for all p, $M(p) \leq p$. Let $m(p) = M'(p)$.

By changing the variable of integration from v to $p = M^{-1}(v)$ in the second integral in (B4), we obtain

$$\Delta = \int_0^{a_s} \left[H(p)^2 - H_s(M(p)) H(p) m(p) \right] dp$$

where $H(p)$ is given by

$$H(p) = H_s(p) + [p - M(p)] h_s(p) \tag{B5}$$

Note that $H_s = H_w$ if and only if $M(p) = p$ for all p.

Consider the integrand in the expression for Δ, that is, the function[17]

$$\Phi(p, M, m) = H^2 - H_s(M) Hm$$

and the perturbation $Z(p) \equiv p - M(p) \geq 0$. Then define

$$\Delta(\varepsilon) = \int_0^{a_s} \Phi(p, M + \varepsilon Z, m + \varepsilon z) dp$$

to be the revenue difference when M is perturbed by ε in the direction of p. Figure 2 shows how an ε-perturbation of M in the direction of p moves both F_w and F closer to F_s; that is, in the direction of increased symmetry. Notice that perturbing M leaves F_s unchanged.

We now use (B1) to evaluate $\Delta'(0)$. For this we need

$$\Phi_M = 2H \frac{\partial H}{\partial M} - h_s(M) Hm - H_s(M) \frac{\partial H}{\partial M} m$$

[17] To ease the notational burden, we suppress the argument p for the remainder of the proof.

and since $\partial H/\partial M = -h_s$, derived from (B5), we have

$$\Phi_M = -2Hh_s - h_s(M)\,Hm + H_s(M)\,h_s m \tag{B6}$$

And since,

$$\Phi_m = -H_s(M)\,H$$

we have

$$\frac{d}{dp}(\Phi_m) = -h_s(M)\,Hm - H_s(M)\,h \tag{B7}$$

Now (B6) and (B7) result in

$$\begin{aligned}
\Phi_M - \frac{d}{dp}(\Phi_m) &= -2Hh_s + H_s(M)\,h_s m + H_s(M)\,h \\
&= -2Hh_s + H_s(M)\,h_s m + H_s(M)\left[2h_s + [p-M]\,h_s' - h_s m\right] \\
&= 2\left[H_s(M) - H\right]h_s + H_s(M)\,[p-M]\,h_s' \\
&= H_s(M)\left(2\left[1 - \frac{H}{H_s(M)}\right]h_s - \left[\frac{H_s - H}{h_s}\right]h_s'\right)
\end{aligned}$$

where the second equality is obtained by substituting $h = 2h_s + [p-M]\,h_s' - h_s m$, which is derived from (B5). The fourth equality is obtained by substituting $[p-M] = -(H_s - H)/h_s$, again from (B5).

But since H_s is decreasing and $M(p) \leq p$, we have $H_s(M(p)) \geq H_s(p)$ and using the fact that $h_s < 0$,

$$\begin{aligned}
\Phi_M - \frac{d}{dp}(\Phi_m) &\leq H_s(M)\left(2\left[1 - \frac{H}{H_s}\right]h_s - \left[\frac{H - H_s}{h_s}\right]h_s'\right) \\
&= H_s(M)\left[\frac{H_s - H}{H_s h_s}\right]\left(2h_s^2 - H_s h_s'\right)
\end{aligned}$$

Since F_s is regular, $2h_s^2 - H_s h_s' > 0$, as in (B3). Together with $H_s > H$ and $h_s < 0$, this implies

$$\Phi_M - \frac{d}{dp}(\Phi_m) < 0$$

whenever $M(p) < p$.

If $M(p) < p$ for all p, then $Z(p) = p - M(p) > 0$, and so (B1) now implies

$$\Delta'(0) = \int_0^{a_s}\left[\Phi_M - \frac{d}{dp}(\Phi_m)\right][p-M]\,dp < 0$$

Now notice that *exactly* the same argument can be replicated at any $\varepsilon > 0$. Evaluating the integral at any $\overline{M} = (1-\varepsilon) M + \varepsilon p$ and $\overline{m} = (1-\varepsilon) m + \varepsilon$, shows that for all ε such that $0 < \varepsilon < 1$,

$$\Delta'(\varepsilon) = \int_0^{a_s} \left[\Phi_M - \frac{d}{dp} (\Phi_m) \right] \left[p - \overline{M} \right] dp < 0$$

where the terms in the first square bracket are all evaluated at $\left(p, \overline{M}, \overline{m}\right)$. Recall that a change from M to $\overline{M}$ leaves H_s unchanged.

We have thus shown that $\Delta(\varepsilon)$ is decreasing for all $\varepsilon \in (0,1)$ and so is minimized at $\varepsilon = 1$. But when $\varepsilon = 1$, $M(p) = p$ and so $H_s = H_w = H$ and in turn $F_s = F_w = F$. In that case, the situation is symmetric and (B4) implies that $\Delta(1) = 0$ (this also follows from the revenue equivalence principle). We have thus shown that for all regular F_s and all F_w, $\Delta \geq 0$.

This completes the proof of Theorem 2.

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