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OBSERVATION AND PREDICTION FOR THE HEAT EQUATION

by

V. J. Mizel and T. I. Seidman

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OBSERVATION AND PREDICTION FOR THE HEAT EQUATION

V. J. Mizel and T.I. Seidman

§1; Consider an insulated uniform rod with unknown temperature distribution. With proper normalization the temperature $u = u(t,x)$ satisfies

$$(1)\qquad \begin{aligned} &u_t = u_{xx} && (t > 0,\ \ 0 < x < 1), \\ &u_x(t,0) = u_x(t,1) = 0 \end{aligned}$$

Suppose it possible to observe $f(t) = u(t,0)$ for $0 < t < T$.

We ask: Is it possible, given f, to determine $w(x) = u(T,x)$ for $0 < x < 1$? Assuming the answer to this first question is 'yes', is this a well-posed problem?

We may formulate the problem more precisely as follows. The boundary data f must satisfy certain consistency conditions, i.e., must lie in a certain manifold to. We are then asking whether the operator A: fi—»w is well-defined from to to $L^2(0,1)$ and whether it is continuous, topologizing to by the L^2 norm on $(0,T)$:

$$\text{iiff} = \int_0^T |f(t)|^2 dt.$$

As is well known, we may write the solution of (1) as

$$(2)\qquad u(t,x) = L\wedge_o c_n e''^{n^2 T} \wedge os\ mrx$$

with appropriate coefficients $\{c_n : n = 0,1,\ldots\}$ such that

$$\Sigma_n |c_n|^2 e^{-n^2\pi^2 t} < \infty$$

for $t > 0$. With the substitution $s = e^{\cdot\cdot\ {}^{2}4-}$, we have

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$$f(t) \qquad\qquad 2$$

for $0 < t < T$ or $a = e^{-7r^2/T} < s < 1$. Then

$$Af = w(x) = E_n\ c_n a^{n^2} \cos mrx$$
$$= \pounds_n K>_n (<p)\ a^{n^2} \cos mrx$$

where each I_N (N = 0,1,...) is the linear functional - assuming it is well-defined - such that

$$V^L n^c n^{s^{n^2}} > - <V$$

Letting

$$[A] = sp\{s^{n^2} : n = 0,1,...\},$$

we have, assuming each $\wedge_N$ is continuous on $M = [\overline{A}]\ \pounds\ L^Q(a,l)$,

(3) $$l|A||^2 \leq ir^2 L_n a^{2n^2} ||t_n||^2.$$

What is needed, therefore, is a suitable estimate for IKMII •
The Hahn-Banach Theorem assures us that if

(4) $$s^N\ jL\ (K\pounds\backslash = iF\{s^{n^2} : n = 0,1,...,n \neq N\}$$

then $\wedge_N$ exists as a continuous linear functional on [M] with

(5) $$IM \sim^{1=} H^{sN2} - [\Lambda_N]\|_1$$

where $||\cdot|_1^U$ is the L^2 norm on (a,l), so that

(3') $$||A||^2 \leq 7r^2 L_n\ a^{2n^2} ||s^{N^2} - [\Lambda_N]\|_1^{-2}.$$

The problem is thus reduced to showing that (4) holds, so that

each t_N is well-defined, and obtaining a lower bound for (5) .

If we had $a = 0$ then Muntz[1] Theorem (for which see, e.g.. [2]) assures (4) since $E n''^2 < oo$. For $0 < a < 1$, a result due to Clarkson and Erdős [1] shows that $[A_N j$ is not dense in $L^2(a,l)$ but does not give (4), much less a lower bound for (5) . We shall obtain a sharpened form of the Clarkson--Erdős result, for a certain class of sequences including $\{n^2 : n = 0,1,...\}$, showing that (4) holds if a is sufficiently close to zero (i.e., for large enough T) and that then $||l_N j| = \&(N \log N)$ so that the right hand side of (3) converges and A is continuous.

§2: Let $A= (A_0, A_1, ...)$ be a sequence of non-negative reals with $0 \leq A_0 < A_x < ...$; let $A^\wedge = A \setminus U_N)$. Let $[A] = sp\{s^n : 7_n \in A, n = 0,1,...\}$. We assume that

$$\Sigma_A \lambda_n^{-1} < oo \tag{6}$$

and set

$$T_N = E_{n>N} \lambda^{-1}.$$

It is convenient to introduce

$$r_n^N = \begin{cases} \lambda_n / \lambda_N & n < N \\ \lambda_N / \lambda_n & n > N \end{cases}, \quad r_N = \max_n \{ r_n^N : n \neq N \} < 1$$

and

$$\varphi(r) = -\log[(1-r)/(1+(1+\lambda_1^{-1})r)].$$

If we set, for $n,N = 0,1,\ldots$ with $n \wedge N$,

$$\varphi_n^N = \begin{cases} -\log[(1-r_n^N)/(1+r_n^N+x_N^{-1})] & n < N, \\ \log[(1-r_n^N)/(1-r_N^{-1})] & n > N, \end{cases}$$

then

$$(7) \qquad 0 < \varphi_n^* \leq \varphi(r_n^*) .$$

It is known (see, e.g., [2]) that, for any sequence A,

$$(8) \qquad \|s_N^{[\lambda_N]}\|^2 = \frac{1}{1+2\lambda_N} \quad (\wedge j \quad ^2$$

where $\|*\|$ denotes the L^2 norm on $(0,1)$. Thus, by (7),

$$(8') \qquad \|s_N^{\lambda_N} - [A]\| = (1 + 2\lambda_N)^{-1/2} \exp[-\Sigma_n \varphi_n^N]$$
$$\geq (1+2X_N)^{-1/2} \exp[-\Sigma_n \varphi(r_n^N)] .$$

<u>Lemma 1</u>: Let A, etc., be as above. Then

$$(9) \qquad \|B^{\lambda_N} - [A]\| \geq (1+2A_N)^{-1/2} \exp[-N\varphi(r_N) - V \wedge] > 0$$

where $a\wedge = \max\{\varphi'(0), \varphi(r_N)/r_N)$.

<u>Proof</u>; We break up the sum on the right of (8') into $\Sigma_{n<M}$ and $\Sigma_{n>N}$. Then as φ is an increasing function of r,

$$\Sigma_{n<N} \varphi(r_n^N) \leq N\varphi(r_N) .$$

From the form of the function φ we have

$$\varphi(r) \leq \wedge r \quad (0 < r \leq r_N)$$

so that

$$\Sigma_{n>N} * \wedge \wedge E_{n>N} V n^{N} = V N V$$

Combining these inequalities with (8') gives (9) . ||

<u>Lemma 2</u>; Let Pe [A^] so that $(s^{A_N} - P) = S \wedge s^{A_n}$ with all but finitely many b^n zero and with $b_N = 1$. Then

(10) $$|b_n| < ||sV P|| / ||s^{\lambda_n} - [A]|| \quad (n = 0,1,...).$$

<u>Proof</u>; Observe that (10) is immediate if n = N or if $b_n = 0$. Otherwise (n ^ N, b^ 0) , note that

$$\|s^{\lambda_N} - P\| = \|\Sigma_\nu b_\nu s^{\lambda_\nu}\| = |b_n| \|s^{\lambda_n} - \Sigma_{\nu \neq n} (b_\nu / b_n) s^{\lambda_\nu}\|$$
$$= |b_n| \|s^{\lambda_n} - P_o\|$$

where $P_Q = \wedge (\wedge A \wedge s^{\lambda_\nu} e [A_n]$. Thus,

$$\|s^{\lambda_N} - P\| \geq |b_n| \|s^{\lambda_n} - [A_n]\|$$

which is just (10). ||

Note that by (6) the sequence A satisfies:

(11) $$T_n a^{A_n} < \infty \quad \text{for } 0 < a < 1.$$

We add the assumption that A satisfies the additional condition:

(12) $$E V_N + (N/A_N)(p(r_N)] \to 0 \text{ as } N \to \infty.$$

<u>Lemma 3</u>: Let A, etc., be as above with A satisfying (12). Set

$$\Gamma = \Sigma_0^\infty \alpha^{\lambda_n} \exp[n\varphi(r_n) + a_n \lambda_n \tau_n].$$

Then, for $0 < a < 1$, the series T converges and

(13) $$||B V P_t \leq \& ^r H s^{AN} - P||$$

for $P \in [A_N]$, where $||\cdot||_Q$ is the L^2 norm on $(0,a)$. Observe that $F \to 0$ as $a \to 0$.

Proof: Let $a < a_1 < 1$. By (12), for large enough n we have

$$a_n T_n + (^n A_n) V(r_n) \leq (\log a_1/\alpha)$$

so that

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and, by (11), the series converges. Observe, now, that

$$\|s^{\lambda_N} - P\|_o = \|\Sigma_o^{\infty} b_n s^{\lambda_n}\| \leq \Sigma_o^{\infty} |b_n| \|s^{\lambda_n}\|_o .$$

Using (10) and then (9) and the evaluation

$$\|s^{\lambda}\|_o^2 = \alpha^{2\lambda+1} / (2\lambda+1),$$

we obtain

$$||sV P||_o \leq \Sigma_o P [\alpha^{2\lambda_n+1}/(2\lambda_n+1)]^{1/2} \|s^{\lambda_N} - P\| / \|s^{\lambda_n} - [A_n]\|$$

$$\leq \sqrt{\alpha}\, \|s^{\lambda_N} - P\| 2\pi a^{\lambda_n} \exp[n(p(r_n) + a_n \lambda_n \tau_n]$$

which is just (13). ||

We are now ready, at last, to obtain the desired lower bound for $||s^{\lambda_N} - [A_N]||_1$.

Theorem 1; Let $A = (7v_0, X_1, \ldots)$ be an increasing sequence $(0 \leq X_0 < \geq_1 u < \ldots)$ of reals satisfying (6) and (12) and let $0 < a < 1$ such that $(1 - oa^{-2}) = c^2 > 0$ (observe that $c = c(a) - 1$ as $a - 0$). Then

$$(14) \qquad \| sV_{\Lambda_N}] \; H \geq cH^{\Lambda^N} - [\Lambda_N \| \|$$

$$\geq c(2A_N + l)^{-1/2} \exp[-N < p(r_N) - a_N \lambda_N r_N] > 0.$$

Proof: For any $Pe[A^\wedge]$ we have

$$\| s^{A_N^T} - P \|^2 = \| s^{A_N^-} - P \|_0^2 + \| s^{A_N^-} - P \|_1^2$$

by the definition of the norms. Using (13) gives

$$\| sV \; P \|_1^2 \geq d - ar^2) \| s^{AN} - P \|^2 \geq C^2 \| S^{AN} - [\Lambda_N] \| .$$

Since this holds for every $Pe[A^\wedge]$,

$$\| s^{\lambda_N} - [\Lambda_N] \|_1 \geq c \| s^{\lambda_N} - [\Lambda_N] \|$$

which, with (9), gives (14). ||

§3: In order to apply the Theorem above to the prediction problem described earlier it is only necessary to show that the sequence $A = (n^2 : n = 0, 1, \ldots)$ satisfies (6) and (12). Certainly (6) holds and, in fact,

$$(15) \qquad T_N = \Sigma_{n+2}^{\infty} n^{-2} = \Theta(N^{-1}).$$

We have

$$\varphi(r) = |\log(1-r)/(1+2r)|, \quad r_n = n/(n+1)$$

so, for large N, it follows that U-r_N) = ©(iT^1),

(16) <P(r_N) = &(log(l-r_N)) = ©(log N)

and

(17) a_N = <P(r_N)/r_N = ©(log N) .

Thus, combining (15), (16) and (17),

a^ + (N/A_N)<p(r_N) = © (N""^{1}log N + (N/N^2) log N)
= ©(N""^{1}log N)

which goes to 0 as N —» oo , satisfying (12) .

It follows, therefore, that Theorem 1 may be applied so that, if *a* is small enough that

(18) $ar^2 < 1$,

we have $c^2 = 1 - alT^2 > 0$ and, by (5) and (14),

(19) 11*^11_< c"^{X}(2N^2 + l)$^{1/2}$exp[N<p(r_N)+ y ^]

= exp [©(N log N)] .

Substituting this in (3) we see that the factor a^{2N^2} = exp[-&(N^2)] dominates and the series converges. We have thus shown the following.

<u>Theorem 2</u>: For T large enough that (18) is satisfied with a = e^{-T}, the mapping

A: f = u(* ,0) i—*u(T, •) = w

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is a well-defined bounded (using L norms) linear map for solutions u of (1) with $0 < t < T$. I.e., the *observation and prediction[1] problem is well-posed. ||

Remark: The above proof does not, of course, show that A is undefined or unbounded for smaller $T > 0$. It would seem of interest to determine whether this notion of a minimal period of observation[1] is, indeed, a genuine phenomenon or whether it is, perhaps, imposed merely by the exigencies of this particular method of proof. Since F—•oo as *a*—*1, (18) is genuinely a restriction on T and it would also be of interest to estimate the minimal T for which (18) is satisfied. The complications involved in estimating the sum for *T* make the task of estimating a suitable T unfortunately difficult.

§4: A couple of generalizations of Theorem 2 suggest themselves. For example, it is clear that the similar problem for a non-uniform bar

$$(1') \qquad u_t = a(x)u_{xx}, \quad u_x(t,0) = u_x(t,l) = 0$$

could be treated the same way using the expansion

$$(2<) \qquad u(t,x) = L_n c_n e^{\ n} v_h(x)$$

where the $\{A_n\}$ and $\{v_n\}$ are the appropriate eigenvalues and eigenfunctions. The asymptotic behavior of the *[7*$_n$}* is known to be similar enough to that of $\{n^2\}$ to ensure the applicability of Theorem 1. The only new problem introduced would be the estimation of an asymptotic lower bound for $\{v_n(o)\}$.

Another example is the treatment of the observation and prediction problem for a solid body, rather than a rod. For a cylindrical body in R^k we have the following result•

<u>Theorem</u> 3: Let & be a 'suitable' region in R^{k-1} and let &* = (0,1)X *. Then, for T large enough (as in Theorem 2), the mapping from $L^2((0,T) \times \&)$ to $L^2(**)$. defined by

$$A!\ f = f(t,y) = u(t,0,y) i\!\!\longrightarrow w = w(x,y) = u(T,x,y)$$

is a well defined, bounded (using L^2 norms) linear map for solutions u of

$$\text{(1")} \qquad u_t = Au(=u_{xx} + A_y u) \qquad (0 < t < T,\ (x,y) \in \mathcal{D}_*),$$
$$\partial u/\partial \nu = 0 \qquad (0 < t < T,\ (x,y) \in \partial\mathcal{D}_*).$$

<u>**Proof**</u>: **We use the expansion**

$$\text{(2")} \qquad u(t,x,y) = E_{m,n}\, c_{m,n}\, e^{x}p[-(n^2 + M_m)\wedge^2 t] v_m(y) \cos nirx$$

where the $[v_m\}$ are normalized solutions of

$$\Delta_y v_m + \mu_m \pi^2 v_m = 0 \qquad (y \in \mathcal{D}),$$
$$\partial v_m/\partial \nu = 0 \qquad (y \in \partial\mathcal{D}).$$

Since the $\{v_{HI}\}$ form an orthonormal sequence we have

$$\langle u(t,0,\cdot), v_m\rangle_{\mathcal{D}} = e^{-u\, ir^{ry} t} \Sigma_n c_{m,n} e^{-2\ 2.} \text{»»*}.$$

Hence, letting I_N be as originally, we have

$$c_{m,N} = \ell_N(s^{-\mu_n} \langle \varphi(s,\cdot), v_m\rangle_{\mathcal{D}})$$

where

$$<p(s,y) = f(t,y) = u(t,0,y)$$

on making the substitution $s = e^{-\pi^2 t}$ as before. Thus

$$Af = E_{m,n} V^{s} \quad N \wedge ') ' 7 \wedge \quad n^{2}+/i \quad m \quad vjyjcos\ mx$$

and

$$\|Af\|^2 \leq \Sigma_n \alpha^{2n^2} \|\ell_n\|^2 \Sigma_m \alpha^{2\mu_m} \|s^{-\mu_m} <\varphi, v_m>_{\mathcal{D}}\|_1^2$$

$$\leq \Sigma_n \alpha^{2n^2} \|\ell_n\|^2 \Sigma_m \|<\varphi, v_m>_{\mathcal{D}}\|_1^2$$

$$= \Sigma_n \alpha^{2n^2} \|\ell_n\|^2 \|\varphi\|^2_{(\alpha,1)\times\mathcal{D}} \leq \pi^2 \Sigma_n \alpha^{2n^2} \|\ell_n\|^2 \|f\|^2.$$

This gives the identical estimate (3) for $\|A\|$ as in Theorem 1 and the same proof now goes through. ||

The result of Theorem 3 suggests a conjecture that a similar prediction problem would be well-posed given observation, for a sufficiently long period, of the restriction to a non-empty relatively open subset Q c: Sfi$_*$ where &$_*$ is a more general domain in R^n {Cl here corresponds to {0)x & in Theorem 3) . The methods of this paper, however, seem to afford no direct mode of attack on this more general problem; the proof of Theorem 3 makes essential use of the special nature, of &^ and Θ .

References

[1] Clarkson, J. A., and Erdős, P., 'Approximation by Polynomials i, Duke Math. J., 10, (1943) pp. 5-11.

[2] Rudin, W., 'Real and Complex Analysis,'