AN ECONOMETRIC METHOD FOR DECOMPOSING TOTAL INPUT PRODUCTIVITY INTO INPUT SPECIFIC PRODUCTIVITIES*

Peter A. Zadrozny

Bureau of Labor Statistics
Division of Price Index Number Research
2 Massachusetts Ave., NE, Room 3105,
Washington, DC, USA
e-mail: zadrozny.peter@bls.gov

For presentation at the 2nd Society for Economic Measurement conference, OECD, Paris, France,
July 22-24, 2015

[Updated July 23, 2015]

*This work represents the author's views and does not represent any official positions of the Bureau of Labor Statistics.
Organization:

1. Introduction.

2. Cobb-Douglas marginal production (CDMP).
   2.1. Statement of CDMP.
   2.2. Optimal input allocation in CDMP.
   2.3. Homogeneity and returns to scale in CDMP.
   2.4. Integrability of CDMP.


4. Estimation of CDMP model.

5. Application to quarterly U.S. QKLEMS data.

6. Conclusions.

7. References.
1. Introduction.

Productivity is a foremost measure of current and predicted future economic conditions of any production unit from a single firm to a whole economy.

Example: Blinder (2015) wrote that the fall in average annual labor productivity growth in U.S. from 2.9% in 1995-2005 to 1.3% since then raises concerns for future U.S. living standards.

The Solow residual $dr$ has been a principal measure of total input productivity:

\[ dr = dq - \mu_1 dx_1 - \ldots - \mu_n dx_n, \]

\[ dq = \text{dln of observed output}, \]

\[ dx_i = \text{dln of observed input } i, \]

\[ \mu_i = \text{observed cost share of input } i. \]
Conventionally, the Solow residual cannot be decomposed into input-specific productivity changes without additional input-specific information.

For example: Only if one knew how much capital had been installed recently and how much more productive it was compared to previously installed and still used capital, could one attribute a change in total-input productivity to capital productivity.

This paper develops and illustrates a method for decomposing the Solow residual into input-specific productivity changes or input-specific Solow residuals, that requires the same data as the computation of the Solow residual, i.e., requires no additional input-specific information.
Consider Cobb-Douglas Marginal Production (CDMP):

\( dq = \mu_1(dx_1 + da_1) + ... + \mu_n(dx_n + da_n) \)

and \( \mu_i \) is defined by eqs. (7) and (10). Eqs. (1)-(2) imply

\( dr = \mu_1da_1 + ... + \mu_nda_n. \)

Productivity analysis suggests: \( da_i \) measures efficiency of input \( dx_i \) or efficiency not captured by \( dx_i \), so that eq. (3) decomposes a Solow residual \( dr \) into input-specific Solow residuals \( da_i \).

Econometric analysis suggests: \( da_i \) should also reflect input measurement errors and model specification errors.

We admit all 3 possibilities but focus on the 1st:

\( da_i = \) unobserved combination of an efficiency measure, a specification error, and a measurement error of input \( i \), which we call the efficiency of input \( i \).
Econometric advantage of this CDMP productivity analysis:

For a sample $t = 1, \ldots, T$, $dr_t$ can be regressed onto input-specific information $z_t$ to determine sources of productivity change.

However, decomposition (3) suggests regressing $da_{it}$ onto $z_t$ to obtain more accurate inferences on productivity change.

Example:

Let $dx_{1t} = \text{capital input}$ and suppose only capital experiences productivity change according to

(4) \hspace{1cm} da_{1t} = \beta z_t + \varepsilon_t,

$z_t = \text{observed source of capital productivity},$

$\varepsilon_t = \text{unobserved disturbance} \sim \text{IID}(0, \sigma_\varepsilon^2)$ and independently of $z_t$. 
Dividing decomposition (3) by $\mu_{1t}$,

\begin{equation}
\frac{d\mu_t}{\mu_{1t}} = \beta z_t + v_t,
\end{equation}

\[ v_t = \varepsilon_t + \left(\frac{\mu_{2t}}{\mu_{1t}}\right)d\alpha_{2t} + \ldots + \left(\frac{\mu_{nt}}{\mu_{1t}}\right)d\alpha_{nt} \]

\[ = \text{disturbance} \sim \text{IID}(0, \sigma_v^2) \text{ and independently of } z_t. \]

Regression analysis of eqs. (4)-(5) yields consistent estimates of $\beta$, with a limiting ratio of t statistics of

\begin{equation}
\sqrt{1 + \left(\frac{\sigma_v^2}{\sigma_\varepsilon^2}\right)} > 1.
\end{equation}

Thus, regressing $d\alpha_{1t}$ onto $z_t$ instead of $d\mu_t/\mu_{1t}$ onto $z_t$ gives higher t statistics and more accurate inferences on the effect of capital productivity on total productivity.
2. Cobb-Douglas marginal production (CDMP).

2.1. **Statement of CDMP.**

**CDMP is based on 3 assumptions, A1 - A3:**

Assumption A1 in original unlogged form: For $i = 1, \ldots, n$,

$$
\frac{\partial q}{\partial x_i} = \left( \frac{A_i}{Q} \right) \prod_{j=1}^{n} x_j^{\delta_{ij} - \beta_{ij}} \equiv \mu_i,
$$

$X = (X_1, \ldots, X_n)^T = n \times 1$ vector of observed inputs,

$A = (A_1, \ldots, A_n)^T = n \times 1$ vector of unobserved variable efficiencies,

$B = [\beta_{ij}] = n \times n$ matrix of unobserved constant elasticities.

$\delta_{ij} = \text{Kronecker delta (} \delta_{ii} = 1; \delta_{ij} = 0, \text{ for } i \neq j).$
Assumption A1 in log-linear-system form:

\[
(8) \begin{bmatrix}
\ln(\partial Q / \partial x_1) \\
\vdots \\
\ln(\partial Q / \partial x_n)
\end{bmatrix} = \begin{bmatrix}
a_1 \\
\vdots \\
a_n
\end{bmatrix} + \begin{bmatrix}
1 - b_{11} & \ldots & - b_{1n} \\
\vdots & \ddots & \vdots \\
- b_{n1} & \ldots & 1 - b_{nn}
\end{bmatrix} \begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix}
\]

\[= a + (I_n - B)x,\]

\[a = (a_1, \ldots, a_n)^T = (\ln(A_1), \ldots, \ln(A_n))^T,\]

\[x = (x_1, \ldots, x_n)^T = (\ln(X_1), \ldots, \ln(X_n))^T.\]

Comments:

1. A1 is the only assumption that can be tested with data.

2. \(A_i\) generalize \(A_s\) and \(A_u\) in Caselli and Coleman (2006).


Assumption A2: For $i = 1, \ldots, n$,

(9) $\frac{\partial^2 Q}{\partial x_i \partial a_i} = \frac{\partial^2 Q}{\partial a_i \partial x_i}$

A2 is a mild and technical "smoothness" assumption.

Assumption A3: For $i = 1, \ldots, n$,

(10) $\frac{\partial q}{\partial a_i} = \frac{A_i}{Q} \prod_{j=1}^{n} x_j^{\delta_{ij}} \equiv \mu_i,$

Comments:

1. A3 is not really a separate assumption but derives from A1 and a first-order approximation of an integral, akin to a Harberger (1971) triangle in applied welfare analysis.

2. Deriving A3 from A1 is crucial, because A3 is crucial for the analysis but cannot be tested with data, because efficiency parameters $A_i$ are unobserved. A3 is arbitrary without this derivation.
CDMP in differential logarithmic form:

Marginal products (7) and (10) imply CDMP in differential logarithmic form,

\[ dq = \mu_1(dx_1 + da_1) + \ldots + \mu_n(dx_n + da_n), \]

\[ \mu_i \equiv (A_i/Q) \prod_{j=1}^{n} X_j^{\delta_{ij}} \beta_{ij}. \]

Comments:

1. The CDMP "relation" is not necessarily a "function" but is the "differential of a function".

2. A CDMP function, \( Q = f(X,A) \), may or may not exist, depending on whether integrability conditions hold.
CDMP roughly corresponds to standard production functions:

1. If $B = I_n$, then, $CDMP \approx$ Cobb-Douglas:
   
   Input-price elasticities depend on fixed $\beta_{ij} = 0, 1$.

2. If $B = \beta I_n$, $\beta > 0$, $\beta$ "moderate", then, $CDMP \approx$ CES:
   
   Input-price elasticities depend on the one $\beta$.

3. If $B = \beta I_n$, $\beta > 0$, $\beta$ "large", then, $CDMP \approx$ fixed proportions:
   
   Input-price elasticities are zero.

4. If $B = \text{diag}(\beta_{ii})$, $\beta_{ii} > 0$, $\beta_{ii}$ moderate, then, $CDMP = \text{generalized CES (GCES)}$:
   
   Input-price elasticities depend on all $\beta_{ii}$ and are illustrated below for a similar consumer-expenditure analysis.
2.2. **Optimal input allocation in CDMP.**

At moment $\tau$ in a sample period, given CDMP, values of efficiencies $A_i$, elasticities $\beta_{ij}$, prices $P = (P_1, \ldots, P_n)^T$, and output $\bar{Q}$:

Allocate inputs in $X = (X_1, \ldots, X_n)^T$ to minimize the cost of production,

\begin{equation}
C = P_1X_1 + \ldots + P_nX_n,
\end{equation}

subject to output being constant at $\bar{Q}$.

**First-order conditions (FOC) in dln form:** For $i = 1, \ldots, n$,

\begin{equation}
\begin{array}{c}
d\lambda + d\ln(\partial Q/\partial X_i) = p_i, \\
dq = 0,
\end{array}
\end{equation}

where $\lambda = \ln(\text{Lagrange multiplier})$, $p_i = \ln(P_i)$, and $q = \ln(Q)$. 
CDMP summary equation:

Combining eqs. (8), (11), and (13), we get the Slutsky-type summary equation

$$
\begin{bmatrix}
- \beta_{11} & \cdots & - \beta_{1n} & 1 \\
\vdots & & \vdots & \vdots \\
- \beta_{n1} & \cdots & - \beta_{nn} & 1 \\
\mu_1 & \cdots & \mu_n & 0
\end{bmatrix}
\begin{bmatrix}
da_1 \\
\vdots \\
da_n \\
d\lambda
\end{bmatrix}
=
\begin{bmatrix}
dp_1 + \sum_{j=1}^{n} \beta_{1j} dx_j \\
\vdots \\
dp_n + \sum_{j=1}^{n} \beta_{nj} dx_j \\
dq - \sum_{j=1}^{n} \mu_j dx_j
\end{bmatrix}.
$$

$$
\mu_i \equiv (A_i/Q) \prod_{j=1}^{n} x_j^{\delta_{ij}} - \beta_{ij}.
$$

Comments:

1. Eq. (14) summarizes the CDMP theory.

2. In eq. (14), unobserved and to be determined $da_i$ and $d\lambda$ are on the left side and observed and given $dp_i$, $dx_i$, and $dq$ are on the right side.
Second-order conditions (SOC):

SOC guarantee a unique local optimum X and are necessary for multi-step perturbation (MSP) computations to work correctly.

Standard SOC for a fixed price vector P (Mann, 1943; Samuelson, 1947) impose alternating signs on principal minors of the system matrix of eq. (14).

However, we want the SOC to be valid for all possible price vectors P and, therefore, assume:

(15) $B > 0$ (positive definite),

which means all eigenvalues of B are real and positive.

Comments:

1. B needn't be symmetric to be positive definite.

2. We want to allow asymmetric B.
Example of price and cost elasticities for B = diagonal:

Consider the dual problem of maximizing output for given cost. Let $c = \ln(C)$ and assume $da_i = 0$. Then, analogous to eq. (14),

$$
\begin{bmatrix}
-\beta_{11} & \cdots & -\beta_{1n} & 1 \\
\vdots & & \vdots & \vdots \\
-\beta_{n1} & \cdots & -\beta_{nn} & 1 \\
\sigma_1 & \cdots & \sigma_n & 0
\end{bmatrix}
\begin{bmatrix}
dx_1 \\
dx_n \\
dl
\end{bmatrix}
= 
\begin{bmatrix}
dp_1 \\
dp_n \\
dc
\end{bmatrix},
$$

(16)

where $s_i = P_iX_i/C = \text{cost share of input } i$.

For B = diagonal, (16) solves analytically as

$$
\frac{dx_i}{dp_i} = -\sigma_i [1 - (s_i\sigma_i/\overline{\sigma})] < 0,
$$

$$
\frac{dx_i}{dp_j} = s_j\sigma_i\sigma_j/\overline{\sigma} > 0 \ (i \neq j),
$$

$$
\frac{dx_i}{dc} = \sigma_i/\overline{\sigma} > 0, \quad \sigma_i = b_{ii}^{-1}, \quad \overline{\sigma} = \sum_{j=1}^{n} s_j\sigma_j.
$$
A similar analysis with $n = 8$ aggregate consumer expenditures resulted in estimated $B = \text{diag}(38, 8, 10, 32, 14, 151, 186, 32)$ and, for this $B$ and $s_i = \text{sample averages}$, elasticities (17) were:

```

<table>
<thead>
<tr>
<th>dq_i</th>
<th>dp_a</th>
<th>dp_e</th>
<th>dp_f</th>
<th>dp_g</th>
<th>dp_h</th>
<th>dp_m</th>
<th>dp_r</th>
<th>dp_t</th>
<th>dc</th>
</tr>
</thead>
<tbody>
<tr>
<td>Appar</td>
<td>-0.123</td>
<td>0.012</td>
<td>0.013</td>
<td>0.015</td>
<td>0.015</td>
<td>0.019</td>
<td>0.006</td>
<td>0.018</td>
<td>1.76</td>
</tr>
<tr>
<td>Educa</td>
<td>0.013</td>
<td>-0.059</td>
<td>0.006</td>
<td>0.007</td>
<td>0.007</td>
<td>0.009</td>
<td>0.003</td>
<td>0.008</td>
<td>0.746</td>
</tr>
<tr>
<td>Food</td>
<td>0.015</td>
<td>0.006</td>
<td>-0.062</td>
<td>0.008</td>
<td>0.008</td>
<td>0.010</td>
<td>0.003</td>
<td>0.010</td>
<td>0.801</td>
</tr>
<tr>
<td>Goods</td>
<td>0.021</td>
<td>0.009</td>
<td>0.009</td>
<td>-0.073</td>
<td>0.011</td>
<td>0.015</td>
<td>0.004</td>
<td>0.013</td>
<td>0.990</td>
</tr>
<tr>
<td>Housi</td>
<td>0.018</td>
<td>0.008</td>
<td>0.008</td>
<td>0.010</td>
<td>-0.071</td>
<td>0.013</td>
<td>0.004</td>
<td>0.012</td>
<td>0.941</td>
</tr>
<tr>
<td>Medic</td>
<td>0.029</td>
<td>0.012</td>
<td>0.013</td>
<td>0.016</td>
<td>0.016</td>
<td>-0.086</td>
<td>0.006</td>
<td>0.019</td>
<td>1.24</td>
</tr>
<tr>
<td>Recre</td>
<td>0.001</td>
<td>0.003</td>
<td>0.003</td>
<td>0.004</td>
<td>0.003</td>
<td>0.004</td>
<td>-0.030</td>
<td>0.004</td>
<td>0.367</td>
</tr>
<tr>
<td>Trans</td>
<td>0.022</td>
<td>0.009</td>
<td>0.010</td>
<td>0.012</td>
<td>0.012</td>
<td>0.016</td>
<td>0.005</td>
<td>-0.084</td>
<td>1.14</td>
</tr>
</tbody>
</table>
```
2.3. Homogeneity and returns to scale in CDMP.

Standard production functions like CD and CES are homogeneous (have linear expansion paths) and usually have constant returns to scale.

**CDMP homogeneity and returns to scale:**

One of Euler's theorems implies CDMP is homogeneous and has returns to scale $\rho > 0$, if and only if, for $i = 1, \ldots, n$,

\[(18) \quad \rho = 1 + \sum_{j=1}^{n} \beta_{ij} > 0.\]

**Comments:**

1. $\rho < 1$, $= 1$, and $> 1$, respectively, indicates decreasing, constant, and increasing returns to scale.

2. Homogeneity is unnecessary for applications with CDMP and can be tested using standard statistical tests.

3. $\rho$ can be estimated without output observations.
2.4. Integrability of CDMP.

Integrability means a production function exists and holds if and only if 2nd-partial derivatives of Q with respect to X and A are equal.

**CDMP is integrable if and only if:** For i, j, k = 1, ..., n, such that i ≠ j and k ≠ i or j,

\[
A_i \beta_{ij} = A_j \beta_{ji}, \quad \beta_{ii} = 1 + \beta_{ji}, \quad \beta_{ik} = \beta_{jk}.
\]

**Comments:**

1. Conditions (19) are quite stringent and hold for variable \(A_i\) if and only if \(B = \text{diagonal}\).

2. Nonintegrability complicates estimation of \(B\) by requiring estimation of most likely continuous paths of observed variables, but doesn't affect computing \(\Delta a_{it}\).

3. Interestingly, nonintegrability says that "history matters", which seems appropriate for development and adaptation of technology.

The CDMP theory has been stated in empirically inapplicable differential and continuous-time form. To apply the theory to discrete-time observations, summary equation (14) must be integrated to first-differenced discrete-time form.

Discrete-time observations on inputs, prices, and output relate to integrated CDMP variables in first differences

\[
\Delta x_{it} = \int_{\tau=t}^{t+1} dx_i(\tau), \quad \Delta p_{it} = \int_{\tau=t}^{t+1} dp_i(\tau), \quad \Delta q_t = \int_{\tau=t}^{t+1} dq(\tau).
\]

Computational problem: Given observed $\Delta x_{it}$, $\Delta p_{it}$, and $\Delta q_t$, numerically solve eq. (14) and, thereby, compute discrete-time first-differenced efficiencies

\[
\Delta a_{it} = \int_{\tau=t}^{t+1} da_i(\tau).
\]

We propose using MSP for doing this because it worked accurately for a similar project (Chen and Zadrozny, 2009).
4. Estimation of CDMP Model.

There are at least 2 possible estimation strategies:

1. A simpler, consistent, but inefficient strategy that doesn't use observations on output.

2. A more complicated but efficient strategy that uses observations on output.

Simpler estimation strategy:

1. Consider the first n equations of FOC (14) in integrated discrete-time form.

2. Restate the equations with observed $\Delta x_{it}$ and $\Delta p_{it}$ on the left side and unobserved $\Delta a_{it}$ and $\Delta \lambda_t$ on the right side.

3. Consider the right side of the equations as an $n \times 1$ vector of unobserved disturbances, $\eta_{xt}$, composed of the $\Delta a_{it}$ and $\Delta \lambda_t$. 
4. Consider the input-price vector as being generated by $\Delta p_t = \eta_{pt}$, where $\eta_{pt}$ is an $n \times 1$ unobserved time-series process.

5. Stack the resulting equations as

$$
\begin{pmatrix}
-B & I_n \\
0_{n \times n} & I_n
\end{pmatrix}
\begin{bmatrix}
\Delta x_t \\
\Delta p_t
\end{bmatrix}
= \begin{bmatrix}
\eta_{xt} \\
\eta_{pt}
\end{bmatrix}
\equiv \eta_t,
$$

$\Delta x_t = (\Delta x_{1t}, \ldots, \Delta x_{nt})^T$, $\Delta p_t = (\Delta p_{1t}, \ldots, \Delta p_{nt})^T$.

6. Specify a vector time-series process for the $2n \times 1$ disturbance vector, $\eta_t$, such that $\eta_{xt}$ and $\eta_{pt}$ are independent, which is an identifying assumption on $B$.

7. Estimate $B = [\beta_{ij}]$ and the parameters of $\eta_t$ by maximum likelihood (ML).
5. **Application to U.S. quarterly QKLEMS data.**

[To be completed]
6. Conclusions.

1. The main point of this project is the CDMP method and not the particular CDMP formulation. Other useful formulations will surely be found.

2. The moral of the story is that applying mathematics in a new way to an old problem can yield new and interesting results.

3. Please suggest some data that could be used to demonstrate regression analysis of eqs. (4) and (5).
7. References.


