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Juan Jorge Schäffer
Carnegie Mellon University, js6n@andrew.cmu.edu

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BOUNDS FOR THE GIRTH OF SPHERES

Juan Jorge Schäffer

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Let $X$ be a real normed linear space with norm $||x||$, and let $T$ be its unit ball, with the boundary $BE$. Assume $\dim X > 2$. These notations and assumptions will be maintained throughout the paper. In [1] we defined the girth of $S$ to be $2m(X)$, where $m(X) = \inf \{6(-p,p) : p \in \mathbb{F}\}$ and $6$ denotes the inner metric of $\mathbb{F}$ induced by the norm or, equivalently, $m(X) = \inf \{L(c) : c$ a rectifiable curve in $5\mathbb{F}$ with antipodal endpoints\}. If $\dim X < \infty$, then these infima are attained, and $m(X) > 2$ [1; Lemma 5.1, Theorem 5.5]. The purpose of this paper is to sharpen this inequality to $m(X) > 2(1+n^{-1})$, where $n = \dim X$, and to remark that this bound is best possible when $\dim X$ is even.

In [4] the following property of a space $X$ was defined, for a given positive integer $n$ and a given real $p$, $0 < p < 1$:

$$\text{(J)}_{n,p}: \text{There exist } x_k \in X, k = 1, \ldots, n \text{ such that }$$

$$\left|\sum_{k=1}^{n} p^{k-1} \right| > n^p \text{ for every sequence } (p_k)_{k=1}^{n} \text{ satisfying (J)_{n,p}}.$$

1. **Lemma** ([4; Theorem 3.2]). If $m(X) < 2/B^n$, then $X$ satisfies (J).

**Proof.** Under the assumption, there exists a rectifiable curve in $5\mathbb{F}$ with endpoints, say, $-p^*p$ and length $t < 2p^*$. Let $g: [0,1] \rightarrow hji$ be its parametrization in terms of arc-length.

For a given positive integer $n$, set $p_k = g(\frac{k}{n} - 1)$, $k = 0, \ldots, n$ so that $P_0 + P_n = -P + P = 0$. Set $x^* = \sum_{k=1}^{n} p_{k-1}$. Then $||x^*|| = n^{1/2}gC kn^{-1/2}U - 1J n^{-1}! > 1$, and $||j-p^* x, j|| = n^{-1}! > n > pn$, $j = 0, \ldots, n$, $3+1/k$.
2. **Theorem.** If \( \dim X = n < \infty \), then \( m^{n.2(l+n^{-\frac{1}{2}})} \).

**Proof.** Assume that \( m(X) < 2(l+n^{\frac{1}{2}}) \); by Lemma 1, \( X \) satisfies \((J, , , \ldots, -1)\). Let then \( x, \epsilon X, k = 0, \ldots, n \), be such that

\[
\sum_{j=0}^{n} x_j^2 > n(n+1)^{-\frac{1}{2}} = n, \quad j = 0, \ldots, n + 1.
\]

Since \( \dim X = n \), there exist real numbers \( a_k, k = 0, \ldots, n \), not all 0, such that

\[
\sum_{k=0}^{n} a_k x_k = 0.
\]

We may assume without loss that

\[
\max\{|a_k| : k \in \{0, \ldots, n\}\} = 1
\]

and that, say, \( |a_h| = 1 \) for some \( h, 0 \leq h \leq n \). Then

\[
\sum_{k=0}^{h-1} a_k + T a_h > 1 \quad \text{for some} \quad j, 0 \leq j \leq n + 1.
\]

Combining (1) for that value of \( j \) with (2),(3),(4), we obtain

\[
\sum_{k=0}^{n} a_k x_k > 1 \quad \text{for some} \quad j, 0 \leq j \leq n + 1.
\]

a contradiction.
In [2] it was shown that, if \( \dim X = n \) and \( j \) is a parallelotope (i.e., \( X \) is congruent to \( I^n \)), then \( m(X) = 2(l+(n-1)^{-1}) \); it was further shown that, if \( n \) is odd, there is a subspace \( Y \) of co-dimension 1 such that \( m(Y) \) is still \( 2(l+(n-1)^{-1}) \). (We remark that exactly the same results obtain if \( X \) is taken to be congruent to \( K^\frac{1}{n} \) instead of to \( I^n \), but we omit the proof.) Thus Theorem 2 yields the best lower bound for even dimension.

This conclusion is best stated in terms of \( m^*(n) = \min\{m(X): \dim X = n\} \), \( n = 2, 3, \ldots \), a sequence of numbers introduced (and shown to exist) in [1].

3. **Theorem.** \( m^*(n) = 2(l+n^{-1}) \) if \( n \) is even,
\[ 2(l+n^{m-1}) \leq m^*(n) \leq 2(l+(n-1)^{-1}) \] if \( n \) is odd.

**Proof.** Theorem 2 and [2; Theorem 7].

This theorem confirms one-half of the conjecture at the end of [2]; the other half, asserting that \( m^*(n) = 2(l+(n-1)^{m-1}) \) when \( n \) is odd, has so far only been confirmed for \( n = 3 \) (see [3]).
References.


4. _____________ and K. Sundaresan: Reflexivity and the girth of spheres.