Nonlinear eigenvalue problems and critical points of functions

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NONLINEAR EIGENVALUE PROBLEMS AND
CRITICAL POINTS OF FUNCTIONS

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1. Introduction

We here study the nonlinear eigenvalue problem

\[ A \tilde{x} + F(\tilde{x}) = \lambda \tilde{x} \quad (1.1) \]

where \( A : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is self-adjoint and linear and \( \nabla \cdot (\cdot) \) is the gradient of a potential \( \psi \); i.e.

\[ \nabla \cdot (\tilde{x}) = \nabla \cdot \psi (\tilde{x}) \quad (1.2) \]

It is well known that the nontrivial solutions of (1.1) of fixed amplitude \( \tilde{r} \) (i.e. \( \tilde{x} \cdot \tilde{x} = \tilde{r}^2 \)) are the critical points of

\[ \phi (\tilde{x}) \equiv \frac{1}{2} A \tilde{x} \cdot \tilde{x} + \psi (\tilde{x}) \text{ on } \tilde{x} \cdot \tilde{x} = \tilde{r}^2. \]

Moreover, if \( \chi \) is such a critical point, then the eigenvalue \( \lambda (\chi) \) is given by

\[ \lambda (\chi) = \frac{(\tilde{\chi} \cdot \nabla \psi (\chi)) \cdot \tilde{\chi}}{\tilde{r}^2} \quad (1.3) \]

It is no loss in generality to assume that

\[ \frac{1}{2} A \tilde{x} \cdot \tilde{x} = \sum_{i=1}^{n} \lambda_i x_i^2 ; \quad (1.4) \]

in this case \( \phi \) takes the form

\[ \phi (\tilde{x}) \equiv \sum_{i=1}^{n} \lambda_i x_i^2 + \psi (\tilde{x}) \quad (1.5) \]

Our interest is in showing that if appropriate conditions are met, then the nontrivial solutions of (1.1) (eigenvectors of \( A + F \)) may be parameterized smoothly by \( \tilde{r} \).

We also obtain results about the maximal extension of a given branch of eigenvectors.
2. **Statement of Results**

Let \( \phi \) be given by (1.4) and assume that the numbers \( \{ \lambda_j, j = 1, 2, \ldots, n \} \) are distinct and indexed in decreasing order \( (\lambda_j > \lambda_{j+1}) \). In addition assume that the map \( x \mapsto \psi(x) : \mathbb{R}^n \to \mathbb{R} \) is smooth \((C^3 \text{ will suffice}) \) and satisfies

\[
\left( \sum_{|\alpha|=j} |D^\alpha \psi(x)|^2 \right)^{1/2} \leq K\|x\|^{3-j}, \ j = 0, 1, 2. \quad (2.1)
\]

In (2.1) \( D^\alpha \) stands for any derivative of order \( j \) and \( \|\cdot\| \) for the Euclidean norm.

The assumption that the \( \lambda_j \)'s are distinct implies that the vectors

\[
\pm r e_j, \ e_j = (0, \ldots, 0, 1, 0, \ldots, 0), \ j = 1, 2, \ldots, n,
\]

are the unique critical points of \( \phi_0 = \sum_{i=1}^n \lambda_i x_i^2 \) on \( x^* x = r^2 \).

For each \( 0 < \epsilon < 1 \) we let

\[
\eta_j^{(+)}(1, \epsilon) = \left\{ \nu \middle| v_j = \frac{+}{(-)} \sqrt{1 - \sum_{k=1}^n v_k^2}, \ \sum_{k=1, k \neq j}^n v_k^2 \leq \epsilon^2 \right\}
\]

For our purposes we will want two numbers \( 0 < \epsilon_1 < \epsilon_2 < 1 \) such that

\[
\eta_j^{(+)}(1, \epsilon_1) \equiv \left\{ \nu \middle| \|\nu\|^2 = 1, \ \|\nu_{\sim j} \|_{\sim} \leq \epsilon_2 \right\}
\]

*or equivalently \( \eta_j^{(+)}(1, \epsilon) \equiv \left\{ \nu \middle| \|\nu\|^2 = 1, \ \|\nu_{\sim j} \|_{\sim} \leq \epsilon_2 \right\} \)
the neighborhoods \( \eta_j^{+(-)}(1, \varepsilon_1) \), \( j = 1, 2, \ldots, n \) are disjoint on the unit sphere while the neighborhoods \( \eta_j^{+(-)}(1, \varepsilon_2) \), \( j = 1, 2, \ldots, n \) cover the unit sphere.

**Theorem 1** (Local Existence and Uniqueness Theorem)

There is an \( r_0 > 0 \) such that for any \( r \in (0, r_0] \) the function \( \phi \) has exactly 2n critical points on the sphere \( \tilde{x} \cdot \tilde{x} = r^2 \). These points may be labeled in pairs \( (\tilde{x}_j^+(r), \tilde{x}_j^-(r)) = r (v_j^+(r), v_j^-(r)) \) according to the scheme

\[
\begin{align*}
v_j^+(r) &\in \eta_j^{+(-)}(1, \varepsilon_1) \quad \text{and} \quad v_j^-(r) \in \eta_j^{+(-)}(1, \varepsilon_1). 
\end{align*}
\] (2.3)

The functions \( r \rightarrow v_j^+(r) \) (respectively \( r \rightarrow v_j^-(r) \)) are \( C^1(0 < r \leq r_0) \) and satisfy

\[
\lim_{r \to 0} v_j^+(r) = e_j \quad \text{and} \quad \lim_{r \to 0} v_j^-(r) = -e_j , \quad j = 1, 2, \ldots, n. \] (2.4)

In order to state the global existence theorem it is necessary to introduce some additional notation. For each \( \tilde{v} \Rightarrow \|\tilde{v}\|^2 = 1 \) we let \( V(\tilde{v}) \) be the \( n-1 \) dimensional vector space

\[
V(\tilde{v}) = \{ u \in \mathbb{R}^n \mid u \cdot \tilde{v} = 0 \} \] (2.5)

For each \( r > 0 \) and \( \tilde{v} \Rightarrow \|\tilde{v}\|^2 = 1 \) we define the symmetric bilinear form \( B(\tilde{v} \tilde{w} ; \cdot, \cdot) : V(\tilde{v}) \times V(\tilde{v}) \rightarrow \mathbb{R} \) by

\[
B(\tilde{v} \tilde{w} ; u, \tilde{w}) = \frac{1}{r^2} \frac{\partial}{\partial s \partial t} 2 \phi \left( r (v + su + tw) \right) \bigg|_{s=t=0}
- \frac{1}{r} (\nabla \tilde{v} \phi(\tilde{v} \tilde{w}) \cdot \tilde{v})(u \tilde{w}), \] (2.6)
\( \mathfrak{B}(rv) : V(\nu) \rightarrow V(\nu) \) is the linear operator associated with the bilinear form \( B(rv ; \cdot , \cdot ) \).

Theorem 2 (Global Existence of a Given Branch of Critical Points)

For each \( j \) it is possible to extend the function \( r \mapsto \nu^+_j(r) \) from \([0, r_0]\) to some maximal interval \([0, R^+_j]\) in such a way that the function \( x^+_j(r) = r \nu^+_j(r) \) is a critical point of \( \phi \) (or \( \nu \cdot \nu = r^2 \)). The function \( \nu^+_j(\cdot) \) is extended as the unique solution of the initial value problem:

\[
\mathfrak{B}(rv) \nu^+(r) = F(r, \nu) , \quad r > r_0
\]

\( \nu^+(r_0) = \nu^+_j(r_0) \) .

The initial data \( r \nu^+_j(r_0) \) is the unique critical point of \( \phi \) on \( \nu \cdot \nu = r_0^2 \) such that \( \nu^+_j(r_0) \in \eta^+_j(1, \varepsilon) \), and \( F(r, \nu) \in V(\nu) \) is defined by

\[
F(r, \nu) = \frac{\partial}{\partial r} \left\{ \frac{1}{r} \left[ \nu \cdot \nu(r, \nu) - (\nu \cdot \nu(r, \nu) \cdot \nu) \nu \right] \right\}.
\]

The number \( R^+_j \) is characterized as the first \( r > 0 \) such that the quadratic form \( B(\nu^+_j(r) ; \nu, \nu) \) has zero as a critical value on the unit sphere \( V(\nu^+_j(r)) \). For all \( r < R^+_j \) the quadratic has \( j-1 \) positive and \( n-j \) negative critical values on the unit sphere \( V(\nu^+_j(r)) \).

The following example shows that Theorems 1 and 2 are the best that may be expected.

* For any \( u \in V(\nu) \) \( B(rv)u = [\nu \cdot \nu \phi(rv)]u - (u \cdot [\nu \cdot \nu \phi(rv)] \nu) \nu \ww - \frac{1}{r} (\nu \cdot \nu \phi(rv) \cdot \nu)u \).
Let
\[ \phi_0 = x^2 + 2y^2 \],
\[ \psi = \frac{2}{3} x^3 - (x^2 + y^2)(x^2 + 2y^2) \],
and
\[ \phi = \phi_0 + \psi \).

If we introduce the polar coordinates
\[ x = r \cos \theta \] and \[ y = r \sin \theta \],
then \( \eta(r, \theta) = \frac{\phi}{r^2} \) (\( r \cos \theta \), \( r \sin \theta \)) takes the form
\[ \eta(r, \theta) = (1-r^2) (\cos \theta + 2 \sin \theta) + \frac{2}{3} r \cos^3 \theta \).

A simple computation shows that the critical points of \( \eta \) on \( x^2 + y^2 = r^2 \) are those numbers \( \theta \in [0,2\pi) \) which satisfy
\[ \frac{\partial \eta}{\partial \theta} = \sin 2\theta (1-r^2 - r \cos \theta) = 0 \).

It is clear that for all \( r > 0 \) the numbers \( \theta = 0, \pi/2, \pi, \) and \( 3\pi/2 \) are critical points of \( \eta \). For
\[ \frac{\sqrt{5} - 1}{2} < r < \frac{\sqrt{5} + 1}{2} \] there are two additional critical points
\( \theta_1 \in (0,\pi) \) and \( \theta_2 \in (\pi,2\pi) \) which satisfy
\[ \cos \theta_i = \frac{1-r^2}{r}, \quad \frac{\sqrt{5} - 1}{2} < r < \frac{\sqrt{5} + 1}{2}, \quad i = 1,2. \]

Finally, for \( r > \frac{\sqrt{5} + 1}{2} \) the numbers \( \theta = 0, \pi, \) and \( \frac{3\pi}{2} \) are again the
only critical points.

We now analyze these critical points in more detail. For $0 < r < \sqrt{5 - \frac{1}{2}}$ the points $\theta = 0$ and $\pi$ correspond to relative minima of $\eta$ while $\theta = \pi/2$ and $3\pi/2$ correspond to relative maxima. At $r = \sqrt{5 - \frac{1}{2}}$ (the point where $\frac{\partial^2 \eta}{\partial \theta^2} | \theta = 0 = 0$) the point $\theta = 0$ becomes an inflection point, and for all $r > \sqrt{5 + \frac{1}{2}}$ is a maximum.

For $\sqrt{5 - \frac{1}{2}} < r < 1$ the points $\theta_1 \in (0, \pi/2)$ and $\theta_2 \in (3\pi/2, 2\pi)$ correspond to relative minima of $\eta$ and the character of the points $\theta = \pi, \pi$, and $3\pi/2$ is as before. At $r = 1$ (the point where $\frac{\partial^2 \eta}{\partial \theta^2} | \theta = \pi/2 = \frac{\partial^2 \eta}{\partial \theta^2} | \theta = 3\pi/2$) coalesces with $\pi/2$ and $\theta_2$ with $3\pi/2$. For $1 < r < \sqrt{5 + \frac{1}{2}}$ $\theta = \pi/2$ and $3\pi/2$ become relative minima, the points $\theta_1 \in (\pi, 3\pi/2)$ relative maxima. $\theta = \pi$ is still a relative minimum. At $r = \sqrt{5 + \frac{1}{2}}$ (the place where $\frac{\partial^2 \eta}{\partial \theta^2} | \theta = \pi = 0$) $\theta_1 = \theta_2 = \pi$ is an inflection point of $\eta$. For $r > \sqrt{5 + \frac{1}{2}}$ the critical points $\theta = 0$ and $\pi$ correspond to relative maxima of $\eta$ while $\theta = \pi/2$ and $3\pi/2$ correspond to relative minima.
3. Proofs

To establish Theorem 1 we look at $\phi$ in neighborhoods of the critical points of $\phi_0 \equiv \sum_{i=1}^{n} \lambda_i x_i^2$ on $x \cdot x = r^2$. Clearly, it suffices to look at $\phi$ in a neighborhoods of $r e_1$.

We now let $0 < \epsilon_1 < \epsilon < 1$ be two numbers such that the neighborhoods $\eta_j^{(\epsilon)}(1, \epsilon_1)$, $j = 1, 2, \ldots, n$ are disjoint on $x \cdot x = 1$ while the neighborhoods $\eta_j^{(\epsilon)}(1, \epsilon_2)$, $j = 1, 2, \ldots, n$ cover $x \cdot x = 1$. We set $x = r \tilde{x}$, $\tilde{y} = (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n)$ and introduce local coordinates:

$$v_1 = \sqrt{1 - \sum_{k=2}^{n} v_k^2}$$

where $\sum_{k=2}^{n} v_k^2 \leq \epsilon_2^2$. (3.1)

The function $\phi$ becomes $r^2 \hat{\phi}(r, v)$ where

$$\hat{\phi}(v_2, v_3, \ldots, v_n; r) = [\lambda_1 + \sum_{k=2}^{n} \mu_k v_k^2]$$

$$+ \frac{1}{r^2} v(r \sqrt{1 - \sum_{k=2}^{n} v_k^2}, rv_2, \ldots, rv_n),$$

(3.2)

and $\mu_k = \lambda_k - \lambda_1$.

To obtain Theorem 1 it suffices to show that for $r$ sufficiently small ($< \text{some } r_0$)

(A) There exists a unique $n$-1 tuple $(v_2^+, v_3^+, \ldots, v_n^+)$ with

$$\sum_{k=2}^{n} v_k^2 \leq \epsilon_1^2$$

such that
\frac{\partial \phi}{\partial v_i}(v_2, v_3, \ldots, v_n; r) = \left\{ 2\mu_i \psi \left( r \sqrt{1 - \frac{\sum_{k=2}^{n} v_k^2}{r}}, r v_2, \ldots, r v_n \right) \right\} v_i \frac{r}{\sqrt{1 - \sum_{k=2}^{n} v_k^2}} + \frac{1}{r} \psi(x_1) r \sqrt{1 - \sum_{k=2}^{n} v_k^2} = 0, i = 2, 3, \ldots, n. \tag{3.3}

(B) The \((n-1)\) tuple \((v_2^+, v_3^+, \ldots, v_n^+)\) satisfying (3.3) is the only solution in the larger sphere \(\sum_{k=1}^{n} v_k^2 \leq \epsilon_2^2\); and

(C) the map \(r \rightarrow (v_2^+(r), v_3^+(r), \ldots, v_n^+(r))\) is \(C^1(0 < r \leq r_0)\) and satisfies

\[ \lim_{r \to 0} (v_2^+(r), v_3^+(r), \ldots, v_n^+(r)) = (0, 0, \ldots, 0). \tag{3.4} \]

The growth condition on \(\psi\) implies that

\[ \frac{1}{r} \psi(x_1) \left( r \sqrt{1 - \frac{\sum_{k=2}^{n} v_k^2}{r}}, r v_2, \ldots, r v_n \right) \leq \frac{K r}{\sqrt{1 - \epsilon_2^2}} \tag{3.5} \]
and
\[
\left( \sum_{i=2}^{n} \psi_{x_i} \left( r \sqrt{1 - \sum_{k=2}^{n} \lambda_k^2, \lambda_2, \ldots, \lambda_n} \right) \right)^{1/2} \leq K r \quad (3.6)
\]

for all \((\lambda_2, \lambda_3, \ldots, \lambda_n)\) satisfying
\[
\sum_{k=2}^{n} \lambda_k^2 \leq \epsilon_2^2.
\] (3.7)

Equation (3.5) implies that if
\[
r \leq r_1 = \min_i \left| \mu_i \right| \frac{1}{\mu_1} \sqrt{1 - \epsilon_2^2} = \frac{1}{\mu_2} \frac{1}{\sqrt{1 - \epsilon_2^2}},
\] (3.8)

and if (3.7) holds, then the operator
\[
P = \text{diag} (p_1, p_2, \ldots, p_n)
\]

with
\[
p_i = 2\mu_i - x_i \left( r \sqrt{1 - \sum_{k=2}^{n} \lambda_k^2, \lambda_2, \ldots, \lambda_n} \right)
\]

is invertible and
\[
\|p^{-1}\| \leq \frac{1}{1/|\mu_2|}
\] (3.9)
It now follows that solving (3.3) is equivalent to solving

\[ v_i = T_i(v_2, v_3, \ldots, v_n; r), \quad i = 2, 3, \ldots, n \]  

(3.10)

where

\[ T_i(v_2, v_3, \ldots, v_n; r) = \frac{1}{2\mu_i - \frac{\psi_{x_i}}{r}} \psi_{x_i} \left( r \sqrt{1 - \frac{n}{k=2}v_k^2}, rv_2, \ldots, rv_n \right) \]  

(3.11)

We now observe that for \( r \leq r_1 \) and \((v_2, v_3, \ldots, v_n)\) satisfying (3.7)

\[ \left( \sum_{k=2}^{n} T_k^2 \right)^{1/2} \leq \frac{K r}{|\mu_2|} \]  

(3.12)

Equation (3.12) implies that for

\[ r \leq \min(r_1, \frac{|\mu_2|}{K} \epsilon_1) \equiv r_2 \]  

(3.13)

\[ T(\cdot, r) : \sum_{k=2}^{n} v_k^2 \leq \epsilon_2^2 \rightarrow \sum_{k=2}^{n} v_k^2 \leq \epsilon_1^2 \]  

(3.14)

hence for \( r \leq r_2 \) any fixed point of (3.10) in \( \sum_{k=2}^{n} v_k^2 \leq \epsilon_2^2 \) must be in \( \sum_{k=2}^{n} v_k^2 \leq \epsilon_1^2 \). The smoothness of \( \psi \) implies that \( T(\cdot, r) \) is \( C^1 \) in \((v_2, v_3, \ldots, v_n)\) and hence Brouwer's Theorem.
guarantees (for all $r \leq r_2$) the existence of at least one solution of (3.10) (and hence (3.3)) in $\sum_{k=2}^{n} v_k^2 \leq \varepsilon_2^2$.

To establish uniqueness, it suffices to show that for some $r_o \leq r_2$ and all $r$ in $[o, r_o]$ the maps $T(\cdot, r)$ are contractions on $\sum_{k=2}^{n} v_k^2 \leq \varepsilon_2^2$. This computation follows from (2.1).

The smoothness of $r \rightarrow v_i(r) = (\sqrt{1 - \sum_{k=2}^{n} v_k^2(r), v_2(r), \ldots, v_n(r)})$ follows from the smoothness of $x \rightarrow \varphi(x)$. We find that for $o < r \leq r_o$ $\dot{v}_i(r) = \frac{d}{dr} v_i(r)$ exists, is continuous, and satisfies

$$ \sum_{j=2}^{n} b_{ij}(v_2, v_3, \ldots, v_n; r) \dot{v}_j(r) = f_i(v_2, v_3, \ldots, v_n; r), \quad i = 2, 3, \ldots, n, \quad (3.15) $$

where

$$ b_{ij}(v_2, v_3, \ldots, v_n; r) = \begin{cases} 2\mu_i 1 - \frac{\psi_{x_1}}{r} \sqrt{1 - \sum_{k=2}^{n} v_k^2} & \text{if } i = j \\ \frac{1}{r} \sqrt{1 - \sum_{k=2}^{n} v_k^2} & \text{if } i \neq j \\ \end{cases} $$

and $f_i(v_2, v_3, \ldots, v_n; r)$ is given by

$$ f_i(v_2, v_3, \ldots, v_n; r) = \left\{ \frac{\psi_{x_1} v_1 + \psi_{x_1} v_j}{\sqrt{1 - \sum_{k=2}^{n} v_k^2}} (r \sqrt{1 - \sum_{k=2}^{n} v_k^2}, rv_2, \ldots, rv_n) \right\}_j $$

$\delta_{ij}$
\[
\begin{align*}
+ \frac{v_i v_j}{(1 - \sum_{k=2}^{n} v_k^2)^{3/2}} & \left\{ \sqrt{1 - \sum_{k=2}^{n} v_k^2} \psi_{x_i x_j} - \frac{\psi_{x_1 x_1}}{r} \right\} \left( r \sqrt{1 - \sum_{k=2}^{n} v_k^2}, r v_2, \ldots, r v_n \right) \\
+ \psi_{x_i x_j} \left( r \sqrt{1 - \sum_{k=2}^{n} v_k^2}, r v_2, \ldots, r v_n \right) & ; i, j = 2, 3, \ldots, n, \quad (3.16)
\end{align*}
\]

and

\[
F_i(v_2, v_3, \ldots, v_n; r) =
\]

\[
\frac{1}{r^2} \left\{ \psi_{x_i x_i} - \frac{\psi_{x_1 x_1} v_i}{\sqrt{1 - \sum_{k=2}^{n} v_k^2}} \right\} \left( r \sqrt{1 - \sum_{k=2}^{n} v_k^2}, r v_2, \ldots, r v_n \right)
\]

\[
\frac{v_i}{r} \left\{ \psi_{x_i x_1} + \frac{\sum_{j=2}^{n} \psi_{x_1 x_j} v_j}{\sqrt{1 - \sum_{k=2}^{n} v_k^2}} \right\} \left( r \sqrt{1 - \sum_{k=2}^{n} v_k^2}, r v_2, \ldots, r v_n \right)
\]

\[
+ \frac{1}{r} \left\{ \sqrt{1 - \sum_{k=2}^{n} v_k^2} \psi_{x_i x_i} + \frac{\sum_{j=2}^{n} \psi_{x_i x_j}}{r} \right\} \left( r \sqrt{1 - \sum_{k=2}^{n} v_k^2}, r v_2, \ldots, r v_n \right),
\]

\[i = 2, 3, \ldots, n. \quad (3.17)\]

To obtain the limiting relation (3.4) we simply make use of the estimate (3.12).

To establish theorem 2 it again suffices to work with a particular branch of critical points of \( \phi \). We shall extend the branch
\( x_1^+(r) = r \, x_1^+(r) , \, r \in [0, r_0] \). It is clear that if we extend

\( x_1^+(r) \) as a solution the initial value problem (2.7), then

\( x_1^+(r) = r x_1^+(r) \) will be a critical point of \( \phi \) on \( x \cdot x = r^2 \).

It is also clear that the initial value problem (or any of its representations) has a (have) unique solution(s) provided the operator \( \Theta(r_1^+(r)) : V(r_1^+(r)) \rightarrow V(r_1^+(r)) \) is invertible.

The condition for the lack of invertibility \( \Theta \) along \( rv_1^+(r) \)

is simply that the quadratic \( B(rv_1^+(r) ; u,u) \) have 0 as a
critical value on the unit sphere \( V(r_1^+(r)) \). That \( B(rv_1^+(r) ; u,u) \)
has no positive and \( n-1 \) negative critical values for \( r < R_1 \)
follows from the fact that \( \Theta(rv_1^+(r)) \) is symmetric and invertible for
\( r < R_1 \) and the fact that \( \Theta_0,1 = \lim_{r \rightarrow 0^+} \Theta(rv_1^+(r)) = \text{diag} (\mu_2^1, \mu_3^1, \ldots, \mu_n^1) \)
maps \( V(e_1) \rightarrow V(e_1) \) and has no positive and \( n-1 \) negative eigenvalues.