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CHARACTERISTIC VALUES ASSOCIATED WITH A CLASS OF NON-LINEAR BOUNDARY VALUE PROBLEMS

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This note is a sequel to [1] and provides the answer to a question left open there. Let $P(x), F(t,x)$ be real valued functions, continuous on $[0,1]$ and $T [0,1]$ respectively, and assume that for some $\epsilon > 0$, $F$ satisfies, for each $x \in [0,1]$,

$$0 < n^\epsilon F(t_1,x) < n^\epsilon F(t_2,x), \quad 0 < r_1 < r_2.$$  

Let

$$G(r,x) = \int F(s,x)ds, \quad 0 < r < \infty, \quad 0 < x < 1,$$

and for $y \in C[0,1]$ let,

$$H(y) = \int \left[ y^2(x)F(y^2(x),x) - G(y^2(x),x) \right] dx.$$

A function $y$ is admissible if it belongs to $C^2 [0,1] = \{ UC^2 [0,1] \mid u(0) = u(1) = 0 \}$, does not vanish identically, and satisfies,

$$\int y^2(x) \left[ P(x) + F(y^2(x),x) \right] dx \geq \int (y')(x)^2 dx.$$

An admissible set is a subset of $C^2_o [0,1]$ consisting entirely of admissible functions. We denote by $\mathcal{Q}_m$ the class of subsets of $C^2_o [0,1]$ which are compact, symmetric, and admissible and have genus (see def. in [1]), $\geq m$. Then, as defined in [1], the characteristic values of the boundary value problem,
The purpose of this note is to show that the characteristic numbers for (1.5) defined by (1.6) are the same as those defined in [2]. Assuming the results of [1], it amounts to the same thing to show that to each non-vanishing characteristic number $\lambda_m$ there corresponds a non-trivial solution $y$ of (1.5) which has precisely $m - 1$ zeros in $[0,1]$ and for which,

\begin{equation}
H(y) = \lambda_m.
\end{equation}

For $y \in C^2[0,1]$, let $n_m = M_m(y)$ denote the $m$th eigenvalue of the linear problem,

\begin{equation}
v'' + iv(P(x) + F(y^2(x),x)) = 0, \quad v(0) = v(1) = 0.
\end{equation}

Lemma 1. Let $y \in C^2[0,1]$ and assume that,

\begin{equation}
M_m(y) < i,
\end{equation}

then

\begin{equation}
H(y) \geq A_m.
\end{equation}

As Theorem 4 of [1] we proved the special case of the above assertion in which $y$ is a solution of (1.5) with precisely $m - 1$ zeros in $(0,1)$. Exactly the same argument works if we assume only (1.9).
Lemma 2. Let \( B \subseteq \mathbb{R}^m \), then there is at least one point \( y \in B \) for which (1.9) holds.

Proof. We observe first that the eigenvalues of (1.8) depend continuously on \( y \in C^2_0[0,1] \). Secondly, if we define
\[
q_k = q_k(x,y),
\]
to be the \( k \)th eigenfunction of (1.8) normalized by
\[
\int_0^1 q_k^2(x,y) (P(x) + F(y^2(x),x)) dx = 1, \quad \forall \in (0,y) > 0,
\]
then \( y \ast q_k^*(x,y) \) is continuous as a map of \( C^2_0(0,1^2 \setminus \{0\}) \) into itself. The Fourier coefficients \( a_k \) of \( y \) with respect to \( \{q_i^{(x)}, (sy)\} \) are computed by,
\[
(1.11) \quad a_k = \int_0^1 y(x)q_k(x,y) (P(x) + F(y^2(x),x)) dx,
\]
and if \( a_k \) vanishes for \( k = 1, \ldots, m - 1 \), then
\[
(1.12) \quad \int_0^1 y^2(x)(P(x) + F(y^2(x),x)) dx \leq \sup_{x} (y(x))^{-1} \int_0^1 (y'(x))^2 dx.
\]
Consider the mapping from \( B \) to \( \mathbb{R}^{m-1} \) given by,
\[
y \mapsto (a_1, \ldots, a_{m-1}),
\]
where the \( a_k \) are given by (1.11). This mapping is odd and continuous, thus since \( B \subseteq \mathbb{R}^m \), there exists a \( y \in B \) for which \( a_1 = \ldots = a_{m-1} = 0 \), and for which consequently (1.12) holds. Together (1.4) and (1.12) imply (1.9), and this completes the proof.
Theorem. If $m$ is a positive integer and if $X_m > 0$ then there exists a solution $y = y_\text{DF}(1.5)$ with precisely $m - 1$ zeros in $(0,1)$ and such that

$$H(y) = X_m.$$ 

Proof. The proof of Theorem 2 of [1] shows that there exists a set $B \in \mathcal{B}$ such that,

$$(1.12) \quad \max_{y \in B} H(y) = X_m'$$

and with the additional property that $H(y) = \nabla m$ for $y^\mathcal{B}$ only if $y$ is a solution of (1.5). (Indeed, in the notation of [1], take $B = 0(0(N^\wedge))$, $c = X^\wedge$). By Lemma 2, there is a $y \in B$ for which (1.9) holds, and by Lemma 1 and (1.12), $H(y) = X_m$, so that $y$ is a solution of (1.5). Since the non-zero characteristic values are simple, (Theorem 3 of [1]), we have $X_{m+1} > X_m$, and thus by Lemma 1, with $m$ replaced by $m + 1$, we have $\nabla m(y) = 1$, and $y$ has precisely $m - 1$ zeros in $(0,1)$.

Corollary. The characteristic values of (1.5), as defined by (1.6), are the same as those defined in [2].

Proof. See the Corollary to Theorem 4 in [1].
References
