1968

A minimum-maximum principle for a class of non-linear integral equations

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A MINIMUM-MAXIMUM PRINCIPLE FOR A CLASS
OF NON-LINEAR INTEGRAL EQUATIONS

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Report 68-26

August, 1968

* This research was supported by NSF GP-7662

1. <u>Introduction</u>• In this paper we treat the non-linear integral equation considered previously in [2], namely,

(1.1) $y(x) = f\ K(x,t)y(t)(P(t) + F(y^2(t),t)dt.$

The hypotheses are as follows. 0 is assumed to.be a bounded region in some Euclidean space. The kernel K is assumed to be real valued, symmetric and measurable on *Q* X 0. Moreover it is assumed that for a certain pair p,q of conjugate indices,

(1.2) $1 < q £ 2 \wedge p, \qquad p"^1 + q"^1 = 1,$

the operator A defined by

(1.3) $[Au](x) = f\ K(x,t)u(t)dt,$

is completely continuous from $IJ^q(0)$ to *lP{£X)* , and positive definite on $IJ^q(0)$ in the sense that,

(1.4) $J\ K(x,t)u(x)u(t)dxdt > o, \qquad u\ e\ L^q(O)\backslash\{O\}$.

We suppose that P(x) is non-negative and that

(1.5) $P\ €\ L^r(ft), \qquad r = p/(p-2).$

The function F is defined on $\overline{R}_+$ X ft and is assumed to satisfy i) the Carathéodory hypothesis: F(*,x) is continuous on $\overline{R}_+$ for almost all x € ft and F(rj, •) is measurable for all *r) e* $\overline{R}_+$, ii) there exists a positive constant 3 > 0 such that for almost all (fixed) x *e Cl,* (1.6) $0 < \eta_1^{-\epsilon} F(\eta_1, x) \leq \eta_2^{-\epsilon} F(\eta_2, x), \qquad 0 < \eta_1 < \eta_2$,

iii) there are positive constants c, γ and there is a non-negative function σ,

(1.7) $$\sigma \in L^r(\Omega), \qquad r = p/(p-2),$$

such that for almost all (fixed) $x \in \Omega$,

(1.8) $F(\eta, x) \leq c\eta^\gamma + \sigma(x), \qquad 0 \leq \eta$.

where

(1.9) $$0 < 2\gamma \leq p-2.$$

A sufficient condition for the complete continuity of A, defined by (1.3), is,

(1.10) $$\iint_{\Omega\Omega} |K(x,t)|^p dt < \infty ,$$

or

(1.11) $$\operatorname*{ess\,sup}_{x\in\Omega} \int_\Omega |K(x,t)|^a dt < \infty, \quad \text{for some} \quad a > p/2 ;$$

see[4]. When $A : L^q(\Omega) \to L^p(\Omega)$ is completely continuous a necessary and sufficient condition for (1.4) is that $A|L^2(\Omega)$ be positive definite and that the range of A be dense in $L^p(\Omega)$; see [2].

The results of this paper are obtained by employing techniques developed in the series of papers [6-9]. In a previous paper, [2], we adapted the methods of [9], which treats the case of (1.1) in which K is a continuous kernel, to treat the case where K is unbounded. The existence theorems of [9] and [2] assert the existence of just a single solution of (1.1). However it is shown in [8] that for P and F continuous, P positive, and F satisfying (1.6), the boundary value problem

$$(1.12) \qquad y'' + P(x)y + yF(y^2,x) = 0\,, \qquad y(0) = y(1) = 0,$$

has infinitely many solutions. In this paper we prove the existence of an infinity of solutions of the integral equation (1.1). For this purpose we have found it necessary to make use of the methods of the Lusternik-Schnirelman theory, however we do not use the so-called Lusternik-Schnirelman category but rather the closely related concept of "genus" which is used in [5].

One question which we have left unanswered concerns the relation between the characteristic numbers for (1.12) defined by the methods of this paper and those defined by Nehari in [8]. When we specialize our results to (1.12) then it can be shown that the $n^{\underline{th}}$ characteristic number λ_n for (1.12), as defined in [8], is not less than the $n^{\underline{th}}$ characteristic number Λ_n, defined by the methods of this paper. It is trivial that $\lambda_1 = \Lambda_1$ in all cases, and for $n > 1$ we will have $\lambda_n = \Lambda_n$, if, to within a factor of -1, (1.12) has at most one solution with precisely $n-1$ zeros in $(0,1)$; conditions for this to be the case are given in [1]. Whether $\lambda_n = \Lambda_n$ in general remains an open question. Some support for conjecturing that the answer is affirmative lies in the fact that both methods of definition lead to certain stability properties of the characteristic numbers. These we hope to study in subsequent work.

We remark finally that our results can be applied to a class of non-linear elliptic boundary value problems. See [2] and [3].

2. Statement of results. In this section we introduce the basic definitions and formulate the main results. It is assumed throughout that the hypotheses stated in the introduction hold. For simplicity of notation we shall omit the range of integration from formulas involving integrals unless that range is other than tt, also L^P, L^q will always mean $L^p(0)$, $L^q(0)$.

If X is a Banach space we shall denote by £(X) the class of subsets of x\{0} which are symmetric through the origin. For B *e* £(X), we define the genus of B, p(B)*9* to be the supremum of the set of non-negative integers n such that every odd continuous map of X\{0} into R^{n-1} has a zero on B; here we understand R^{o} = *{0}* * R^{-1} -- *f**

A function y e *iP* will be said to be admissible if

(2.1) $\quad y(x) = JK(x,t)u(t)dt,\quad$ where u *e* L^q\{0},

and

(2.2) $\quad J\ y^2(x)\ (P(x)+F(y^2(x),x))dx \wedge J\ \ y(x)u(x)dx.$

An admissible set will be a subset of L^P consisting of admissible elements. (By (2.2) and Lemmas 1 and A.I below a bounded admissible set has compact closure). The class of all compact symmetric admissible sets will be denoted by (Q. Finally, for each non-negative integer m, we put,

$$B_m = \{B\ e\ B\ |\ p(B) \wedge m).$$

Let the functional H(y) be defined in *lP* by

$$\mathbf{H(y) = J[y^2(x)F(y^2(x),x)-G(y^2(x),x]dx,}$$

where G(rj,x) has the same domain of definition as F(?7; x) and is given by,

$$G(rj,x) = \int_{o} F(s,x)ds.$$

The characteristic values of the problem (1.1) are the numbers

$$A_m = \inf_{B\in B_m} \max_{y\in B} H(y).$$

Theorem 1. For each positive integer m the class *H* is non-empty. The characteristic values $\{A_m\}$ form a non-decreasing sequence of non-negative real numbers and

$$\lim_{m\to\infty} A_m = \infty .$$

Let $A_1 \le A_2 \le \cdots$ be the eigenvalues of the linear integral equation

(2.3) y(x) = A[K(k,t)P(t)y(t)dt,

then for a given integer m > 1, A > 0 if and only if A > 1.

In this assertion and throughout the paper we follow the convention that v(x) is an eigenfunction of (2.3) corresponding to the eigenvalue +<> if P(x)v(x) = 0a.e. on Q. In particular $A_1 = A_2 = \cdots = +\infty$ if P(x) = 0 a.e- on 0. Let m be a positive integer such that $A_m > A_{m-1}$, if m > l. If £ > 1, and if

$$A_m = \backslash n+<t-l < \backslash +l \ '$$

then we shall say that A_m has multiplicity *I.* The set of X^p-solutions y of (1.1) which satisfy

$$H(y) = \lambda_m$$

will be denoted by E_m.

Theorem 2. *For each* $m = 1, 2, \ldots,$ E_m *is a symmetric compact subset of* L^p. *If* $\Lambda_m > 1$ *then* E_m *is not empty and*

$$\rho(E_m) \geq \text{multiplicity of } \lambda_m.$$

3. With the exception of the theory of the genus, the basic machinery for the proof of Theorems 1 and 2 will be developed in the following sequence of lemmas.

Lemma 1. The functionals $\int y^2(x)P(x)dx$, $\int y^2(x)F(y^2(x),x)dx$, $\int G(y^2(x),x)dx$, and $H(y)$ are continuous on L^p and bounded on bounded subsets of L^p.

Proof. See Lemma 1 and its proof in [2].

If $y \in L^p$, and y admits representation in the form (2.1), i.e. if y is in the range of A, then, because of (1.4), u in (2.1) is uniquely determined by y. Accordingly, when $y \in AL^q$ we shall write $\int y(x)u(x)dx$ always with the understanding that y and u are related by (2.1). When $u \neq 0$ then, by (1.4), $\int y(x)u(x)dx > 0$. For y given by (2.1), $y \neq 0$, we define real valued functions $\varkappa, \varkappa_1$ as follows

(3.1) $$\varkappa(y) = \int y^2(x)P(x)dx \Big/ \int y(x)u(x)dx,$$

(3.2) $$\varkappa_1(y) = \int y^2(x)(P(x)+F(y^2(x),x)dx \Big/ \int y(x)u(x)dx;$$

$\varkappa A$ and $\varkappa_1 A$ will be denoted respectively by $\bar{\varkappa}$ and $\bar{\varkappa}_1$. We observe that $\bar{\varkappa}$ and $\bar{\varkappa}_1$ are continuous on $L^q \setminus \{0\}$.

Lemma 2. For $y \in AL^q$, $y \neq 0$, if

(3.3) $$\varkappa(y) < 1$$

then there exists a unique positive $\alpha = \alpha(y)$ such that

(3.4) $$\varkappa_1(\alpha y) = 1.$$

The function $\bar{a} = {}_a A$ JLs. continuous on the open subset of $L^q \setminus \{0\}$ determined by $\bar{x}(u) < 1$.

Proof. Since $y / 0$ in L^, it follows from (1.6) that $\int y^2(x)F(a^2y^2(x),x)dx$ is strictly increasing as a function of a and tends to ∞ as $a \to \infty$ and to 0 as $a \to 0$. It follows that $x_1(ay)$ is a strictly increasing function of a and that $x_n(ay) \to \infty$ as $a \to \infty$ and $x_n(ay) \to x(y)$ as $a \to 0$. Thus, provided (3.3) holds, there exists a unique a corresponding to y such that (3.4) holds. The fact that the subset of L^q {0) determined by $x(u) < 1$ is open and the continuity of a A on that set follow from the continuity of x and x_1 •

Lemma 3. ~~Let~~ $A > 1$, ~~there exists a constant~~ $v = v(A) > 0$ ~~such that~~

$$(3.5) \qquad \int y^2(x)F(y^2(x),x)dx > v,$$

~~for all admissible~~ y ~~with~~ $x(y) < A^{-1}$.

Proof. If y is admissible then y can be represented in the form (2.1), thus

$$\left(\int y^2(x)F(y^2(x),x)dx\right)^2 = \left(\iint K(x,t)u(t)y(x)F(y^2(x),x)\,dt\,dx\right)^2$$

$$\leq \left(\iint K(x,t)u(x)u(t)\,dx\,dt\right)\left(\iint K(x,t)y(x)F(y^2(x),x)y(t)F(y^2(t),t)\,dx\,dt\right),$$

or

$$(3.6) \quad \left(\int y^2(x)F(y^2(x),x)dx\right)^2 \leq \left(\int y(x)u(x)\,dx\right)\left(\iint K(x,t)y(x)F(y^2(x),x)y(t)F(y^2(t),t)\,dx\,dt\right)$$

By (2.2), for $x(y) < A^{-1}$,

$$(3.7) \quad \int y^2(x)F(y^2(x),x)dx \geq \int y(x)u(x)dx - \int y^2(x)P(x)\,dx,$$

$$\geq (1 - A^{-1})\int y(x)u(x)dx.$$

Combined with (3.6) this gives

$$\int y^2(x)F(y^2(x),x)dx \le (1-A^{-1})^{-1}\iint K(x,t)y(x)F(y^2(x),x)y(t)F(y^2(t),t)\,dxdt,$$

from whence, by the continuity of $A : L^q \to L^p$,

$$\text{(3.8)} \quad \int y^2(x)F(y^2(x),x)dx \le M(1-A^{-1})^{-1}\left(\int |y(x)F(y^2(x),x)|^q dx\right)^{2/q}.$$

By Hölder's inequality

$$\text{(3.9)} \quad \int |y(x)F(y^2(x),x)|^q dx \le \left(\int |F(y^2(x),x)|^r dx\right)^{1-q/2}\left(\int y^2(x)F(y^2(x),x)\,dx\right)^{q/2},$$

where, as before, $r = p/(p-2)$. Together (3.8) and (3.9) give, for admissible y with $x(y) < A^{-1}$,

$$\text{(3.10)} \qquad 1 \le M(1-A^{-1})^{-1}\int |F(y^2(x),x)|^r dx,$$

Suppose now that

$$\text{(3.11)} \quad \inf\{\int y^2(x)F(y^2(x),x)dx \mid y \text{ admissible, } x(y) < A^{-1}\} = 0,$$

Then by (3.7)

$$\inf\{\int y(x)\,u(x)dx \mid y \text{ admissible, } x(y) < A^{-1}\} = 0$$

It then follows from A.2 of Lemma A.I (Appendix) that 0 is a cluster point of $\{y \mid y \text{ admissible, } x(y) < A^{-1}\}$. Since $\int |F(y^2(x),x)|^r dx$ is continuous on L^p, this contradicts (3.10) so (3.11) is false and the lemma is proved.

Let the non-linear transformation $y \longrightarrow \int K(\cdot,t)y(t)(P(t)+F(y^2(t),t))dt$ of L^p into L^p be denoted by T. We define

$$\phi_0(y) = a(T(y))\,T(y), \quad \text{for } y \in L^p\setminus\{0\},\ x(T(y)) < 1.$$

Lemma 4. <u>For</u> y <u>admissible,</u> $\varkappa(T(y)) < 1$,

$$H(\phi_o(y)) \leq H(y), \tag{3.12}$$

<u>and equality holds in</u> (3.12) <u>only if</u> y <u>satisfies</u> (1.1). <u>For each</u> $\Lambda > 1$ <u>there exists</u> $c_o = c_o(\Lambda) > 0$ <u>such that, for admissible</u> y, <u>when</u> $\varkappa(y) \leq \Lambda^{-1}$, <u>then</u>

$$\|\phi_o(y)\|_p \leq c_o(\Lambda)(1+H(y))^{(2+q)/2q} . \tag{3.13}$$

<u>Proof</u>. Let $y \in N$ be given by (2.1), let $\alpha = \alpha(T(y))$, and let $v = \phi_o(y)$. Since $\varkappa_1(v) = 1$ we have,

$$\int v^2(x)F_1(v^2(x),x)\,dx = \alpha\int v(x)y(x)F_1(y^2(x),x)\,dx, \tag{3.14}$$

where $F_1(\eta,x) = P(x) + F(\eta,x)$, and thus by Schwarz's inequality

$$\left(\int v^2(x)F_1(v^2(x),x)\,dx\right)^2 \leq \alpha^2\left(\int v^2(x)F_1(y^2(x),x)\,dx\right)\left(\int y^2(x)F_1(y^2(x),x)\,dx\right). \tag{3.15}$$

By (2.1)

$$\int y^2(x)F_1(y^2(x),x)\,dx = \iint K(x,t)u(t)y(t)F_1(y^2(x),x)\,dt\,dx,$$

from which, by Schwarz's inequality,

$$\left(\int y^2(x)F_1(y^2(x),x)\,dx\right)^2 \leq \left(\iint K(x,t)y(x)F_1(y^2(x),x)y(t)F_1(y^2(t),t)\,dx\,dt\right)\left(\iint K(x,t)u(x)u(t)\,dx\,dt\right). \tag{3.16}$$

Since y is admissible

$$\iint K(x,t)u(x)u(t)\,dx\,dt = \int y(x)u(x)\,dx \leq \int y^2(x)F_1(y^2(x),x)\,dx,$$

and from the definition of v,

$$\alpha\iint K(x,t)y(x)F_1(y^2(x),x)y(t)F_1(y^2(t),t)\,dx\,dt = \int v(x)y(x)F_1(y^2(x),x)\,dx.$$

From the above we obtain,

$$aJy^2(x)F_1(y^2(x),x)dx \wedge Jv(x)y(x)F_1(y^2(x),x)dx.$$

This last inequality, together with (3.14), gives,

$$(3.17)\qquad {}_a{}^2Jy^2(x)F_1(y^2(x),x)dx \pounds Jv^2(x)F\wedge v^2(x),x)dx,$$

which, combined with (3.15), yields,

$$(3.18)\qquad Jv^2(x)F_1(y^2(x),x)dx \wedge Jv^2(x)F_{\pm}\{v^2(x),x)dx.$$

However, for any y, v e L^P, as is shown in [9],

$$(3.19)\qquad H(v) \pounds H(y) + Jv^2(x)[F_{\pm}(v^2(x),x) - F\wedge y^2(x),x)dx.$$

Thus, as v = $<f>_o(y)$, (3.12) follows from (3.19) and (3.20). In order for equality to hold in (3.12), it must hold in each of the inequalities in the proof. In particular we must have equality in (2.2), (i.e. y e N), and in (3.16). The equality can hold in (3.16) only if u(x) and $y(x)F_{\pm}(y^2(x),x)$ are colinear, but then, since equality holds in (2.2), we must have

$$u(x) = y(x)F_x(y^2(x),x),$$

in which case y is a solution of (1.1).

We now prove (3.13). For y e L^P,

$$(3.20)\qquad H(y) \wedge e\ (1+e)\,Jy^2(x)F(y^2(x),x)dx,$$

where C is the constant in (1.6); see [9].

From (3.17) and (3.5)

$$(3.21)\quad a^2 \pounds i/"^1Jv^2(x)F_1(v^2(x),x)dx \pounds cjv^2(x)F(v^2(x),x)dx$$

where $v = v(x(y))$ and $C = (1-0\ (T(y)))^{-3}$ i/""1. Since $v = \theta_o(y)$, bY (3.12), (3.20) and (3.21) we have,

(3.22) $a^2 \wedge C\epsilon(1+e)H(v) \wedge C\epsilon(1+\epsilon)H(y)$.

In Lemma 2 of [2] is proved the existence of constants R_1 and R_2 such that

$$(J|J K(x,t)y(t)F(y^2(t),t)dt|^p dx)^{\frac{1}{p}} \pounds (k \wedge kj y^2(t)F(y^2(t),t)dt)^{\frac{1}{q}} .$$

From this inequality, the definition of ϕ_o, and (3.20) and (3.21) readily follows (3.13).

The mapping ϕ_o defined above is adequate for our purpose only when $A_1 > 1$, in which case the domain of ϕ_o is iP {0}. Our next objective is to construct, in the general case, a mapping $\phi : L^p\backslash\{0\} \to L^p\backslash\{0\}$ such that: for admissible $y \in L^p$ either

a) $\phi(y) \in N$ and

(3.23) $H(\phi(y)) \pounds H(y)$

with equality only if y is a solution of (1.1) ; or

(3.24) $H(f(y)) < e$, $k = 1,2,...,$

where e is a preassigned positive constant.

To this end we first prove the following.

Lemma 5. Let $S \subset A(L^q\backslash\{0\})$ and suppose that for $y \in S$,

$$0 < C \pounds x(y) < x_x(y) \pounds M < \bullet$$

where C,M are constants. Then $\bar{S}$ is compact in L^P. The functional $\int u(x)y(x)dx$ is bounded on S.

Proof. The general element $y \in S$ can be represented in the form (2.1). It follows from Lemma A.1 that $\bar{S}$ is compact if $\int y(x)u(x)dx$ is uniformly bounded for $y \in S$. Suppose therefore that there is a sequence $\{y_n\}$ in S such that $\int y_n(x)u_n(x)dx \to \infty$ as $n \to \infty$, (where u_n has the obvious meaning). Let the constants ρ_n be determined by

$$(3.25) \qquad \rho_n^2 \int y_n(x)u_n(x)dx = 1,$$

we shall assume that $\rho_n < 1$ for all n and that $\rho_n y_n \to y_o \neq 0$ in L^p; the second part of the assumption is justified by Lemma A.1. Under this assumption, by Lemma 1,

$$(3.26) \qquad \rho_n^2 \int y_n^2(x)F(\rho_n^2 y_n^2(x),x)dx \to \int y_o^2(x)F(y_o^2(x),x)dx, \quad \text{as} \quad n\to\infty.$$

By (3.1), (3.2) and (3.25)

$$\kappa_1(y_n)-\kappa(y_n) = \rho_n^2 \int y_n^2(x)F(y_n^2(x),x)dx,$$

and by (1.6), since $\rho_n < 1$,

$$\rho_n^2 \int y_n^2(x)F(y_n^2(x),x)dx \geq \rho_n^{-2\epsilon}[\rho_n^2 \int y_n^2(x)F(\rho_n^2 y_n^2(x),x)dx].$$

Since $\rho_n \to 0$, as $n \to \infty$ and $y_o \neq 0$ these last two relations, together with (3.26), imply that $\kappa_1(y_n) \to \infty$ as $n \to \infty$, $\kappa(y_n)$ being bounded by Λ_1^{-1} (see Lemma A.3). This yields a contradiction, and consequently $\int y(x)u(x)dx$ must be bounded on S. In view of the observation made at the beginning of the proof, the assertion of the lemma follows.

For A > 1 put

$$H_A = \sup\{H(y)|y \text{ e } N, \ x(y) \wedge A"^{1}],$$

and

$$h_\Lambda = \inf\{H(y)|y \text{ e. } N, \ x(y) \text{ £ } A7^{1}\},$$

(the finiteness of H_Λ follows from Lemma 5).

Lemma 6. <u>For</u> A > 1,

$$0 \ll h_A \lhd H_\Lambda \cdot$$

<u>Moreover</u>,

$$\lim_{\Lambda \to 1-} H_A = 0.$$

Proof. The positivity of h_Λ follows from (3.20) and Lemma 3; the inequality $h_\Lambda < H_\Lambda$ is obvious. The second assertion follows from Lemma 5 since for y e N,

$$H(y) \text{ £ } Jy^2(x)F(y^2(x),x)dx = (1 - x(y))Jy(x)u(x)dx,$$

and thus

$$H_A \wedge \text{const.} (1 - A"^{"1})^\wedge \qquad \text{for } 1 < A. \text{ £ const.}$$

By (1.6) and the definition of H, for 1 > p > 0

$$H(py) \wedge p^2\int y^2(x)F(\beta^2y^2(x),x)dx$$

$$\text{£ } p^2 Jy^2(x)F(y^2(x),x)dx.$$

Let e > 0 be preassigned, since^ by Lemma 1, $Jy^2(x)F(y^2(x),x)dx$ is bounded on the unit sphere in L^ there follows the existence of a positive constant p such that,

$$H(p||y||_p^{-1}y) \wedge e, \qquad y \text{e } L^P\backslash\{o\}.$$

We now define, for admissible y,

$$\bar{\alpha}(y) = \max \{\alpha(y),\ \beta\|y\|_p^{-1}\},$$

where for convenience we take $\alpha(y) = 0$ when $\varkappa(y) \geq 1$. Since this extension of α preserves the continuity of $\alpha \circ A$ on $L^q\setminus\{0\}$, as is easily verified, we have also that $\bar{\alpha} \circ A$ is continuous on $L^q\setminus\{0\}$. We now define $\phi(y)$, for $y \in L^p\setminus\{0\}$, by,

$$\phi(y) = \bar{\alpha}(Ty)Ty.$$

Clearly the image by ϕ of $L^p\setminus\{0\}$ is the set $\{y \in L^p\setminus\{0\} \mid \bar{\alpha}(y) = 1\}$ which we shall denote by N'.

Lemma 7. _If_ $y \in N'$ _then_

$$\|y\|_p \geq \beta, \tag{3.27}$$

thus $0 \notin \overline{N'}$. _There exists a_ $\Lambda' > 1$ _such that equality holds in_ (3.27) _for_ $\varkappa(y) \geq \Lambda'^{-1}$.

Proof. By definition $\bar{\alpha}(y) \geq \beta\|y\|_p^{-1}$, thus $\bar{\alpha}(y) = 1$ implies (3.27). To prove the second assertion we need only show that $\sup\{\|y\|_p \mid y \in N,\ \varkappa(y) \geq \Lambda^{-1}\} \to 0$ as $\Lambda \to 1-$. This follows from Lemmas 5 and 6 and the fact that, in view of (3.20) and (1.6), $H(y) = 0$ for $y \in L^p$ if and only if $y = 0$.

Lemma 8. _The mapping_ ϕ _is an odd continuous transformation of_ $L^p\setminus\{0\}$ _into_ N'. _For any admissible_ y _either_

$$e < H(\phi(y)) \leq H(y) \tag{3.28}$$

with equality only if y _is a solution of_ (1.1), _or_

$$H(\phi^k(y)) \leq e \qquad k = 1,2,\ldots\ . \tag{3.29}$$

Finally there exists ja constant $C_0 > 0$ such that, for admissible y,

(3.30) $$||fy)||_p \leq C_0(1 + H(y))^{(2+q)/2<*} .$$

Proof, It is clear that *(f>* is odd. The continuity of *f>* follows from the continuity of T and $\bar{a} \cdot T$, which in turn follow from the continuity of $A : IJ^q \to L^P$, $\bar{a} \cdot A : L^2\backslash\{0\} \to R$

and the continuity of the transformation $y(x) \to y(x)F(y^2(x),x)$ from L^p to L^q. The continuity of this last mapping follows from (1.8), (1.9) and Theorem 19.1, [10]. The second assertion of the lemma follows immediately from the first assertion of Lemma 4 and the construction of *(f>.*

To prove (3.30) we note that, by Lemma 7, either $||0(y)|!^p$ = p, or $x(T(y)) < A <1$ and $0(y) = <p^0(y)$. Choose (by Lemma 6) $A^M > 1$ so that $H_{\hat{M}} < h_{\hat{,}}$, then when the second alternative holds, we have by Lemma 4,

$$H(y) \gtrsim H(*(y)) \gtrsim h_{At} > H_{AM} ,$$

and thus $x(y) < A^{11"1}$. By Lemma 4, (3.30) holds in this case with $C_Q = C_0(A^{TI})$. If we take $C_Q = \max(p, C_Q(A"))$ then (3.30) will hold for all admissible y.

Lemma 9. Let y e *iP*. J[f

$$e_0 = \liminf_{n\to\infty} H((f>^n(y)) > e,$$

then the cluster points of the sequence $\{<j)^n(y)\}$ form a. non-empty connected set of solutions of (1.1) .

Proof. By (3.28) of Lemma 8, the limit inferior in the assertion above is actually a limit. Using Lemma 8 again we see that the sequence $\{\phi^n(y)\}$ is bounded. Since $\phi^n(y) \in N$, $n = 1,2,\ldots$, the sequence has cluster points, in fact by Lemma 1, (2.2) and Lemma A.1, any bounded admissible set has compact closure. By the continuity of ϕ and H if y_o is one of these cluster points then $H(y_o) = e_o$. If we suppose that $y_o = \lim_{k\to\infty} \phi^{n_k}(y)$ then $\phi(y_o) = \lim_{k\to\infty} \phi^{n_k+1}(y)$, from which we conclude that $H(\phi(y_o)) = e_o = H(y_o)$ and therefore y_o is a solution of (1.1).

The connectedness of the set of cluster points follows by a standard argument from the fact that $\|\phi^n(y)-\phi^{n-1}(y)\|_p \to 0$ as $n \to \infty$.

4. Proof of Theorems 1 and 2. We begin by listing the relevant properties of the genus. Let $\overset{o}{\mathcal{C}}(X) = \{B \in \mathcal{C}(X) \mid B \text{ compact}\}$.

1) If $G_1, G_2 \in \overset{o}{\mathcal{C}}(X)^*$ and if there exists a continuous odd map $f : G_1 \to G_2$ then $p(G_1) \leq p(G_2)$, in particular, if $G_1 \subset G_2$ then $p(G_1) \leq p(G_2)$.

2) For $G_1, G_2 \in \mathcal{C}(X)$, $p(G_1 \cup G_2) \leq p(G_1) + p(G_2)$.

3) If $G \in \overset{o}{\mathcal{C}}(X)$ then $p(G) < \infty$, and G has a neighborhood U such that $U \in \mathcal{C}(X)$ and $p(U) = p(G)$.

4) If $G \in \mathcal{C}(X)$ and G is a homeomorphic image of an n-sphere then $p(G) = n+1$.

We note first that, in view of Tietze's extension theorem, $p(B) \leq n$, for $B \in \overset{o}{\mathcal{C}}(X)^*$ if there exists a zero free odd continuous map of B into R^n; assertion 1) is then obvious. Assertion 2 and the first part of 3) follow from Lemma 10 below, the proof of which will be omitted. The second part of 3) follows from standard extension theorems and 4) follows from the Borsuk-Ulam theorem. For a more complete discussion of the genus see [5].

Lemma 10. Let $G \in \mathcal{C}(X)^*$ then $p(G) \leq n$ if and only if there exist n sets $G_1, \dots, G_n \in \mathcal{C}(X)$ with $p(G_i) = 1$, $i = 1, \dots, n$ and $G \subset \bigcup_{i=1}^{n} G_i$.

Let M be an m-dimensional subspace of $\mathcal{H}(Q)$ and let

$$\Gamma_m = \{u \in M \mid \|u\|_q = 1\}.$$

By the results of section 3 the transformation $u(x) \to y(x) = \bar{a}\int K(x,t)u(t)dt$, where $\bar{a} > 0$ is chosen so that $y \in N^1$, is a continuous mapping of E_m onto a compact symmetric subset of N^1. Since this mapping is one to one, it is a homeomorphism (E_m being compact), and therefore $\gamma(B) = m+1$. It follows that for an arbitrary positive integer the class $\mathcal{B}_m$, defined in section 2, is non-empty, as asserted in Theorem 1.

The non-negative and non-decreasing character of the $\{\lambda_m\}$ is evident, we pass now to the last assertion of Theorem 1. Suppose that $\lambda_{m_0} \le 1 < \lambda_{m_0+1}$ $(0 \le m_0)$ let $\mathfrak{M}$ be the subspace of L^p spanned by the first m_0 eigenfunctions of (2.3). Any non-zero element of $\mathfrak{M}$ is admissible, hence any symmetric subset of $\mathfrak{M}\setminus\{0\}$ is admissible. It readily follows, since $H(y) \to 0$ as $y \to 0$ that $\lambda_m = 0$ for $1 \le m \le m_0$. Let P denote the projection onto $\mathfrak{M}$ which annihilates all the other eigenfunctions of (2.3). If B is set of genus $\ge m_0+1$ then $0 \in PB$, for otherwise we would have $PB \in \Sigma(L^p)$ with $\gamma(PB) \ge \gamma(B) \ge m_0 + 1$, and this is impossible since $PB \subset \mathfrak{M}$. If $y \in L^p$, $y \ne 0$ and $Py = 0$ then $\chi(y) \le \mu_{m_0+1} < 1$ (see Lemma A.3); if in addition y is admissible we have by Lemma 3 and (3.20),

$$H(y) \ge \epsilon(1+\epsilon)\nu(\lambda_{m_0+1}).$$

It follows that for an admissible set B with $\gamma(B) \ge m_0+1$,

$$\max_{y\in B} H(y) \ge \epsilon(1+\epsilon)\nu(\lambda_{m_0+1}), \tag{4.1}$$

and thus,

$$\lambda_{m_o+1} \geq \epsilon(1+\epsilon)\nu(\Lambda_{m_o+1}) > 0. \tag{4.2}$$

This completes the proof of the last assertion of Theorem 1.

We come back to the second assertion of Theorem 1, namely that,

$$\lim_{n\to\infty} \lambda_m = \infty. \tag{4.3}$$

We assume once and for all that the number e which figures in the construction of ϕ in section 3 satisfies,

$$\epsilon(1+\epsilon)\nu(\Lambda_{m_o+1}) > e > 0, \tag{4.4}$$

where $\Lambda_{m_o} \leq 1 < \Lambda_{m_o+1}$. Consider the mapping ψ defined, for y admissible, by $\psi(y) = \overline{\alpha}(y)y$. It is easily seen that for y admissible $\overline{\alpha}(y) \leq 1$ whenever $H(y) > e$. For an admissible set B with $\rho(B) \geq m_o+1$ we have from (4.1) and (4.4) that

$$\max_{y\in B} H(y) > e$$

and thus

$$\max_{y\in\psi(B)} H(y) \leq \max_{y\in B} H(y).$$

Consequently, for $m \geq m_o + 1$,

$$\lambda_m = \inf_{B\in\mathcal{B}'_m} \max_{y\in B} H(y) \tag{4.5}$$

where

$$\mathcal{B}'_m = \{B\in\mathcal{B}_m \mid B \subseteq N'\}.$$

Now for $c > 0$ let

$$N'(c) = \{y\in N' \mid H(y) \leq c\}.$$

By Lemma 8, 0 maps NM (c) into a bounded subset of itself, thus <^(N^f (c)) has compact closure. It follows from property 1 of the genus, since <*f*> is odd, that N^f (c) can contain no symmetric set of genus greater than p($\overline{\text{^fNHc))}$) . As we have shown N^f contains symmetric sets of arbitrarily large genus, thus (4.3) follows from (4.5). This completes the proof of Theorem 1.

Let m be a positive integer and assume that m J> m_o +1 and $A_m > A_{m-1}$; let c be a real number, with $c > A_m$ and such that none of the characteristic values of (1.1) lie in the interval (A_m, c). Let

S_m = {y € "?TF7 |H(<f(y)) J> A_m, n = 1,2,...}-

Since <*f*> is continuous and odd and $\overline{0(N_c)}$ is compact it follows that S_m is symmetric and compact. Let E_m be as defined in section 2. It is clear that E_m^* is symmetric and since it is obviously a closed subset of S^m it is compact. For 6 > 0, put

U(E_m,6) = [y e N|0 £dist (y,E_m) < 6}

where

dist (y,E_m) = inf(||u|||y + u € ,E_m),

so that U(E_m,6) is empty if E_m is empty. Choose 6 > 0 so that U(E_m,6) € £(:L^P) and

p(U(E_m,6)) = o(E_m).

For $y \in S_m$, $H(y) > \lambda_m$ unless $y \in E_m$, therefore, since $S_m \backslash U(E_m, \frac{\delta}{2})$ is compact there exists a constant $c_1 > \lambda_m$ such that

$$H(y) \geq c_1, \qquad \text{for } y \in S_m \backslash U(E_m, \tfrac{\delta}{2}).$$

From (4.5) and the definition of the multiplicity of λ_m it follows that there exists a set $B \in \mathcal{B}'_{m+\ell-1}$, where ℓ = multiplicity of λ_m, with

$$\lambda_m \leq \max_{y \in B} H(y) < c_1.$$

From the definition of c_1,

$$B \cap S_m \subseteq U(E_m, \tfrac{\delta}{2}),$$

thus, for every $y \in B \backslash U(E_m, \delta)$,

$$\lim_{n \to \infty} H(\phi^n(y)) < \lambda_m.$$

By a standard argument it follows from the compactness of $B \backslash U(E_m, \delta)$ and the monotone decreasing character of $\phi^n(y)$ that there exists an integer n_o such that

$$H(\phi^{n_o}(y)) < \lambda_m,$$

for all $y \in B \backslash U(E_m, \delta)$. Consequently

$$\rho(B \backslash U(E_m, \delta)) \leq \rho(\phi^{n_o}(B \backslash U(E_m, \delta)) < m.$$

It follows from the subadditivity of the genus (Property 2)) that

$$\rho(E_m) = \rho(U(E_m, \delta)) \geq \rho(B) - \rho(B \backslash U(E_m, \delta)) \geq \ell.$$

Thus $\rho(E_m) \geq \ell$, in particular E_m is not empty. This completes the proof of Theorem 2.

JRemark 1. Let Y be a Banach space stronger than L^p, (i.e. $Y \subseteq$ I/P, $||\cdots||_p \wedge$ const. $||\ldots||_Y$), and let some power, say T^n, of T map $I\mathcal{J}^p$ continuously into Y. Then it is clear first of all that every I/P-solution of (1.1) actually belongs to Y. Moreover, if i denotes the inclusion map Y - L^p, then, for a given m, $T^n|E_m$ = ^ l 1 3 ^ * ™ 1 thus the L^p and Y topologies on E_m coincide. In particular E_m ^ J c has the same genus as a subset of Y as it does as a subset of L^P. Also, if n has the same meaning as above, then $(j>^n$ maps $L^p\backslash(0\}$ continuously into Y. Thus, by (3.28), A^m is given by,

$$A_m = \inf_{B \in \mathcal{B}^\wedge} \max_{y \in B} H(y) ,$$

where

$$\mathcal{B}''_m = \{B \in \mathcal{B}_m | B \subseteq Y\}.$$

Remark 2. When P, F and K are continuous on $\bar{\theta}_9$ $\bar{R}_+ \times \bar{\theta}$ and $\overline{\theta \times \theta}$ respectively then one can use the above techniques in the space of continuous functions on $\bar{\theta}$ to obtain the same results as above but without assuming (1.8). The derivations of certain of the required bounds may be slightly different in this case, but most of these are obtained in [9], or can be derived as here by formally putting $q = l$, $p = »$, $r = l$.

5. Application to (1.12). We suppose that P and F are continuous on $[0,1]$ and on $\overline{R}_+ \times [0,1]$ respectively and that F satisfies (1.6). We shall use a somewhat restricted notion of admissibility and say that y is admissible only if it belongs to

$$C_0^2[0,1] = \{y \in C^2[0,1] \mid y(0) = y(1) = 0\}, \quad \text{and}$$

$$\int y^2(x)(P(x) + F(y^2(x),x))dx \geq \int (y'(x))^2 dx,$$

the characteristic values for (1.12) are defined as in the general case except that we take the infimum only over compact symmetric admissible sets in $C_0^2[0,1]$. This, as well as the omission of (1.8) is justified by the remarks at the end of section 5.

Theorem 3. Each positive characteristic number of (1.12) is simple, (i.e. has multiplicity 1).

Proof. Suppose $\lambda_m > 0$, and $\lambda_m > \lambda_{m-1}$ if $m > 1$. Then E_m is a compact symmetric non-empty subset of $C_0^2[0,1]\setminus\{0\}$ and $y \to y'(0)$ defines a continuous odd map of E_m into $R\setminus\{0\}$. By Tietze's theorem this extends to an odd continuous map of $C_0^2[0,1]$ into R, thus $\rho(E_m) = 1$, so it follows from the last assertion of Theorem 2 that the multiplicity of λ_m is one.

Theorem 4. Let y be a non-trivial solution of (1.12) with precisely $m-1$ zeros in $(0,1)$. Then

$$0 < \lambda_m \leq H(y).$$

If we let $\tilde{\lambda}_m$ denote the $m^{\underline{th}}$ characteristic number of (1.12) as defined in [8] then Theorem 4 implies the following.

Corollary. <u>For each</u> m,

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<u>there is equality if and only if</u> E_m <u>contains a solution</u> y <u>of</u> (1.12) <u>with exactly</u> m-1 <u>zeros in</u> (0,1). <u>If</u> A_m <u>is the first non-vanishing characteristic</u> number of (1.12) then $A = A \cdot$

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Proof of Theorem 4. If (1.12) has a non-trivial solution with precisely m-1 zeros in (0,1) then by the Sturm comparison theorem the m^{th} eigenvalue Λ_m of

(5.1) $u'' + \Lambda p(x)u = 0, \qquad u(0) = u(1) = 0,$

is strictly larger than 1 and thus by Theorem 1, $A_m > 0$.

If y_m is a solution of (1.12) with just m-1 zeros in (0,1) then the linear problem

$$v'' + \mu v F_1(y^2(x),x) = 0, \qquad v(0) = v(1) = 0,$$

has eigenvalues $\mu_1 < \mu_2 < \cdots < \mu_m = 1$. Let M be the subspace of $C_0^2[0,1]$ spanned by the corresponding eigenfunctions. Then for $v \in M$, $v \neq 0$

$$(5.2) \qquad \int_0^1 v^2(x)F_1(y^2(x),x)dx \Big/ \int_0^1 (v'(x))^2dx \geq 1.$$

We now consider the set $M \cap N'$, which is compact (in $C_0^2[0,1]$), symmetric, admissible and of genus m. Since $A_m > 1$ the proof of Theorem 2 shows that by a suitable choice of the number e (in the construction of φ), we can be assured that the maximum value of H on $M \cap N'$ is attained on N. However for $v \in N$

$$\int_0^1 v^2(x)F_1(v^2(x),x)dx \Big/ \int_0^1 (v'(x))^2dx = 1,$$

thus from (5.2), for $v \in M \cap N$,

$$\int_0^1 v^2(x)F_1(v^2(x),x)\,dx \le \int_0^1 y^2(x)F_1(y^2(x),x)\,dx.$$

It then follows from (3.19), for $v \in M \cap N$, that

$$H(v) \le H(y)\ .$$

We have therefore

$$\max_{v \in M \cap N'} H(v) = \max_{v \in M \cap N} H(v) \le H(y)\ .$$

Since $\rho(M \cap N') = m$, this completes the proof of Theorem 4.

Theorem 7.1 of [8] implies that when the m^{th} eigenvalue of (5.1) is strictly greater than 1 then (1.12) has a solution y with precisely $m-1$ zeros in $(0,1)$, and, as we have already observed, this condition is necessary for the existence of such a solution. The m^{th} characteristic number of (1.12) as defined in [8] can be characterized as the minimum value of H over all non-trivial solutions of (1.12) with exactly $m-1$ zeros in $(0,1)$. Taking these facts into account, the proof of the corollary is immediate.

<u>Appendix</u>. This section contains several results for linear integral operators and integral equations which are used in the preceeding sections.

We begin by quoting a result from [5].

(*) <u>Let the operator</u> A, <u>given by</u> (1.3), <u>be completely continuous from</u> L^q <u>to</u> L^p, $(1 < q \leq 2 \leq p, \frac{1}{p} + \frac{1}{q} = 1)$. <u>Then</u>,

$$A = HH^*, \tag{A.1}$$

<u>where</u> H <u>is a completely continuous operator from</u> L^2 <u>to</u> L^p; $H^* : L^q \to L^2$ <u>is the adjoint of</u> H.

See Theorem 4.4, [5].

As a corollary of the above assertion we have the following result, we assume throughout that A is as in (*).

Lemma A.1. <u>Let</u> y <u>be given by</u> (2.1), <u>then</u>

$$\|y\|_p \leq \text{const.} \int y(x)u(x)\,dx. \tag{A.2}$$

<u>Moreover, if</u> B <u>is subset of</u> L^p <u>lying in the range of</u> A <u>and if</u> $\int y(x)u(x)\,dx$ <u>is bounded on</u> B <u>then</u> $\overline{B}$ <u>is compact</u>.

<u>Proof</u>. By (*) we have,

$$\int y(x)u(x)\,dx = \|H^*u\|_2^2 , \tag{A.3}$$

thus (A.2) follows from (2.1), (A.1) and the continuity of H. If $B \subseteq L^p$ lies in the range of A then $B = AB' = HB''$, where $B' \subseteq L^q$ and $B'' = H^*B' \subseteq L^2$. By (A.3), B'' is a bounded set in L^2 if $\int y(x)u(x)\,dx$ is bounded on B. The compactness of $\overline{B}$, when the latter is the case, follows from the compactness of H.

We consider next the problem (2.3) where K and P are as in section 1, and P does not vanish almost everywhere.

Lemma A.2. <u>The problem</u> (2.3) <u>has a sequence</u> $0 < \lambda_1 \le \lambda_2 \le \cdots$ <u>of eigenvalues of finite multiplicity, moreover</u> $\lambda_m \to \infty$ <u>as</u> $m \to \infty$. <u>If</u> m_0 <u>is a positive integer then there exists a projection</u> P <u>of</u> L^p <u>onto the subspace spanned by</u> the first m_0 <u>eigenfunctions of</u> (2.3) (i.e. those corresponding to $\lambda_1, \ldots, \lambda_{m_0}$) <u>and which annihilates all other eigenfunctions of</u> (2.3).

<u>Proof</u>. We shall use P to denote the operation of multiplication by $P(x)$. By (1.5) it follows that the operator P is continuous from L^p to L^q. By (*) it then follows that the symmetric operator H^*PH is completely continuous on L^2. The assertions of the lemma then follow from the theory of compact self-adjoint operators. In particular, if L^2 is represented as a direct sum, $L^2 = X_1 \oplus X_2$ where X_1 and X_2 are invariant subspaces for H^*PH and X_1 is finite dimensional, then $L^p = Y_1 \oplus Y_2$ where $Y_1 = HX_1$, $Y_2 = HX_2$ and Y_1 and Y_2 are invariant subspaces for AP.

We observe also that if $v = H^*u$, $u \in L^q\setminus\{0\}$, and $y = Au$, then

$$\chi(y) = \int p(x)y^2(x)dx \Big/ \int u(x)y(x)dx = (v, H^*PHv)/\|v\|_2^2 .$$

As a consequence we have the following.

Lemma A.3. For $y \in A(L^q\setminus\{0\})$,

$$\chi(y) \le \lambda_1^{-1} ,$$

<u>if</u> P <u>is as in</u> Lemma A.2 <u>and</u> $Py = 0$ <u>then</u>

$$\chi(y) \le \lambda_{m_0+1}^{-1} .$$

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