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Deepening the Automated Search for Gödel's Proofs

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Abstract

Gödel's incompleteness theorems establish the stunning result that mathematics cannot be fully formalized and, further, that any formal system containing a modicum of number or set theory cannot establish its own consistency. Wilfried Sieg and Clinton Field, in their paper *Automated Search for Gödel's Proofs*, presented automated proofs of Gödel's theorems at an abstract axiomatic level; they used an appropriate expansion of the strategic considerations that guide the search of the automated theorem prover AProS. The representability conditions that allow the syntactic notions of the metalanguage to be represented inside the object language were taken as axioms in the automated proofs. The concrete task I am taking on in this project is to extend the search by formally verifying these conditions. Using a formal metatheory defined in the language of binary trees, the syntactic objects of the metatheory lend themselves naturally to a direct encoding in Zermelo's theory of sets. The metatheoretic notions can then be inductively defined and shown to be representable in the object-theory using appropriate inductive arguments. Formal verification of the representability conditions is the first step towards an automated proof thereof which, in turn, brings the automated verification of Gödel's theorems one step closer to completion.

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1 Preliminaries

1.1 Background

In 1931, Kurt Gödel established in his seminal paper *On formally undecidable propositions of Principia Mathematica and related systems I* that any consistent formal system containing a modicum of number theory is inherently incomplete, in that there exist true propositions of the formal system that are not provable in the formal system. He also proved that any such system cannot prove its own consistency. The proofs of Gödel's incompleteness theorems rely on the representation of metatheoretic notions like "formula", "proof", and "theorem" inside the formal system, allowing the system to effectively prove statements about itself. A sentence can then be constructed that says of itself that it is not provable in the formal system. Such a sentence is undecidable as neither it nor its negation is provable, thus rendering the system incomplete.

Wilfried Sieg and Clinton Field, in their paper *Automated Search for Gödel's Proofs*, presented automated proofs of Gödel's theorems at an abstract axiomatic level that were generated by AProS, an automated theorem prover that searches for natural deduction proofs. With an appropriate expansion of the strategic considerations guiding the search procedure of AProS, they were able to generate concise and structurally intelligible proofs of the theorems. The representability of the metatheory, the construction of the self-referential sentence, and the derivability conditions needed for the second incompleteness theorem were taken for granted in their abstract presentation [2]. The present work seeks to provide foundational support to the proofs of Gödel's theorems at the abstract level by formally verifying the representability conditions that permit the representation of the metatheory inside the object theory.

1.2 Description of Metatheory

The metatheory will provide a rigorous buildup of notions leading up to the theorem predicate, which is used to prove incompleteness. Instead of formalizing the metatheoretic notions as primitive recursive functions and then using natural numbers to encode them as Gödel did [1], we attempt to bypass the arithmetization by formalizing the metatheory in the language of binary trees. This approach was originally explored by Wilfried Sieg in [4] and elucidated on in his joint paper [3] with Ingrid and Sten Lindström. In their paper *Gödel's Incompleteness Theorem: A computer-based course in elementary proof theory*, they describe the development of a formal theory for elementary metamathematics TEM that is the starting point for the formal metatheory described next. The advantage to using a formal metatheory defined in the language of binary trees is that the coding of the metatheoretic notions is immediate in ZF (which will be used as the object-theory); each binary tree can be uniquely mapped to an ordered pair in ZF. The codings then directly reflect the structure of the objects being coded.

The syntax of the metatheory includes:

- Logical connectives: $\&, \vee, \rightarrow, \leftrightarrow$

- Quantifiers: $\forall, \exists$
- Variables ranging over all binary trees: $X, X_1, X_2, \ldots$

For convenience we may use also subscripted Y's and Z's for variables, but note that they are used only for readability. Note also that the same logical symbols are used in the metatheory as the object-theory. Ambiguity is avoided in proofs by a clear distinction between statements being proved in the metatheory and statements being proved in the object-theory, which will be further explained in the next section. There is also no ambiguity in the definitions as the metatheoretic notions are defined separately from the object-theoretic ones.

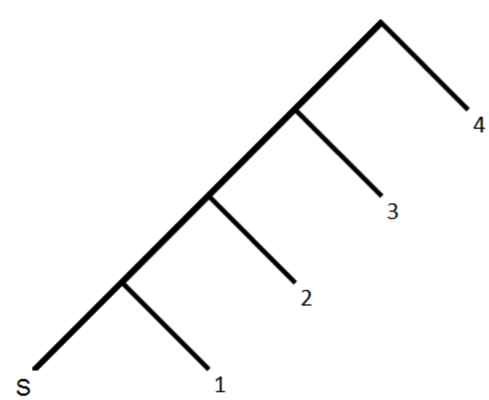
We also require an appropriate syntax to represent binary trees. The empty tree is denoted by S, and more complex trees are denoted by $X = [X_1, X_2]$ where X_1 is the left sub-tree of X and X_2 the right sub-tree. The axioms for binary trees are stated as follows:

- $B_1 : (\forall X, Y) S \neq [X, Y]$
- $B_2 : (\forall X_1, X_2, Y_1, Y_2)([X_1, Y_1] = [X_2, Y_2] \longleftrightarrow (X_1 = X_2 \,\&\, Y_1 = Y_2))$
- Ind: $(\varphi(S) \,\&\, (\forall X_1, X_2)((\varphi(X_1) \,\&\, \varphi(X_2)) \rightarrow \varphi([X_1, X_2]))) \rightarrow (\forall X)\varphi(X)$

A key notion that will be used extensively in the metatheoretic definitions is that of projection. We define left and right projections inductively as follows:

- $(S)_1 = S$ and $(S)_2 = S$
- For $X = [X_1, X_2]$
  - $(X)_1 = X_1$
  - $(X)_2 = X_2$

We will also need a binary tree representation for arbitrary sequences of n objects. We adopt the convention of using a binary tree that branches off to the left, where the right branches store the elements of the sequence. For example, if we had the sequence $\{1, 2, 3, 4\}$, the binary tree would look like:



and this would be represented in our tree notation as $X = [[[[S, 1], 2], 3], 4]$. This representation will be used both in representing n-ary predicates and terms, as well as sequences of assumptions. In the latter case, we use the $\in$-relation to express membership of a sequence. For example, $1 \in X$ from above, but $5 \notin X$.

Note that the $\in$ sign is used also in set theory to denote set membership; we overbook the symbol so that the intutition is the same in the metatheory for an element being a member of a sequence. The ambiguity is resolved by the object that the membership is ascirbed to; sequence membership is only in the metatheory and set-membership is only in the object-theory.

Within this system we will be able to formulate inductive definitions for the notions of "formula", "proof", and "theorem" and will represent each of them as a binary tree exhibiting a particular structure.

1.3 Description of Object-theory

For the object theory we use a slightly modified axiomatic formulation of Zermelo-Fraenkel set theory (ZF). Such a formulation allows us to prove representability in a reasoned way while adhering to the required conditions of the incompleteness theorems and avoiding the need to arithmetize the syntax. The language of ZF is as follows:

- Logical connectives: $\&, \vee, \rightarrow, \leftrightarrow$
- Quantifiers: $\forall, \exists$
- Variables: $x, x_1, x_2, \ldots$
- Function symbols: $f, f_1, f_2, \ldots$
- Terms: $x, x_1, x_2, \ldots, f_i(x_m, \ldots, x_n), \ldots$
- Predicates: $P, P_1, P_2, \ldots$

We have as axioms:

- (Extensionality) $(\forall x)(\forall y)((\forall z)(z \in x \longleftrightarrow z \in y) \longrightarrow x = z)$
- (Powerset) $(\forall x)(\exists y)(\forall z)(z \in y \longleftrightarrow z \subseteq x)$
- (Union) $(\forall x)(\exists y)(\forall z)(z \in y \longleftrightarrow (\exists w \in x) z \in w)$
- (Infinity) $(\exists y)(\emptyset \in y \,\&\, (\forall x_1, x_2 \in y)\langle x_1, x_2 \rangle \in y)$

Note that the last axiom is a slightly modified version of the traditional Axiom of Infinity; it uses ordered-pair generation as its successor operation in place of the unary successor function that is typically used. This modification guarantees the existence of a set v that contains the emptyset and all of its binary tree successors and makes the following definition of the set of ordered pairs well-defined:

$$\mathbb{B} = \bigcap\{z \in \wp(v) \,|\, \emptyset \in z \,\&\, (\forall x_1, x_2 \in z)\langle x_1, x_2\rangle \in z\}$$

In the spirit of Dedekind, the intersection over all sets satisfying the defining conditions produces the smallest such set. This intersection is well-defined since there is at least one set satisfying the conditions, namely v, and thus we are not taking the intersection of the empty set. The set $\mathbb{B}$ will be used as the set of all codes of binary trees from the metatheory.

We also have two axiom schemata that generate infinitely many axioms. They are the following:

- (Replacement) $(\forall z)(Sv(\varphi(x, y), z) \longrightarrow (\exists v)(\forall y)(y \in v \longleftrightarrow (\exists x \in z)\varphi(x, y)))$ where $\varphi(x, y)$ is a formula with free variables x and y and $Sv(\varphi(x, y), z)$ is an abbreviation for $(\forall x \in z)((\forall y_1)(\forall y_2)(\varphi(x, y_1) \,\&\, \varphi(x, y_2)) \longrightarrow y_1 = y_2)$
- ($\in$-Induction) $(\forall x)((\forall y \in x)\varphi(y) \longrightarrow \varphi(x)) \longrightarrow (\forall x)\varphi(x)$

A theorem that will be particularly useful in the representability proofs is the Fundamental Theorem of Ordered Pairs (FTOP), stated as:

$$(\forall x_1, x_2, y_1, y_2)(\langle x_1, y_1\rangle = \langle x_2, y_2\rangle \longleftrightarrow (x_1 = x_2 \,\&\, y_1 = y_2))$$

The proof is not difficult to show inside of ZF.

1.4 Conventions/Notation

Formal proofs will be presented in Fitch diagrams using the same Intercalation Calculi used in AProS (a standard first-order theory that uses introduction and elimination rules for a fully specified language). As was mentioned earlier, Wilfried Sieg and Clinton Field used an appropriate expansion of the the strategies guiding the search procedure of AProS. One such strategy permitted movement between the metatheory and object-theory. To be more precise, when presented with a goal of the form $ZF \vdash \varphi$, one can move into the object theory and try to prove φ inside ZF, justifying it by the inverted rule ProvI. Similarly, if one has as an assumption $ZF \vdash \varphi$, one can use the rule ProvE to make the assumption inside ZF and proceed from there. In the Fitch diagrams, statements being proved inside of ZF will be denoted by an asterisk (*) preceding the line in the derivation. Some lines in the proofs will be justified by numbered lemmata; an index of lemmata with brief justifications can be found at the end of the document.

The representability conditions were also used in a strategic way, and were captured by the introduction and elimination rules of RepI and RepE respectively. If one is faced with a statement in the metatheory that is a representable notion in ZF, one can move the coded object into ZFusing the rule RepE. For RepI, if one proves something about a representable notion inside ZF, one may move the notion to the metatheory. These rules are of course justified by the reprentability conditions that are to be verified in what follows, but we will still utilize them in the inductive cases when we have previously shown or assumed some notion to be representable.

Every notion in the metatheory that is shown to be representable is first formalized in the metatheory using inductive definitions and represented as an object by assigning it a binary tree representation. This can be confusing as there is a distinction between saying something is, for example, a proof, and some binary tree X is a proof. The former might be the explicit sequence of steps in the proof, whereas the latter is a binary tree meant to represent the proof as an object. The object is to include information about what formula is being proved, how it is being proved, and what was used to establish the proof. For each object I will indicate the general binary tree structure, and then the inductive definitions in the metatheory will implement the structure with particular conditions that each component of the tree must satisfy.

2 Representability

The general formulation of the representability conditions that must be satisfied is as follows:

$$R_1 : \mathrm{P}(X_1, \ldots, X_n) \longrightarrow ZF \vdash p(\lfloor X_1 \rfloor, \ldots, \lfloor X_n \rfloor)$$
$$R_2 : \textsc{Not}\mathrm{P}(X_1, \ldots, X_n) \longrightarrow ZF \vdash \neg p(\lfloor X_1 \rfloor, \ldots, \lfloor X_n \rfloor)$$

where P is some metatheoretic notion, $X_1, \ldots, X_n$ are objects of the metatheory, and p and $\lfloor X_1 \rfloor, \ldots, \lfloor X_n \rfloor$ are the object-theoretic codes of those respective objects. We would in the end like to verify these conditions for the theorem predicate, as this is the notion used in the construction of the undecidable sentence of ZF. We first require the representation of the notions of "formula" and "proof", which themselves require the representation of a number of metatheoretic notions. We will begin the process by showing that the formal sytax of the metatheory can be represented inside of ZF.

2.1 Representability of Syntax

Each symbol in the language of ZF is assigned a binary tree representation in the metatheory. The actual assignment is arbitrary, so long as each symbol is uniquely represented. We give an iterative specification, representing each symbol in terms of some previously defined symbol, beginning with the empty tree and branching off to the right. The coding into ZF is then immediate, as we can define the following mapping that takes advantage of the structural identity that holds between the representation of binary trees and ordered pairs:

$$S \longmapsto \emptyset$$
$$[X_1, X_2] \longmapsto \langle \lfloor X_1 \rfloor, \lfloor X_2 \rfloor \rangle$$

where $\lfloor X_1 \rfloor$ and $\lfloor X_2 \rfloor$ are the codes for X_1 and X_2 respectively, i.e. the ordered pairs that are assigned to the binary trees X_1 and X_2. The codings are defined as follows:

Symbol	Binary Tree	Code in ZF
$\neg$	S	$\emptyset$
$\&$	$[S, \neg]$	$\langle \emptyset, \lfloor \neg \rfloor \rangle$
$\vee$	$[S, \&]$	$\langle \emptyset, \lfloor \& \rfloor \rangle$
$\rightarrow$	$[S, \vee]$	$\langle \emptyset, \lfloor \vee \rfloor \rangle$
$\leftrightarrow$	$[S, \rightarrow]$	$\langle \emptyset, \lfloor \rightarrow \rfloor \rangle$
$=$	$[S, \leftrightarrow]$	$\langle \emptyset, \lfloor \leftrightarrow \rfloor \rangle$
$\forall$	$[S, =]$	$\langle \emptyset, \lfloor = \rfloor \rangle$
$\exists$	$[S, \forall]$	$\langle \emptyset, \lfloor \forall \rfloor \rangle$
x	$[S, \exists]$	$\langle \emptyset, \lfloor \exists \rfloor \rangle$
x_1	$[S, x]$	$\langle \emptyset, \lfloor x \rfloor \rangle$
x_n	$[S, x_{n-1}]$	$\langle \emptyset, \lfloor x_{n-1} \rfloor \rangle$
f	$[S, [x, x]]$	$\langle \emptyset, \langle \lfloor x \rfloor, \lfloor x \rfloor \rangle \rangle$
f_1	$[S, f]$	$\langle \emptyset, \lfloor f \rfloor \rangle$
f_n	$[S, f_{n-1}]$	$\langle \emptyset, \lfloor f_{n-1} \rfloor \rangle$
P	$[S, [f, f]]$	$\langle \emptyset, \langle \lfloor f \rfloor, \lfloor f \rfloor \rangle \rangle$
P_1	$[S, P]$	$\langle \emptyset, \lfloor P \rfloor \rangle$
P_n	$[S, P_{n-1}]$	$\langle \emptyset, \lfloor P_{n-1} \rfloor \rangle$

(Coding is based on Wilfried Sieg's formulation in his initial work on the formalization of the metatheory in [4])

2.2 Representability of Equality

The first metatheoretic notion to represent in ZF is equality. We would like to prove:

$$(\forall X)(\forall Y)[X = Y \longrightarrow ZF \vdash \lfloor X \rfloor = \lfloor Y \rfloor] \quad (1)$$

$$(\forall X)(\forall Y)[X \neq Y \longrightarrow ZF \vdash \lfloor X \rfloor \neq \lfloor Y \rfloor] \quad (2)$$

Proof. (1) follows immediately from the coding defined in the metatheory: if two binary trees X and Y are equal, then their ordered pair counterparts $\lfloor X \rfloor$ and $\lfloor Y \rfloor$ will be syntactically identical and the equality $\lfloor X \rfloor = \lfloor Y \rfloor$ is then provable in ZF.

We prove (2) by induction on X.

Base Case: $X = S$.

1	$S \neq Y$	Assume
2	$(\exists Y_1, Y_2) Y = [Y_1, Y_2]$	Lemma(3): 1
3	$Y = [Y_1, Y_2]$	Assume
4	$* \quad \lfloor S \rfloor = \lfloor Y \rfloor$	Assume
5	$* \quad \lfloor S \rfloor = \lfloor [Y_1, Y_2] \rfloor$	RepE(=): 3,4
6	$* \quad \emptyset = \langle \lfloor Y_1 \rfloor, \lfloor Y_2 \rfloor \rangle$	DefE(Code): 5
7	$* \quad \bot$	Lemma(2): 6
8	$* \quad \lfloor S \rfloor \neq \lfloor Y \rfloor$	¬I: 7
9	$ZF \vdash \lfloor S \rfloor \neq \lfloor Y \rfloor$	ProvI: 8
10	$ZF \vdash \lfloor S \rfloor \neq \lfloor Y \rfloor$	∃E: 2,9
11	$S \neq Y \longrightarrow ZF \vdash \lfloor S \rfloor \neq \lfloor Y \rfloor$	→I: 10
12	$(\forall Y)[S \neq Y \longrightarrow ZF \vdash \lfloor S \rfloor \neq \lfloor Y \rfloor]$	∀I: 11

Induction Case: For the Induction Hypothesis, assume for arbitrary X_1 and X_2 that

$$(\forall Y)[X_i \neq Y \longrightarrow ZF \vdash \lfloor X_i \rfloor \neq \lfloor Y \rfloor]$$

We wish to show true for $X = [X_1, X_2]$.

1	$[X_1, X_2] \neq Y$	Assume
2	$Y = S \vee (\exists Y_1, Y_2) Y = [Y_1, Y_2]$	Lemma(1): 1
3	$Y = S$	Assume
4	$ZF \vdash \lfloor [X_1, X_2] \rfloor \neq \lfloor Y \rfloor$	Base Case: 3
5	$(\exists Y_1, Y_2) Y = [Y_1, Y_2]$	Assume
6	$Y = [Y_1, Y_2]$	Assume
7	$[X_1, X_2] \neq [Y_1, Y_2]$	=E: 1,6
8	$X_1 \neq Y_1 \vee X_2 \neq Y_2$	Lemma(4): 7
9	$X_1 \neq Y_1$	Assume
10	$ZF \vdash \lfloor X_1 \rfloor \neq \lfloor Y_1 \rfloor$	IH: 9
11	$* \quad \lfloor X_1 \rfloor \neq \lfloor Y_1 \rfloor$	ProvE: 10
12	$* \quad \langle \lfloor X_1 \rfloor, \lfloor X_2 \rfloor \rangle \neq \langle \lfloor Y_1 \rfloor, \lfloor Y_2 \rfloor \rangle$	Lemma(5): 11
13	$* \quad \lfloor [X_1, X_2] \rfloor \neq \lfloor [Y_1, Y_2] \rfloor$	DefI(Code): 12
14	$* \quad \lfloor [X_1, X_2] \rfloor \neq \lfloor Y \rfloor$	RepE(=): 13,6
15	$ZF \vdash \lfloor [X_1, X_2] \rfloor \neq \lfloor Y \rfloor$	ProvI: 14
16	$X_2 \neq Y_2$	Assume
17	$ZF \vdash \lfloor X_2 \rfloor \neq \lfloor Y_2 \rfloor$	IH: 16
18	$* \quad \lfloor X_2 \rfloor \neq \lfloor Y_2 \rfloor$	ProvE: 17
19	$* \quad \langle \lfloor X_1 \rfloor, \lfloor X_2 \rfloor \rangle \neq \langle \lfloor Y_1 \rfloor, \lfloor Y_2 \rfloor \rangle$	Lemma(5): 18
20	$* \quad \lfloor [X_1, X_2] \rfloor \neq \lfloor [Y_1, Y_2] \rfloor$	DefI(Code): 19
21	$* \quad \lfloor [X_1, X_2] \rfloor \neq \lfloor Y \rfloor$	RepE(=): 20,6
22	$ZF \vdash \lfloor [X_1, X_2] \rfloor \neq \lfloor Y \rfloor$	ProvI: 21
23	$ZF \vdash \lfloor [X_1, X_2] \rfloor \neq \lfloor Y \rfloor$	$\vee$E: 8,15,22
24	$ZF \vdash \lfloor [X_1, X_2] \rfloor \neq \lfloor Y \rfloor$	$\exists$E: 5,23
25	$ZF \vdash \lfloor [X_1, X_2] \rfloor \neq \lfloor Y \rfloor$	$\vee$E: 2,4,24
26	$[X_1, X_2] \neq Y \longrightarrow ZF \vdash \lfloor [X_1, X_2] \rfloor \neq \lfloor Y \rfloor$	$\rightarrow$I: 25
27	$(\forall Y)([X_1, X_2] \neq Y \longrightarrow ZF \vdash \lfloor [X_1, X_2] \rfloor \neq \lfloor Y \rfloor)$	$\forall$I: 26

We conclude, by the principle of induction for binary trees, that

$$(\forall X)(\forall Y)[X \neq Y \longrightarrow ZF \vdash \lfloor X \rfloor \neq \lfloor Y \rfloor]$$

. $\square$

We have thus established that the metatheoretic notion of equality can be represented inside of ZF.

2.3 Representability of Variables

Though we have a syntactic specification of variables, we would like to justify the existence of the set of variables in ZF.

Metatheory

Based on the binary tree assignment in the metatheory, variables will be inductively defined as:

$$(\forall X)((X = [S, \exists] \vee (X = [S, (X)_2] \,\&\, \textsc{Var}((X)_2))) \longrightarrow \textsc{Var}(X))$$

Such an inductive definition justifies also an induction principle that is formulated as follows:

$$(\varphi([S, \exists]) \,\&\, (\forall X)(\varphi((X)_2) \longrightarrow \varphi([S, (X)_2]))) \longrightarrow (\forall X)(\textsc{Var}(X) \longrightarrow \varphi(X))$$

Informally this principle states that if one can show a property φ to hold for the object $X = [S, \exists]$ and, assuming for some arbitrary object X of the form $[S, (X)_2]$ that φ holds for $(X)_2$, one can prove that it holds also for X, then one may conclude the property φ holds for all variables.

Object-theory

The same considerations that went into defining the set of binary trees will go into defining the set of variables. We would like to obtain the smallest set z satisfying the following conditions:

- $\langle \emptyset, \lfloor \exists \rfloor \rangle \in z$
- $(\forall x \in z)\langle \emptyset, x \rangle \in z$

For convenience in the proofs that follow, we define v to be:

$$v = \{z \in \wp(\mathbb{B}) \mid \langle \emptyset, \lfloor \exists \rfloor \rangle \in z \,\&\, (\forall x \in z)\langle \emptyset, x \rangle \in z\}$$

and then define the set of variables $\mathbb{V}$ as:

$$\mathbb{V} = \bigcap v$$

We can now give a quick proof by induction on variables in the metatheory that

$$(\forall X)(\textsc{Var}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathbb{V})$$

Proof. Base Case: $X = [S, \exists]$.

1	$X = [S, \exists]$	Premise
2	$*$ | | $z \in v$	Assume
3	$*$ | | $\langle \emptyset, \lfloor \exists \rfloor \rangle \in z$	DefE(v): 2
4	$*$ | | $\lfloor [S, \exists] \rfloor \in z$	DefI(Code): 3
5	$*$ | | $\lfloor X \rfloor \in z$	RepE(=): 4,1
6	$*$ | $(\forall z \in v) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 5
7	$*$ | $\lfloor X \rfloor \in \bigcap v$	DefI($\bigcap$): 6
8	$*$ | $\lfloor X \rfloor \in \mathbb{V}$	DefI($\mathbb{V}$): 7
9	$ZF \vdash \lfloor X \rfloor \in \mathbb{V}$	ProvI: 8

Induction Case: $X = [S, (X)_2] \,\&\, \textsc{Var}((X)_2)$.

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{Var}((X)_2) \longrightarrow ZF \vdash \lfloor (X)_2 \rfloor \in \mathbb{V}$$

1	$X = [S, (X)_2]$	Premise
2	$\textsc{Var}((X)_2)$	Premise
3	$ZF \vdash \lfloor (X)_2 \rfloor \in \mathbb{V}$	IH: 2
4	$*$ | $\lfloor (X)_2 \rfloor \in \mathbb{V}$	ProvE: 3
5	$*$ | | $z \in v$	Assume
6	$*$ | | $(\forall x \in z) \langle \emptyset, x \rangle \in z$	DefE(v): 5
7	$*$ | | $\lfloor (X)_2 \rfloor \in v$	Lemma(6): 4
8	$*$ | | $\langle \emptyset, \lfloor (X)_2 \rfloor \rangle \in z$	$\forall_{\in}$E: 6,7
9	$*$ | | $\lfloor [S, (X)_2] \rfloor \in z$	DefI(Code): 8
10	$*$ | | $\lfloor X \rfloor \in z$	RepE(=): 9,1
11	$*$ | $(\forall z \in v) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 10
12	$*$ | $\lfloor X \rfloor \in \bigcap v$	DefI($\bigcap$): 11
13	$*$ | $\lfloor X \rfloor \in \mathbb{V}$	DefI($\mathbb{V}$): 12
14	$ZF \vdash \lfloor X \rfloor \in \mathbb{V}$	ProvI: 13

Thus by the principle of induction on variables, we have

$$(\forall X)(\textsc{Var}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathbb{V})$$

$\square$

We would also like to establish R_2 for variables, i.e. show:

$$(\forall X)(\textsc{NotVar}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \notin \mathbb{V})$$

Equivalently we may define the set of all non-variables $\mathbb{NV}$ in ZF and show

1. $(\forall X)(\textsc{NotVar}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathbb{NV})$
2. $ZF \vdash (\forall x)(x \in \mathbb{V} \longrightarrow x \notin \mathbb{NV})$
3. $ZF \vdash (\forall x)(x \in \mathbb{NV} \longrightarrow x \notin \mathbb{V})$

(2) and (3) prove that $\mathbb{V}$ and $\mathbb{NV}$ are disjoint. We will first provide an inductive definition for non-variables in the metatheory and an appropriate definition in the object theory for the set of all non-variables.

Metatheory

$$\begin{aligned}(\forall X)((&X = S \\ &\vee (X = [(X)_1, (X)_2] \,\&\, (X)_1 \neq S) \\ &\vee (X = [S, (X)_2] \,\&\, (X)_2 \neq (\exists) \,\&\, \textsc{NotVar}((X)_2))) \longrightarrow \textsc{NotVar}(X))\end{aligned}$$

This definition admits an induction principle for non-variables, which will be utilized in proving the above claim.

Object-theory

We define the set nv as

$$\begin{aligned} nv = \{z \in \wp(\mathbb{B}) \mid\, &\emptyset \in z \\ &\& \,(\forall x \neq \emptyset)(\forall y)\langle x, y\rangle \in z \\ &\& \,(\forall x \in z)(x \neq \lfloor \exists \rfloor \longrightarrow \langle \emptyset, x\rangle \in z)\} \end{aligned}$$

and then the set of non-variables is defined as

$$\mathbb{NV} = \bigcap nv$$

We will now complete the representability of variables by proving (1),(2), and (3) above.

Proof. (1) will be established by the metatheoretic principle of induction on non-variables.

Base Case 1: $X = S$.

1	$X = S$	Premise
2	$*$ $z \in nv$	Assume
3	$*$ $\emptyset \in z$	DefE(nv): 2
4	$*$ $\lfloor S \rfloor \in z$	DefI(Code): 3
5	$*$ $\lfloor X \rfloor \in z$	RepE(=): 4,1
6	$*$ $(\forall z \in nv)\lfloor X \rfloor \in z$	$\forall_{\in}$I: 5
7	$*$ $\lfloor X \rfloor \in \bigcap nv$	DefI($\cap$): 6
8	$*$ $\lfloor X \rfloor \in \mathbb{NV}$	DefI($\mathbb{NV}$): 7
9	$ZF \vdash \lfloor X \rfloor \in \mathbb{NV}$	ProvI: 8

Base Case 2: $X = [(X)_1, (X)_2]$ and $(X)_1 \neq S$.

1	$X = [(X)_1, (X)_2]$	Premise
2	$(X)_1 \neq S$	Premise
3	$*$ $z \in nv$	Assume
4	$*$ $(\forall x \neq \emptyset)(\forall y)\langle x, y\rangle \in z$	DefE(nv): 3
5	$*$ $\lfloor (X)_1 \rfloor \neq \emptyset$	RepE(=): 2
6	$*$ $\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle \in z$	$\forall_{\in}$E: 4,5
7	$*$ $\lfloor [(X)_1, (X)_2] \rfloor \in z$	DefI(Code): 6
8	$*$ $\lfloor X \rfloor \in z$	RepE(=): 7,1
9	$*$ $(\forall z \in nv)\lfloor X \rfloor \in z$	$\forall_{\in}$I: 8
10	$*$ $\lfloor X \rfloor \in \bigcap nv$	DefI($\cap$): 9
11	$*$ $\lfloor X \rfloor \in \mathbb{NV}$	DefI($\mathbb{NV}$): 10
12	$ZF \vdash \lfloor X \rfloor \in \mathbb{NV}$	ProvI: 11

Inductive Case: $X = [S, (X)_2]$ and $(X)_2 \neq \exists$ and $\textsc{NotVar}((X)_2)$.

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{NotVar}((X)_2) \longrightarrow ZF \vdash \lfloor (X)_2 \rfloor \in \mathbb{NV}$$

1		$X = [S, (X)_2]$	Premise
2		$(X)_2 \neq \exists$	Premise
3		$\textsc{NotVar}((X)_2)$	Premise
4		$ZF \vdash \lfloor (X)_2 \rfloor \in \mathbb{NV}$	IH: 3
5	*	$\lfloor (X)_2 \rfloor \in \mathbb{NV}$	ProvE: 4
6	*	$z \in nv$	Assume
7	*	$(\forall x \in z)(x \neq \lfloor \exists \rfloor \longrightarrow \langle \emptyset, x \rangle \in z)$	DefE(nv): 6
8	*	$\lfloor (X)_2 \rfloor \in z$	Lemma(6): 5
9	*	$\lfloor (X)_2 \rfloor \neq \lfloor \exists \rfloor \longrightarrow \langle \emptyset, \lfloor (X)_2 \rfloor \rangle \in z$	$\forall_{\in}$E: 7,8
10	*	$\lfloor (X)_2 \rfloor \neq \lfloor \exists \rfloor$	RepE(=): 2
11	*	$\langle \emptyset, \lfloor (X)_2 \rfloor \rangle \in z$	$\rightarrow$E: 9,10
12	*	$\lfloor [S, (X)_2] \rfloor$	DefI(Code): 11
13	*	$\lfloor X \rfloor \in z$	RepE(=): 12
14	*	$(\forall z \in nv) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 13
15	*	$\lfloor X \rfloor \in \bigcap nv$	DefI($\cap$): 14
16	*	$\lfloor X \rfloor \in \mathbb{NV}$	DefI($\mathbb{NV}$): 15
17		$ZF \vdash \lfloor X \rfloor \in \mathbb{NV}$	ProvI: 16

Thus by the metatheoretic principle of induction on non-variables, we have

$$(\forall X)(\textsc{NotVar}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathbb{NV})$$

It remains to show that ZF proves $\mathbb{V}$ and $\mathbb{NV}$ are disjoint. The definitions of $\mathbb{V}$ and $\mathbb{NV}$ in ZF justify the specification of an induction principle for each (as the sets have some collection of base elements and are closed under a finite list of generating operations). These induction principles will be utilized to establish the claims guaranteeing $\mathbb{V}$ and $\mathbb{NV}$ to be disjoint.

We first prove $ZF \vdash (\forall x)(x \in \mathbb{V} \longrightarrow x \notin \mathbb{NV})$ using the induction principle for $\mathbb{V}$ and the following lemma that is jusified by the definition of $\mathbb{NV}$:

$$(\forall x)(x \in \mathbb{NV} \longleftrightarrow (x = \emptyset \vee (\exists y \neq \emptyset)(\exists z)x = \langle y, z \rangle \vee (\exists z \in \mathbb{NV})(z \neq \lfloor \exists \rfloor \,\&\, x = \langle \emptyset, z \rangle)))$$

Base Case: $x = \langle \emptyset, \lfloor \exists \rfloor \rangle$.

1	$*$	$\langle \emptyset, \lfloor \exists \rfloor \rangle \in \mathbb{NV}$	Assume
2	$*$	$\langle \emptyset, \lfloor \exists \rfloor \rangle = \emptyset$	Assume
3	$*$	$\bot$	Lemma(2): 2
4	$*$	$(\exists y \neq \emptyset)(\exists z)\langle \emptyset, \lfloor \exists \rfloor \rangle = \langle y, z \rangle$	Assume
5	$*$	$y \neq \emptyset$	Assume
6	$*$	$\langle \emptyset, \lfloor \exists \rfloor \rangle = \langle y, z \rangle$	Assume
7	$*$	$\emptyset = y$	FTOP: 6
8	$*$	$\bot$	$\bot$I: 7,5
9	$*$	$\bot$	$\exists$E: 4,8
10	$*$	$(\exists z \in \mathbb{NV})(z \neq \lfloor \exists \rfloor \,\&\, \langle \emptyset, \lfloor \exists \rfloor \rangle = \langle \emptyset, z, \rangle)$	Assume
11	$*$	$z \in \mathbb{NV}$	Assume
12	$*$	$z \neq \lfloor \exists \rfloor$	Assume
13	$*$	$\langle \emptyset, \lfloor \exists \rfloor \rangle = \langle \emptyset, z, \rangle$	Assume
14	$*$	$\lfloor \exists \rfloor = z$	FTOP: 13
15	$*$	$\bot$	$\bot$I: 14,12
16	$*$	$\bot$	$\exists_{\in}$E: 10,15
17	$*$	$\bot$	Lemma(7): 3,9,16
18	$*$	$\langle \emptyset, \lfloor \exists \rfloor \rangle \notin \mathbb{NV}$	$\neg$I: 17
19		$ZF \vdash \langle \emptyset, \lfloor \exists \rfloor \rangle \notin \mathbb{NV}$	ProvI: 18

Inductive Case: For arbitrary $z \in \mathbb{V}$, $x = \langle \emptyset, z \rangle$.

Induction Hypothesis: Assume for z that

$$ZF \vdash z \in \mathbb{V} \longrightarrow z \notin \mathbb{NV}$$

1	$z \in \mathbb{V}$	Premise
2	$* \quad \langle \emptyset, z \rangle \in \mathbb{NV}$	Assume
3	$* \quad \langle \emptyset, z \rangle = \emptyset$	Assume
4	$* \quad \bot$	Lemma(2): 3
5	$* \quad (\exists y \neq \emptyset)(\exists w)\langle \emptyset, z \rangle = \langle y, w \rangle$	Assume
6	$* \quad y \neq \emptyset$	Assume
7	$* \quad \langle \emptyset, z \rangle = \langle y, w \rangle$	Assume
8	$* \quad \emptyset = y$	FTOP: 7
9	$* \quad \bot$	$\bot$I: 8,6
10	$* \quad \bot$	$\exists$E: 5,9
11	$* \quad (\exists w \in \mathbb{NV})(w \neq \lfloor \exists \rfloor \ \& \ \langle \emptyset, z \rangle = \langle \emptyset, w, \rangle)$	Assume
12	$* \quad w \in \mathbb{NV}$	Assume
13	$* \quad w \neq \lfloor \exists \rfloor$	Assume
14	$* \quad \langle \emptyset, z \rangle = \langle \emptyset, w, \rangle$	Assume
15	$* \quad z = w$	FTOP: 14
16	$* \quad z \in \mathbb{NV}$	=E: 12,15
17	$* \quad z \in \mathbb{V} \longrightarrow z \notin \mathbb{NV}$	IH
18	$* \quad z \notin \mathbb{NV}$	$\rightarrow$E: 17,1
19	$* \quad \bot$	$\bot$I: 16,18
20	$* \quad \bot$	$\exists$E: 11,19
21	$* \quad \bot$	Lemma(7): 4,9,20
22	$* \quad \langle \emptyset, z \rangle \notin \mathbb{NV}$	$\neg$I: 21
23	$ZF \vdash \langle \emptyset, z \rangle \notin \mathbb{NV}$	ProvI: 22

Thus $ZF \vdash (\forall x)(x \in \mathbb{V} \longrightarrow x \notin \mathbb{NV})$ holds from the induction principle for $\mathbb{V}$. It only remains to show $ZF \vdash (\forall x)(x \in \mathbb{NV} \longrightarrow x \notin \mathbb{V})$, this time using the induction principle for $\mathbb{NV}$ and a similar lemma for $\mathbb{V}$ stated as:

$$(\forall x)(x \in \mathbb{V} \longleftrightarrow x = \langle \emptyset, \lfloor \exists \rfloor \rangle \vee (\exists z \in \mathbb{V})x = \langle \emptyset, z \rangle)$$

Base Case 1: $x = \emptyset$.

1	$* \quad \emptyset \in \mathbb{V}$	Assume
2	$* \quad \emptyset = \langle \emptyset, \lfloor \exists \rfloor \rangle$	Assume
3	$* \quad \bot$	Lemma(2): 2
4	$* \quad (\exists z \in \mathbb{V})\emptyset = \langle \emptyset, z \rangle$	Assume
5	$* \quad z \in \mathbb{V}$	Assume
6	$* \quad \emptyset = \langle \emptyset, z \rangle$	Assume
7	$* \quad \bot$	Lemma(2): 6
8	$* \quad \bot$	$\exists_{\in}$E: 4,7
9	$* \quad \bot$	Lemma(7): 3,8
10	$* \quad \emptyset \notin \mathbb{V}$	$\neg$I: 9
11	$ZF \vdash \emptyset \notin \mathbb{V}$	ProvI: 10

Base Case 2: For arbitrary y, z, $x = \langle y, z \rangle$ and $y \neq \emptyset$.

1	$y \neq \emptyset$	Premise
2	$* \quad \langle y, z \rangle \in \mathbb{V}$	Assume
3	$* \quad \langle y, z \rangle = \langle \emptyset, \lfloor \exists \rfloor \rangle$	Assume
4	$* \quad y = \emptyset$	FTOP: 3
5	$* \quad \bot$	$\bot$I: 4,1
6	$* \quad (\exists z \in \mathbb{V})\langle y, z \rangle = \langle \emptyset, z \rangle$	Assume
7	$* \quad z \in \mathbb{V}$	Assume
8	$* \quad \langle y, z \rangle = \langle \emptyset, z \rangle$	Assume
9	$* \quad y = \emptyset$	FTOP: 8
10	$* \quad \bot$	$\bot$I: 9,1
11	$* \quad \bot$	$\exists_{\in}$E: 6,10
12	$* \quad \bot$	Lemma(7): 5,11
13	$* \quad \langle y, z \rangle \notin \mathbb{V}$	$\neg$I: 12
14	$ZF \vdash \langle y, z \rangle \notin \mathbb{V}$	ProvI: 13

Inductive Case: For arbitrary $z \in \mathbb{NV}$ such that $z \neq \lfloor\exists\rfloor$, $x = \langle\emptyset, z\rangle$.

Induction Hypothesis: Assume for z that

$$ZF \vdash z \in \mathbb{NV} \longrightarrow z \notin \mathbb{V}$$

1		$z \in \mathbb{NV}$	Premise
2		$z \neq \lfloor\exists\rfloor$	Premise
3	*	$\langle\emptyset, z\rangle \in \mathbb{V}$	Assume
4	*	$\langle\emptyset, z\rangle = \langle\emptyset, \lfloor\exists\rfloor\rangle$	Assume
5	*	$z = \lfloor\exists\rfloor$	FTOP: 4
6	*	$\bot$	$\bot$I: 5,2
7	*	$(\exists w \in \mathbb{V})\langle\emptyset, z\rangle = \langle\emptyset, w\rangle$	Assume
8	*	$w \in \mathbb{V}$	Assume
9	*	$\langle\emptyset, z\rangle = \langle\emptyset, w\rangle$	Assume
10	*	$z = w$	FTOP: 9
11	*	$z \in \mathbb{V}$	=E: 8,10
12	*	$z \in \mathbb{NV} \longrightarrow z \notin \mathbb{V}$	IH
13	*	$z \notin \mathbb{V}$	$\rightarrow$E: 12,1
14	*	$\bot$	$\bot$I: 11,13
15	*	$\bot$	$\exists_{\in}$E: 7,14
16	*	$\langle\emptyset, z\rangle \notin \mathbb{V}$	$\neg$I: 15
17		$ZF \vdash \langle\emptyset, z\rangle \notin \mathbb{V}$	ProvI: 16

Thus by the induction principle for $\mathbb{NV}$, $ZF \vdash (\forall x)(x \in \mathbb{NV} \longrightarrow x \notin \mathbb{V})$, and thus concludes the proof of representability for variables. □

(Note: the proof for non-variables will serve as an archetype for all subsequent proofs of "negative" representability for the other metatheoretic notions. We will show most of the formal proof for the representability of non-formulae in full, but the others will only be indicated as they all follow the same line of reasoning. These proofs also become increasingly impractical as the definitions of the metatheoretic notions become more complex, since the definitions generate a large number of sub-cases that each need to be verified. Without a more general specification of inductive definitions, it is almost necessary to use a machine to verify these proofs; doing them by hand is far too time-consuming and adds very little to one's understanding of the overall procedure of proving representability.)

2.4 Representability of Function Symbols and Predicates

The definitions in both the metatheory and ZF will be given for function symbols and predicates, but the representability proofs will be omitted as they follow the same argument used for the representability for variables, with the appropriate replacement of initial syntactic codings.

Metatheory

$$(\forall X)((X = [S,[x,x]] \vee (X = [S,(X)_2] \,\&\, \textsc{Func}((X)_2))) \longrightarrow \textsc{Func}(X))$$
$$(\forall X)((X = [S,[f,f]] \vee (X = [S,(X)_2] \,\&\, \textsc{Pred}((X)_2))) \longrightarrow \textsc{Pred}(X))$$

Object-theory

$$func = \bigcap\{z \in \wp(\mathbb{B}) \mid \langle\emptyset,\langle\lfloor x\rfloor,\lfloor x\rfloor\rangle\rangle \in z \,\&\, (\forall x \in z)\langle\emptyset,x\rangle \in z\}$$
$$pred = \bigcap\{z \in \wp(\mathbb{B}) \mid \langle\emptyset,\langle\lfloor f\rfloor,\lfloor f\rfloor\rangle\rangle \in z \,\&\, (\forall x \in z)\langle\emptyset,x\rangle \in z\}$$

2.5 Representability of Terms

The informal specification of terms is as follows:

- If x is a variable, then x is a term.
- If f is a function symbol and $t_1,\ldots,t_n$ are terms, $f(t_1,\ldots,t_n)$ is a term.

Metatheory

We need a way of representing arbitrary terms as binary trees. For the case where a term t is a variable, this is clear, as we have already indicated the binary tree structure of variables. We thus require a binary tree structure for terms of the form $f(t_1,\ldots,t_n)$ where f is a function symbol and $t_1,\ldots,t_n$ are themselves terms. X is considered to be a term if it exhibits the following structure:

$$[f,[t_1,[t_2,\ldots,[t_{n-1},t_n]\cdots]]]$$

where f and $t_1,\ldots,t_n$ are specified as before.

Terms are then defined inductively in the metatheory as

$$\begin{aligned}(\forall X)((&\textsc{Var}(X) \vee (X = [(X)_1,[(X)_2,\ldots,[(X)_{2\cdots 21},(X)_{2\cdots 22}]\cdots]]\\ &\&\,\textsc{Func}((X)_1)\,\&\,\textsc{Term}((X)_{21})\,\&\cdots\&\,\textsc{Term}((X)_{2\cdots 21})\,\&\,\textsc{Term}((X)_{2\cdots 22})))\\ &\longrightarrow \textsc{Term}(X))\end{aligned}$$

with an induction principle that allows one to conclude $(\forall X)(\textsc{Term}(X) \longrightarrow \varphi(X))$, defined in the same way the induction principle for variables was (one must show φ to hold for the base case and then assume φ to hold for arbitrary objects an show that φ holds also for the composition of those objects specified by the definition).

Object-theory

The set of terms is defined as $\mathbb{T} = \bigcap t$, where t is

$$t = \{z \in \wp(\mathbb{B}) \mid \mathbb{V} \subseteq z \,\&\, (\forall x \in func)(\forall y_1, \ldots, y_n \in z)\langle z, \langle y_1, \ldots, \langle y_{n-1}, y_n\rangle \cdots\rangle\rangle \in z\}$$

We will show by induction on terms that

$$(\forall X)(\textsc{Term}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathbb{T})$$

Proof. Base Case: X is a variable.

1		$\textsc{Var}(X)$	Premise
2	$*$	$\quad z \in t$	Assume
3	$*$	$\quad \mathbb{V} \subseteq z$	DefE(t): 2
4	$*$	$\quad \lfloor X \rfloor \in \mathbb{V}$	RepE(Var): 1
5	$*$	$\quad \lfloor X \rfloor \in z$	DefE($\subseteq$): 3,4
6	$*$	$(\forall z \in t)\lfloor X \rfloor \in z$	$\forall_{\in}$I: 5
7	$*$	$\lfloor X \rfloor \in \bigcap t$	DefI($\cap$): 6
8	$*$	$\lfloor X \rfloor \in \mathbb{T}$	DefI($\mathbb{T}$): 7
9		$ZF \vdash \lfloor X \rfloor \in \mathbb{T}$	ProvI: 8

Inductive Case: X is built up from arbitrary terms.

Inductive Hypotheses: Assume for arbitrary X that

$$\textsc{Term}((X)_{21}) \longrightarrow ZF \vdash \lfloor (X)_{21} \rfloor \in \mathbb{T}$$
$$\vdots$$
$$\textsc{Term}((X)_{2\cdots21}) \longrightarrow ZF \vdash \lfloor (X)_{2\cdots21} \rfloor \in \mathbb{T}$$
$$\textsc{Term}((X)_{2\cdots22}) \longrightarrow ZF \vdash \lfloor (X)_{2\cdots22} \rfloor \in \mathbb{T}$$

1	$X = [(X)_1, [(X)_2, \ldots, [(X)_{2\cdots21}, (X)_{2\cdots22}] \cdots]]$	Premise
2	$\textsc{Func}((X)_1)$	Premise
3	$\textsc{Term}((X)_{21})$	Premise
4	$\vdots$	Premise
5	$\textsc{Term}((X)_{2\cdots21})$	Premise
6	$\textsc{Term}((X)_{2\cdots22})$	Premise
7	$ZF \vdash \lfloor (X)_{21} \rfloor \in \mathbb{T}$	IH: 3
8	$\vdots$	
9	$ZF \vdash \lfloor (X)_{2\cdots21} \rfloor \in \mathbb{T}$	IH: 5
10	$ZF \vdash \lfloor (X)_{2\cdots22} \rfloor \in \mathbb{T}$	IH: 6
11	$* \quad \lfloor (X)_{21} \rfloor \in \mathbb{T}$	ProvE: 7
12	$* \quad \vdots$	
13	$* \quad \lfloor (X)_{2\cdots21} \rfloor \in \mathbb{T}$	ProvE: 9
14	$* \quad \lfloor (X)_{2\cdots22} \rfloor \in \mathbb{T}$	ProvE: 10
15	$* \quad \lfloor (X)_{21} \rfloor \in z$	Lemma(6): 11
16	$* \quad \vdots$	
17	$* \quad \lfloor (X)_{2\cdots21} \rfloor \in z$	Lemma(6): 13
18	$* \quad \lfloor (X)_{2\cdots22} \rfloor \in z$	Lemma(6): 14
19	$* \quad \quad z \in t$	Assume
20	$* \quad \quad (\forall x \in func)(\forall y_1, \ldots, y_n \in z)$	
	$* \quad \quad \quad \langle z, \langle y_1, \ldots, \langle y_{n-1}, y_n \rangle \cdots \rangle\rangle \in z$	DefE(t): 19
21	$* \quad \quad \lfloor (X)_1 \rfloor \in func$	RepE(Func): 2
22	$* \quad \quad \langle \lfloor (X)_1 \rfloor, \langle \lfloor (X)_{21} \rfloor, \ldots, \langle \lfloor (X)_{2\cdots21} \rfloor, \lfloor (X)_{2\cdots22} \rfloor \rangle \cdots \rangle\rangle \in z$	$\forall_\in$E: 20,21,15-18
23	$* \quad \quad \lfloor [(X)_1, [(X)_2, \ldots, [(X)_{2\cdots21}, (X)_{2\cdots22}] \cdots]] \rfloor \in z$	DefI(Code): 22
24	$* \quad \quad \lfloor X \rfloor \in z$	RepE(=): 23,1
25	$* \quad (\forall z \in t) \lfloor X \rfloor \in z$	$\forall_\in$I: 24
26	$* \quad \lfloor X \rfloor \in \bigcap t$	DefI($\cap$): 25
27	$* \quad \lfloor X \rfloor \in \mathbb{T}$	DefI($\mathbb{T}$): 26
28	$ZF \vdash \lfloor X \rfloor \in \mathbb{T}$	ProvI: 27

Thus by the induction principle for terms,

$$(\forall X)(\textsc{Term}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathbb{T})$$

□

2.6 Representability of Formulae

Formulae are defined informally as:

- All atomic formulae are formulae.
- If φ is a formula, then $\neg\varphi$ is a formula.
- If φ and ψ are both formulae, then $\varphi \,\Box\, \psi$ is a formula (where '$\Box$' is an arbitrary binary logical connective, i.e. one of $\&, \vee, \rightarrow, \leftrightarrow$).
- If φ is a formula and x is a variable, then $Qx\varphi$ is a formula (where 'Q' is one of $\forall, \exists$).

We will first show the representability of atomic formulae.

2.6.1 Atomic Formulae

Atomic formulae consist of all equalities between two terms t_1 and t_2 of the form $t_1 = t_2$ and all n-ary predicates $P(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n$ are terms. Atomic formulae are slightly different from the other notions we have been and will be considering as they do not need to be inductively defined; their construction depends only on terms and predicates and does not require any atomic formulae to be previously shown to be representable. We can thus use an equivalence definition in the metatheory and show that the definition specifies the same objects as the defined set does in the object-theory.

Metatheory

Similar to terms, n-ary predicates will exhibit the following structure:

$$[P, [x_1, [x_2, \ldots, [x_{n-1}, x_n] \cdots]]]$$

where P is a predicate symbol and $x_1, \ldots, x_n$ are variables.

Atomic formulae are then specified as

$$\begin{aligned}(\forall X)(\textsc{Atom}(X) \longleftrightarrow ((&X = [=, [(X)_{21}, (X)_{22}]] \,\&\, \textsc{Term}((X)_{2i}))\\ &\vee X = [(X)_1, [(X)_{21}, \ldots, [(X)_{2\cdots 21}, (X)_{2\cdots 22}] \cdots]]\\ &\&\, \textsc{Pred}((X)_1) \,\&\, \textsc{Term}((X)_{21}) \,\&\, \cdots \,\&\, \textsc{Term}((X)_{2\cdots 21})\\ &\quad \&\, \textsc{Term}((X)_{2\cdots 22})))\end{aligned}$$

Object-theory

The set of all atomic formulae $\mathbb{A}$ is defined as $\mathbb{A} = \bigcap a$ where a is

$$a = \{z \in \wp(\mathbb{B}) \mid (\forall x, y \in \mathbb{T})\langle \lfloor = \rfloor, \langle x, y \rangle\rangle \in z$$
$$\&(\forall x \in \mathit{pred})(\forall y_1, \ldots, y_n \in z)\langle x, \langle y_1, \ldots, \langle y_{n-1}, y_n\rangle \cdots\rangle\rangle \in z\}$$

We will first show $(\forall X)(\textsc{Atom}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathbb{A})$ directly by making a case distinction for when X is considered atomic, i.e. in one case we will assume it is an equality of terms, and in the other an n-ary predicate.

Proof. Case 1: X is an equality of terms.

1		$X = [=, [(X)_{21}, (X)_{22}]]$	Premise
2		$\textsc{Term}((X)_{2i})$	Premise
3	$*$	$z \in a$	Assume
4	$*$	$(\forall x, y \in \mathbb{T})\langle \lfloor = \rfloor, \langle x, y\rangle\rangle \in z$	DefE(a): 3
5	$*$	$\lfloor (X)_{2i} \rfloor \in \mathbb{T}$	RepE(Term): 2
6	$*$	$\langle \lfloor = \rfloor, \langle \lfloor (X)_{21} \rfloor, \lfloor (X)_{22} \rfloor\rangle\rangle \in z$	$\forall_\in$E: 4,5
7	$*$	$\lfloor [=, [(X)_{21}, (X)_{22}]] \rfloor \in z$	DefI(Code): 6
8	$*$	$\lfloor X \rfloor \in z$	RepE(=): 7,1
9	$*$	$(\forall z \in a)\lfloor X \rfloor \in z$	$\forall_\in$I: 8
10	$*$	$\lfloor X \rfloor \in \bigcap a$	DefI($\cap$): 9
11	$*$	$\lfloor X \rfloor \in \mathbb{A}$	DefI($\mathbb{A}$): 10
12		$ZF \vdash \lfloor X \rfloor \in \mathbb{A}$	ProvI: 11

Case 2: X is an n-ary predicate.

1	$X = [(X)_1, [(X)_{21}, \ldots, [(X)_{2\cdots21}, (X)_{2\cdots22}] \cdots]]$	Premise
2	$\textsc{Pred}((X)_1)$	Premise
3	$\textsc{Term}((X)_{21})$	Premise
4	$\vdots$	Premise
5	$\textsc{Term}((X)_{2\cdots21})$	Premise
6	$\textsc{Term}((X)_{2\cdots22})$	Premise
7	$*$ $z \in a$	Assume
8	$*$ $(\forall x \in \mathit{pred})(\forall y_1, \ldots, y_n \in z)$	
	$*$ $\langle x, \langle y_1, \ldots, \langle y_{n-1}, y_n \rangle \cdots \rangle\rangle \in z$	DefE(a): 7
9	$*$ $\lfloor (X)_1 \rfloor \in \mathit{pred}$	RepE(Pred): 2
10	$*$ $\lfloor (X)_{21} \rfloor \in \mathbb{T}$	RepE(Term): 3
11	$*$ $\vdots$	
12	$*$ $\lfloor (X)_{2\cdots21} \rfloor \in \mathbb{T}$	RepE(Term): 5
13	$*$ $\lfloor (X)_{2\cdots22} \rfloor \in \mathbb{T}$	RepE(Term): 6
14	$*$ $\lfloor (X)_{21} \rfloor \in z$	Lemma(6): 10
15	$*$ $\vdots$	
16	$*$ $\lfloor (X)_{2\cdots21} \rfloor \in z$	Lemma(6): 12
17	$*$ $\lfloor (X)_{2\cdots22} \rfloor \in z$	Lemma(6): 13
18	$*$ $\langle \lfloor (X)_1 \rfloor, \langle \lfloor (X)_{21} \rfloor, \ldots, \langle \lfloor (X)_{2\cdots21} \rfloor, \lfloor (X)_{2\cdots22} \rfloor \rangle \cdots \rangle\rangle \in z$	$\forall_\in$E: 8,9,14-17
19	$*$ $\lfloor [(X)_1, [(X)_{21}, \ldots, [(X)_{2\cdots21}, (X)_{2\cdots22}] \cdots]] \rfloor \in z$	DefI(Code): 18
20	$*$ $\lfloor X \rfloor \in z$	RepE(=): 19,1
21	$*$ $(\forall z \in a) \lfloor X \rfloor \in z$	$\forall_\in$I: 20
22	$*$ $\lfloor X \rfloor \in \bigcap a$	DefI($\cap$): 21
23	$*$ $\lfloor X \rfloor \in \mathbb{A}$	DefI($\mathbb{A}$): 22
24	$ZF \vdash \lfloor X \rfloor \in \mathbb{A}$	ProvI: 23

Thus we have shown $(\forall X)(\textsc{Atom}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathbb{A})$. $\square$

The representability of non-atomic formulae will be included in the proof of the representability of all non-formulae.

2.6.2 Formulae

Formulae will be inductively defined so as to satisfy the informal specification given initially.

Metatheory

$$\begin{aligned}(\forall X)((&\textsc{Atom}(X)\\ &\vee [X = [\neg, (X)_2] \,\&\, \textsc{Form}((X)_2)]\\ &\vee [X = [\Box, [(X)_{21}, (X)_{22}]] \,\&\, \textsc{Form}((X)_{2i})]\\ &\vee [X = [[Q, (X)_{12}], (X)_2] \,\&\, \textsc{Var}((X)_{12}) \,\&\, \textsc{Form}((X)_2)]) \longrightarrow \textsc{Form}(X))\end{aligned}$$

Such an inductive definition justifies an induction principle for formulae that allows one to conclude

$$(\forall X)(\textsc{Form}(X) \longrightarrow \varphi(X))$$

and is defined in a similar fashion to the induction principles for variables and terms.

Object-theory

We again define the smallest set satisfying a finite list of closure conditions. For formulae, the conditions each set z in the intersection must satisfy are:

- $\mathbb{A} \subseteq z$
- $(\forall x \in z)\langle\lfloor\neg\rfloor, x\rangle \in z$
- $(\forall x_1, x_2 \in z)\langle\lfloor\Box\rfloor, \langle x_1, x_2\rangle\rangle \in z$
- $(\forall x_1 \in z)(\forall x_2 \in \mathbb{V})\langle\langle\lfloor Q\rfloor, x_2\rangle, x_1\rangle \in z$

The existence of at least one set satisfying these conditions is guaranteed by the Axiom of Separation using $\mathbb{B}$ as the set from which elements are taken. We can then take the intersection of all sets satisfying the conditions to obtain the smallest such set. We define the set f as:

$$\begin{aligned}f = \{z \in \wp(\mathbb{B}) | \mathbb{A} \subseteq z &\\ &\& \,(\forall x \in z)\langle\lfloor\neg\rfloor, x\rangle \in z\\ &\& \,(\forall x_1, x_2 \in z)\langle\lfloor\Box\rfloor, \langle x_1, x_2\rangle\rangle \in z\\ &\& \,(\forall x_1 \in z)(\forall x_2 \in \mathbb{V})\langle\langle\lfloor Q\rfloor, x_2\rangle, x_1\rangle \in z\}\end{aligned}$$

and then define the set of formulae $\mathbb{F}$ as

$$\mathbb{F} = \bigcap f$$

With these definitions in place, we can now prove the first representability condition for formulae, namely:

$$(\forall X)(\textsc{Form}(X) \longrightarrow ZF \vdash \lfloor X\rfloor \in \mathbb{F})$$

Proof. By the metatheoretic induction principle for formulae.

Base Case: Atomic formulae.

Shown previously.

Inductive Case 1: Negation.

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{Form}((X)_2) \longrightarrow ZF \vdash (X)_2 \in \mathbb{F}$$

1			$X = [\neg, (X)_2]$	Premise
2			$\textsc{Form}((X)_2)$	Premise
3			$ZF \vdash (X)_2 \in \mathbb{F}$	IH: 2
4	*		$\lfloor (X)_2 \rfloor \in \mathbb{F}$	ProvE: 3
5	*		$z \in f$	Assume
6	*		$(\forall x \in z)\langle \lfloor \neg \rfloor, x \rangle \in z$	DefE(f): 5
7	*		$\lfloor (X)_2 \rfloor \in z$	Lemma(6): 4
8	*		$\langle \lfloor \neg \rfloor, \lfloor (X)_2 \rfloor \rangle \in z$	$\forall_{\in}$E: 6,7
9	*		$\lfloor [\neg, (X)_2] \rfloor \in z$	DefI(Code): 8
10	*		$\lfloor X \rfloor \in z$	RepE(=): 9,1
11	*		$(\forall z \in f)\lfloor X \rfloor \in z$	$\forall_{\in}$I: 10
12	*		$\lfloor X \rfloor \in \bigcap f$	DefI($\cap$): 11
13	*		$\lfloor X \rfloor \in \mathbb{F}$	DefI($\mathbb{F}$): 12
14			$ZF \vdash \lfloor X \rfloor \in \mathbb{F}$	ProvI: 13

Inductive Case 2: Binary Connectives.

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{Form}((X)_{2i}) \longrightarrow ZF \vdash \lfloor (X)_{2i} \rfloor \in \mathbb{F}$$

1	$X = [\Box, [(X)_{21}, (X)_{22}]]$	Premise
2	$\textsc{Form}((X)_{2i})$	Premise
3	$ZF \vdash \lfloor (X)_{2i} \rfloor \in \mathbb{F}$	IH: 2
4	$* \quad \lfloor (X)_{2i} \rfloor \in \mathbb{F}$	ProvE: 3
5	$* \quad\quad z \in f$	Assume
6	$* \quad\quad (\forall x_1, x_2 \in z)\langle \lfloor \Box \rfloor, \langle x_1, x_2 \rangle \rangle \in z$	DefE(f): 5
7	$* \quad\quad \lfloor (X)_{2i} \rfloor \in z$	Lemma(6): 4
8	$* \quad\quad \langle \lfloor \Box \rfloor, \langle \lfloor (X)_{21} \rfloor, \lfloor (X)_{22} \rfloor \rangle \rangle \in z$	$\forall_{\in}$E: 6,7
9	$* \quad\quad \lfloor [\Box, [(X)_{21}, (X)_{22}]] \rfloor \in z$	DefI(Code): 8
10	$* \quad\quad \lfloor X \rfloor \in z$	RepE(=): 9,1
11	$* \quad (\forall z \in f) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 10
12	$* \quad \lfloor X \rfloor \in \bigcap f$	DefI($\cap$): 11
13	$* \quad \lfloor X \rfloor \in \mathbb{F}$	DefI($\mathbb{F}$): 12
14	$ZF \vdash \lfloor X \rfloor \in \mathbb{F}$	ProvI: 13

Inductive Case 3: Quantified Formulae.

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{Form}((X)_2) \longrightarrow ZF \vdash \lfloor (X)_2 \rfloor \in \mathbb{F}$$

1	$X = [[Q, (X)_{12}], (X)_2]$	Premise
2	$\textsc{Var}((X)_{12})$	Premise
3	$\textsc{Form}((X)_2)$	Premise
4	$ZF \vdash \lfloor (X)_2 \rfloor \in \mathbb{F}$	IH: 3
5	$* \quad \lfloor (X)_2 \rfloor \in \mathbb{F}$	ProvE: 4
6	$* \quad\quad z \in f$	Assume
7	$* \quad\quad (\forall x_1 \in z)(\forall x_2 \in \mathbb{V})\langle\langle \lfloor Q \rfloor, x_2\rangle, x_1\rangle \in z$	DefE(f): 6
8	$* \quad\quad \lfloor (X)_{12} \rfloor \in \mathbb{V}$	RepE(Var): 2
9	$* \quad\quad \lfloor (X)_2 \rfloor \in z$	Lemma(6): 5
10	$* \quad\quad \langle\langle \lfloor Q \rfloor, \lfloor (X)_{12} \rfloor\rangle, \lfloor (X)_2 \rfloor\rangle \in z$	$\forall_\in$E: 7,9,8
11	$* \quad\quad \lfloor [[Q, (X)_{12}], (X)_2] \rfloor \in z$	DefI(Code): 10
12	$* \quad\quad \lfloor X \rfloor \in z$	RepE(=): 11,1
13	$* \quad (\forall z \in f)\lfloor X \rfloor \in z$	$\forall_\in$I: 12
14	$* \quad \lfloor X \rfloor \in \bigcap f$	DefI($\cap$): 13
15	$* \quad \lfloor X \rfloor \in \mathbb{F}$	DefI($\mathbb{F}$): 14
16	$ZF \vdash \lfloor X \rfloor \in \mathbb{F}$	ProvI: 15

Thus by the metatheoretic principle of induction for formulas, we conclude

$$(\forall X)(\textsc{Form}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathbb{F})$$

□

To establish R_2 for formulae, we must prove

$$(\forall X)(\textsc{NotForm}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \notin \mathbb{F})$$

Similar to the representability proof for non-variables, we may equivalently define the set of non-formulae $\mathbb{NF}$ in ZF and show

1. $(\forall X)(\textsc{NotForm}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathbb{NF})$
2. $ZF \vdash (\forall x \in \mathbb{F} \longrightarrow x \notin \mathbb{NF})$

3. $ZF \vdash (\forall x \in \mathbb{NF} \longrightarrow x \notin \mathbb{F})$

We will first prove (1) by providing a metatheoretic and object-theoretic definition for non-formulae.

Metatheory

$$
\begin{aligned}
(\forall X)((X = S \\
&\vee X = [=, S] \\
&\vee X = [\Box, S] \\
&\vee (X = [=, [(X)_{21}, (X)_{22}]] \,\&\, (\textsc{NotTerm}((X)_{21}) \vee \textsc{NotTerm}((X)_{22}))) \\
&\vee (X = [(X)_1, [(X)_{21}, \ldots, [(X)_{2\cdots 21}, (X)_{2\cdots 22}] \cdots]] \,\&\, (\textsc{NotPred}((X)_1) \\
&\quad \vee \textsc{NotTerm}((X)_{21}) \vee \cdots \vee \textsc{NotTerm}((X)_{2\cdots 21}) \vee \textsc{NotTerm}((X)_{2\cdots 22}))) \\
&\vee (X = [(X)_1, (X)_2] \,\&\, \textsc{NotPred}((X)_1) \,\&\, (X)_1 \neq (\neg) \,\&\, (X)_1 \neq (\Box) \\
&\quad \& \, (X)_1 \neq (=) \,\&\, (X)_{11} \neq Q) \\
&\vee (X = [\neg, (X)_2] \,\&\, \textsc{NotForm}((X)_2)) \\
&\vee (X = [\Box, [(X)_{21}, (X)_{22}]] \,\&\, (\textsc{NotForm}((X)_{21}) \vee \textsc{NotForm}((X)_{22}))) \\
&\vee (X = [[Q, (X)_{12}], (X)_2] \,\&\, (\textsc{NotVar}((X)_{12}) \vee \textsc{NotForm}((X)_2)))) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \longrightarrow \textsc{NotForm}(X))
\end{aligned}
$$

with an appropriate induction principle for non-formulae.

Object-theory

We define the set of non-formulae $\mathbb{NF}$ as $\mathbb{NF} = \bigcap nf$, where nf is

$$
\begin{aligned}
nf = \{z \in \wp(\mathbb{B}) \mid\ & \emptyset \in z \,\&\, \langle \lfloor = \rfloor, \emptyset \rangle \in z \,\&\, \langle \lfloor \Box \rfloor, \emptyset \rangle \in z \\
& \& \, (\forall x)(\forall y \in \mathbb{NT})(\langle \lfloor = \rfloor, \langle x, y \rangle \rangle \in z \,\&\, \langle \lfloor = \rfloor, \langle y, x \rangle \rangle \in z) \\
& \& \, (\forall x_1, \ldots, x_n)(\forall x \in \mathit{pred})(\forall y \in \mathbb{NT})(\langle x, \langle y, \ldots, \langle x_1, x_n \rangle \cdots \rangle \rangle \in z \\
& \quad \& \\
& \quad \vdots \\
& \quad \& \langle x, \langle x_1, \ldots, \langle x_n, y \rangle \cdots \rangle \rangle \in z) \\
& \& \, (\forall x)(\forall y \in \mathbb{NV})\langle \langle \lfloor Q \rfloor, y \rangle, x \rangle \in z \\
& \& \, (\forall x)(\forall y \in \mathit{npred})((y \neq \lfloor \neg \rfloor \,\&\, y \neq \lfloor \Box \rfloor \,\&\, y \neq \lfloor = \rfloor \,\&\, (y)_1 \neq \lfloor Q \rfloor) \\
& \quad \longrightarrow \langle y, x \rangle \in z) \\
& \& \, (\forall x \in z)\langle \lfloor \neg \rfloor, x \rangle \in z \\
& \& \, (\forall x)(\forall y \in z)(\langle \lfloor \Box \rfloor, \langle x, y \rangle \in z \,\&\, \langle \lfloor \Box \rfloor, \langle y, x \rangle \rangle \in z) \\
& \& \, (\forall x)(\forall y \in z)\langle \langle \lfloor Q \rfloor, x \rangle, y \rangle \in z\}
\end{aligned}
$$

We can now prove the representability of non-formulae by proving (1),(2), and (3) above.

Proof. We will show (1) by the principle of induction for non-formulae.

Base Case 1: $X = S$.

1	$X = S$	Premise
2	$*$ $\quad z \in nf$	Assume
3	$*$ $\quad \emptyset \in z$	DefE(nf): 2
4	$*$ $\quad \lfloor S \rfloor \in z$	DefI(Code): 3
5	$*$ $\quad \lfloor X \rfloor \in z$	RepE(=): 4,1
6	$*$ $(\forall z \in nf)\lfloor X \rfloor \in z$	$\forall_{\in}$I: 5
7	$*$ $\lfloor X \rfloor \in \bigcap nf$	DefI($\cap$): 6
8	$*$ $\lfloor X \rfloor \in \mathbb{NF}$	DefI($\mathbb{NF}$): 7
9	$ZF \vdash \lfloor X \rfloor \in \mathbb{NF}$	ProvI: 8

Base Case 2: $X = [=, S]$.

1	$X = [=, S]$	Premise
2	$*$ $\quad z \in nf$	Assume
3	$*$ $\quad \langle \lfloor = \rfloor, \emptyset \rangle \in z$	DefE(nf): 2
4	$*$ $\quad \lfloor [=, S] \rfloor \in z$	DefI(Code): 3
5	$*$ $\quad \lfloor X \rfloor \in z$	RepE(=): 4,1
6	$*$ $(\forall z \in nf)\lfloor X \rfloor \in z$	$\forall_{\in}$I: 5
7	$*$ $\lfloor X \rfloor \in \bigcap nf$	DefI($\cap$): 6
8	$*$ $\lfloor X \rfloor \in \mathbb{NF}$	DefI($\mathbb{NF}$): 7
9	$ZF \vdash \lfloor X \rfloor \in \mathbb{NF}$	ProvI: 8

Base Case 3: $[\Box, S]$.

1		$X = [\Box, S]$	Premise
2	$*$	$z \in nf$	Assume
3	$*$	$\langle \lfloor \Box \rfloor, \emptyset \rangle \in z$	DefE(nf): 2
4	$*$	$\lfloor [\Box, S] \rfloor \in z$	DefI(Code): 3
5	$*$	$\lfloor X \rfloor \in z$	RepE(=): 4,1
6	$*$	$(\forall z \in nf) \lfloor X \rfloor \in z$	$\forall_\in$I: 5
7	$*$	$\lfloor X \rfloor \in \bigcap nf$	DefI($\cap$): 6
8	$*$	$\lfloor X \rfloor \in \mathbb{NF}$	DefI($\mathbb{NF}$): 7
9		$ZF \vdash \lfloor X \rfloor \in \mathbb{NF}$	ProvI: 8

Base Case 4: $X = [=, [(X)_{21}, (X)_{22}]]$ either of $(X)_{21}$ or $(X)_{22}$ is not a term.

1	$X = [=, [(X)_{21}, (X)_{22}]]$	Premise
2	$\textsc{NotTerm}((X)_{21}) \vee \textsc{NotTerm}((X)_{22})$	Premise
3	$\textsc{NotTerm}((X)_{21})$	Assume
4	$*$ $z \in nf$	Assume
5	$*$ $(\forall x)(\forall y \in \mathbb{NT})(\langle \lfloor = \rfloor, \langle x, y \rangle \rangle \in z \,\&$	
	$*$ $\langle \lfloor = \rfloor, \langle y, x \rangle \rangle \in z)$	DefE(nf): 4
6	$*$ $\lfloor (X)_{21} \rfloor \in \mathbb{NT}$	RepE(NotTerm): 3
7	$*$ $\langle \lfloor = \rfloor, \langle \lfloor (X)_{21} \rfloor, \lfloor (X)_{22} \rfloor \rangle \rangle \in z$	$\forall_{\in}$E: 5,6
8	$*$ $\lfloor [=, [(X)_{21}, (X)_{22}]] \rfloor \in z$	DefI(Code): 7
9	$*$ $\lfloor X \rfloor \in z$	RepE(=): 8,1
10	$*$ $(\forall z \in nf) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 9
11	$*$ $\lfloor X \rfloor \in \bigcap nf$	DefI($\cap$): 10
12	$*$ $\lfloor X \rfloor \in \mathbb{NF}$	DefI($\mathbb{NF}$): 11
13	$ZF \vdash \lfloor X \rfloor \in \mathbb{NF}$	ProvI: 12
14	$\textsc{NotTerm}((X)_{22})$	Assume
15	$*$ $z \in nf$	Assume
16	$*$ $(\forall x)(\forall y \in \mathbb{NT})(\langle \lfloor = \rfloor, \langle x, y \rangle \rangle \in z \,\&$	
	$*$ $\langle \lfloor = \rfloor, \langle y, x \rangle \rangle \in z)$	DefE(nf): 14
17	$*$ $\lfloor (X)_{22} \rfloor \in \mathbb{NT}$	RepE(NotTerm): 15
18	$*$ $\langle \lfloor = \rfloor, \langle \lfloor (X)_{21} \rfloor, \lfloor (X)_{22} \rfloor \rangle \rangle \in z$	$\forall_{\in}$E: 16,17
19	$*$ $\lfloor [=, [(X)_{21}, (X)_{22}]] \rfloor \in z$	DefI(Code): 18
20	$*$ $\lfloor X \rfloor \in z$	RepE(=): 19,1
21	$*$ $(\forall z \in nf) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 20
22	$*$ $\lfloor X \rfloor \in \bigcap nf$	DefI($\cap$): 21
23	$*$ $\lfloor X \rfloor \in \mathbb{NF}$	DefI($\mathbb{NF}$): 22
24	$ZF \vdash \lfloor X \rfloor \in \mathbb{NF}$	ProvI: 23
25	$ZF \vdash \lfloor X \rfloor \in \mathbb{NF}$	$\vee$E: 2,13,24

Base Case 5: $X = [(X)_1, [(X)_{21}, \ldots, [(X)_{2\cdots21}, (X)_{2\cdots22}] \cdots]]$ where either the first component is not a predicate symbol or one of the other components fails to be a term.

Similar to Base Case 4, with arbitrarily many subderivations, each corresponding to one of the disjuncts above where either the first component fails to be a prediate or one of the other components fails to be a term.

Base Case 6: The initial component of X fails to be a logical symbol or predicate.

1	$X = [(X)_1, (X)_2]$	Premise
2	$\textsc{NotPred}((X)_1)$	Premise
3	$(X)_1 \neq (\neg) \,\&\, (X)_1 \neq (\Box) \,\&\, (X)_1 \neq (=) \,\&\, (X)_{11} \neq Q$	Premise
4	$*$ $\quad$ $z \in nf$	Assume
5	$*$ $\quad$ $(\forall x)(\forall y \in npred)((y \neq \lfloor\neg\rfloor \,\&\, y \neq \lfloor\Box\rfloor \,\&\, y \neq \lfloor=\rfloor \,\&$	
	$*$ $\quad\quad$ $(y)_1 \neq \lfloor Q\rfloor) \longrightarrow \langle y, x\rangle \in z)$	DefE(nf): 4
6	$*$ $\quad$ $\lfloor(X)_1\rfloor \in npred$	RepE(NotPred): 2
7	$*$ $\quad$ $(\lfloor(X)_1\rfloor \neq \lfloor\neg\rfloor \,\&\, \lfloor(X)_1\rfloor \neq \lfloor\Box\rfloor \,\&\, \lfloor(X)_1\rfloor \neq \lfloor=\rfloor \,\&$	
	$*$ $\quad\quad$ $\lfloor(X)_{11}\rfloor \neq \lfloor Q\rfloor) \longrightarrow \langle\lfloor(X)_1\rfloor, \lfloor(X)_2\rfloor\rangle \in z$	$\forall_\in$E: 5,6
8	$*$ $\quad$ $\lfloor(X)_1\rfloor \neq \lfloor\neg\rfloor \,\&\, \lfloor(X)_1\rfloor \neq \lfloor\Box\rfloor \,\&\, \lfloor(X)_1\rfloor \neq \lfloor=\rfloor \,\&$	
	$*$ $\quad\quad$ $\lfloor(X)_{11}\rfloor \neq \lfloor Q\rfloor$	RepE(=): 3
9	$*$ $\quad$ $\langle\lfloor(X)_1\rfloor, \lfloor(X)_2\rfloor\rangle \in z$	$\to$E: 7,8
10	$*$ $\quad$ $\lfloor[(X)_1, (X)_2]\rfloor$	DefI(Code): 9
11	$*$ $\quad$ $\lfloor X\rfloor \in z$	RepE(=): 10,1
12	$*$ $(\forall z \in nf)\lfloor X\rfloor \in z$	$\forall_\in$I: 11
13	$*$ $\lfloor X\rfloor \in \bigcap nf$	DefI($\cap$): 12
14	$*$ $\lfloor X\rfloor \in \mathbb{NF}$	DefI($\mathbb{NF}$): 13
15	$ZF \vdash \lfloor X\rfloor \in \mathbb{NF}$	ProvI: 14

Inductive Case 1: $X = [\neg, (X)_2]$ and $(X)_2$ is not a formula.

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{NotForm}((X)_2) \longrightarrow ZF \vdash \lfloor (X)_2 \rfloor \in \mathbb{NF}$$

1		$X = [\neg, (X)_2]$	Premise
2		$\textsc{NotForm}((X)_2)$	Premise
3		$ZF \vdash \lfloor (X)_2 \rfloor \in \mathbb{NF}$	IH: 2
4	*	$\lfloor (X)_2 \rfloor \in \mathbb{NF}$	ProvE: 3
5	*	$z \in nf$	Assume
6	*	$(\forall x \in z)\langle \lfloor \neg \rfloor, x \rangle \in z$	DefE(nf): 5
7	*	$\lfloor (X)_2 \rfloor \in z$	Lemma(6): 4
8	*	$\langle \lfloor \neg \rfloor, \lfloor (X)_2 \rfloor \rangle \in z$	$\forall_\in$E: 6,7
9	*	$\lfloor [\neg, (X)_2] \rfloor \in z$	DefI(Code): 8
10	*	$\lfloor X \rfloor \in z$	RepE(=): 9,1
11	*	$(\forall z \in nf)\lfloor X \rfloor \in z$	$\forall_\in$I: 10
12	*	$\lfloor X \rfloor \in \bigcap nf$	DefI($\cap$): 11
13	*	$\lfloor X \rfloor \in \mathbb{NF}$	DefI($\mathbb{NF}$): 12
14		$ZF \vdash \lfloor X \rfloor \in \mathbb{NF}$	ProvI: 13

Inductive Case 2: $X = [\Box, [(X)_{21}, (X)_{22}]]$ and ≥ 1 component not a formula.

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{NotForm}((X)_{2i}) \longrightarrow ZF \vdash \lfloor (X)_{2i} \rfloor \in \mathbb{NF}$$

1	$X = [\square, [(X)_{21}, (X)_{22}]]$	Premise
2	$\textsc{NotForm}((X)_{21}) \vee \textsc{NotForm}((X)_{22})$	Premise
3	$\textsc{NotForm}((X)_{21})$	Assume
4	$ZF \vdash \lfloor (X)_{21} \rfloor \in \mathbb{NF}$	IH: 3
5	$* \quad \lfloor (X)_{21} \rfloor \in \mathbb{NF}$	ProvE: 4
6	$* \quad z \in nf$	Assume
7	$* \quad (\forall x)(\forall y \in z)(\langle \lfloor \square \rfloor, \langle x, y \rangle \in z \,\&\, \langle \lfloor \square \rfloor, \langle y, x \rangle \rangle \in z)$	DefE(nf): 6
8	$* \quad \lfloor (X)_{21} \rfloor \in z$	Lemma(6): 5
9	$* \quad \langle \lfloor \square \rfloor, \langle \lfloor (X)_{21} \rfloor, \lfloor (X)_{22} \rfloor \rangle \in z$	$\forall_{\in}$E: 7,8
10	$* \quad \lfloor [\square, [(X)_{21}, (X)_{22}]] \rfloor \in z$	DefI(Code): 9
11	$* \quad \lfloor X \rfloor \in z$	RepE(=): 10,1
12	$* \quad (\forall z \in nf) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 11
13	$* \quad \lfloor X \rfloor \in \bigcap nf$	DefI($\cap$): 12
14	$* \quad \lfloor X \rfloor \in \mathbb{NF}$	DefI($\mathbb{NF}$): 13
15	$ZF \vdash \lfloor X \rfloor \in \mathbb{NF}$	ProvI: 14
16	$\textsc{NotForm}((X)_{22})$	Assume
17	$ZF \vdash \lfloor (X)_{22} \rfloor \in \mathbb{NF}$	IH: 16
18	$* \quad \lfloor (X)_{22} \rfloor \in \mathbb{NF}$	ProvE: 17
19	$* \quad z \in nf$	Assume
20	$* \quad (\forall x)(\forall y \in z)(\langle \lfloor \square \rfloor, \langle x, y \rangle \in z \,\&\, \langle \lfloor \square \rfloor, \langle y, x \rangle \rangle \in z)$	DefE(nf): 19
21	$* \quad \lfloor (X)_{22} \rfloor \in z$	Lemma(6): 18
22	$* \quad \langle \lfloor \square \rfloor, \langle \lfloor (X)_{21} \rfloor, \lfloor (X)_{22} \rfloor \rangle \in z$	$\forall_{\in}$E: 20,21
23	$* \quad \lfloor [\square, [(X)_{21}, (X)_{22}]] \rfloor \in z$	DefI(Code): 22
24	$* \quad \lfloor X \rfloor \in z$	RepE(=): 23,1
25	$* \quad (\forall z \in nf) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 24
26	$* \quad \lfloor X \rfloor \in \bigcap nf$	DefI($\cap$): 25
27	$* \quad \lfloor X \rfloor \in \mathbb{NF}$	DefI($\mathbb{NF}$): 26
28	$ZF \vdash \lfloor X \rfloor \in \mathbb{NF}$	ProvI: 27
29	$ZF \vdash \lfloor X \rfloor \in \mathbb{NF}$	$\vee$E: 2,15,28

Inductive Case 3: $X = [[Q, (X)_{12}], (X)_2]$ and $(X)_{12}$ is not a variable or $(X)_2$ is not a formula.

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{NotForm}((X)_2) \longrightarrow ZF \vdash \lfloor (X)_2 \rfloor \in \mathbb{NF}$$

1	$X = [[Q, (X)_{12}], (X)_2]$	Premise
2	$\textsc{NotVar}((X)_{12}) \vee \textsc{NotForm}((X)_2)$	Premise
3	$\quad \textsc{NotVar}((X)_{12})$	Assume
4	$\quad * \quad z \in nf$	Assume
5	$\quad * \quad (\forall x)(\forall y \in \mathbb{NV})\langle\langle\lfloor Q \rfloor, y\rangle, x\rangle \in z$	DefE(nf): 4
6	$\quad * \quad \lfloor (X)_{12} \rfloor \in \mathbb{NV}$	RepE(NotVar): 3
7	$\quad * \quad \langle\langle\lfloor Q \rfloor, \lfloor (X)_{12} \rfloor\rangle, \lfloor (X)_2 \rfloor\rangle \in z$	$\forall_\in$E: 5,6
8	$\quad * \quad \lfloor [[Q, (X)_{12}], (X)_2] \rfloor \in z$	DefI(Code): 7
9	$\quad * \quad \lfloor X \rfloor \in z$	RepE(=): 8,1
10	$\quad * \quad (\forall z \in nf)\lfloor X \rfloor \in z$	$\forall_\in$I: 9
11	$\quad * \quad \lfloor X \rfloor \in \bigcap nf$	DefI($\cap$): 10
12	$\quad * \quad \lfloor X \rfloor \in \mathbb{NF}$	DefI($\mathbb{NF}$): 11
13	$\quad ZF \vdash \lfloor X \rfloor \in \mathbb{NF}$	ProvI: 12

(cont'd)

1	$\text{NotForm}((X)_2)$	Assume
15	$ZF \vdash \lfloor (X)_2 \rfloor \in \mathbb{NF}$	IH: 14
16	$* \quad \lfloor (X)_2 \rfloor \in \mathbb{NF}$	ProvE: 15
17	$* \quad z \in nf$	Assume
18	$* \quad (\forall x)(\forall y \in z)\langle\langle\lfloor Q\rfloor, x\rangle, y\rangle \in z$	DefE(nf): 17
19	$* \quad \lfloor (X)_2 \rfloor \in z$	Lemma(6): 16
20	$* \quad \langle\langle\lfloor Q\rfloor, \lfloor (X)_{12}\rfloor\rangle, \lfloor (X)_2\rfloor\rangle \in z$	$\forall_\in$E: 18,19
21	$* \quad \lfloor[[Q, (X)_{12}], (X)_2]\rfloor \in z$	DefI(Code): 20
22	$* \quad \lfloor X \rfloor \in z$	RepE(=): 21,1
23	$* \quad (\forall z \in nf)\lfloor X \rfloor \in z$	$\forall_\in$I: 22
24	$* \quad \lfloor X \rfloor \in \bigcap nf$	DefI($\cap$): 23
25	$* \quad \lfloor X \rfloor \in \mathbb{NF}$	DefI($\mathbb{NF}$): 24
26	$ZF \vdash \lfloor X \rfloor \in \mathbb{NF}$	ProvI: 25
27	$ZF \vdash \lfloor X \rfloor \in \mathbb{NF}$	$\vee$E: 2,13,26

Thus by the principle of induction for non-formulae, we have

$$(\forall X)(\text{NotForm}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathbb{NF})$$

We must now show that ZF can prove that $\mathbb{F}$ and $\mathbb{NF}$ are disjoint by showing (2) and (3) from earlier. As in the case for variables, the object-theoretic definitions for $\mathbb{F}$ and $\mathbb{NF}$ in ZF justify principles of induction for each.

We will first show $ZF \vdash (\forall x)(x \in \mathbb{F} \longrightarrow x \notin \mathbb{NF})$ using the induction principle for $\mathbb{F}$ just described and the following lemma that is justified by the definition of $\mathbb{NF}$:

$$\begin{aligned}
(\forall w)(w \in \mathbb{NF} \longleftrightarrow\ & w = \emptyset \vee w = \langle\lfloor=\rfloor, \emptyset\rangle \vee w = \langle\lfloor\Box\rfloor, \emptyset\rangle \\
& \vee (\exists x)(\exists y \in \mathbb{NT})(w = \langle\lfloor=\rfloor, \langle x, y\rangle\rangle \vee w = \langle\lfloor=\rfloor, \langle y, x\rangle\rangle) \\
& \vee (\exists x_1, \ldots, x_n)(\exists x \in \mathit{pred})(\exists y \in \mathbb{NT})(w = \langle x, \langle y, \ldots, \langle x_1, x_n\rangle \cdots\rangle\rangle \\
& \quad \vee \cdots \vee w = \langle x, \langle x_1, \ldots, \langle x_n, y\rangle \cdots\rangle\rangle) \\
& \vee (\exists x)(\exists y \in \mathbb{NV})w = \langle\langle\lfloor Q\rfloor, y\rangle, x\rangle \\
& \vee (\exists x)(\exists y \in \mathit{npred})(y \neq \lfloor\neg\rfloor \& y \neq \lfloor\Box\rfloor \& y \neq \lfloor=\rfloor \& (y)_1 \neq \lfloor Q\rfloor \& w = \langle y, x\rangle) \\
& \vee (\exists x \in z)w = \langle\lfloor\neg\rfloor, x\rangle \vee (\exists x)(\exists y \in z)(w = \langle\lfloor\Box\rfloor, \langle x, y\rangle\rangle \\
& \vee w = \langle\lfloor\Box\rfloor, \langle y, x\rangle\rangle) \vee (\exists x)(\exists y \in z)w = \langle\langle\lfloor Q\rfloor, x\rangle, y\rangle)
\end{aligned}$$

The proofs for the base cases (where x is some atomic formula) will not be demonstrated explicitly, as they involve only the representation of explicitly coded objects and are thus not any different from the cases shown in the proof of the representability of non-variables. We will, however, provide the formal proofs for each of the inductive cases.

Inductive Case 1: For arbitrary $z \in \mathbb{F}$, $x = \langle \lfloor\neg\rfloor, z\rangle$.

Induction Hypothesis: Assume for z that

$$ZF \vdash z \in \mathbb{F} \longrightarrow z \notin \mathbb{NF}$$

1		$z \in \mathbb{F}$	Premise
2	*	$z \in \mathbb{F} \longrightarrow z \notin \mathbb{NF}$	IH
3	*	$\langle \lfloor\neg\rfloor, z\rangle \in \mathbb{NF}$	Assume
4	*	$\langle \lfloor\neg\rfloor, z\rangle = \emptyset$	Assume
5	*	$\bot$	Lemma(2): 4
6	*	$\langle \lfloor\neg\rfloor, z\rangle = \langle \lfloor=\rfloor, \emptyset\rangle$	Assume
7	*	$z = \emptyset$	FTOP: 6
8	*	$z \in \mathbb{NF}$	Lemma(8): 7
9	*	$z \notin \mathbb{NF}$	→E: 2,8
10	*	$\bot$	⊥I: 8,9
11	*	$\langle \lfloor\neg\rfloor, z\rangle = \langle \lfloor\Box\rfloor, \emptyset\rangle$	Assume
12	*	$\bot$	Previous Case: 11
13	*	$(\exists x)(\exists y \in \mathbb{NT})(\langle \lfloor\neg\rfloor, z\rangle = \langle \lfloor=\rfloor, \langle x, y\rangle\rangle$	
	*	$\vee \langle \lfloor\neg\rfloor, z\rangle = \langle \lfloor=\rfloor, \langle y, x\rangle\rangle)$	Assume
14	*	$y \in \mathbb{NT}$	Assume
15	*	$\langle \lfloor\neg\rfloor, z\rangle = \langle \lfloor=\rfloor, \langle x, y\rangle\rangle \vee \langle \lfloor\neg\rfloor, z\rangle = \langle \lfloor=\rfloor, \langle y, x\rangle\rangle$	Assume
16	*	$\langle \lfloor\neg\rfloor, z\rangle = \langle \lfloor=\rfloor, \langle x, y\rangle\rangle$	Assume
17	*	$\lfloor\neg\rfloor = \lfloor=\rfloor$	FTOP: 16
18	*	$\emptyset = \langle \emptyset\lfloor, \leftrightarrow\rfloor\rangle$	DefE(Code): 17
19	*	$\bot$	Lemma(2): 18

(cont'd)

1	*	$\langle\lfloor\neg\rfloor, z\rangle = \langle\lfloor=\rfloor, \langle y, x\rangle\rangle$	Assume
21	*	$\bot$	Previous Case: 20
22	*	$\bot$	$\vee$E: 15,19,21
23	*	$\bot$	$\exists_\in$E: 13,22
24	*	$(\exists x_1, \ldots, x_n)(\exists x \in pred)(\exists y \in \mathbb{NT})$	
	*	$(\langle\lfloor\neg\rfloor, z\rangle = \langle x, \langle y, \ldots, \langle x_1, x_n\rangle \cdots\rangle\rangle \vee$	
	*	$\cdots \vee \langle\lfloor\neg\rfloor, z\rangle = \langle x, \langle x_1, \ldots, \langle x_n, y\rangle \cdots\rangle\rangle)$	Assume
25	*	$x \in pred$	Assume
26	*	$y \in \mathbb{NT}$	Assume
27	*	$\langle\lfloor\neg\rfloor, z\rangle = \langle x, \langle y, \ldots, \langle x_1, x_n\rangle \cdots\rangle\rangle \vee$	
	*	$\cdots \vee \langle\lfloor\neg\rfloor, z\rangle = \langle x, \langle x_1, \ldots, \langle x_n, y\rangle \cdots\rangle\rangle$	Assume
28	*	$\langle\lfloor\neg\rfloor, z\rangle = \langle x, \langle y, \ldots, \langle x_1, x_n\rangle \cdots\rangle\rangle$	Assume
29	*	$\lfloor\neg\rfloor = x$	FTOP
30	*	$(\exists y_1)\emptyset = \langle\emptyset, y_1\rangle$	Lemma(9): 29
31	*	$\bot$	Lemma(2): 30
32	*	$\vdots$	
33	*	$\langle\lfloor\neg\rfloor, z\rangle = \langle x, \langle x_1, \ldots, \langle x_n, y\rangle \cdots\rangle\rangle$	Assume
34	*	$\bot$	Previous cases: 33
35	*	$\bot$	$\vee$E: 27,31,...,34
36	*	$\bot$	$\exists_\in$E: 24,35
37	*	$(\exists x)(\exists y \in \mathbb{NV})\langle\lfloor\neg\rfloor, z\rangle = \langle\langle\lfloor Q\rfloor, y\rangle, x\rangle$	Assume
38	*	$y \in \mathbb{NV}$	Assume
39	*	$\langle\lfloor\neg\rfloor, z\rangle = \langle\langle\lfloor Q\rfloor, y\rangle, x\rangle$	Assume
40	*	$\lfloor\neg\rfloor = \langle\lfloor Q\rfloor, y\rangle$	FTOP: 39
41	*	$\emptyset = \langle\lfloor Q\rfloor, y\rangle$	DefE(Code): 40
42	*	$\bot$	Lemma(2): 41
43	*	$\bot$	$\exists_\in$E: 37,42

(cont'd)

1	$*$	$(\exists x)(\exists y \in npred)(y \neq \lfloor\neg\rfloor \,\&\, y \neq \lfloor\Box\rfloor \,\&\, y \neq \lfloor=\rfloor \,\&$	
	$*$	$(y)_1 \neq \lfloor Q\rfloor \,\&\, \langle\lfloor\neg\rfloor, z\rangle = \langle y, x\rangle)$	Assume
45	$*$	$y \in npred$	Assume
46	$*$	$y \neq \lfloor\neg\rfloor \,\&\, y \neq \lfloor\Box\rfloor \,\&\, y \neq \lfloor=\rfloor \,\&\, (y)_1 \neq \lfloor Q\rfloor$	Assume
47	$*$	$\langle\lfloor\neg\rfloor, z\rangle = \langle y, x\rangle$	Assume
48	$*$	$\lfloor\neg\rfloor = y$	FTOP: 47
49	$*$	$y \neq \lfloor\neg\rfloor$	&E: 46
50	$*$	$\bot$	$\bot$I: 48,49
51	$*$	$\bot$	$\exists_\in$E: 44,50
52	$*$	$(\exists x \in \mathbb{NF})\langle\lfloor\neg\rfloor, z\rangle = \langle\lfloor\neg\rfloor, x\rangle$	Assume
53	$*$	$x \in \mathbb{NF}$	Assume
54	$*$	$\langle\lfloor\neg\rfloor, z\rangle = \langle\lfloor\neg\rfloor, x\rangle$	Assume
55	$*$	$z = x$	FTOP: 54
56	$*$	$z \in \mathbb{NF}$	=E: 53,55
57	$*$	$z \notin \mathbb{NF}$	$\rightarrow$E: 2,1
58	$*$	$\bot$	$\bot$I: 56,57
59	$*$	$\bot$	$\exists_\in$E: 52,58

(cont'd)

1	$(\exists x)(\exists y \in z)(\langle\lfloor\neg\rfloor, z\rangle = \langle\lfloor\Box\rfloor, \langle x, y\rangle\rangle \vee$	
	$\langle\lfloor\neg\rfloor, z\rangle = \langle\lfloor\Box\rfloor, \langle y, x\rangle\rangle)$	Assume
61	$y \in z$	Assume
62	$\langle\lfloor\neg\rfloor, z\rangle = \langle\lfloor\Box\rfloor, \langle x, y\rangle\rangle \vee \langle\lfloor\neg\rfloor, z\rangle = \langle\lfloor\Box\rfloor, \langle y, x\rangle\rangle$	Assume
63	$\langle\lfloor\neg\rfloor, z\rangle = \langle\lfloor\Box\rfloor, \langle x, y\rangle\rangle$	Assume
64	$\lfloor\neg\rfloor = \lfloor\Box\rfloor$	FTOP: 63
65	$(\exists y_1)\emptyset = \langle\emptyset, y_1\rangle$	Lemma(9): 64
66	$\bot$	Lemma(2): 65
67	$\langle\lfloor\neg\rfloor, z\rangle = \langle\lfloor\Box\rfloor, \langle y, x\rangle\rangle$	Assume
68	$\bot$	Previous Case: 67
69	$\bot$	$\vee$E: 62
70	$\bot$	$\exists_\in$E: 60,69
71	$(\exists x)(\exists y \in z)\langle\lfloor\neg\rfloor, z\rangle = \langle\langle\lfloor Q\rfloor, x\rangle, y\rangle$	Assume
72	$y \in z$	Assume
73	$\langle\lfloor\neg\rfloor, z\rangle = \langle\langle\lfloor Q\rfloor, x\rangle, y\rangle$	Assume
74	$\lfloor\neg\rfloor = \langle\lfloor Q\rfloor, x\rangle$	FTOP: 73
75	$\emptyset = \langle\lfloor Q\rfloor, x\rangle$	DefE(Code): 74
76	$\bot$	Lemma(2): 75
77	$\bot$	$\exists$E: 71,76
78	$\bot$	Lemma(8):
		5,10,12,23,
		36,43,51,59,70,77
79	$\langle\lfloor\neg\rfloor, z\rangle \notin \mathbb{NF}$	$\neg$I: 78
80	$ZF \vdash \langle\lfloor\neg\rfloor, z\rangle \notin \mathbb{NF}$	ProvI: 79

Inductive Case 2: For arbitrary $z_1, z_2 \in \mathbb{F}$, $x = \langle \lfloor \Box \rfloor, \langle z_1, z_2 \rangle \rangle$.

Induction Hypothesis: Assume for z_1, z_2 that

$$ZF \vdash z_i \in \mathbb{F} z_i \notin \mathbb{NF}$$

The proof for this case will follow the same procedure as in Case 1, i.e. each case will fail in the same way with $\lfloor \Box \rfloor$ being the term that will generate the contradictions in place of the term $\lfloor \neg \rfloor$ above. We will thus not show every case. The only two cases that will be different (and for which formal proofs will be provided) are:

- $(\exists x \in \mathbb{NF})\langle \lfloor \Box \rfloor, \langle z_1, z_2 \rangle \rangle = \langle \lfloor \neg \rfloor, x \rangle$
- $(\exists x)(\exists y \in \mathbb{NF})(\langle \lfloor \Box \rfloor, \langle z_1, z_2 \rangle \rangle = \langle \lfloor \Box \rfloor, \langle x, y \rangle \rangle \vee \langle \lfloor \Box \rfloor, \langle z_1, z_2 \rangle \rangle = \langle \lfloor \Box \rfloor, \langle y, x \rangle \rangle$

To be clear, the following two subderivations are part of a larger derivation, the larger derivation intended to establish $ZF \vdash \langle \lfloor \Box \rfloor, \langle z_1, z_2 \rangle \rangle \notin \mathbb{NF}$ by using the same lemma that was utilized in Case 1. The argument is indirect, so each subderivation generated by the lemma is to establish a contradiction. Thus the two subderivations given below, corresponding to the two cases described, will each try to establish a contradiction.

1	*	$(\exists x \in \mathbb{NF})\langle \lfloor \Box \rfloor, \langle z_1, z_2 \rangle \rangle = \langle \lfloor \neg \rfloor, x \rangle$	Assume
2	*	$x \in \mathbb{NF}$	Assume
3	*	$\langle \lfloor \Box \rfloor, \langle z_1, z_2 \rangle \rangle = \langle \lfloor \neg \rfloor, x \rangle$	Assume
4	*	$\lfloor \Box \rfloor = \lfloor \neg \rfloor$	FTOP: 3
5	*	$(\exists y_1)\langle \emptyset, y_1 \rangle = \emptyset$	Lemma(9): 4
6	*	$\bot$	Lemma(2): 5
7	*	$\bot$	$\exists_{\in}$E: 1,6

1	*	$(\exists x)(\exists y \in \mathbb{NF})(\langle\langle \lfloor \Box \rfloor, \langle z_1, z_2 \rangle\rangle =$	
	*	$\langle \lfloor \Box \rfloor, \langle x, y \rangle\rangle \vee \langle \lfloor \Box \rfloor, \langle z_1, z_2 \rangle\rangle = \langle \lfloor \Box \rfloor, \langle y, x \rangle\rangle)$	Assume
2	*	$y \in \mathbb{NF}$	Assume
3	*	$\langle \lfloor \Box \rfloor, \langle z_1, z_2 \rangle\rangle = \langle \lfloor \Box \rfloor, \langle x, y \rangle\rangle \vee$	
	*	$\langle \lfloor \Box \rfloor, \langle z_1, z_2 \rangle\rangle = \langle \lfloor \Box \rfloor, \langle y, x \rangle\rangle$	Assume
4	*	$\langle \lfloor \Box \rfloor, \langle z_1, z_2 \rangle\rangle = \langle \lfloor \Box \rfloor, \langle x, y \rangle\rangle$	Assume
5	*	$z_2 = y$	FTOP: 4
6	*	$z_2 \in \mathbb{NF}$	=E: 2,5
7	*	$z_2 \notin \mathbb{NF}$	IH
8	*	$\bot$	$\bot$I: 6,7
9	*	$\langle \lfloor \Box \rfloor, \langle z_1, z_2 \rangle\rangle = \langle \lfloor \Box \rfloor, \langle y, x \rangle\rangle$	Assume
10	*	$z_1 = y$	FTOP: 9
11	*	$z_1 \in \mathbb{NF}$	=E: 2,10
12	*	$z_1 \notin \mathbb{NF}$	IH
13	*	$\bot$	$\bot$I: 11,12
14	*	$\bot$	$\vee$E: 3,8,13
15	*	$\bot$	$\exists$E: 1,14

Inductive Case 3: For arbitrary $z \in \mathbb{F}$ and $y \in \mathbb{V}$, $x = \langle\langle \lfloor Q \rfloor, y\rangle, z\rangle$.

Induction Hypothesis: Assume for z that

$$ZF \vdash z \in \mathbb{F} z \notin \mathbb{NF}$$

As in the previous case, it is unnecessary to show every sub-case. Most of the cases allow one to immediately infer that two different basic codings are equal to each other, and since all of the basic codings must be unique, we get a contradiction trivially. The case that will be explicitly carried out is the last case where we assume $(\exists x)(\exists y \in z)\langle\langle \lfloor Q \rfloor, y\rangle, z\rangle = \langle\langle \lfloor Q \rfloor, x\rangle, y)$ and try to derive a contradicion (this is again just one subderivation of a larger derivation that is trying to establish that $\langle\langle \lfloor Q \rfloor, y\rangle, z\rangle \notin \mathbb{NF}$ using the same main lemma from the previous cases. The proof is by refutation, so each subderivation, including the one shown below, should try to establish some contradiction).

1	*	$(\exists x)(\exists y \in \mathbb{NF})\langle\langle\lfloor Q\rfloor, y\rangle, z\rangle = \langle\langle\lfloor Q\rfloor, x\rangle, y$	Assume
2	*	$\quad y \in \mathbb{NF}$	Assume
3	*	$\quad \langle\langle\lfloor Q\rfloor, y\rangle, z\rangle = \langle\langle\lfloor Q\rfloor, x\rangle, y$	Assume
4	*	$\quad z = y$	FTOP: 3
5	*	$\quad z \in \mathbb{NF}$	=E: 2,4
6	*	$\quad z \notin \mathbb{NF}$	IH
7	*	$\quad \bot$	$\bot$I: 5,6
8	*	$\bot$	$\exists_{\in}$E: 7

This is the final inductive case, and thus we conclude by the principle of induction for the set $\mathbb{F}$ that

$$ZF \vdash (\forall x)(x \in \mathbb{F} \longrightarrow x \notin \mathbb{NF})$$

To conclude the representability of non-formulae, and thus the complete representability of formulae, it remains to show that

$$ZF \vdash (\forall x)(x \in \mathbb{NF} \longrightarrow x \notin \mathbb{F})$$

so that it is guaranteed that the set containing all coded formulae and the set containing all coded non-formulae are disjoint. The proof of this claim will only be indicated as it follows nearly identically to the proof carried out for the previous claim. We use the induction principle for $\mathbb{NF}$ and use the following lemma justified by the definition of formulae:

$$\begin{aligned}(\forall x)(x \in \mathbb{F} \longleftrightarrow x \in \mathbb{A} &\vee (\exists z \in \mathbb{F})x = \langle\lfloor\neg\rfloor z\rangle \\ &\vee (\exists z_1, z_2 \in \mathbb{F})x = \langle\lfloor\square\rfloor, \langle z_1, z_2\rangle\rangle \vee (\exists x \in \mathbb{V})(\exists y \in \mathbb{F})x = \langle\langle\lfloor Q\rfloor, x\rangle, y\rangle)\end{aligned}$$

For each case in the inductive argument, we assume that x is of some particular structure and try to show that $ZF \vdash x \notin \mathbb{F}$. Each case proceeds indirectly, assuming $x \in \mathbb{F}$ and using the lemma described above to derive a series of subderivations, each establishing a contradiction.

From this we conclude that $ZF \vdash (\forall x)(x \in \mathbb{NF} \longrightarrow x \notin \mathbb{F})$ by the principle of induction for $\mathbb{NF}$.

We have shown

1. $(\forall X)(\textsc{Form}(X) \longrightarrow ZF \vdash \lfloor X\rfloor \in \mathbb{F})$
2. $(\forall X)(\textsc{NotForm}(X) \longrightarrow ZF \vdash \lfloor X\rfloor \in \mathbb{NF})$
3. $ZF \vdash (\forall x)(x \in \mathbb{F} \longrightarrow x \notin \mathbb{NF})$
4. $ZF \vdash (\forall x)(x \in \mathbb{NF} \longrightarrow x \notin \mathbb{F})$

(1) is the first representability condition, R_1. (3) and (4) together show that $\mathbb{F}$ and $\mathbb{NF}$ are disjoint, and these together with (2) show

$$(\forall X)(\textsc{NotForm}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \notin \mathbb{F})$$

which is the second representability condition R_2, and thus we have shown that formulae are representable inside ZF. □

2.7 Representability of Free Variables

Informally, free variables are specified as

- All variables are free in all terms and atomic formulae.
- If x is a free variable of formula φ, then x is a free variable of $\neg\varphi$
- If x is a free variable of formulae φ and ψ, then x is a free variable of $\varphi \,\Box\, \psi$.
- If x_1 is a free variable of formula φ and x_2 is a variable different from x_1, then x_1 is a free variable of $Qx_2\varphi$

(informal specification is based largely on Van Dalen's formulation in *Logic and Structure* [5])

Metatheory

We again require the objects representing free variables to be of a particular binary tree structure:

$$X = [[[X_1, \ldots, X_n], V], F]$$

where $X_1, \ldots, X_n$ are some other free variable objects that are appealed in inductive cases (this list may be empty, for base cases), V is the free variable itself, and F is the formula for which the variable is free.

The metatheoretic definition is as follows:

$$\begin{aligned}
(\forall X)(&[[X = [(X)_1, (X)_2] \,\&\, \textsc{Var}((X)_1) \,\&\, \textsc{Term}((X)_2)] \\
&\vee [X = [(X)_1, (X)_2] \,\&\, \textsc{Var}((X)_1) \,\&\, \textsc{Atom}((X)_2)] \\
&\vee [X = [[(X)_{1211}, (X)_{12}], [\neg, (X)_{122}]] \,\&\, \textsc{Free}((X)_{12}) \,\&\, \textsc{Form}((X)_{122})] \\
&\vee [X = [[(X)_{12111}, [(X)_{121}, (X)_{122}]], [\Box, [(X)_{1212}, (X)_{1222}]]] \,\&\, \textsc{Free}((X)_{12i}) \\
&\quad \,\&\, \textsc{Form}((X)_{12i2}) \,\&\, (X)_{12111} = (X)_{12211}] \\
&\vee [X = [[(X)_{1211}, (X)_{12}], [[Q, (X)_{212}], (X)_{122}]] \,\&\, \textsc{Free}((X)_{12}) \,\&\, \textsc{Form}((X)_{122}) \\
&\quad \,\&\, \textsc{Var}((X)_{212}) \,\&\, (X)_{1211} \neq (X)_{212}]] \longrightarrow \textsc{Free}(X))
\end{aligned}$$

also with an induction schema for proving $(\forall X)(\textsc{Free}(X) \longrightarrow \varphi(X))$.

Object-theory

As in the other cases, we define the smallest set satisfying the appropriate conditions, in this case for free variables. We first define fv as

$$\begin{aligned} fv = \{z \in \wp(\mathbb{B}) \mid \quad & \{x \in \mathbb{B} \mid (x)_1 \in \mathbb{V} \,\&\, (x)_2 \in \mathbb{T}\} \subseteq z \\ & \&\{x \in \mathbb{B} \mid (x)_1 \in \mathbb{V} \,\&\, (x)_2 \in \mathbb{A}\} \subseteq z \\ & \&(\forall x \in z)((x)_2 \in \mathbb{F} \longrightarrow \langle\langle (x)_{11}, x\rangle, \langle \lfloor \neg \rfloor, (x)_2\rangle\rangle \in z) \\ & \&(\forall x, y \in z)([(x)_2, (y)_2 \in \mathbb{F} \,\&\, (x)_{11} = (y)_{11}] \longrightarrow \\ & \quad \langle\langle (x)_{11}, \langle x, y\rangle\rangle, \langle \lfloor \Box \rfloor, \langle (x)_2, (y)_2\rangle\rangle\rangle \in z) \\ & \&(\forall x \in z)(\forall y \in \mathbb{V})([(x)_2 \in \mathbb{F} \,\&\, (x)_{11} \neq y] \longrightarrow \\ & \quad \langle\langle (x)_{11}, x\rangle, \langle\langle \lfloor Q \rfloor, y\rangle, (x)_2\rangle\rangle \in z)\} \end{aligned}$$

and then define the set of free variables $\mathcal{F}$ as

$$\mathcal{F} = \bigcap fv$$

We can now show by the principle of induction for free variables that

$$(\forall X)[\textsc{Free}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathcal{F}]$$

Proof. Base Case 1: Terms.

1	$X = [(X)_1, (X)_2]$	Premise
2	$\textsc{Var}((X)_1)$	Premise
3	$\textsc{Term}((X)_2)$	Premise
4	$*$ $\quad$ $z \in fv$	Assume
5	$*$ $\quad$ $\{x \in \mathbb{B} \mid (x)_1 \in \mathbb{V} \,\&\, (x)_2 \in \mathbb{T}\} \subseteq z$	DefE(fv): 4
6	$*$ $\quad\quad$ $y \in b$	Assume
7	$*$ $\quad\quad$ $(\forall x_1, x_2 \in y)\langle x_1, x_2\rangle \in y$	DefE(b): 6
8	$*$ $\quad\quad$ $\lfloor (X)_1 \rfloor \in \mathbb{V}$	RepE(Var): 2
9	$*$ $\quad\quad$ $\lfloor (X)_1 \rfloor \in \mathbb{B}$	DefE($\mathbb{V}$): 8
10	$*$ $\quad\quad$ $\lfloor (X)_1 \rfloor \in y$	Lemma(6): 9
11	$*$ $\quad\quad$ $\lfloor (X)_2 \rfloor \in \mathbb{T}$	RepE(Term): 3
12	$*$ $\quad\quad$ $\lfloor (X)_2 \rfloor \in \mathbb{B}$	DefE($\mathbb{T}$): 11
13	$*$ $\quad\quad$ $\lfloor (X)_2 \rfloor \in y$	Lemma(6): 12
14	$*$ $\quad\quad$ $\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle \in y$	$\forall_\in$E: 7,10,13
15	$*$ $\quad$ $(\forall y \in b)\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle \in y$	$\forall_\in$I: 14
16	$*$ $\quad$ $\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle \in \bigcap b$	DefI($\cap$): 15
17	$*$ $\quad$ $\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle \in \mathbb{B}$	DefI($\mathbb{B}$): 16
18	$*$ $\quad$ $(\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle)_1 \in \mathbb{V}$	DefI(Proj): 8
19	$*$ $\quad$ $(\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle)_2 \in \mathbb{T}$	DefI(Proj): 11
20	$*$ $\quad$ $\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle \in \{x \in \mathbb{B} \mid (x)_1 \in \mathbb{V} \,\&\, (x)_2 \in \mathbb{T}\}$	DefI(Comp): 17,18,19
21	$*$ $\quad$ $\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle \in z$	DefE($\subseteq$): 5,20
22	$*$ $\quad$ $\lfloor [(X)_1, (X)_2] \rfloor \in z$	DefI(Code): 21
23	$*$ $\quad$ $\lfloor X \rfloor \in z$	RepE(=): 22,1
24	$*$ $(\forall z \in fv)\lfloor X \rfloor \in z$	$\forall_\in$I: 23
25	$*$ $\lfloor X \rfloor \in \bigcap fv$	DefI($\cap$): 24
26	$*$ $\lfloor X \rfloor \in \mathcal{F}$	DefI($\mathcal{F}$): 25
27	$ZF \vdash \lfloor X \rfloor \in \mathcal{F}$	ProvI: 26

Base Case 2: Atomic Formulae.

1	$X = [(X)_1, (X)_2]$	Premise
2	$\textsc{Var}((X)_1)$	Premise
3	$\textsc{Atom}((X)_2)$	Premise
4	* $z \in fv$	Assume
5	* $\{x \in \mathbb{B} \mid (x)_1 \in \mathbb{V} \,\&\, (x)_2 \in \mathbb{A}\} \subseteq z$	DefE(fv): 4
6	* $y \in b$	Assume
7	* $(\forall x_1, x_2 \in y)\langle x_1, x_2\rangle \in y$	DefE(b): 6
8	* $\lfloor (X)_1 \rfloor \in \mathbb{V}$	RepE(Var): 2
9	* $\lfloor (X)_1 \rfloor \in \mathbb{B}$	DefE($\mathbb{V}$): 8
10	* $\lfloor (X)_1 \rfloor \in y$	Lemma(6): 9
11	* $\lfloor (X)_2 \rfloor \in \mathbb{A}$	RepE(Atom): 3
12	* $\lfloor (X)_2 \rfloor \in \mathbb{B}$	DefE($\mathbb{A}$): 11
13	* $\lfloor (X)_2 \rfloor \in y$	Lemma(6): 12
14	* $\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle \in y$	$\forall_\in$E: 7,10,13
15	* $(\forall y \in b)\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle \in y$	$\forall_\in$I: 14
16	* $\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle \in \bigcap b$	DefI($\cap$): 15
17	* $\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle \in \mathbb{B}$	DefI($\mathbb{B}$): 16
18	* $(\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle)_1 \in \mathbb{V}$	DefI(Proj): 8
19	* $(\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle)_2 \in \mathbb{A}$	DefI(Proj): 11
20	* $\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle \in \{x \in \mathbb{B} \mid (x)_1 \in \mathbb{V} \,\&\, (x)_2 \in \mathbb{A}\}$	DefI(Comp): 17,18,19
21	* $\langle \lfloor (X)_1 \rfloor, \lfloor (X)_2 \rfloor \rangle \in z$	DefE($\subseteq$): 5,20
22	* $\lfloor [(X)_1, (X)_2] \rfloor \in z$	DefI(Code): 21
23	* $\lfloor X \rfloor \in z$	RepE(=): 22,1
24	* $(\forall z \in fv)\lfloor X \rfloor \in z$	$\forall_\in$I: 23
25	* $\lfloor X \rfloor \in \bigcap fv$	DefI($\cap$): 24
26	* $\lfloor X \rfloor \in \mathcal{F}$	DefI($\mathcal{F}$): 25
27	$ZF \vdash \lfloor X \rfloor \in \mathcal{F}$	ProvI: 26

Induction Case 1: Negation.

Assume for arbitrary X that

$$\textsc{Free}((X)_{11}) \longrightarrow ZF \vdash \lfloor (X)_{11} \rfloor \in \mathcal{F}$$

1	$X = [[(X)_{11}, (X)_{1112}], [\neg, (X)_{112}]]$	Premise
2	$\textsc{Free}((X)_{11})$	Premise
3	$\textsc{Form}((X)_{112})$	Premise
4	$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathcal{F}$	IH: 2
5	$* \quad \lfloor (X)_{11} \rfloor \in \mathcal{F}$	ProvE: 4
6	$* \quad z \in fv$	Assume
7	$* \quad (\forall x \in z)((x)_2 \in \mathbb{F} \longrightarrow \langle \langle x, (x)_{12} \rangle, \langle \lfloor \neg \rfloor, (x)_2 \rangle \rangle \in z)$	DefE(fv): 6
8	$* \quad \lfloor (X)_{11} \rfloor \in z$	Lemma(6): 5
9	$* \quad \lfloor (X)_{112} \rfloor \in \mathbb{F} \longrightarrow \langle \langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor \rangle,$	
	$* \qquad \langle \lfloor \neg \rfloor, \lfloor (X)_{112} \rfloor \rangle \rangle \in z$	$\forall_{\in}$E: 7,8
10	$* \quad \lfloor (X)_{112} \rfloor \in \mathbb{F}$	RepE(Form): 3
11	$* \quad \langle \langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor \rangle, \langle \lfloor \neg \rfloor, \lfloor (X)_{112} \rfloor \rangle \rangle \in z$	$\rightarrow$E: 9,10
12	$* \quad \lfloor [[(X)_{11}, (X)_{1112}], [\neg, (X)_{112}]] \rfloor \in z$	DefI(Code): 11
13	$* \quad \lfloor X \rfloor \in z$	RepE(=): 12,1
14	$* \quad (\forall z \in fv) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 13
15	$* \quad \lfloor X \rfloor \in \bigcap fv$	DefI($\cap$): 14
16	$* \quad \lfloor X \rfloor \in \mathcal{F}$	DefI($\mathcal{F}$): 15
17	$ZF \vdash \lfloor X \rfloor \in \mathcal{F}$	ProvI: 16

Induction Case 2: Binary Connectives.

Assume for arbitrary X that

$$\textsc{Free}((X)_{11i}) \longrightarrow ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathcal{F}$$

1	$X = [[[(X)_{111}, (X)_{112}], (X)_{11112}], [\Box, [(X)_{1112}, (X)_{1122}]]]$	Premise
2	$\textsc{Free}((X)_{11i})$	Premise
3	$\textsc{Form}((X)_{11i2})$	Premise
4	$(X)_{11112} = (X)_{11212}$	Premise
5	$ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathcal{F}$	IH: 2
6	$* \quad \lfloor (X)_{11i} \rfloor \in \mathcal{F}$	ProvE: 5
7	$* \quad\quad z \in fv$	Assume
8	$* \quad\quad (\forall x, y \in z)([(x)_2, (y)_2 \in \mathbb{F} \,\&\, (x)_{12} = (y)_{12}] \longrightarrow$	
	$* \quad\quad\quad \langle\langle\langle x, y\rangle, (x)_{12}\rangle,$	
	$* \quad\quad\quad \langle\lfloor\Box\rfloor, \langle (x)_2, (y)_2\rangle\rangle\rangle \in z)$	DefE(fv): 7
9	$* \quad\quad \lfloor (X)_{11i} \rfloor \in z$	Lemma(6): 6
10	$* \quad\quad [\lfloor (X)_{1112}\rfloor, \lfloor (X)_{1122}\rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11112}\rfloor = \lfloor (X)_{11212}\rfloor] \longrightarrow$	
	$* \quad\quad\quad \langle\langle\langle \lfloor (X)_{111}\rfloor, \lfloor (X)_{112}\rfloor\rangle, (X)_{11112}\rangle,$	
	$* \quad\quad\quad \langle\lfloor\Box\rfloor, \langle \lfloor (X)_{1112}\rfloor, \lfloor (X)_{1122}\rfloor\rangle\rangle\rangle \in z$	$\forall_\in$E: 8,9
11	$* \quad\quad \lfloor (X)_{1112}\rfloor, \lfloor (X)_{1122}\rfloor \in \mathbb{F}$	RepE(Form): 3
12	$* \quad\quad \lfloor (X)_{11112}\rfloor = \lfloor (X)_{11212}\rfloor$	RepE(=): 4
13	$* \quad\quad \lfloor (X)_{1112}\rfloor, \lfloor (X)_{1122}\rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11112}\rfloor = \lfloor (X)_{11212}\rfloor$	&I: 11,12
14	$* \quad\quad \langle\langle\langle \lfloor (X)_{111}\rfloor, \lfloor (X)_{112}\rfloor\rangle, (X)_{11112}\rangle,$	
	$* \quad\quad\quad \langle\lfloor\Box\rfloor, \langle \lfloor (X)_{1112}\rfloor, \lfloor (X)_{1122}\rfloor\rangle\rangle\rangle \in z$	$\rightarrow$E: 10,13
15	$* \quad\quad \lfloor [[[(X)_{111}, (X)_{112}], (X)_{11112}], [\Box, [(X)_{1112}, (X)_{1122}]]] \rfloor \in z$	DefI(Code): 14
16	$* \quad\quad \lfloor X \rfloor \in z$	RepE(=): 15,1
17	$* \quad (\forall z \in fv)\lfloor X \rfloor \in z$	$\forall_\in$I: 16
18	$* \quad \lfloor X \rfloor \in \bigcap fv$	DefI($\cap$): 17
19	$* \quad \lfloor X \rfloor \in \mathcal{F}$	DefI($\mathcal{F}$): 18
20	$ZF \vdash \lfloor X \rfloor \in \mathcal{F}$	ProvI: 19

Induction Case 3: Quantified Formulae.

Assume for arbitrary X that

$$\textsc{Free}((X)_{11}) \longrightarrow ZF \vdash \lfloor (X)_{11} \rfloor \in \mathcal{F}$$

1	$X = [[(X)_{11}, (X)_{1112}], [[Q, (X)_{212}], (X)_{112}]]$	Premise
2	$\textsc{Free}((X)_{11})$	Premise
3	$\textsc{Form}((X)_{112})$	Premise
4	$\textsc{Var}((X)_{212})$	Premise
5	$(X)_{1112} \neq (X)_{212}$	Premise
6	$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathcal{F}$	IH: 2
7	$* \quad \lfloor (X)_{11} \rfloor \in \mathcal{F}$	ProvE: 6
8	$* \quad\quad z \in fv$	Assume
9	$* \quad\quad (\forall x \in z)(\forall y \in \mathbb{V})([(x)_2 \in \mathbb{F} \,\&\, (x)_{12} \neq y] \longrightarrow$	
	$* \quad\quad\quad \langle\langle x, (x)_{12}\rangle, \langle\langle \lfloor Q \rfloor, y\rangle, (x)_2\rangle\rangle \in z)$	DefE(fv): 8
10	$* \quad\quad \lfloor (X)_{11} \rfloor \in z$	Lemma(6): 7
11	$* \quad\quad \lfloor (X)_{212} \rfloor \in \mathbb{V}$	RepE(Var): 4
12	$* \quad\quad [\lfloor (X)_{112} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{1112} \rfloor \neq \lfloor (X)_{212} \rfloor] \longrightarrow$	
	$* \quad\quad\quad \langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \langle\langle \lfloor Q \rfloor, \lfloor (X)_{212} \rfloor\rangle, \lfloor (X)_{112} \rfloor\rangle\rangle \in z$	$\forall_\in$E: 9,10,11
13	$* \quad\quad \lfloor (X)_{112} \rfloor \in \mathbb{F}$	RepE(Form): 3
14	$* \quad\quad \lfloor (X)_{1112} \rfloor \neq \lfloor (X)_{212} \rfloor$	RepE(=): 5
15	$* \quad\quad \lfloor (X)_{112} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{1112} \rfloor \neq \lfloor (X)_{212} \rfloor$	&I: 13,14
16	$* \quad\quad \langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \langle\langle \lfloor Q \rfloor, \lfloor (X)_{212} \rfloor\rangle, \lfloor (X)_{112} \rfloor\rangle\rangle \in z$	→E: 12,15
17	$* \quad\quad \lfloor [[(X)_{11}, (X)_{1112}], [[Q, (X)_{212}], (X)_{112}]] \rfloor \in z$	DefI(Code): 16
18	$* \quad\quad \lfloor X \rfloor \in z$	RepE(=): 17,1
19	$* \quad (\forall z \in fv)\lfloor X \rfloor \in z$	$\forall_\in$I: 18
20	$* \quad \lfloor X \rfloor \in \bigcap fv$	DefI($\cap$): 19
21	$* \quad \lfloor X \rfloor \in \mathcal{F}$	DefI($\mathcal{F}$): 20
22	$ZF \vdash \lfloor X \rfloor \in \mathcal{F}$	ProvI: 21

Thus by the principle of induction for free variables,

$$(\forall X)[\textsc{Free}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathcal{F}]$$

□

We must also show

$$(\forall X)[\textsc{NotFree}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \notin \mathcal{F}]$$

The proof of this claim proceeds in the same way that the proof for the representability of non-variables did. As was mentioned before, we will not carry out the proof for this direction as the general structure follows that of the proof of non-variables exactly.

With the proof of R_1, and R_2 being taken for granted (though straightforwardly provable), we conclude the representability of free variables.

2.8 Representability of Bound Variables

The informal specification of bound variables is as follows:

- If x is a free variable of φ, then x is a bound variable of $Qx\varphi$
- If x is a bound variable of φ, then x is a bound variable of $\neg\varphi$
- If x is a bound variable of φ or of ψ, then x is a bound variable of $\varphi \,\Box\, \psi$
- If x is a bound variable of φ, then x is a bound variable of $Qx\varphi$

(also based on Van Dalen's formulation [5]).

Metatheory

The binary tree objects representing bound variables will be of a same structure as free variables:

$$X = [[[X_1, \ldots, X_n], V], F]$$

where $X_1, \ldots, X_n$ are some other bound variable objects, V is the bound variable itself, and F is the formula in which the variable occurs.

The metatheoretic definition is formulated as:

$$\begin{aligned}
(\forall X)([[(\forall x \in \mathcal{F})((x)_2 \in \mathbb{F} \longrightarrow \langle\langle (x)_{11}, x\rangle, \langle\langle \lfloor Q \rfloor, (x)_{11}\rangle, (x)_2\rangle\rangle \in z)]\\
&\vee [X = [[(X)_{1211}, (X)_{12}], [\neg, (X)_{122}]] \,\&\, \textsc{Bound}((X)_{12})]\\
&\vee [X = [[(X)_{12111}, (X)_{121}], [\Box, [(X)_{1212}, (X)_{1222}]]] \,\&\, \textsc{Bound}((X)_{121}) \,\&\, \textsc{Form}((X)_{122})]\\
&\vee [X = [[(X)_{12111}, (X)_{122}], [\Box, [(X)_{1212}, (X)_{1222}]]] \,\&\, \textsc{Bound}((X)_{122}) \,\&\, \textsc{Form}((X)_{121})]\\
&\vee [X = [[(X)_{1211}, (X)_{12}], [[Q, (X)_{212}], (X)_{122}]] \,\&\, \textsc{Bound}((X)_{12}) \,\&\, \textsc{Var}((X)_{212})]]\\
&\qquad\qquad\qquad\qquad \longrightarrow \textsc{Bound}(X))
\end{aligned}$$

also with an induction principle for proving $(\forall X)(\textsc{Bound}(X) \longrightarrow \varphi(X))$.

Object-theory

We define the set bv to be

$$\begin{aligned} bv = \{z \in \wp(\mathbb{B}) \mid & (\forall x \in \mathcal{F})\langle\langle (x)_{11}, x\rangle, \langle\langle \lfloor Q \rfloor, (x)_{11}\rangle, (x)_2\rangle\rangle \in z \\ & \&(\forall x \in z)\langle\langle (x)_{11}, x\rangle, \langle \lfloor \neg \rfloor, (x)_2\rangle \in z \\ & \&(\forall x \in z)(\forall y \in \mathbb{F})\langle\langle (x)_{11}, x\rangle, \langle \lfloor \Box \rfloor, \langle (x)_2, y\rangle\rangle\rangle \in z \\ & \&(\forall x \in z)(\forall y \in \mathbb{F})\langle\langle (x)_{11}, x\rangle, \langle \lfloor \Box \rfloor, \langle y, (x)_2\rangle\rangle\rangle \in z \\ & \&(\forall x \in z)(\forall y \in \mathbb{V})\langle\langle (x)_{11}, x\rangle, \langle\langle \lfloor Q \rfloor, y\rangle, (x)_2\rangle\rangle \in z\} \end{aligned}$$

and then define the set of bound variables $\mathcal{B}$ to be

$$\mathcal{B} = \bigcap bv$$

We can now prove R_1 for bound variables. We will show

$(\forall X)[\textsc{Bound}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathcal{B}]$ by induction on X.

Proof. Base Case: Formulae containing free variables.

1		$X = [[(X)_{11}, (X)_{1112}], [[Q, (X)_{1112}], (X)_{112}]]$	Premise
2		$\textsc{Free}((X)_{11})$	Premise
3		$\textsc{Form}((X)_{112})$	Premise
4	*	$z \in bv$	Assume
5	*	$(\forall x \in \mathcal{F})((x)_2 \in \mathbb{F} \longrightarrow$	
	*	$\langle\langle x, (x)_{12}\rangle, \langle\langle \lfloor Q \rfloor, (x)_{12}\rangle, (x)_2\rangle\rangle \in z)$	DefE(bv): 4
6	*	$\lfloor (X)_{11} \rfloor \in \mathcal{F}$	RepE(Free): 2
7	*	$\lfloor (X)_{112} \in \mathbb{F} \longrightarrow$	
	*	$\langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \langle\langle \lfloor Q \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \lfloor (X)_{112} \rfloor\rangle\rangle \in z$	$\forall_{\in}$E: 5,6
8	*	$\lfloor (X)_{112} \rfloor \in \mathbb{F}$	RepE(Form): 3
9	*	$\langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \langle\langle \lfloor Q \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \lfloor (X)_{112} \rfloor\rangle\rangle \in z$	$\rightarrow$E: 7,8
10	*	$\lfloor [[(X)_{11}, (X)_{1112}], [[Q, (X)_{1112}], (X)_{112}]] \rfloor \in z$	DefE(Code): 9
11	*	$\lfloor X \rfloor \in z$	RepE(=): 10,1
12	*	$(\forall z \in bv)\lfloor X \rfloor \in z$	$\forall_{\in}$I: 11
13	*	$\lfloor X \rfloor \in \bigcap bv$	DefI($\cap$): 12
14	*	$\lfloor X \rfloor \in \mathcal{B}$	DefI($\mathcal{B}$): 13
15		$ZF \vdash \lfloor X \rfloor \in \mathcal{B}$	ProvI: 14

Induction Case 1: Negation.

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{Bound}((X)_{11}) \longrightarrow ZF \vdash \lfloor (X)_{11} \rfloor \in \mathcal{B}$$

1	$X = [[(X)_{11}, (X)_{1112}], [\neg, (X)_{112}]]$	Premise
2	$\textsc{Bound}((X)_{11})$	Premise
3	$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathcal{B}$	IH: 2
4	$*$ $\lfloor (X)_{11} \rfloor \in \mathcal{B}$	ProvE: 3
5	$*$ $z \in bv$	Assume
6	$*$ $(\forall x \in z)\langle\langle x, (x)_{12}\rangle, \langle \lfloor \neg \rfloor, (x)_2\rangle \in z$	DefE(bv): 5
7	$*$ $\lfloor (X)_{11} \rfloor \in z$	Lemma(6): 4
8	$*$ $\langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \langle \lfloor \neg \rfloor, \lfloor (X)_{112} \rfloor\rangle \in z$	$\forall_{\in}$E: 6,7
9	$*$ $\lfloor [[(X)_{11}, (X)_{1112}], [\neg, (X)_{112}]] \rfloor \in z$	DefE(Code): 8
10	$*$ $\lfloor X \rfloor \in z$	RepE(=): 9,1
11	$*$ $(\forall z \in bv)\lfloor X \rfloor \in z$	$\forall_{\in}$I: 10
12	$*$ $\lfloor X \rfloor \in \bigcap bv$	DefI($\cap$): 11
13	$*$ $\lfloor X \rfloor \in \mathcal{B}$	DefI($\mathcal{B}$): 12
14	$ZF \vdash \lfloor X \rfloor \in \mathcal{B}$	ProvI: 13

Induction Case 2.1: Binary Connectives (left component).

Induction Hypothesis:

$$\textsc{Bound}((X)_{11}) \longrightarrow ZF \vdash \lfloor (X)_{11} \rfloor \in \mathcal{B}$$

1		$X = [[(X)_{11}, (X)_{1112}], [\Box, [(X)_{112}, (X)_{222}]]]$	Premise
2		$\textsc{Bound}((X)_{11})$	Premise
3		$\textsc{Form}((X)_{222})$	Premise
4		$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathcal{B}$	IH: 2
5	*	$\lfloor (X)_{11} \rfloor \in \mathcal{B}$	ProvE: 4
6	*	$\quad z \in bv$	Assume
7	*	$\quad (\forall x \in z)(\forall y \in \mathbb{F})\langle\langle x, (x)_{12}\rangle, \langle \lfloor \Box \rfloor, \langle (x)_2, y\rangle\rangle\rangle \in z$	DefE(bv): 6
8	*	$\quad \lfloor (X)_{11} \rfloor \in z$	Lemma(6): 5
9	*	$\quad \lfloor (X)_{222} \rfloor \in \mathbb{F}$	RepE(Form): 3
10	*	$\quad \langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \langle \lfloor \Box \rfloor, \langle \lfloor (X)_{112} \rfloor, \lfloor (X)_{222} \rfloor\rangle\rangle\rangle \in z$	$\forall_{\in}$E: 7,9
11	*	$\quad \lfloor [[(X)_{11}, (X)_{1112}], [\Box, [(X)_{112}, (X)_{222}]]] \rfloor \in z$	DefI(Code): 10
12	*	$\quad \lfloor X \rfloor \in z$	RepE(=): 11
13	*	$(\forall z \in bv) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 12
14	*	$\lfloor X \rfloor \in \bigcap bv$	DefI($\cap$): 13
15	*	$\lfloor X \rfloor \in \mathcal{B}$	DefI($\mathcal{B}$): 14
16		$ZF \vdash \lfloor X \rfloor \in \mathcal{B}$	ProvI: 15

Induction Case 2.2: Binary Connectives (right component).

Induction Hypothesis:

$$\textsc{Bound}((X)_{11}) \longrightarrow ZF \vdash \lfloor (X)_{11} \rfloor \in \mathcal{B}$$

1	$X = [[(X)_{11}, (X)_{1112}], [\Box, [(X)_{221}, (X)_{112}]]]$	Premise
2	$\textsc{Bound}((X)_{11})$	Premise
3	$\textsc{Form}((X)_{221})$	Premise
4	$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathcal{B}$	IH: 2
5	$* \quad \lfloor (X)_{11} \rfloor \in \mathcal{B}$	ProvE: 4
6	$* \quad\quad z \in bv$	Assume
7	$* \quad\quad (\forall x \in z)(\forall y \in \mathbb{F})\langle\langle x, (x)_{12}\rangle, \langle \lfloor \Box \rfloor, \langle y, (x)_2 \rangle\rangle\rangle \in z$	DefE(bv): 6
8	$* \quad\quad \lfloor (X)_{11} \rfloor \in z$	Lemma(6): 5
9	$* \quad\quad \lfloor (X)_{221} \rfloor \in \mathbb{F}$	RepE(Form): 3
10	$* \quad\quad \langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \langle \lfloor \Box \rfloor, \langle \lfloor (X)_{221} \rfloor, \lfloor (X)_{112} \rfloor \rangle\rangle\rangle \in z$	$\forall_{\in}$E: 7,9
11	$* \quad\quad \lfloor [[(X)_{11}, (X)_{1112}], [\Box, [(X)_{221}, (X)_{112}]]] \rfloor \in z$	DefI(Code): 10
12	$* \quad\quad \lfloor X \rfloor \in z$	RepE(=): 11,1
13	$* \quad (\forall z \in bv) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 12
14	$* \quad \lfloor X \rfloor \in \bigcap bv$	DefI($\cap$): 13
15	$* \quad \lfloor X \rfloor \in \mathcal{B}$	DefI($\mathcal{B}$): 14
16	$ZF \vdash \lfloor X \rfloor \in \mathcal{B}$	ProvI: 15

Induction Case 3: Quantified Formlae.

Induction Hypothesis:

$$\textsc{Bound}((X)_{11}) \longrightarrow ZF \vdash \lfloor (X)_{11} \rfloor \in \mathcal{B}$$

1	$X = [[(X)_{11}, (X)_{1112}], [[Q, (X)_{212}], (X)_{112}]]$	Premise
2	$\textsc{Bound}((X)_{11})$	Premise
3	$\textsc{Var}((X)_{212})$	Premise
4	$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathcal{B}$	IH: 2
5	$* \quad \lfloor (X)_{11} \rfloor \in \mathcal{B}$	ProvE: 4
6	$* \quad\quad z \in bv$	Assume
7	$* \quad\quad (\forall x \in z)(\forall y \in \mathbb{V})\langle\langle x, (x)_{12}\rangle, \langle\langle \lfloor Q \rfloor, y\rangle, (x)_2\rangle\rangle \in z$	DefE(bv): 6
8	$* \quad\quad \lfloor (X)_{11} \rfloor \in z$	Lemma(6): 5
9	$* \quad\quad \lfloor (X)_{212} \rfloor \in \mathbb{V}$	RepE(Var): 3
10	$* \quad\quad \langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \langle\langle \lfloor Q \rfloor, \lfloor (X)_{212} \rfloor\rangle, \lfloor (X)_{112} \rfloor\rangle\rangle \in z$	$\forall_\in$E: 7,8,9
11	$* \quad\quad \lfloor [[(X)_{11}, (X)_{1112}], [[Q, (X)_{212}], (X)_{112}]] \rfloor \in z$	DefI(Code): 10
12	$* \quad\quad \lfloor X \rfloor \in z$	RepE(=): 11,1
13	$* \quad (\forall z \in bv)\lfloor X \rfloor \in z$	$\forall_\in$I: 12
14	$* \quad \lfloor X \rfloor \in \bigcap bv$	DefI($\cap$): 13
15	$* \quad \lfloor X \rfloor \in \mathcal{B}$	DefI($\mathcal{B}$): 14
16	$ZF \vdash \lfloor X \rfloor \in \mathcal{B}$	ProvI: 15

Thus by the principle of induction for bound variables, we have

$$(\forall X)[\textsc{Bound}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathcal{B}]$$

□

Again the proof of R_2 for bound variables, i.e.

$$(\forall X)[\textsc{NotBound}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \notin \mathcal{B}]$$

will be taken for granted, though it is directly provable by following the same procedure as in the case of non-variables and non-formulae.

The proofs of R_1 and R_2 together establish the representability of bound variables.

2.9 Representability of Substitution Instances

$\varphi[t/x]$ will denote a substitution instance of φ where all free instances of x are replaced by a term t. Substitution instances are then specified as:

- $y[t/y] = t$
- $y[t/x] = y$ for $x \neq y$
- $f(t_1, \ldots, t_n)[t/x] = f(t_1[t/x], \ldots, t_n[t/x])$
- $(t_1 = t_2)[t/x] = \;\; t_1[t/x] = t_2[t/x]$
- $P(x_1, \ldots, x_n)[t/x] = P(t_1[t/x], \ldots, t_n[t/x])$
- $(\neg\varphi)[t/x] = \neg\varphi[t/x]$
- $(\varphi \,\square\, \psi)[t/x] = \varphi[t/x] \,\square\, \psi[t/x]$
- $(Qx\varphi)[t/x] = Qx\varphi$
- If $x \neq y$, then $(Qx\varphi)[t/y] = Qx\varphi[t/y]$

(based on Van Dalen [5])

Metatheory

Substitution instances will have the following binary tree structure:

$$X = [[[X_1, \ldots, X_n], [T, V]], F]$$

where $X_1, \ldots, X_n$ are some other substitution instance objects, T is the term being substituted in for the variable V, and F is the formula that the substitution instance is of.

The metatheoretic definition is formulated as:

$$
\begin{aligned}
&(\forall X)(([X = [[(X)_{11}, [(X)_{11}, (X)_{122}]], (X)_{122}] \,\&\, \textsc{Term}((X)_{11}) \,\&\, \textsc{Var}((X)_{122})] \\
&\quad \vee [X = [[(X)_{11}, [(X)_{121}, (X)_{122}]], (X)_{11}] \,\&\, \textsc{Term}((X)_{121}) \,\&\, \textsc{Var}((X)_{11}) \,\&\, \\
&\quad\quad \textsc{Var}((X)_{122}) \,\&\, (X)_{11} \neq (X)_{122}] \\
&\quad \vee [X = [[S, [(X)_{121}, (X)_{212}]], (X)_2] \,\&\, \textsc{Form}((X)_2) \,\&\, \textsc{Term}((X)_{121}) \\
&\quad\quad \,\&\, (X)_{211} = Q] \\
&\quad \vee [X = [[[(X)_{111}, \ldots, [(X)_{112\cdots 21}, (X)_{112\cdots 22}] \cdots], [(X)_{111121}, (X)_{111122}]], \\
&\quad\quad [(X)_{21}, [(X)_{1112}, \ldots, [(X)_{112\cdots 212}, (X)_{112\cdots 222}] \cdots]]] \,\&\, \textsc{Subst}((X)_{111}) \,\&\, \cdots \\
&\quad\quad \,\&\, \textsc{Subst}((X)_{112\cdots 2i}) \,\&\, \textsc{Term}((X)_{1112}) \,\&\, \cdots \,\&\, \textsc{Term}((X)_{1112i2}) \,\&\, \\
&\quad\quad (X)_{11112i} = \cdots = (X)_{112\cdots 2112i} = (X)_{112\cdots 2212i} \,\&\, \textsc{Func}((X)_{21})] \\
&\quad \vee [X = [[[(X)_{111}, (X)_{112}], [(X)_{111121}, (X)_{111122}]], [=, [(X)_{1112}, (X)_{1122}]]] \,\&\, \\
&\quad\quad \textsc{Subst}((X)_{11i}) \,\&\, \textsc{Term}((X)_{11i2}) \,\&\, (X)_{11112i} = (X)_{11212i}] \\
&\quad \vee [X = [[(X)_{11}, [(X)_{11121}, (X)_{11122}]], [\neg, (X)_{112}]] \,\&\, \textsc{Subst}((X)_{11}) \,\&\, \\
&\quad\quad \textsc{Form}((X)_{112})] \\
&\quad \vee [X = [[[(X)_{111}, (X)_{112}], [(X)_{111121}, (X)_{111122}]], [\Box[(X)_{1112}, (X)_{1122}]]] \,\&\, \\
&\quad\quad \textsc{Subst}((X)_{11i}) \,\&\, \textsc{Form}((X)_{11i2}) \,\&\, (X)_{11112i} = (X)_{11212i}] \\
&\quad \vee [X = [[(X)_{11}, [(X)_{11121}, (X)_{11122}]], [[Q, (X)_{212}], (X)_{112}]] \,\&\, \textsc{Subst}((X)_{11}) \,\&\, \\
&\quad\quad \textsc{Form}((X)_{112}) \,\&\, \textsc{Var}((X)_{212}) \,\&\, (X)_{11122} \neq (X)_{212}]) \longrightarrow \textsc{Subst}(X))
\end{aligned}
$$

with an appropriate induction principle that establishes $(\forall X)(\textsc{Subst}(X) \longrightarrow \varphi(X))$.

Object-theory

We define s to be the set

$$
\begin{aligned}
s = \{z \in \wp(\mathbb{B}) | &(\forall x \in \mathbb{T})(\forall y \in \mathbb{V})\langle\langle x, \langle x, y\rangle, y\rangle \in z \\
&\,\&\, (\forall x \in \mathbb{T})(\forall y_1, y_2 \in \mathbb{V})(y_1 \neq y_2 \longrightarrow \langle\langle y_1, \langle x, y_2\rangle\rangle, y_1\rangle \in z) \\
&\,\&\, (\forall x \in \mathbb{F})(\forall y \in \mathbb{T})((x)_{11} = \lfloor Q \rfloor \longrightarrow \langle\langle \emptyset, \langle y, (x)_{12}\rangle\rangle, x\rangle \in z) \\
&\,\&\, (\forall x_1, \ldots, x_n \in z)(\forall y \in \mathit{func})(((x_1)_2, \ldots (X_n)_2 \in \mathbb{T} \,\&\, \\
&\quad (x)_{12i} = \cdots = (x_{n-1})_{12i} = (x_n)_{12i}) \longrightarrow \langle y, \langle x_1, \ldots, \langle x_{n-1}, x_n\rangle \cdots\rangle\rangle \in z) \\
&\,\&\, (\forall x, y \in z)((((x)_2, (y)_2 \in \mathbb{T} \,\&\, (x)_{12i} = (y)_{12i}) \longrightarrow \\
&\quad \langle\langle\langle x, y\rangle, \langle (x)_{121}, (x)_{122}\rangle\rangle, \langle \lfloor = \rfloor, \langle (x)_2, (y)_2\rangle\rangle\rangle \in z) \\
&\,\&\, (\forall x \in z)((x)_2 \in \mathbb{F} \longrightarrow \langle\langle x, \langle (x)_{121}, (x)_{122}\rangle\rangle, \langle, \lfloor \neg \rfloor, (x)_2\rangle\rangle \in z) \\
&\,\&\, (\forall x, y \in z)([(x)_2 \in \mathbb{F} \,\&\, (y)_2 \in \mathbb{F} \,\&\, (x)_{12i} = (y)_{12i}] \longrightarrow \\
&\quad \langle\langle\langle x, y\rangle, \langle (x)_{121}, (x)_{122}\rangle\rangle, \langle \lfloor \Box \rfloor, \langle (x)_2, (y)_2\rangle\rangle\rangle \in z) \\
&\,\&\, (\forall x \in z)(\forall y \in \mathbb{V})([(x)_2 \in \mathbb{F} \,\&\, (x)_{122} \neq y] \longrightarrow \\
&\quad \langle\langle x, \langle (x)_{121}, (x)_{122}\rangle\rangle, \langle\langle \lfloor Q \rfloor, y\rangle, (x)_2\rangle\rangle \in z)\}
\end{aligned}
$$

and then define the set of substitution instances $\mathbb{S}$ to be

$$
\mathbb{S} = \bigcap s
$$

By the principle of induction for substitution instances, we would like to prove the following statement:

$$(\forall X)[\textsc{Subst}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathbb{S}]$$

Proof. Base Case 1: Term in for same variable.

1	$X = [[(X)_{11}, [(X)_{11}, (X)_{122}]], (X)_{122}]$	Premise
2	$\textsc{Term}((X)_{11})$	Premise
3	$\textsc{Var}((X)_{122})$	Premise
4	$*$ $\quad z \in s$	Assume
5	$*$ $\quad (\forall x \in \mathbb{T})(\forall y \in \mathbb{V})\langle\langle x, \langle x, y\rangle, y\rangle \in z$	DefE(s): 4
6	$*$ $\quad \lfloor (X)_{11} \rfloor \in \mathbb{T}$	RepE(Term): 2
7	$*$ $\quad \lfloor (X)_{122} \rfloor \in \mathbb{V}$	RepE(Var): 3
8	$*$ $\quad \langle\langle \lfloor (X)_{11} \rfloor, \langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{122} \rfloor\rangle, \lfloor (X)_{122} \rfloor\rangle \in z$	$\forall_{\in}$E: 5,6,7
9	$*$ $\quad \lfloor [[(X)_{11}, [(X)_{11}, (X)_{122}]], (X)_{122}] \rfloor \in z$	DefI(Code): 8
10	$*$ $\quad \lfloor X \rfloor \in z$	RepE(=): 9,1
11	$*$ $(\forall z \in s)\lfloor X \rfloor \in z$	$\forall_{\in}$I: 10
12	$*$ $\lfloor X \rfloor \in \bigcap s$	DefI($\bigcap$): 11
13	$*$ $\lfloor X \rfloor \in \mathbb{S}$	DefI($\mathbb{S}$): 12
14	$ZF \vdash \lfloor X \rfloor \in \mathbb{S}$	ProvI: 13

Base Case 2: Term in for non-identical variables.

1		$X = [[(X)_{11}, [(X)_{121}, (X)_{122}]], (X)_{11}]$	Premise
2		$\text{TERM}((X)_{121})$	Premise
3		$\text{VAR}((X)_{11})$	Premise
4		$\text{VAR}((X)_{122})$	Premise
5		$(X)_{11} \neq (X)_{122}$	Premise
6	*	$z \in s$	Assume
7	*	$(\forall x \in \mathbb{T})(\forall y_1, y_2 \in \mathbb{V})(y_1 \neq y_2 \longrightarrow$	
	*	$\langle\langle y_1, \langle x, y_2\rangle\rangle, y_1\rangle \in z)$	DefE(s): 6
8	*	$\lfloor (X)_{121} \rfloor \in \mathbb{T}$	RepE(Term): 2
9	*	$\lfloor (X)_{11} \rfloor \in \mathbb{V}$	RepE(Var): 3
10	*	$\lfloor (X)_{122} \rfloor \in \mathbb{V}$	RepE(Var): 4
11	*	$\lfloor (X)_{11} \rfloor \neq \lfloor (X)_{122} \rfloor \longrightarrow$	
	*	$\langle\langle \lfloor (X)_{11} \rfloor, \langle \lfloor (X)_{121} \rfloor, \lfloor (X)_{122} \rfloor\rangle\rangle, \lfloor (X)_{11} \rfloor\rangle \in z$	$\forall_\in$E: 7,8,9,10
12	*	$\lfloor (X)_{11} \rfloor \neq \lfloor (X)_{122} \rfloor$	RepE(=): 5
13	*	$\langle\langle \lfloor (X)_{11} \rfloor, \langle \lfloor (X)_{121} \rfloor, \lfloor (X)_{122} \rfloor\rangle\rangle, \lfloor (X)_{11} \rfloor\rangle \in z$	→E: 11,12
14	*	$\lfloor [[(X)_{11}, [(X)_{121}, (X)_{122}]], (X)_{11}] \rfloor$	DefI(Code): 13
15	*	$\lfloor X \rfloor \in z$	RepE(=): 14,1
16	*	$(\forall z \in s)\lfloor X \rfloor \in z$	$\forall_\in$I: 15
17	*	$\lfloor X \rfloor \in \bigcap s$	DefI($\bigcap$): 16
18	*	$\lfloor X \rfloor \in \mathbb{S}$	DefI($\mathbb{S}$): 17
19		$ZF \vdash \lfloor X \rfloor \in \mathbb{S}$	ProvI: 18

Base Case 3: Quantified Formulae (substitution into bound variable)

1	$X = [[S, [(X)_{121}, (X)_{212}]], (X)_2]$	Premise
2	$\textsc{Form}((X)_2)$	Premise
3	$\textsc{Term}((X)_{121})$	Premise
4	$(X)_{211} = Q$	Premise
5	$* \quad z \in s$	Assume
6	$* \quad (\forall x \in \mathbb{F})(\forall y \in \mathbb{T})((x)_{11} = \lfloor Q \rfloor \longrightarrow$	
	$* \quad\quad \langle\langle \emptyset, \langle y, (x)_{12} \rangle\rangle, x \rangle \in z)$	DefE(s): 5
7	$* \quad \lfloor (X)_2 \rfloor \in \mathbb{F}$	RepE(Form): 2
8	$* \quad \lfloor (X)_{121} \rfloor \in \mathbb{T}$	RepE(Term): 3
9	$* \quad \lfloor (X)_{211} \rfloor = \lfloor Q \rfloor \longrightarrow$	
	$* \quad\quad \langle\langle \emptyset, \langle \lfloor (X)_{121} \rfloor, \lfloor (X)_{212} \rfloor \rangle\rangle, \lfloor (X)_2 \rfloor \rangle \in z$	$\forall_{\in}$E: 6,7,8
10	$* \quad \lfloor (X)_{211} \rfloor = \lfloor Q \rfloor$	RepE(=): 4
11	$* \quad \langle\langle \emptyset, \langle \lfloor (X)_{121} \rfloor, \lfloor (X)_{212} \rfloor \rangle\rangle, \lfloor (X)_2 \rfloor \rangle \in z$	$\rightarrow$E: 9,10
12	$* \quad \lfloor [[S, [(X)_{121}, (X)_{212}]], (X)_2] \rfloor \in z$	DefI(Code): 11
13	$* \quad \lfloor X \rfloor \in z$	RepE(=): 12,1
14	$* \quad (\forall z \in s) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 13
15	$* \quad \lfloor X \rfloor \in \bigcap s$	DefI($\bigcap$): 14
16	$* \quad \lfloor X \rfloor \in \mathbb{S}$	DefI($\mathbb{S}$): 15
17	$ZF \vdash \lfloor X \rfloor \in \mathbb{S}$	ProvI: 16

Induction Case 1: n-ary functions.

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{Subst}((X)_{111}) \longrightarrow ZF \vdash \lfloor (X)_{111} \rfloor \in \mathbb{S}$$

$$\vdots$$

$$\textsc{Subst}((X)_{112\cdots 2i}) \longrightarrow ZF \vdash \lfloor (X)_{112\cdots 2i} \rfloor \in \mathbb{S}$$

1	$X = [[[(X)_{111}, \ldots, [(X)_{112\cdots21}, (X)_{112\cdots22}] \cdots],$	
	$[(X)_{111121}, (X)_{111122}]],$	
	$[(X)_{21}, [(X)_{1112}, \ldots, [(X)_{112\cdots212}, (X)_{112\cdots222}] \cdots]]]$	Premise
2	$\textsc{Subst}((X)_{111}) \,\&\, \cdots \,\&\, \textsc{Subst}((X)_{112\cdots2i})$	Premise
3	$\textsc{Term}((X)_{1112}) \,\&\, \cdots \,\&\, \textsc{Term}((X)_{1112i2})$	Premise
4	$(X)_{11112i} = \cdots = (X)_{112\cdots2112i} = (X)_{112\cdots2212i}$	Premise
5	$\textsc{Func}((X)_{21})$	Premise
6	$ZF \vdash \lfloor(X)_{111}\rfloor \in \mathbb{S}$	IH: 2
7	$\vdots$	
8	$ZF \vdash \lfloor(X)_{112\cdots2i}\rfloor \in \mathbb{S}$	IH: 2
9	$* \quad \lfloor(X)_{111}\rfloor \in \mathbb{S}, \ldots, \lfloor(X)_{112\cdots2i}\rfloor \in \mathbb{S}$	ProvE: 6-8
10	$* \quad z \in s$	Assume
11	$* \quad (\forall x_1, \ldots, x_n \in z)(\forall y \in func)(((x_1)_2, \ldots (X_n)_2 \in \mathbb{T} \,\&$	
	$* \quad (x)_{12i} = \cdots = (x_{n-1})_{12i} = (x_n)_{12i}) \longrightarrow$	
	$* \quad \langle y, \langle x_1, \ldots, \langle x_{n-1}, x_n\rangle \cdots\rangle\rangle \in z)$	DefE(s): 10
12	$* \quad \lfloor(X)_{111}\rfloor \in z, \ldots, \lfloor(X)_{112\cdots2i}\rfloor \in z$	Lemma: 9
13	$* \quad \lfloor(X)_{21}\rfloor \in func$	RepE(Func): 5
14	$* \quad (\lfloor(X)_{1112}\rfloor, \ldots \lfloor(X)_{112\cdots2212i}\rfloor \in \mathbb{T} \,\&$	
	$* \quad \lfloor(X)_{11112i}\rfloor = \cdots = \lfloor(X)_{112\cdots2112i}\rfloor = \lfloor(X)_{112\cdots2212i}\rfloor)$	
	$* \quad \longrightarrow \langle\lfloor(X)_{21}\rfloor, \langle\lfloor(X)_{111}\rfloor, \ldots,$	
	$* \quad \langle\lfloor(X)_{112\cdots21}\rfloor, \lfloor(X)_{112\cdots22}\rfloor\rangle \cdots\rangle\rangle \in z$	$\forall_\in$E: 11,12,13
15	$* \quad \lfloor(X)_{1112}\rfloor, \ldots \lfloor(X)_{112\cdots2212i}\rfloor \in \mathbb{T}$	RepE(Term): 3
16	$* \quad \lfloor(X)_{11112i}\rfloor = \cdots = \lfloor(X)_{112\cdots2112i}\rfloor = \lfloor(X)_{112\cdots2212i}\rfloor$	RepE(=): 4
17	$* \quad \lfloor(X)_{1112}\rfloor, \ldots \lfloor(X)_{112\cdots2212i}\rfloor \in \mathbb{T} \,\&$	
	$* \quad \lfloor(X)_{11112i}\rfloor = \cdots = \lfloor(X)_{112\cdots2112i}\rfloor = \lfloor(X)_{112\cdots2212i}\rfloor$	&I: 15,16
18	$* \quad \langle\lfloor(X)_{21}\rfloor, \langle\lfloor(X)_{111}\rfloor, \ldots,$	
	$* \quad \langle\lfloor(X)_{112\cdots21}\rfloor, \lfloor(X)_{112\cdots22}\rfloor\rangle \cdots\rangle\rangle \in z$	→E: 14,17

(cont'd)

1	$* \quad \lfloor[[[(X)_{111},\ldots,[(X)_{112\cdots 21},(X)_{112\cdots 22}]\cdots],$	
	$* \quad [(X)_{111121},(X)_{111122}]],[(X)_{21},[(X)_{1112},\ldots,$	
	$* \quad [(X)_{112\cdots 212},(X)_{112\cdots 222}]\cdots]]]\rfloor \in z$	DefI(Code): 18
20	$* \quad \lfloor X\rfloor \in z$	RepE(=): 19,1
21	$* \quad (\forall z \in s)\lfloor X\rfloor \in z$	$\forall_{\in}$I: 20
22	$* \quad \lfloor X\rfloor \in \bigcap s$	DefI($\bigcap$): 21
23	$* \quad \lfloor X\rfloor \in \mathbb{S}$	DefI($\mathbb{S}$): 22
24	$ZF \vdash \lfloor X\rfloor \in \mathbb{S}$	ProvI: 23

Induction Case 2: Equality of terms.

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{Subst}((X)_{11i}) \longrightarrow ZF \vdash \lfloor (X)_{11i}\rfloor \in \mathbb{S}$$

1	$X = [[[(X)_{111}, (X)_{112}], [(X)_{111121}, (X)_{111122}]],$	
	$\quad [=, [(X)_{1112}, (X)_{1122}]]]$	Premise
2	$\textsc{Subst}((X)_{11i})$	Premise
3	$\textsc{Term}((X)_{11i2})$	Premise
4	$(X)_{11112i} = (X)_{11212i}$	Premise
5	$ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathbb{S}$	IH: 2
6	$* \quad \lfloor (X)_{11i} \rfloor \in \mathbb{S}$	ProvE: 5
7	$* \quad z \in s$	Assume
8	$* \quad (\forall x, y \in z)((((x)_2, (y)_2 \in \mathbb{T} \,\&\, (x)_{12i} = (y)_{12i}) \longrightarrow$	
	$* \quad\quad \langle\langle\langle x, y\rangle, \langle (x)_{121}, (x)_{122}\rangle\rangle, \langle \lfloor = \rfloor, \langle (x)_2, (y)_2\rangle\rangle\rangle \in z)$	DefE(s): 7
9	$* \quad \lfloor (X)_{11i} \rfloor \in z$	Lemma(6): 6
10	$* \quad (\lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{T} \,\&$	
	$* \quad\quad \lfloor (X)_{11112i} \rfloor = \lfloor (X)_{11212i} \rfloor) \longrightarrow$	
	$* \quad\quad \langle\langle\langle \lfloor (X)_{111} \rfloor, y\rangle, \langle \lfloor (X)_{111121} \rfloor, \lfloor (X)_{111122} \rfloor\rangle\rangle,$	
	$* \quad\quad \langle \lfloor = \rfloor, \langle \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor\rangle\rangle\rangle \in z$	$\forall_\in$E: 8,9
11	$* \quad \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{T}$	RepE(Term): 3
12	$* \quad \lfloor (X)_{11112i} \rfloor = \lfloor (X)_{11212i} \rfloor$	RepE(=): 4
13	$* \quad \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{T} \,\&\, \lfloor (X)_{11112i} \rfloor = \lfloor (X)_{11212i} \rfloor$	&I: 11,12
14	$* \quad \langle\langle\langle \lfloor (X)_{111} \rfloor, y\rangle, \langle \lfloor (X)_{111121} \rfloor, \lfloor (X)_{111122} \rfloor\rangle\rangle,$	
	$* \quad \langle \lfloor = \rfloor, \langle \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor\rangle\rangle\rangle \in z$	$\to$E: 10,13
15	$* \quad \lfloor [[[(X)_{111}, (X)_{112}], [(X)_{111121}, (X)_{111122}]],$	
	$* \quad\quad [=, [(X)_{1112}, (X)_{1122}]]] \rfloor \in z$	DefI(Code): 14
16	$* \quad \lfloor X \rfloor \in z$	RepE(=): 15,1
17	$* \quad (\forall z \in s) \lfloor X \rfloor \in z$	$\forall_\in$I: 16
18	$* \quad \lfloor X \rfloor \in \bigcap s$	DefI($\bigcap$): 17
19	$* \quad \lfloor X \rfloor \in \mathbb{S}$	DefI($\mathbb{S}$): 18
20	$ZF \vdash \lfloor X \rfloor \in \mathbb{S}$	ProvI: 19

Induction Case 3: Negation.

Inductive Hypothesis: Assume for arbitrary X that

$$\textsc{Subst}((X)_{11}) \longrightarrow ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{S}$$

1	$X = [[(X)_{11}, [(X)_{11121}, (X)_{11122}]], [\neg, (X)_{112}]]$	Premise
2	$\textsc{Subst}((X)_{11})$	Premise
3	$\textsc{Form}((X)_{112})$	Premise
4	$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{S}$	IH: 2
5	$*\ \ \lfloor (X)_{11} \rfloor \in \mathbb{S}$	ProvE: 4
6	$*\ \ \ \ z \in s$	Assume
7	$*\ \ \ \ (\forall x \in z)((x)_2 \in \mathbb{F} \longrightarrow \langle\langle x, \langle (x)_{121}, (x)_{122} \rangle\rangle,$	
	$*\ \ \ \ \ \ \langle \lfloor \neg \rfloor, (x)_2 \rangle\rangle \in z)$	DefE(s): 6
8	$*\ \ \ \ \lfloor (X)_{11} \rfloor \in z$	Lemma(6): 5
9	$*\ \ \ \ \lfloor (X)_{112} \rfloor \in \mathbb{F} \longrightarrow$	
	$*\ \ \ \ \ \ \langle\langle \lfloor (X)_{11} \rfloor, \langle \lfloor (X)_{11121} \rfloor, \lfloor (X)_{11122} \rfloor \rangle\rangle,$	
	$*\ \ \ \ \ \ \langle \lfloor \neg \rfloor, \lfloor (X)_{112} \rfloor \rangle\rangle \in z$	$\forall_\in$E: 7,8
10	$*\ \ \ \ \lfloor (X)_{112} \rfloor \in \mathbb{F}$	RepE(Form): 3
11	$*\ \ \ \ \langle\langle \lfloor (X)_{11} \rfloor, \langle \lfloor (X)_{11121} \rfloor, \lfloor (X)_{11122} \rfloor \rangle\rangle,$	
	$*\ \ \ \ \ \ \langle \lfloor \neg \rfloor, \lfloor (X)_{112} \rfloor \rangle\rangle \in z$	$\rightarrow$E: 9,10
12	$*\ \ \ \ \lfloor [[(X)_{11}, [(X)_{11121}, (X)_{11122}]], [\neg, (X)_{112}]] \rfloor \in z$	DefI(Code): 11
13	$*\ \ \ \ \lfloor X \rfloor \in z$	RepE(=): 12,1
14	$*\ \ (\forall z \in s) \lfloor X \rfloor \in z$	$\forall_\in$I: 13
15	$*\ \ \lfloor X \rfloor \in \bigcap s$	DefI($\bigcap$): 14
16	$*\ \ \lfloor X \rfloor \in \mathbb{S}$	DefI($\mathbb{S}$): 15
17	$ZF \vdash \lfloor X \rfloor \in \mathbb{S}$	ProvI: 16

Induction Case 4: Binary Connectives.

Inductive Hypothesis: Assume for Arbitrary X that

$$\textsc{Subst}((X)_{11i}) \longrightarrow ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathbb{S}$$

1	$X = [[[(X)_{111}, (X)_{112}], [(X)_{111121}, (X)_{111122}]], [\Box[(X)_{1112}, (X)_{1122}]]]$	Premise
2	$\textsc{Subst}((X)_{11i})$	Premise
3	$\textsc{Form}((X)_{11i2})$	Premise
4	$(X)_{11112i} = (X)_{11212i}$	Premise
5	$ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathbb{S}$	IH: 2
6	* $\lfloor (X)_{11i} \rfloor \in \mathbb{S}$	ProvE: 5
7	* $z \in s$	Assume
8	* $(\forall x, y \in z)([(x)_2 \in \mathbb{F} \,\&\, (y)_2 \in \mathbb{F} \,\&\, (x)_{12i} = (y)_{12i}] \longrightarrow$ * $\langle\langle\langle x, y\rangle, \langle (x)_{121}, (x)_{122}\rangle\rangle, \langle \lfloor \Box \rfloor, \langle (x)_2, (y)_2\rangle\rangle\rangle \in z)$	DefE(s): 7
9	* $\lfloor (X)_{11i} \rfloor \in z$	Lemma(6): 6
10	* $[\lfloor (X)_{1112} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{1122} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11112i} \rfloor = \lfloor (X)_{11212i} \rfloor]$ * $\longrightarrow \langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor\rangle, \langle \lfloor (X)_{111121} \rfloor, \lfloor (X)_{111122} \rfloor\rangle\rangle,$ * $\langle \lfloor \Box \rfloor, \langle \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor\rangle\rangle\rangle \in z$	$\forall_\in$E: 8,9
11	* $\lfloor (X)_{1112} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{1122} \rfloor \in \mathbb{F}$	RepE(Form): 3
12	* $\lfloor (X)_{11112i} \rfloor = \lfloor (X)_{11212i} \rfloor$	RepE(=): 4
13	* $\lfloor (X)_{1112} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{1122} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11112i} \rfloor = \lfloor (X)_{11212i} \rfloor$	&I: 11,12
14	* $\langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor\rangle, \langle \lfloor (X)_{111121} \rfloor, \lfloor (X)_{111122} \rfloor\rangle\rangle,$ * $\langle \lfloor \Box \rfloor, \langle \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor\rangle\rangle\rangle \in z$	$\rightarrow$E: 10,13
15	* $\lfloor [[[(X)_{111}, (X)_{112}], [(X)_{111121}, (X)_{111122}]],$ * $[\Box[(X)_{1112}, (X)_{1122}]]] \rfloor \in z$	DefI(Code): 14
16	* $\lfloor X \rfloor \in z$	RepE(=): 15,1
17	* $(\forall z \in s) \lfloor X \rfloor \in z$	$\forall_\in$I: 16
18	* $\lfloor X \rfloor \in \bigcap s$	DefI($\bigcap$): 17
19	* $\lfloor X \rfloor \in \mathbb{S}$	DefI($\mathbb{S}$): 18
20	$ZF \vdash \lfloor X \rfloor \in \mathbb{S}$	ProvI: 19

Induction Case 5: Quantified Formulae (substitution into free variable)

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{Subst}((X)_{22}) \longrightarrow ZF \vdash \lfloor (X)_{22} \rfloor \in \mathbb{S}$$

1	$X = [[(X)_{11}, [(X)_{11121}, (X)_{11122}]], [[Q, (X)_{212}], (X)_{112}]]$	Premise
2	$\textsc{Subst}((X)_{11})$	Premise
3	$\textsc{Form}((X)_{112})$	Premise
4	$\textsc{Var}((X)_{212})$	Premise
5	$(X)_{11122} \neq (X)_{212}$	Premise
6	$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{S}$	IH: 2
7	$* \quad \lfloor (X)_{11} \rfloor \in \mathbb{S}$	ProvE: 6
8	$* \quad z \in s$	Assume
9	$* \quad (\forall x \in z)(\forall y \in \mathbb{V})([(x)_2 \in \mathbb{F} \,\&\, (x)_{122} \neq y] \longrightarrow$	
	$* \quad \langle\langle x, \langle (x)_{121}, (x)_{122} \rangle\rangle, \langle\langle \lfloor Q \rfloor, y \rangle, (x)_2 \rangle\rangle \in z)$	DefE(s): 8
10	$* \quad \lfloor (X)_{11} \rfloor \in z$	Lemma(6): 7
11	$* \quad \lfloor (X)_{212} \rfloor \in \mathbb{V}$	RepE(Var): 4
12	$* \quad [\lfloor (X)_{112} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11122} \rfloor \neq \lfloor (X)_{212} \rfloor] \longrightarrow$	
	$* \quad \langle\langle \lfloor (X)_{11} \rfloor, \langle \lfloor (X)_{11121} \rfloor, \lfloor (X)_{11122} \rfloor \rangle\rangle,$	
	$* \quad \langle\langle \lfloor Q \rfloor, \lfloor (X)_{212} \rfloor \rangle, \lfloor (X)_{112} \rfloor \rangle\rangle \in z$	$\forall_\in$E: 9,10,11
13	$* \quad \lfloor (X)_{11122} \rfloor \neq \lfloor (X)_{212} \rfloor$	RepE(=): 5
14	$* \quad \lfloor (X)_{112} \rfloor \in \mathbb{F}$	RepE(Form): 3
15	$* \quad \lfloor (X)_{112} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11122} \rfloor \neq \lfloor (X)_{212} \rfloor$	&I: 14,13
16	$* \quad \langle\langle \lfloor (X)_{11} \rfloor, \langle \lfloor (X)_{11121} \rfloor, \lfloor (X)_{11122} \rfloor \rangle\rangle,$	
	$* \quad \langle\langle \lfloor Q \rfloor, \lfloor (X)_{212} \rfloor \rangle, \lfloor (X)_{112} \rfloor \rangle\rangle \in z$	$\rightarrow$E: 12,15
17	$* \quad \lfloor [[(X)_{11}, [(X)_{11121}, (X)_{11122}]], [[Q, (X)_{212}], (X)_{112}]] \rfloor \in z$	DefI(Code): 16
18	$* \quad \lfloor X \rfloor \in z$	RepE(=): 17,1
19	$* \quad (\forall z \in s) \lfloor X \rfloor \in z$	$\forall_\in$I: 18
20	$* \quad \lfloor X \rfloor \in \bigcap s$	DefI($\bigcap$): 19
21	$* \quad \lfloor X \rfloor \in \mathbb{S}$	DefI($\mathbb{S}$): 20
22	$ZF \vdash \lfloor X \rfloor \in \mathbb{S}$	ProvI: 21

Thus by the principle of induction for substitution instances:

$$(\forall X)[\textsc{Subst}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathbb{S}]$$

$\square$

R_1 was just established, and we again take R_2 for granted, and thus concludes the complete representability of substitution instances in ZF.

2.10 Representability of Terms Free for a Variable

The informal specification for a term t being free for a variable x is as follows:

- If φ is an atomic formula, then t is free for x in φ.
- If t is free for x in φ, then t is free for x in $\neg\varphi$.
- If t is free for x in both φ_1 and φ_2, then t is free for x in $\varphi_1 \square \varphi_2$.
- If x is a free variable in $Qy\varphi$ and y is not free in t, then t is free for x in $Qy\varphi$.

(based on Van Dalen [5])

Metatheory

A term free for a variable will be represented by the binary tree structure

$$X = [[[X_1, \ldots, X_n], [T, V]], F]$$

where $X_1, \ldots, X_n$ are some other objects representing terms free for a variable, T is the term free for the variable V, and F is the formula in which it occurs.

We define terms free for a variable inductively as follows:

$$\begin{aligned}
(\forall X)(([&X = [[S, [(X)_{121}, (X)_{122}]], (X)_2] \,\&\, \textsc{Atom}((X)_2) \,\&\, \textsc{Term}((X)_{121}) \,\&\, \textsc{Var}((X)_{122})] \\
&\vee [X = [[S, [(X)_{121}, (X)_{122}]], (X)_2] \,\&\, \textsc{Form}((X)_2) \,\&\, (X)_{211} = (Q) \,\&\, \\
&\quad \textsc{Term}((X)_{121}) \,\&\, \textsc{Free}((X)_{122}) \,\&\, (X)_{1222} = (X)_2 \,\&\, \textsc{NotFree}((X)_{212}) \,\&\, \\
&\quad (X)_{2122} = (X)_{121}] \\
&\vee [X = [[(X)_{11}, [(X)_{11121}, (X)_{11122}]], [\neg, (X)_{112}]] \,\&\, \textsc{FreeFor}((X)_{11})] \\
&\vee [X = [[[(X)_{111}, (X)_{112}], [(X)_{111121}, (X)_{111122}]], [\square, [(X)_{1112}, (X)_{1122}]]] \,\&\, \\
&\quad \textsc{FreeFor}((X)_{11i}) \,\&\, (X)_{11112i} = (X)_{11212i}]) \longrightarrow \textsc{FreeFor}(X))
\end{aligned}$$

with an induction principle that allows one to infer $(\forall X)(\textsc{FreeFor}(X) \longrightarrow \varphi(X))$ and is defined in the same way the other induction principles have been.

Object-theory

We will define the set of terms free for a variable $\mathcal{FF}$ by $\mathcal{FF} = \bigcap ff$, where ff is defined as

$$\begin{aligned} ff = \{z \in \wp(\mathbb{B}) \mid & (\forall x \in \mathbb{A})(\forall y \in \mathbb{T})(\forall w \in \mathbb{V})\langle\langle\emptyset, \langle y, w\rangle\rangle, x\rangle \in z \\ & \& \, (\forall x \in \mathbb{F})(\forall y \in \mathbb{T})(\forall w \in \mathcal{F})(((x)_{11} = \lfloor Q \rfloor \,\&\, (w)_2 = x \,\& \\ & \quad (x)_{12} \in \mathcal{NF} \,\&\, (x)_{122} = y) \longrightarrow \langle\langle\emptyset, \langle y, w\rangle\rangle, x\rangle \in z) \\ & \& \, (\forall x \in z)\langle\langle x, \langle (x)_{121}, (x)_{122}\rangle\rangle, \langle\lfloor\neg\rfloor, (x)_2\rangle\rangle \in z \\ & \& \, (\forall x, y \in z)((x)_{12i} = (y)_{12i} \longrightarrow \langle\langle\langle x, y\rangle, \langle (x)_{121}, (x)_{122}\rangle\rangle, \\ & \quad \langle\lfloor\Box\rfloor, \langle (x)_2, (y)_2\rangle\rangle\rangle \in z)\} \end{aligned}$$

We will prove

$$(\forall X)[\text{FreeFor}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathcal{FF}]$$

by the principle of induction for terms free for a variable.

Base Case 1: Atomic Formulae.

1	$X = [[S, [(X)_{121}, (X)_{122}]], (X)_2]$	Premise
2	$\text{Atom}((X)_2)$	Premise
3	$\text{Term}((X)_{121})$	Premise
4	$\text{Var}((X)_{122})$	Premise
5	$*$ $z \in ff$	Assume
6	$*$ $(\forall x \in \mathbb{A})(\forall y \in \mathbb{T})(\forall w \in \mathbb{V})\langle\langle\emptyset, \langle y, w\rangle\rangle, x\rangle \in z$	DefE(ff): 5
7	$*$ $\lfloor (X)_2 \rfloor \in \mathbb{A}$	RepE(Atom): 2
8	$*$ $\lfloor (X)_{121} \rfloor \in \mathbb{T}$	RepE(Term): 3
9	$*$ $\lfloor (X)_{122} \rfloor \in \mathbb{V}$	RepE(Var): 4
10	$*$ $\langle\langle\emptyset, \langle\lfloor (X)_{121}\rfloor \in \mathbb{T}, \lfloor (X)_{122}\rfloor \in \mathbb{V}\rangle\rangle, \lfloor (X)_2 \rfloor \in \mathbb{A}\rangle \in z$	$\forall_\in$E: 6,7,8,9
11	$*$ $\lfloor [[S, [(X)_{121}, (X)_{122}]], (X)_2] \rfloor \in z$	DefI(Code): 10
12	$*$ $\lfloor X \rfloor \in z$	RepE(=): 11,1
13	$*$ $(\forall z \in ff)\lfloor X \rfloor \in z$	$\forall_\in$I: 12
14	$*$ $\lfloor X \rfloor \in \bigcap ff$	DefI($\cap$): 13
15	$*$ $\lfloor X \rfloor \in \mathcal{FF}$	DefI($\mathcal{FF}$): 14
16	$ZF \vdash \lfloor X \rfloor \in \mathcal{FF}$	ProvI: 15

Base Case 2: Quantified Formulae.

1		$X = [[S, [(X)_{121}, (X)_{122}]], (X)_2]$	Premise
2		$\textsc{Form}((X)_2)$	Premise
3		$(X)_{211} = Q$	Premise
4		$\textsc{Term}((X)_{121})$	Premise
5		$\textsc{Free}((X)_{122})$	Premise
6		$(X)_{1222} = (X)_2$	Premise
7		$\textsc{NotFree}((X)_{212})$	Premise
8		$(X)_{2122} = (X)_{121}$	Premise
9	*	$z \in ff$	Assume
10	*	$(\forall x \in \mathbb{F})(\forall y \in \mathbb{T})(\forall w \in \mathcal{F})(((x)_{11} = \lfloor Q \rfloor\,\&$	
	*	$(w)_2 = x\,\&\,(x)_{12} \in \mathcal{NF}\,\&\,(x)_{122} = y)$	
	*	$\longrightarrow \langle\langle \emptyset, \langle y, w\rangle\rangle, x\rangle \in z)$	DefE(ff): 9
11	*	$\lfloor (X)_2 \rfloor \in \mathbb{F}$	RepE(Form): 2
12	*	$\lfloor (X)_{121} \rfloor \in \mathbb{T}$	RepE(Term): 4
13	*	$\lfloor (X)_{122} \rfloor \in \mathcal{F}$	RepE(Free): 5
14	*	$(\lfloor (X)_{211} \rfloor = \lfloor Q \rfloor\,\&\,\lfloor (X)_{1222} \rfloor = \lfloor (X)_2 \rfloor\,\&$	
	*	$(X)_{212} \in \mathcal{NF}\,\&\,\lfloor (X)_{2122} \rfloor = \lfloor (X)_{121} \rfloor) \longrightarrow$	
	*	$\langle\langle \emptyset, \langle \lfloor (X)_{121} \rfloor, \lfloor (X)_{122} \rfloor \rangle\rangle, \lfloor (X)_2 \rfloor \rangle \in z$	$\forall_{\in}$E: 10,11,12,13
15	*	$\lfloor (X)_{211} \rfloor = \lfloor Q \rfloor$	RepE(=): 3
16	*	$\lfloor (X)_{1222} \rfloor = \lfloor (X)_2 \rfloor$	RepE(=): 6
17	*	$(X)_{212} \in \mathcal{NF}$	RepE(NotFree): 7
18	*	$\lfloor (X)_{2122} \rfloor = \lfloor (X)_{121} \rfloor$	RepE(=): 8
19	*	$\lfloor (X)_{211} \rfloor = \lfloor Q \rfloor\,\&\,\lfloor (X)_{1222} \rfloor = \lfloor (X)_2 \rfloor\,\&$	
	*	$(X)_{12} \in \mathcal{NF}\,\&\,\lfloor (X)_{2122} \rfloor = \lfloor (X)_{121} \rfloor$	&I: 15,16,17,18
20	*	$\langle\langle \emptyset, \langle \lfloor (X)_{121} \rfloor, \lfloor (X)_{122} \rfloor \rangle\rangle, \lfloor (X)_2 \rfloor \rangle \in z$	$\rightarrow$E: 14,19

(cont'd)

1	$* \mid\ \mid \lfloor[[S,[(X)_{121},(X)_{122}]],(X)_2]\rfloor \in z$	DefI(Code): 20
22	$* \mid\ \mid \lfloor X\rfloor \in z$	RepE(=): 21,1
23	$* \mid (\forall z \in ff)\lfloor X\rfloor \in z$	$\forall_{\in}$I: 22
24	$* \mid \lfloor X\rfloor \in \bigcap ff$	DefI($\cap$): 23
25	$* \mid \lfloor X\rfloor \in \mathcal{FF}$	DefI($\mathcal{FF}$): 24
26	$ZF \vdash \lfloor X\rfloor \in \mathcal{FF}$	ProvI: 25

Inductive Case 1: Negation.

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{FreeFor}((X)_{11}) \longrightarrow ZF \vdash \lfloor (X)_{11} \rfloor \in \mathcal{FF}$$

1	$X = [[(X)_{11}, [(X)_{11121}, (X)_{11122}]], [\neg, (X)_{112}]]$	Premise
2	$\textsc{FreeFor}((X)_{11})$	Premise
3	$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathcal{FF}$	IH: 2
4	$*\quad \lfloor (X)_{11} \rfloor \in \mathcal{FF}$	ProvE: 3
5	$*\quad\quad z \in ff$	Assume
6	$*\quad\quad (\forall x \in z)\langle\langle x, \langle (x)_{121}, (x)_{122}\rangle\rangle, \langle \lfloor \neg \rfloor, (x)_2\rangle\rangle \in z$	DefE(ff): 5
7	$*\quad\quad \lfloor (X)_{11} \rfloor \in z$	Lemma(6): 4
8	$*\quad\quad \langle\langle \lfloor (X)_{11} \rfloor, \langle \lfloor (X)_{11121} \rfloor, \lfloor (X)_{11122} \rfloor\rangle\rangle,$	
	$*\quad\quad\quad \langle \lfloor \neg \rfloor, \lfloor (X)_{112} \rfloor\rangle\rangle \in z$	$\forall_{\in}$E: 6,7
9	$*\quad\quad \lfloor [[(X)_{11}, [(X)_{11121}, (X)_{11122}]], [\neg, (X)_{112}]] \rfloor$	DefI(Code): 8
10	$*\quad\quad \lfloor X \rfloor \in z$	RepE(=): 9,1
11	$*\quad (\forall z \in ff)\lfloor X \rfloor \in z$	$\forall_{\in}$I: 10
12	$*\quad \lfloor X \rfloor \in \bigcap ff$	DefI($\cap$): 11
13	$*\quad \lfloor X \rfloor \in \mathcal{FF}$	DefI($\mathcal{FF}$): 12
14	$ZF \vdash \lfloor X \rfloor \in \mathcal{FF}$	ProvI: 13

Inductive Case 2: Binary Connectives.

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{FreeFor}((X)_{11i}) \longrightarrow ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathcal{FF}$$

1		$X = [[[(X)_{111}, (X)_{112}], [(X)_{111121}, (X)_{111122}]],$	
		$[\Box, [(X)_{1112}, (X)_{1122}]]]$	Premise
2		$\textsc{FreeFor}((X)_{11i})$	Premise
3		$(X)_{11112i} = (X)_{11212i}$	Premise
4		$ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathcal{FF}$	IH: 2
5	$*$	$\lfloor (X)_{11i} \rfloor \in \mathcal{FF}$	ProvE: 4
6	$*$	$z \in ff$	Assume
7	$*$	$(\forall x, y \in z)((x)_{12i} = (y)_{12i} \longrightarrow$	
	$*$	$\langle\langle\langle x, y\rangle, \langle (x)_{121}, (x)_{122}\rangle\rangle, \langle \lfloor \Box \rfloor, \langle (x)_2, (y)_2\rangle\rangle\rangle \in z)$	DefE(ff): 6
8	$*$	$\lfloor (X)_{11i} \rfloor \in z$	Lemma(6): 5
9	$*$	$\lfloor (X)_{11112i} \rfloor = \lfloor (X)_{11212i} \rfloor \longrightarrow$	
	$*$	$\langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor\rangle, \langle \lfloor (X)_{111121} \rfloor, \lfloor (X)_{111122} \rfloor\rangle\rangle,$	
	$*$	$\langle \lfloor \Box \rfloor, \langle \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor\rangle\rangle\rangle \in z$	$\forall_\in$E: 7,8
10	$*$	$(X)_{11112i} \rfloor = \lfloor (X)_{11212i} \rfloor$	RepE(=): 3
11	$*$	$\langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor\rangle, \langle \lfloor (X)_{111121} \rfloor, \lfloor (X)_{111122} \rfloor\rangle\rangle,$	
	$*$	$\langle \lfloor \Box \rfloor, \langle \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor\rangle\rangle\rangle \in z$	$\rightarrow$E: 9,10
12	$*$	$\lfloor [[[(X)_{111}, (X)_{112}], [(X)_{111121}, (X)_{111122}]],$	
	$*$	$[\Box, [(X)_{1112}, (X)_{1122}]]] \rfloor \in z$	DefI(Code): 11
13	$*$	$\lfloor X \rfloor \in z$	RepE(=): 12,1
14	$*$	$(\forall z \in ff) \lfloor X \rfloor \in z$	$\forall_\in$I: 13
15	$*$	$\lfloor X \rfloor \in \bigcap ff$	DefI($\cap$): 14
16	$*$	$\lfloor X \rfloor \in \mathcal{FF}$	DefI($\mathcal{FF}$): 15
17		$ZF \vdash \lfloor X \rfloor \in \mathcal{FF}$	ProvI: 16

Thus by the induction principle for terms free for a variable, we have

$$(\forall X)[\textsc{FreeFor}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathcal{FF}]$$

R_2 is again taken for granted and thus, together with the claim just established, we conclude the representability of terms free for a variable.

2.11 Representability of Proofs

We will define proofs in the context of a natural deduction system. It is common to use the minimal number of logical operators possible and then define the other operators in terms of the few basic ones (e.g. defining all other connectives in terms of only negation and disjunction). This greatly limits what one can do though in terms of rule applications in proofs. We thus decided to use a fully specified language, having introduction and elimination rules for each. The full language requires significantly more cases to be verified, which is why the formal proof for representability of proofs is too comlex to show explicitly here. The full language is desired though so that the proof-theoretic framework in the object-theory is rich enough to produce short and intelligible proofs.

Using Gentzen's notion of a sequent, we indicate a formula φ depending on a set of assumptions Γ by $\Gamma \supset \varphi$. If D_1 and D_2 are proofs, we denote the combination of these two proofs by $D_1 \cup D_2$, and if D is a proof and φ a formula, $D \cup \varphi$ denotes a new proof obtained by appending φ to D and is then a proof of φ. Proofs are then specified informally as follows:

- Each axiom is a proof of itself.
- Every assumption is a proof of itself.
- If D_1 is a proof of $\Gamma_1 \supset \varphi_1$ and D_2 is a proof of $\Gamma_2 \supset \varphi_2$, then $D_1 \cup D_2 \cup (\varphi_1 \,\&\, \varphi_2)$ is a proof of $\Gamma_1 \cup \Gamma_2 \supset \varphi_1 \,\&\, \varphi_2$.
- If D is a proof $\Gamma \supset \varphi_1 \,\&\, \varphi_2$, then $D \cup \varphi_1$ is a proof of $\Gamma \supset \varphi_1$ and $D \cup \varphi_2$ is a proof of $\Gamma \supset \varphi_2$.
- If D is a proof of $\Gamma \cup \{\varphi_1\} \supset \varphi_2$, then $D \cup (\varphi_1 \to \varphi_2)$ is a proof of $\Gamma \supset \varphi_1 \to \varphi_2$.
- If D_1 is a proof of $\Gamma_1 \supset \varphi_1 \to \varphi_2$ and D_2 is a proof of $\Gamma_2 \supset \varphi_1$, then $D_1 \cup D_2 \cup \varphi_2$ is a proof of $\Gamma_1 \cup \Gamma_2 \supset \varphi_2$.
- If D is a proof of $\Gamma \supset \varphi_1$, then $D \cup (\varphi_1 \vee \varphi_2)$ is a proof of $\Gamma \supset \varphi_1 \vee \varphi_2$.
- If D is a proof of $\Gamma \supset \varphi_2$, then $D \cup (\varphi_1 \vee \varphi_2)$ is a proof of $\Gamma \supset \varphi_1 \vee \varphi_2$.
- If D_1 is a proof of $\Gamma \supset \varphi_1 \vee \varphi_2$, D_2 is a proof of $\Gamma \cup \{\varphi_1\} \supset \varphi_3$, and D_3 is a proof of $\Gamma \cup \{\varphi_2\} \supset \varphi_3$, then $D_1 \cup D_2 \cup D_3 \cup \varphi_3$ is a proof of $\Gamma \supset \varphi_3$.
- If D_1 is a proof of $\Gamma \cup \{\varphi_1\} \supset \varphi_2$ and D_2 is a proof of $\Gamma \cup \{\varphi_2\} \supset \varphi_1$, then $D_1 \cup D_2 \cup (\varphi_1 \leftrightarrow \varphi_2)$ is a proof of $\Gamma \supset \varphi_1 \leftrightarrow \varphi_2$.
- If D_1 is a proof of $\Gamma_1 \supset \varphi_1 \leftrightarrow \varphi_2$ and D_2 is a proof of $\Gamma_2 \supset \varphi_1$, then $D_1 \cup D_2 \cup \varphi_2$ is a proof of $\Gamma_1 \cup \Gamma_2 \supset \varphi_2$.
- If D_1 is a proof of $\Gamma_1 \supset \varphi_1 \leftrightarrow \varphi_2$ and D_2 is a proof of $\Gamma_2 \supset \varphi_2$, then $D_1 \cup D_2 \cup \varphi_1$ is a proof of $\Gamma_1 \cup \Gamma_2 \supset \varphi_1$.
- If D_1 is a proof of $\Gamma_1 \supset \varphi$ and D_2 is a proof of $\Gamma_2 \supset \neg\varphi$, then $D_1 \cup D_2 \cup \bot$ is a proof of $\Gamma_1 \cup \Gamma_2 \supset \bot$.

- If D is a proof of $\Gamma \cup \{\varphi\} \supset \perp$, then $D \cup \neg\varphi$ is a proof of $\Gamma \supset \neg\varphi$.
- If D is a proof of $\Gamma \cup \{\neg\varphi\} \supset \perp$, then $D \cup \varphi$ is a proof of $\Gamma \supset \varphi$.
- If D is a proof of $\Gamma \supset \varphi[t/x]$ and x is a variable not free in any formula in Γ, then $D \cup \forall x\varphi$ is a proof of $\Gamma \supset \forall x\varphi$.
- If D is a proof of $\Gamma \supset \forall x\varphi$ and t is a term free for x in φ, then $D \cup \varphi[t/x]$ is a proof of $\Gamma \supset \varphi[t/x]$.
- If t is term free for variable x in φ and D is a proof of $\Gamma \supset \varphi[t/x]$, then $D \cup \exists x\varphi$ is a proof of $\Gamma \supset \exists x\varphi$.
- If D_1 is a proof of $\Gamma \supset \exists x\varphi_1$, D_2 is a proof of $\Gamma \cup \{\varphi_1[v/x]\} \supset \varphi_2$ and v is a variable not free in any formula in Γ or in φ_2, then $D_1 \cup \varphi_2$ is a proof of $\Gamma \supset \varphi_2$.

Metatheory

Proofs will have the following binary tree structure:

$$X = [[[X_1, \ldots, X_n], G], F]$$

where $X_1, \ldots, X_n$ are the binary tree proof objects that represent the proofs of the assumptions on which the inference rule depends, G is the set of assumptions on which the proof of F (represented by sequences), where F is the formula that results from the application of the inference rule.

We define proofs in the metatheory as follows:

$$
\begin{aligned}
&(\forall X)((\textsc{Axiom}(X) \\
&\vee [X = [[[(X)_{111}, (X)_{112}], (X)_{11112} \cup (X)_{11212}], [\&, [(X)_{1112}, (X)_{1122}]]] \\
&\quad \& \textsc{Proof}((X)_{11i}) \& \textsc{Form}((X)_{11i2})] \\
&\vee [X = [[(X)_{11}, (X)_{1112}], (X)_{1122i}] \& \textsc{Proof}((X)_{11}) \& \textsc{Form}((X)_{112}) \& (X)_{1121} = (\&)] \\
&\vee [X = [[(X)_{11}, (X)_{1112} \setminus (X)_{221}], [\rightarrow, [(X)_{221}, (X)_{112}]]] \& \textsc{Proof}((X)_{11}) \\
&\quad \& \textsc{Form}((X)_{112}) \& \textsc{Form}((X)_{221})] \\
&\vee [X = [[[(X)_{111}, (X)_{112}], (X)_{11112} \cup (X)_{11212}], (X)_{111222}] \& \textsc{Proof}((X)_{11i}) \\
&\quad \& \textsc{Form}((X)_{11i2}) \& (X)_{11121} = (\rightarrow) \& (X)_{111221} = (X)_{1122}] \\
&\vee [X = [[(X)_{11}, (X)_{1112}], [\vee, [(X)_{112}, (X)_{222}]]] \& \textsc{Proof}((X)_{11}) \\
&\quad \& \textsc{Form}((X)_{112}) \& \textsc{Form}((X)_{222})] \\
&\vee [X = [[(X)_{11}, (X)_{1112}], [\vee, [(X)_{221}, (X)_{112}]]] \& \textsc{Proof}((X)_{11}) \\
&\quad \& \textsc{Form}((X)_{112}) \& \textsc{Form}((X)_{221})] \\
&\vee [X = [[[(X)_{111}, [(X)_{1121}, (X)_{1122}]], (X)_{11112}], (X)_{11212}] \& \textsc{Proof}((X)_{111}) \\
&\quad \& \textsc{Proof}((X)_{112i})\textsc{Form}((X)_{1112}) \& \textsc{Form}((X)_{112i2}) \& (X)_{11121} = (\vee) \\
&\quad \& (X)_{112i12} = (X)_{11112} \cup (X)_{11122i} \& (X)_{11212} = (X)_{11222}] \\
&\vee [X = [[[(X)_{111}, (X)_{112}], (X)_{11112} \setminus (X)_{1122}], [\leftrightarrow, [(X)_{1122}, (X)_{1112}]]] \& \\
&\quad \textsc{Proof}((X)_{11i}) \& \textsc{Form}((X)_{11i2}) \& (X)_{11212} = ((X)_{11112} \setminus (X)_{1122}) \cup (X)_{1112}] \\
&\vee [X = [[[(X)_{111}, (X)_{112}], (X)_{11112} \cup (X)_{11212}], (X)_{111221}] \& \textsc{Proof}((X)_{11i}) \& \\
&\quad \textsc{Form}((X_{11i2}) \& (X)_{11121} = (\leftrightarrow) \& (X)_{111222} = (X)_{1122}] \\
&\vee [X = [[[(X)_{111}, (X)_{112}], (X)_{11112} \cup (X)_{11212}], (X)_{111222}] \& \textsc{Proof}((X)_{11i}) \& \textsc{Form}((X_{11i2}) \\
&\quad \& (X)_{11121} = \leftrightarrow) \& (X)_{111222} = (X)_{1122}] \\
&\vee [X = [[[(X)_{111}, (X)_{112}], (X)_{11112} \cup (X)_{11212}], \bot] \& \textsc{Proof}((X)_{11i}) \& \textsc{Form}((X)_{11i2})) \\
&\quad \& (X)_{11221} = (\neg) \& (X)_{1112} = (X)_{11222}] \\
&\vee [X = [[(X)_{11}, (X)_{1112} \setminus (X)_{22}], [\neg, (X)_{22}]] \& \textsc{Proof}((X)_{11}) \& (X)_{112} = (\bot) \& \textsc{Form}((X)_{22})] \\
&\vee [X = [[(X)_{11}, (X)_{1112} \setminus [\neg, (X)_{2}]], (X)_{2}] \& \textsc{Proof}((X)_{11}) \& (X)_{112} = (\bot) \& \textsc{Form}((X)_{2})] \\
&\vee [X = [[(X)_{11}, (X)_{1112}], [[\forall, (X)_{112122}], (X)_{1122}]], \& \textsc{Proof}((X)_{11}) \& \textsc{Subst}((X)_{112}) \& \\
&\quad \textsc{Form}((X)_{1122}) \& (\forall Y \in (X)_{1112})(\exists W)(\textsc{NotFree}(W) \& (W)_2 = Y)] \\
&\vee [X = [[(X)_{11}, (X)_{1112}], (X)_2] \& \textsc{Proof}((X)_{11}) \& \textsc{Form}((X)_{112}) \& (X)_{11211} = \forall \& \\
&\quad \textsc{Subst}((X)_2) \& (X)_{2122} = (X)_{11212} \& \textsc{FreeFor}((X)_{2121}) \& (X)_{21212} = (X)_{1122} \& \\
&\quad (X)_{212112} = (X)_{11212}] \\
&\vee [X = [[(X)_{11}, (X)_{1112}], [[\exists, (X)_{212}], (X)_{1122}]] \& \textsc{Proof}((X)_{11}) \& \textsc{Subst}((X)_{112}) \\
&\quad \& \textsc{Form}((X)_{1122}) \& \textsc{Var}((X)_{212})] \\
&\vee [X = [[[(X)_{111}, (X)_{112}], (X)_{11112}], (X)_{1122}] \& \textsc{Proof}((X)_{11i}) \& \textsc{Form}((X)_{11i2}) \& \\
&\quad (X)_{111211} = \exists \& (X)_{11212} = (X)_{11112} \cup (X)_{112122} \& \textsc{Subst}((X)_{112122}) \& \\
&\quad (X)_{112122122} = (X)_{111212} \&, \textsc{Var}((X)_{112122121}) \& (\exists Y)(\textsc{NotFree}(Y) \& \\
&\quad (Y)_2 = (X)_{1122}) \& (\forall W_1 \in (X)_{11112})(\exists W)(\textsc{NotFree}(W) \& (W)_2 = W_1)]) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \longrightarrow \textsc{Proof}(X))
\end{aligned}
$$

with an induction principle that establishes $(\forall X)(\textsc{Proof}(X) \longrightarrow \varphi(X))$.

Object-theory

The set p is defined as

$$
\begin{aligned}
&p = \{z \in \wp(\mathbb{B}) | \mathbb{AX} \subseteq z \\
&\& \, (\forall x, y \in z)([(x)_2, (y)_2 \in \mathbb{F}] \longrightarrow \langle\langle\langle x, y\rangle, (x)_{12} \cup (y)_{12}\rangle, \langle\lfloor\&\rfloor, \langle(x)_2, (y)_2\rangle\rangle\rangle \in z) \\
&\& \, (\forall x \in z)([(x)_2 \in \mathbb{F} \,\&\, (x)_{21} = \lfloor\&\rfloor] \longrightarrow \langle\langle x, (x)_{12}\rangle, (x)_{22i}\rangle \in z) \\
&\& \, (\forall x \in z)(\forall y \in \mathbb{F})((x)_2 \in \mathbb{F} \longrightarrow \langle\langle x, (x)_{12} \setminus y\rangle, \langle\lfloor\rightarrow\rfloor, \langle y, (x)_2\rangle\rangle\rangle \in z) \\
&\& \, (\forall x, y \in z)(((x)_2, (y)_2 \in \mathbb{F} \,\&\, (x)_{21} = \lfloor\rightarrow\rfloor \,\&\, (x)_{221} = (y)_2) \longrightarrow \\
&\quad \langle\langle\langle x, y\rangle, (x)_{12} \cup (y)_{12}\rangle, (x)_{222}\rangle \in z) \\
&\& \, (\forall x \in z)(\forall y \in \mathbb{F})((x)_2 \in \mathbb{F} \longrightarrow \langle\langle x, (x)_{12}\rangle, \langle\lfloor\vee\rfloor, \langle(x)_2, y\rangle\rangle\rangle \in z) \\
&\& \, (\forall x \in z)(\forall y \in \mathbb{F})((x)_2 \in \mathbb{F} \longrightarrow \langle\langle x, (x)_{12}\rangle, \langle\lfloor\vee\rfloor, \langle y, (x)_2\rangle\rangle\rangle \in z) \\
&\& \, (\forall w, x, y \in z)([(w)_2, (x)_2, (y)_2 \in \mathbb{F} \,\&\, (w)_{21} = \lfloor\vee\rfloor \,\&\, (x)_{12} = (w)_{12} \cup (w)_{221} \\
&\quad \& \, (y)_{12} = (w)_{12} \cup (w)_{222} \,\&\, (x)_2 = (y)_2] \longrightarrow \langle\langle\langle w, \langle x, y\rangle\rangle, (w)_{12}\rangle, (x)_2\rangle \in z) \\
&\& \, (\forall x, y \in z)(((x)_2, (y)_2 \in \mathbb{F} \,\&\, (y)_{12} = ((x)_{12} \setminus (y)_2) \cup (x)_2) \longrightarrow \\
&\quad \langle\langle\langle x, y\rangle, (x)_{12} \setminus (y)_2\rangle, \langle\lfloor\leftrightarrow\rfloor, \langle(y)_2, (x)_2\rangle\rangle\rangle \in z) \\
&\& \, (\forall x, y \in z)(((x)_2, (y)_2 \in \mathbb{F} \,\&\, (x)_{21} = \lfloor\leftrightarrow\rfloor \,\&\, (x)_{222} = (y)_2) \longrightarrow \\
&\quad \langle\langle\langle x, y\rangle, (x)_{12} \cup (y)_{12}\rangle (x)_{221}\rangle \in z) \\
&\& \, (\forall x, y \in z)(((x)_2, (y)_2 \in \mathbb{F} \,\&\, (x)_{21} = \lfloor\leftrightarrow\rfloor \,\&\, (x)_{221} = (y)_2) \longrightarrow \\
&\quad \langle\langle\langle x, y\rangle, (x)_{12} \cup (y)_{12}\rangle (x)_{222}\rangle \in z) \\
&\& \, (\forall x, y \in z)(((x)_2, (y)_2 \in \mathbb{F} \,\&\, (y)_{21} = \lfloor\neg\rfloor \,\&\, (x)_2 = (y)_{22}) \longrightarrow \\
&\quad \langle\langle\langle x, y\rangle, (x)_{12} \cup (y)_{12}\rangle, \lfloor\bot\rfloor\rangle \in z) \\
&\& \, (\forall x \in z)(\forall y \in \mathbb{F})((x)_2 = \lfloor\bot\rfloor \longrightarrow \langle\langle x, (x)_{12} \setminus y\rangle, \langle\lfloor\neg\rfloor, y\rangle\rangle \in z) \\
&\& \, (\forall x \in z)(\forall y \in \mathbb{F})((x)_2 = \lfloor\bot\rfloor \longrightarrow \langle\langle x, (x)_{12} \setminus \langle\lfloor\neg\rfloor, y\rangle\rangle, y\rangle \in z) \\
&(\forall x \in z)(((x)_2 \in \mathbb{S} \,\&\, (x)_{22} \in \mathbb{F} \,\&\, (\forall y \in (x)_{12})(\exists w \in \mathcal{NF})(w)_2 = y) \longrightarrow \\
&\quad \langle\langle x, (x)_{12}\rangle, \langle\langle\lfloor\forall\rfloor, (x)_{2122}\rangle, (x)_{22}\rangle\rangle \in z) \\
&(\forall x \in z)(\forall y \in \mathbb{S})(((x)_2 \in \mathbb{F} \,\&\, (x)_{211} = \lfloor\forall\rfloor \,\&\, (y)_{121} \in \mathcal{FF} \,\&\, (y)_{1212} = (x)_{22} \,\& \\
&\quad (y)_{12112} = (x)_{212}) \longrightarrow \langle\langle x, (x)_{12}\rangle, y\rangle \in z) \\
&(\forall x \in z)(\forall y \in \mathbb{V})([(x)_2 \in \mathbb{S} \,\&\, (x)_{22} \in \mathbb{F}] \longrightarrow \langle\langle x, (x)_{12}\rangle, \langle\langle\lfloor\exists\rfloor, y\rangle, (x)_{22}\rangle\rangle \in z) \\
&(\forall x, y \in z)(((x)_2, (y)_2 \in \mathbb{F} \,\&\, (x)_{211} = \lfloor\exists\rfloor \,\&\, (y)_{12} = (x)_{12} \cup (y)_{122} \,\&\, (y)_{122} \in \mathbb{S} \,\& \\
&\quad (y)_{122122} = (x)_{212} \,\&\, (y)_{122121} \in \mathbb{V} \,\&\, (\exists w \in \mathcal{NF})(w)_2 = (y)_2 \,\& \\
&\quad (\forall v_1 \in (x)_{12})(\exists v_2 \in \mathcal{NF})(v)_2 = v_1) \longrightarrow \langle\langle\langle x, y\rangle, (x)_{12}\rangle, (y)_2\rangle \in z)\}
\end{aligned}
$$

We then define the set of Proofs $\mathbb{P}$ as

$$\mathbb{P} = \bigcap p$$

We will show by the principle of induction for proofs that

$$(\forall X)[\textsc{Proof}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathbb{P}]$$

Proof. Base Case: Axioms.

The representability of the axioms is immediate, as the coding of the basic syntax can be used to code each axiom and axiom schema without any need for an inductive definition or argument.

Inductive Case 1: Conjunction Introduction

Induction Hypothesis: For arbitrary X, assume

$$\textsc{Proof}((X)_{11i}) \longrightarrow ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathbb{P}$$

1	$X = [[[(X)_{111}, (X)_{112}], (X)_{11112} \cup (X)_{11212}],$	
	$\quad [\&, [(X)_{1112}, (X)_{1122}]]]$	Premise
2	$\textsc{Proof}((X)_{11i})$	Premise
3	$\textsc{Form}((X)_{11i2})$	Premise
4	$ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathbb{P}$	IH: 2
5	$* \quad \lfloor (X)_{11i} \rfloor \in \mathbb{P}$	ProvE: 4
6	$* \quad\quad z \in p$	Assume
7	$* \quad\quad (\forall x, y \in z)([(x)_2, (y)_2 \in \mathbb{F}] \longrightarrow$	
	$* \quad\quad\quad \langle\langle\langle x, y\rangle, (x)_{12} \cup (y)_{12}\rangle, \langle \lfloor \& \rfloor, \langle (x)_2, (y)_2\rangle\rangle\rangle \in z)$	DefE(p): 6
8	$* \quad\quad \lfloor (X)_{11i} \rfloor \in z$	Lemma(6): 5
9	$* \quad\quad [\lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F}] \longrightarrow$	
	$* \quad\quad\quad \langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor\rangle, \lfloor (X)_{11112} \rfloor \cup \lfloor (X)_{11212} \rfloor\rangle$	
	$* \quad\quad\quad , \langle \lfloor \& \rfloor, \langle \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor\rangle\rangle\rangle \in z$	$\forall_\in$E: 7,8
10	$* \quad\quad \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F}$	RepE(Form): 3
11	$* \quad\quad \langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor\rangle, \lfloor (X)_{11112} \rfloor \cup \lfloor (X)_{11212} \rfloor\rangle$	
	$* \quad\quad\quad , \langle \lfloor \& \rfloor, \langle \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor\rangle\rangle\rangle \in z$	$\rightarrow$E: 9,10
12	$* \quad\quad \lfloor [[(X)_{111}, (X)_{112}], [(X)_{11112} \cup (X)_{11212}$	
	$* \quad\quad\quad , [\&, [(X)_{1112}, (X)_{1122}]]]] \rfloor \in z$	DefI(Code): 11
13	$* \quad\quad \lfloor X \rfloor \in z$	RepE(=): 12,1
14	$* \quad (\forall z \in p) \lfloor X \rfloor \in z$	$\forall_\in$I: 13
15	$* \quad \lfloor X \rfloor \in \bigcap p$	DefI($\cap$): 14
16	$* \quad \lfloor X \rfloor \in \mathbb{P}$	DefI($\mathbb{P}$): 15
17	$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 16

Inductive Case 2: Conjunction Elimination

Induction Hypothesis: For arbitrary X, assume

$$\textsc{Proof}((X)_{11}) \longrightarrow ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{P}$$

1	$X = [[(X)_{11}, (X)_{1112}], (X)_{1122i}]$	Premise
2	$\textsc{Proof}((X)_{11})$	Premise
3	$\textsc{Form}((X)_{112})$	Premise
4	$(X)_{1121} = \&$	Premise
5	$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{P}$	IH: 2
6	$* \quad \lfloor (X)_{11} \rfloor \in \mathbb{P}$	ProvE: 5
7	$* \quad\quad z \in p$	Assume
8	$* \quad\quad (\forall x \in z)([(x)_2 \in \mathbb{F} \,\&\, (x)_{21} = \lfloor \& \rfloor] \longrightarrow$	
	$* \quad\quad\quad \langle\langle x, (x)_{12}\rangle, (x)_{22i}\rangle \in z)$	DefE(p): 7
9	$* \quad\quad \lfloor (X)_{11} \rfloor \in z$	Lemma(6): 6
10	$* \quad\quad [\lfloor (X)_{112} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{1121} \rfloor = \lfloor \& \rfloor] \longrightarrow$	
	$* \quad\quad\quad \langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \lfloor (X)_{1122i} \rfloor\rangle \in z$	$\forall_{\in}$E: 8,9
11	$* \quad\quad \lfloor (X)_{112} \rfloor \in \mathbb{F}$	RepE(Form): 3
12	$* \quad\quad \lfloor (X)_{1121} \rfloor = \lfloor \& \rfloor$	RepE(=): 4
13	$* \quad\quad \lfloor (X)_{112} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{1121} \rfloor = \lfloor \& \rfloor$	&I: 11,12
14	$* \quad\quad \langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \lfloor (X)_{1122i} \rfloor\rangle \in z$	$\rightarrow$E: 10,13
15	$* \quad\quad \lfloor [(X)_{11}, [(X)_{1112}, (X)_{1122i}]] \rfloor \in z$	DefI(Code): 14
16	$* \quad\quad \lfloor X \rfloor \in z$	RepE(=): 15,1
17	$* \quad (\forall z \in p) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 16
18	$* \quad \lfloor X \rfloor \in \bigcap p$	DefI($\cap$): 17
19	$* \quad \lfloor X \rfloor \in \mathbb{P}$	DefI($\mathbb{P}$): 18
20	$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 19

Inductive Case 3: Conditional Introduction

Induction Hypothesis: For arbitrary X, assume

$$\textsc{Proof}((X)_{11}) \longrightarrow ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{P}$$

1	$X = [[(X)_{11}, (X)_{1112} \setminus (X)_{221}], [\rightarrow, [(X)_{221}, (X)_{112}]]]$	Premise
2	$\textsc{Proof}((X)_{11})$	Premise
3	$\textsc{Form}((X)_{112})$	Premise
4	$\textsc{Form}((X)_{221})$	Premise
5	$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{P}$	IH: 2
6	$* \quad \lfloor (X)_{11} \rfloor \in \mathbb{P}$	ProvE: 5
7	$* \quad\quad z \in p$	Assume
8	$* \quad\quad (\forall x \in z)(\forall y \in \mathbb{F})((x)_2 \in \mathbb{F} \longrightarrow$	
	$* \quad\quad\quad \langle\langle x, (x)_{12} \setminus y\rangle, \langle \lfloor \rightarrow \rfloor, \langle y, (x)_2\rangle\rangle\rangle \in z)$	DefE(p): 7
9	$* \quad\quad \lfloor (X)_{11} \rfloor \in z$	Lemma(6): 6
10	$* \quad\quad \lfloor (X)_{221} \rfloor \in \mathbb{F}$	RepE(Form): 4
11	$* \quad\quad \lfloor (X)_{112} \rfloor \in \mathbb{F} \longrightarrow$	
	$* \quad\quad\quad \langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor \setminus \lfloor (X)_{221} \rfloor\rangle,$	
	$* \quad\quad\quad \langle \lfloor \rightarrow \rfloor, \langle \lfloor (X)_{221} \rfloor, \lfloor (X)_{112} \rfloor\rangle\rangle\rangle \in z$	$\forall_\in$E: 8,9,10
12	$* \quad\quad \lfloor (X)_{112} \rfloor \in \mathbb{F}$	RepE(Form): 3
13	$* \quad\quad \langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor \setminus \lfloor (X)_{221} \rfloor\rangle,$	
	$* \quad\quad\quad \langle \lfloor \rightarrow \rfloor, \langle \lfloor (X)_{221} \rfloor, \lfloor (X)_{112} \rfloor\rangle\rangle\rangle \in z$	$\rightarrow$E: 11,12
14	$* \quad\quad \lfloor [[(X)_{11}, (X)_{1112} \setminus (X)_{221}], [\rightarrow, [(X)_{221}, (X)_{112}]]] \rfloor \in z$	DefI(Code): 13
15	$* \quad\quad \lfloor X \rfloor \in z$	RepE(=): 14,1
16	$* \quad (\forall z \in p) \lfloor X \rfloor \in z$	$\forall_\in$I: 15
17	$* \quad \lfloor X \rfloor \in \bigcap p$	DefI($\cap$): 16
18	$* \quad \lfloor X \rfloor \in \mathbb{P}$	DefI($\mathbb{P}$): 17
19	$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 18

Inductive Case 4: Conditional Elimination

Induction Hypothesis: For arbitrary X, assume

$$\textsc{Proof}((X)_{1i}) \longrightarrow ZF \vdash \lfloor (X)_{1i} \rfloor \in \mathbb{P}$$

1	$X = [[[(X)_{111}, (X)_{112}], (X)_{11112} \cup (X)_{11212}], (X)_{111222}]$	Premise
2	$\textsc{Proof}((X)_{11i})$	Premise
3	$\textsc{Form}((X)_{11i2}$	Premise
4	$(X)_{11121} = \rightarrow$	Premise
5	$(X)_{111221} = (X)_{1122}$	Premise
6	$ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathbb{P}$	IH: 2
7	$* \quad \lfloor (X)_{11i} \rfloor \in \mathbb{P}$	ProvE: 6
8	$* \quad\quad z \in p$	Assume
9	$* \quad\quad (\forall x, y \in z)((x)_2, (y)_2 \in \mathbb{F} \,\&\, (x)_{21} = \lfloor \rightarrow \rfloor \,\&$	
	$* \quad\quad\quad (x)_{221} = (y)_2) \longrightarrow$	
	$* \quad\quad\quad \langle\langle\langle x, y\rangle, (x)_{12} \cup (y)_{12}\rangle, (x)_{222}\rangle \in z)$	DefE(p): 8
10	$* \quad\quad \lfloor (X)_{11i} \rfloor \in z$	Lemma(6): 7
11	$* \quad\quad (\lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11121} \rfloor = \lfloor \rightarrow \rfloor$	
	$* \quad\quad\quad \& \lfloor (X)_{111221} \rfloor = \lfloor (X)_{1122} \rfloor) \longrightarrow$	
	$* \quad\quad\quad \langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor\rangle, \lfloor (X)_{11112} \rfloor \cup \lfloor (X)_{11212} \rfloor\rangle, \vert$	
	$* \quad\quad\quad \lfloor (X)_{111222} \rfloor\rangle\rangle\rangle \in z$	$\forall_\in$E: 9,10
12	$* \quad\quad \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F}$	RepE(Form): 3
13	$* \quad\quad \lfloor (X)_{11121} \rfloor = \lfloor \rightarrow \rfloor$	RepE(=): 4
14	$* \quad\quad \lfloor (X)_{111221} \rfloor = \lfloor (X)_{1122} \rfloor$	RepE(=): 5
15	$* \quad\quad \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11121} \rfloor = \lfloor \rightarrow \rfloor$	
	$* \quad\quad\quad \& \lfloor (X)_{111221} \rfloor = \lfloor (X)_{1122} \rfloor$	&I: 13,14
16	$* \quad\quad \langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor\rangle, \lfloor (X)_{11112} \rfloor \cup \lfloor (X)_{11212} \rfloor\rangle,$	
	$* \quad\quad\quad \lfloor (X)_{111222} \rfloor\rangle\rangle\rangle \in z$	$\rightarrow$E: 11,15
17	$* \quad\quad \lfloor [[[(X)_{111}, (X)_{112}], (X)_{11112} \cup (X)_{11212}], (X)_{111222}] \rfloor \in z$	DefI(Code): 16
18	$* \quad\quad \lfloor X \rfloor \in z$	RepE(=): 17,1
19	$* \quad (\forall z \in p) \lfloor X \rfloor \in z$	$\forall_\in$I: 18
20	$* \quad \lfloor X \rfloor \in \bigcap p$	DefI($\cap$): 19
21	$* \quad \lfloor X \rfloor \in \mathbb{P}$	DefI($\mathbb{P}$): 20
22	$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 21

Inductive Case 5: Disjunction Introduction Right

Induction Hypothesis: For arbitrary X, assume

$$\textsc{Proof}((X)_{11}) \longrightarrow ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{P}$$

1	$X = [[(X)_{11}, (X)_{1112}], [\vee, [(X)_{112}, (X)_{222}]]]$	Premise
2	$\textsc{Proof}((X)_{11})$	Premise
3	$\textsc{Form}((X)_{112})$	Premise
4	$\textsc{Form}((X)_{222})$	Premise
5	$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{P}$	IH: 2
6	$*$ $\lfloor (X)_{11} \rfloor \in \mathbb{P}$	ProvE: 5
7	$*$ $z \in p$	Assume
8	$*$ $(\forall x \in z)(\forall y \in \mathbb{F})((x)_2 \in \mathbb{F} \longrightarrow$	
	$*$ $\langle\langle x, (x)_{12}\rangle, \langle \lfloor \vee \rfloor, \langle (x)_2, y\rangle\rangle\rangle \in z)$	DefE(p): 7
9	$*$ $\lfloor (X)_{11} \rfloor \in z$	Lemma(6): 6
10	$*$ $\lfloor (X)_{222} \rfloor \in \mathbb{F}$	RepE(Form): 4
11	$*$ $\lfloor (X)_{112} \rfloor \in \mathbb{F} \longrightarrow$	
	$*$ $\langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \langle \lfloor \vee \rfloor, \langle \lfloor (X)_{112} \rfloor, \lfloor (X)_{222} \rfloor\rangle\rangle\rangle \in z$	$\forall_{\in}$E: 8,9,10
12	$*$ $\lfloor (X)_{112} \rfloor \in \mathbb{F}$	RepE(Form): 3
13	$*$ $\langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \langle \lfloor \vee \rfloor, \langle \lfloor (X)_{112} \rfloor, \lfloor (X)_{222} \rfloor\rangle\rangle\rangle \in z$	$\rightarrow$E: 11,12
14	$*$ $\lfloor [[(X)_{11}, (X)_{1112}], [\vee, [(X)_{112}, (X)_{222}]]] \rfloor \in z$	DefI(Code): 13
15	$*$ $\lfloor X \rfloor \in z$	RepE(=): 14,1
16	$*$ $(\forall z \in p) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 15
17	$*$ $\lfloor X \rfloor \in \bigcap p$	DefI($\cap$): 16
18	$*$ $\lfloor X \rfloor \in \mathbb{P}$	DefI($\mathbb{P}$): 17
19	$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 18

Inductive Case 6: Disjunction Introduction Left

Induction Hypothesis: For arbitrary X, assume

$$\textsc{Proof}((X)_{11}) \longrightarrow ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{P}$$

1		$X = [[(X)_{11}, (X)_{1112}], [\vee, [(X)_{221}, (X)_{112}]]]$	Premise
2		$\textsc{Proof}((X)_{11})$	Premise
3		$\textsc{Form}((X)_{112})$	Premise
4		$\textsc{Form}((X)_{221})$	Premise
5		$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{P}$	IH: 2
6	*	$\lfloor (X)_{11} \rfloor \in \mathbb{P}$	ProvE: 5
7	*	$z \in p$	Assume
8	*	$(\forall x \in z)(\forall y \in \mathbb{F})((x)_2 \in \mathbb{F} \longrightarrow$	
	*	$\langle\langle x, (x)_{12}\rangle, \langle \lfloor \vee \rfloor, \langle y, (x)_2\rangle\rangle\rangle \in z)$	DefE(p): 7
9	*	$\lfloor (X)_{11} \rfloor \in z$	Lemma(6): 6
10	*	$\lfloor (X)_{221} \rfloor \in \mathbb{F}$	RepE(Form): 4
11	*	$\lfloor (X)_{112} \rfloor \in \mathbb{F} \longrightarrow$	
	*	$\langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \langle \lfloor \vee \rfloor, \langle \lfloor (X)_{221} \rfloor, \lfloor (X)_{112} \rfloor\rangle\rangle\rangle \in z$	$\forall_{\in}$E: 8,9,10
12	*	$\lfloor (X)_{112} \rfloor \in \mathbb{F}$	RepE(Form): 3
13	*	$\langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \langle \lfloor \vee \rfloor, \langle \lfloor (X)_{221} \rfloor, \lfloor (X)_{112} \rfloor\rangle\rangle\rangle \in z$	$\rightarrow$E: 11,12
14	*	$\lfloor [[(X)_{11}, (X)_{1112}], [\vee, [(X)_{221}, (X)_{112}]]] \rfloor \in z$	DefI(Code): 13
15	*	$\lfloor X \rfloor \in z$	RepE(=): 14,1
16	*	$(\forall z \in p) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 15
17	*	$\lfloor X \rfloor \in \bigcap p$	DefI($\cap$): 16
18	*	$\lfloor X \rfloor \in \mathbb{P}$	DefI($\mathbb{P}$): 17
19		$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 18

Inductive Case 7: Disjunction Elimination

Inductive Hypothesis: For arbitrary X, assume

$$\textsc{Proof}((X)_{111}) \longrightarrow ZF \vdash \lfloor (X)_{111} \rfloor \in \mathbb{P}$$
$$\textsc{Proof}((X)_{112i}) \longrightarrow ZF \vdash \lfloor (X)_{112i} \rfloor \in \mathbb{P}$$

1	$X = [[[(X)_{111}, [(X)_{1121}, (X)_{1122}]], (X)_{11112}], (X)_{11212}]$	Premise
2	$\textsc{Proof}((X)_{111})$	Premise
3	$\textsc{Proof}((X)_{112i})$	Premise
4	$\textsc{Form}((X)_{1112})$	Premise
5	$\textsc{Form}((X)_{112i2})$	Premise
6	$(X)_{11121} = \vee$	Premise
7	$(X)_{112i12} = (X)_{11112} \cup (X)_{11122i}$	Premise
8	$(X)_{11212} = (X)_{11222}$	Premise
9	$ZF \vdash \lfloor (X)_{111} \rfloor \in \mathbb{P}$	IH: 2
10	$ZF \vdash \lfloor (X)_{112i} \rfloor \in \mathbb{P}$	IH: 3
11	* $\lfloor (X)_{111} \rfloor \in \mathbb{P}$	ProvE: 9
12	* $\lfloor (X)_{112i} \rfloor \in \mathbb{P}$	ProvE: 10
13	* $z \in p$	Assume
14	* $(\forall w, x, y \in z)([(w)_2, (x)_2, (y)_2 \in \mathbb{F} \,\&\, (w)_{21} = \lfloor \vee \rfloor$	
	* $\&\, (x)_{12} = (w)_{12} \cup (w)_{221} \,\&\, (y)_{12} = (w)_{12} \cup (w)_{222}$	
	* $\&\, (x)_2 = (y)_2] \longrightarrow \langle\langle\langle w, \langle x, y\rangle\rangle, (w)_{12}\rangle, (x)_2\rangle \in z)$	DefE(p): 13
15	* $\lfloor (X)_{111} \rfloor \in z$	Lemma(6): 11
16	* $\lfloor (X)_{112i} \rfloor \in z$	Lemma(6): 12
17	* $[\lfloor (X)_{1112} \rfloor, \lfloor (X)_{11212} \rfloor, \lfloor (X)_{11222} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11121} \rfloor = \lfloor \vee \rfloor$	
	* $\&\, \lfloor (X)_{112112} \rfloor = \lfloor (X)_{11112} \rfloor \cup \lfloor (X)_{111221} \rfloor$	
	* $\&\, \lfloor (X)_{112212} \rfloor = \lfloor (X)_{11112} \rfloor \cup \lfloor (X)_{111222} \rfloor \,\&$	
	* $\lfloor (X)_{11212} \rfloor = \lfloor (X)_{11222} \rfloor]$	
	* $\longrightarrow \langle\langle\langle \lfloor (X)_{111} \rfloor, \langle \lfloor (X)_{1121} \rfloor, \lfloor (X)_{1122} \rfloor\rangle\rangle, \lfloor (X)_{11112} \rfloor\rangle,$	

(cont'd)

1	$* \quad \lfloor (X)_{11212} \rfloor\rangle \in z$	$\forall_{\in}$E: 14,15,16
19	$* \quad \lfloor (X)_{1112} \rfloor, \lfloor (X)_{11212} \rfloor, \lfloor (X)_{11222} \rfloor \in \mathbb{F}$	RepE(Form): 4,5
20	$* \quad \lfloor (X)_{11121} \rfloor = \lfloor \vee \rfloor$	RepE(=): 6
21	$* \quad \lfloor (X)_{112112} \rfloor = \lfloor (X)_{11112} \rfloor \cup \lfloor (X)_{111221} \rfloor$	RepE(=): 7
22	$* \quad \lfloor (X)_{112212} \rfloor = \lfloor (X)_{11112} \rfloor \cup \lfloor (X)_{111222} \rfloor$	RepE(=): 7
23	$* \quad \lfloor (X)_{11212} \rfloor = \lfloor (X)_{11222} \rfloor$	RepE(=): 8
24	$* \quad \lfloor (X)_{1112} \rfloor, \lfloor (X)_{11212} \rfloor, \lfloor (X)_{11222} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11121} \rfloor = \lfloor \vee \rfloor$	
	$* \quad \&\, \lfloor (X)_{112112} \rfloor = \lfloor (X)_{11112} \rfloor \cup \lfloor (X)_{111221} \rfloor$	
	$* \quad \&\, \lfloor (X)_{112212} \rfloor = \lfloor (X)_{11112} \rfloor \cup \lfloor (X)_{111222} \rfloor \,\&$	
	$* \quad \lfloor (X)_{11212} \rfloor = \lfloor (X)_{11222} \rfloor$	&I: 19,20,21,22,23
25	$* \quad \langle\langle\langle \lfloor (X)_{111} \rfloor, \langle \lfloor (X)_{1121} \rfloor, \lfloor (X)_{1122} \rfloor \rangle\rangle, \lfloor (X)_{11112} \rfloor \rangle,$	
	$* \quad \lfloor (X)_{11212} \rfloor\rangle \in z$	$\to$E: 17,30
26	$* \quad \lfloor [[[(X)_{111}, [(X)_{1121}, (X)_{1122}]], (X)_{11112}], (X)_{11212}] \rfloor \in z$	DefI(Code): 31
27	$* \quad \lfloor X \rfloor \in z$	RepE(=): 32,1
28	$* \quad (\forall z \in p) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 33
29	$* \quad \lfloor X \rfloor \in \bigcap p$	DefI($\bigcap$): 34
30	$* \quad \lfloor X \rfloor \in \mathbb{P}$	DefI($\mathbb{P}$): 35
31	$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 36

Inductive Case 8: Biconditional Introduction

Induction Hypothesis: For arbitrary X assume that

$$\textsc{Proof}((X)_{11i}) \longrightarrow ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathbb{P}$$

1		$X = [[[(X)_{111}, (X)_{112}], (X)_{11112} \setminus (X)_{1122}], [\leftrightarrow, [(X)_{1122}, (X)_{1112}]]]$	Premise
2		$\textsc{Proof}((X)_{11i})$	Premise
3		$\textsc{Form}((X)_{11i2})$	Premise
4		$(X)_{11212} = ((X)_{11112} \setminus (X)_{1122}) \cup (X)_{1112}$	Premise
5		$ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathbb{P}$	IH: 2
6	*	$\lfloor (X)_{11i} \rfloor \in \mathbb{P}$	ProvE: 5
7	*	$z \in p$	Assume
8	*	$(\forall x, y \in z)(((x)_2, (y)_2 \in \mathbb{F}\,\&$	
	*	$(y)_{12} = ((x)_{12} \setminus (y)_2) \cup (x)_2) \longrightarrow$	
	*	$\langle\langle\langle x, y\rangle, (x)_{12} \setminus (y)_2\rangle, \langle\lfloor\leftrightarrow\rfloor, \langle (y)_2, (x)_2\rangle\rangle\rangle \in z)$	DefE(p): 7
9	*	$\lfloor (X)_{11i} \rfloor \in z$	Lemma(6): 6
10	*	$(\lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F}\,\&$	
	*	$\lfloor (X)_{11212} \rfloor = (\lfloor (X)_{11112} \rfloor \setminus \lfloor (X)_{1122} \rfloor) \cup \lfloor (X)_{1112} \rfloor) \longrightarrow$	
	*	$\langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor\rangle, \lfloor (X)_{11112} \rfloor \setminus \lfloor (X)_{1122} \rfloor\rangle,$	
	*	$\langle \lfloor\leftrightarrow\rfloor, \langle \lfloor (X)_{1122} \rfloor, \lfloor (X)_{1112} \rfloor\rangle\rangle\rangle \in z$	$\forall_\in$E: 8,9
11	*	$\lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F}$‘	RepE(Form): 3
12	*	$\lfloor (X)_{11212} \rfloor = (\lfloor (X)_{11112} \rfloor \setminus \lfloor (X)_{1122} \rfloor) \cup \lfloor (X)_{1112} \rfloor$	RepE(=): 4
13	*	$\lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F}\,\&$	
	*	$\lfloor (X)_{11212} \rfloor = (\lfloor (X)_{11112} \rfloor \setminus \lfloor (X)_{1122} \rfloor) \cup \lfloor (X)_{1112} \rfloor$	&I: 11,12
14	*	$\langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor\rangle, \lfloor (X)_{11112} \rfloor \setminus \lfloor (X)_{1122} \rfloor\rangle,$	
	*	$\langle \lfloor\leftrightarrow\rfloor, \langle \lfloor (X)_{1122} \rfloor, \lfloor (X)_{1112} \rfloor\rangle\rangle\rangle \in z$	$\to$E: 10,13
15	*	$\lfloor [[[(X)_{111}, (X)_{112}], (X)_{11112} \setminus (X)_{1122}],$	
	*	$[\leftrightarrow, [(X)_{1122}, (X)_{1112}]]] \rfloor \in z$	DefI(Code): 14
16	*	$\lfloor X \rfloor \in z$	RepE(=): 15,1
17	*	$(\forall z \in p) \lfloor X \rfloor \in z$	$\forall_\in$I: 16
18	*	$\lfloor X \rfloor \in \bigcap p$	DefI($\cap$):17
19	*	$\lfloor X \rfloor \in \mathbb{P}$	DefI($\mathbb{P}$): 18
20		$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 19

Inductive Case 9: Biconditional Elimination Left

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{Proof}((X)_{11i}) \longrightarrow ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathbb{P}$$

1		$X = [[[(X)_{111}, (X)_{112}], (X)_{11112} \cup (X)_{11212}], (X)_{111221}]$	Premise
2		$\textsc{Proof}((X)_{11i})$	Premise
3		$\textsc{Form}((X)_{11i2})$	Premise
4		$(X)_{11121} = \leftrightarrow$	Premise
5		$(X)_{111222} = (X)_{1122}$	Premise
6		$ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathbb{P}$	IH: 2
7	*	$\lfloor (X)_{11i} \rfloor \in \mathbb{P}$	ProvE: 6
8	*	$z \in p$	Assume
9	*	$(\forall x, y \in z)(((x)_2, (y)_2 \in \mathbb{F} \,\&\, (x)_{21} = \lfloor \leftrightarrow \rfloor \,\&$	
	*	$(x)_{222} = (y)_2) \longrightarrow \langle\langle\langle x, y\rangle, (x)_{12} \cup (y)_{12}\rangle (x)_{221}\rangle \in z)$	DefE(p): 8
10	*	$\lfloor (X)_{11i} \rfloor \in z$	Lemma(6): 7
11	*	$(\lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11121} \rfloor = \lfloor \leftrightarrow \rfloor \,\&$	
	*	$\lfloor (X)_{111222} \rfloor = \lfloor (X)_{1122} \rfloor) \longrightarrow$	
	*	$\langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor\rangle, \lfloor (X)_{11112} \rfloor \cup$	
	*	$\lfloor (X)_{11212} \rfloor\rangle \lfloor (X)_{111221} \rfloor\rangle \in z$	$\forall_{\in}$E: 9,10
12	*	$\lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F}$	RepE(Form): 3
13	*	$\lfloor (X)_{11121} \rfloor = \lfloor \leftrightarrow \rfloor$	RepE(=): 4
14	*	$\lfloor (X)_{111222} \rfloor = \lfloor (X)_{1122} \rfloor$	RepE(=): 5
15	*	$\lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11121} \rfloor = \lfloor \leftrightarrow \rfloor \,\&$	
	*	$\lfloor (X)_{111222} \rfloor = \lfloor (X)_{1122} \rfloor$	$\&I$: 12,13,14
16	*	$\langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor\rangle, \lfloor (X)_{11112} \rfloor \cup$	
	*	$\lfloor (X)_{11212} \rfloor\rangle \lfloor (X)_{111221} \rfloor\rangle \in z$	$\to$E: 11,15
17	*	$\lfloor [[[(X)_{111}, (X)_{112}], (X)_{11112} \cup (X)_{11212}], (X)_{111221}] \rfloor$	DefI(Code): 16
18	*	$\lfloor X \rfloor \in z$	RepE(=): 17,1
19	*	$(\forall z \in p) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 18
20	*	$\lfloor X \rfloor \in \bigcap p$	DefI($\cap$): 19
21	*	$\lfloor X \rfloor \in \mathbb{P}$	DefI($\mathbb{P}$): 20
22		$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 21

Inductive Case 10: Biconditional Elimination Right

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{Proof}((X)_{11i}) \longrightarrow ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathbb{P}$$

1	$X = [[[(X)_{111}, (X)_{112}], (X)_{11112} \cup (X)_{11212}], (X)_{111222}]$	Premise
2	$\textsc{Proof}((X)_{11i})$	Premise
3	$\textsc{Form}((X)_{11i2})$	Premise
4	$(X)_{11121} = \leftrightarrow$	Premise
5	$(X)_{111222} = (X)_{1122}$	Premise
6	$ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathbb{P}$	IH: 2
7	$* \quad \lfloor (X)_{11i} \rfloor \in \mathbb{P}$	ProvE: 6
8	$* \quad z \in p$	Assume
9	$* \quad (\forall x, y \in z)(((x)_2, (y)_2 \in \mathbb{F} \,\&\, (x)_{21} = \lfloor \leftrightarrow \rfloor \,\&$	
	$* \qquad (x)_{221} = (y)_2) \longrightarrow \langle\langle\langle x, y\rangle, (x)_{12} \cup (y)_{12}\rangle (x)_{222}\rangle \in z$	DefE(p): 8
10	$* \quad \lfloor (X)_{11i} \rfloor \in z)$	Lemma(6): 7
11	$* \quad (\lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11121} \rfloor = \lfloor \leftrightarrow \rfloor \,\&$	
	$* \qquad \lfloor (X)_{111221} \rfloor = \lfloor (X)_{1122} \rfloor) \longrightarrow$	
	$* \qquad \langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor\rangle, \lfloor (X)_{11112} \rfloor \cup$	
	$* \qquad \lfloor (X)_{11212} \rfloor\rangle \lfloor (X)_{111222} \rfloor\rangle \in z$	$\forall_{\in}$E: 9,10
12	$* \quad \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F}$	RepE(Form): 3
13	$* \quad \lfloor (X)_{11121} \rfloor = \lfloor \leftrightarrow \rfloor$	RepE(=): 4
14	$* \quad \lfloor (X)_{111221} \rfloor = \lfloor (X)_{1122} \rfloor$	RepE(=): 5
15	$* \quad \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11121} \rfloor = \lfloor \leftrightarrow \rfloor \,\&$	
	$* \qquad \lfloor (X)_{111222} \rfloor = \lfloor (X)_{1122} \rfloor$	$\&I$: 12,13,14
16	$* \quad \langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor\rangle, \lfloor (X)_{11112} \rfloor \cup$	
	$* \qquad \lfloor (X)_{11212} \rfloor\rangle \lfloor (X)_{111222} \rfloor\rangle \in z$	$\rightarrow$E: 11,15
17	$* \quad \lfloor [[[(X)_{111}, (X)_{112}], (X)_{11112} \cup (X)_{11212}], (X)_{111222}] \rfloor$	DefI(Code): 16
18	$* \quad \lfloor X \rfloor \in z$	RepE(=): 17,1
19	$* \quad (\forall z \in p) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 18
20	$* \quad \lfloor X \rfloor \in \bigcap p$	DefI($\cap$): 19
21	$* \quad \lfloor X \rfloor \in \mathbb{P}$	DefI($\mathbb{P}$): 20
22	$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 21

Inductive Case 11: Falsum Introduction

Induction Hypothesis: For arbitrary X, assume

$$\textsc{Proof}((X)_{11i}) \longrightarrow ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathbb{P}$$

1	$X = [[[(X)_{111}, (X)_{112}], (X)_{11112} \cup (X)_{11212}], \bot]$	Premise
2	$\textsc{Proof}((X)_{11i})$	Premise
3	$\textsc{Form}((X)_{11i2})$	Premise
4	$(X)_{11221} = \neg$	Premise
5	$(X)_{1112} = (X)_{11222}$	Premise
6	$ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathbb{P}$	IH: 2
7	$*\ \lfloor (X)_{11i} \rfloor \in \mathbb{P}$	ProvE: 6
8	$*\ z \in p$	Assume
9	$*\ (\forall x, y \in z)(((x)_2, (y)_2 \in \mathbb{F} \,\&\, (y)_{21} = \lfloor \neg \rfloor \,\&\, (x)_2 = (y)_{22}) \longrightarrow$	
	$*\ \langle\langle\langle x, y \rangle, (x)_{12} \cup (y)_{12} \rangle, \lfloor \bot \rfloor \rangle \in z)$	DefE(p): 8
10	$*\ \lfloor (X)_{11i} \rfloor \in z$	Lemma(6): 7
11	$*\ [\lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11221} \rfloor = \lfloor \neg \rfloor$	
	$*\ \&\, \lfloor (X)_{1112} \rfloor = \lfloor (X)_{11222} \rfloor] \longrightarrow$	
	$*\ \langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor \rangle, \lfloor (X)_{11112} \rfloor \cup \lfloor (X)_{11212} \rfloor \rangle, \lfloor \bot \rfloor \rangle \in z$	$\forall_{\in}$E: 9,10
12	$*\ \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F}$	RepE(Form): 3
13	$*\ \lfloor (X)_{11221} \rfloor = \lfloor \neg \rfloor$	RepE(=): 4
14	$*\ \lfloor (X)_{1112} \rfloor = \lfloor (X)_{11222} \rfloor$	RepE(=): 5
15	$*\ \lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11221} \rfloor = \lfloor \neg \rfloor$	
	$*\ \&\, \lfloor (X)_{1112} \rfloor = \lfloor (X)_{11222} \rfloor$	&I: 13,14
16	$*\ \langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor \rangle, \lfloor (X)_{11112} \rfloor \cup \lfloor (X)_{11212} \rfloor \rangle, \lfloor \bot \rfloor \rangle \in z$	$\rightarrow$E: 11,15
17	$*\ \lfloor [[[(X)_{111}, (X)_{112}], (X)_{11112} \cup (X)_{11212}], \bot] \rfloor$	DefI(Code): 16
18	$*\ \lfloor X \rfloor \in z$	RepE(=): 17,1
19	$*\ (\forall z \in p) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 18
20	$*\ \lfloor X \rfloor \in \bigcap p$	DefI($\cap$): 19
21	$*\ \lfloor X \rfloor \in \mathbb{P}$	DefI($\mathbb{P}$): 20
22	$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 21

Inductive Case 12: Negation Introduction

Induction Hypothesis: For arbitrary X, assume

$$\textsc{Proof}((X)_{11}) \longrightarrow ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{P}$$

1	$X = [[(X)_{11}, (X)_{1112} \setminus (X)_{22}], [\neg, (X)_{22}]]$	Premise
2	$\textsc{Proof}((X)_{11})$	Premise
3	$(X)_{112} = \perp$	Premise
4	$\textsc{Form}((X)_{22})$	Premise
5	$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{P}$	IH: 2
6	$*$ $\lfloor (X)_{11} \rfloor \in \mathbb{P}$	ProvE: 5
7	$*$ $z \in p$	Assume
8	$*$ $(\forall x \in z)(\forall y \in \mathbb{F})((x)_2 = \lfloor \perp \rfloor \longrightarrow$	
	$*$ $\langle \langle x, (x)_{12} \setminus y \rangle, \langle \lfloor \neg \rfloor, y \rangle \rangle \in z)$	DefE(p): 7
9	$*$ $\lfloor (X)_{11} \rfloor \in z$	Lemma(6): 6
10	$*$ $\lfloor (X)_{22} \rfloor \in \mathbb{F}$	RepE(Form): 4
11	$*$ $\lfloor (X)_{112} \rfloor = \lfloor \perp \rfloor \longrightarrow$	
	$*$ $\langle \langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor \setminus \lfloor (X)_{22} \rfloor \rangle, \langle \lfloor \neg \rfloor, \lfloor (X)_{22} \rfloor \rangle \rangle \in z$	$\forall_{\in}$E: 8,9,10
12	$*$ $\lfloor (X)_{112} \rfloor = \lfloor \perp \rfloor$	RepE(=): 3
13	$*$ $\langle \langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor \setminus \lfloor (X)_{22} \rfloor \rangle, \langle \lfloor \neg \rfloor, \lfloor (X)_{22} \rfloor \rangle \rangle \in z$	$\rightarrow$E: 11,12
14	$*$ $\lfloor [[(X)_{11}, (X)_{1112} \setminus (X)_{22}], [\neg, (X)_{22}]] \rfloor \in z$	DefI(Code): 13
15	$*$ $\lfloor X \rfloor \in z$	RepE(=): 14,1
16	$*$ $(\forall z \in p) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 15
17	$*$ $\lfloor X \rfloor \in \bigcap p$	DefI($\cap$): 16
18	$*$ $\lfloor X \rfloor \in \mathbb{P}$	DefI($\mathbb{P}$): 17
19	$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 18

Inductive Case 13: Negation Elimination

Induction Hypothesis: For arbitrary X, assume

$$\textsc{Proof}((X)_1) \longrightarrow ZF \vdash \lfloor (X)_1 \rfloor \in \mathbb{P}$$

1	$X = [[(X)_{11}, (X)_{1112} \setminus [\neg, (X)_2]], (X)_2]$	Premise
2	$\textsc{Proof}((X)_{11})$	Premise
3	$(X)_{112} = \bot$	Premise
4	$\textsc{Form}((X)_2)$	Premise
5	$ZF \vdash \lfloor (X)_1 \rfloor \in \mathbb{P}$	IH: 2
6	$*$ $\lfloor (X)_{11} \rfloor \in \mathbb{P}$	ProvE: 5
7	$*$ $z \in p$	Assume
8	$*$ $(\forall x \in z)(\forall y \in \mathbb{F})((x)_2 = \lfloor \bot \rfloor \longrightarrow$	
	$*$ $\langle\langle x, (x)_{12} \setminus \langle \lfloor \neg \rfloor, y \rangle\rangle, y \rangle \in z)$	DefE(p): 7
9	$*$ $\lfloor (X)_{11} \rfloor \in z$	Lemma(6): 6
10	$*$ $\lfloor (X)_{22} \rfloor \in \mathbb{F}$	RepE(Form): 4
11	$*$ $\lfloor (X)_{112} \rfloor = \lfloor \bot \rfloor \longrightarrow$	
	$*$ $\langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor \setminus \langle \lfloor \neg \rfloor, \lfloor (X)_2 \rfloor \rangle\rangle, \lfloor (X)_2 \rfloor \rangle \in z$	$\forall_\in$E: 8,9,10
12	$*$ $\lfloor (X)_{112} \rfloor = \lfloor \bot \rfloor$	RepE(=): 3
13	$*$ $\langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor \setminus \langle \lfloor \neg \rfloor, \lfloor (X)_2 \rfloor \rangle\rangle, \lfloor (X)_2 \rfloor \rangle \in z$	$\rightarrow$E: 11,12
14	$*$ $\lfloor [[(X)_{11}, (X)_{1112} \setminus [\neg, (X)_2]], (X)_2] \rfloor \in z$	DefI(Code): 13
15	$*$ $\lfloor X \rfloor \in z$	RepE(=): 14,1
16	$*$ $(\forall z \in p) \lfloor X \rfloor \in z$	$\forall_\in$I: 15
17	$*$ $\lfloor X \rfloor \in \bigcap p$	DefI($\cap$): 16
18	$*$ $\lfloor X \rfloor \in \mathbb{P}$	DefI($\mathbb{P}$): 17
19	$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 18

Inductive Case 14: Universal Introduction

Induction Hypothesis: For arbitrary X, assume

$$\textsc{Proof}((X)_{11}) \longrightarrow ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{P}$$

1	$X = [[(X)_{11}, (X)_{1112}], [[\forall, (X)_{112122}], (X)_{1122}]]$	Premise
2	$\textsc{Proof}((X)_{11})$	Premise
3	$\textsc{Subst}((X)_{112})$	Premise
4	$\textsc{Form}((X)_{1122})$	Premise
5	$(\forall Y \in (X)_{1112})(\exists W)(\textsc{NotFree}(W) \,\&\, (W)_2 = Y)$	Premise
6	$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{P}$	IH: 2
7	$* \quad \lfloor (X)_{11} \rfloor \in \mathbb{P}$	ProvE: 6
8	$* \quad\quad z \in p$	Assume
9	$* \quad\quad (\forall x \in z)((((x)_2 \in \mathbb{S} \,\&\, (x)_{22} \in \mathbb{F} \,\&$	
	$* \quad\quad\quad (\forall y \in (x)_{12})(\exists w \in \mathcal{NF})(w)_2 = y) \longrightarrow$	
	$* \quad\quad\quad \langle\langle x, (x)_{12}\rangle, \langle\langle \lfloor\forall\rfloor, (x)_{2122}\rangle, (x)_{22}\rangle\rangle \in z)$	DefE(p): 8
10	$* \quad\quad \lfloor (X)_{11} \rfloor \in z$	Lemma(6): 7
11	$* \quad\quad (\lfloor (X)_{112} \rfloor \in \mathbb{S} \,\&\, \lfloor (X)_{1122} \rfloor \in \mathbb{F} \,\&$	
	$* \quad\quad\quad (\forall y \in \lfloor (X)_{1112} \rfloor)(\exists w \in \mathcal{NF})(w)_2 = y) \longrightarrow$	
	$* \quad\quad\quad \langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle,$	
	$* \quad\quad\quad \langle\langle \lfloor\forall\rfloor, \lfloor (X)_{112122} \rfloor\rangle, \lfloor (X)_{1122} \rfloor\rangle\rangle \in z$	$\forall_{\in}$E: 9,10
12	$* \quad\quad \lfloor (X)_{112} \rfloor \in \mathbb{S}$	RepE(Subst): 3
13	$* \quad\quad \lfloor (X)_{1122} \rfloor \in \mathbb{F}$	RepE(Form): 4

(cont'd)

1	*	$\lfloor Y \rfloor \in \lfloor (X)_{1112} \rfloor$	Assume
15		$Y \in (X)_{1112}$	RepI(∈): 14
16		$(\exists W)(\textsc{NotFree}(W) \,\&\, (W)_2 = Y)$	$\forall_{\in}$E: 5,15
17		$\textsc{NotFree}(W)$	Assume
18		$(W)_2 = Y$	Assume
19	*	$\lfloor W \rfloor \in \mathcal{NF}$	RepI(NotFree): 17
20	*	$\lfloor (W)_2 \rfloor = \lfloor Y \rfloor$	RepI(=): 18
21	*	$(\exists w \in \mathcal{NF})(w)_2 = \lfloor Y \rfloor$	$\exists_{\in}$I: 19,20
22		$ZF \vdash (\exists w \in \mathcal{NF})(w)_2 = \lfloor Y \rfloor$	ProvI: 21
23		$ZF \vdash (\exists w \in \mathcal{NF})(w)_2 = \lfloor Y \rfloor$	∃E: 16,22
24	*	$(\exists w \in \mathcal{NF})(w)_2 = \lfloor Y \rfloor$	ProvE: 23
25	*	$(\forall y \in \lfloor (X)_{1112} \rfloor)(\exists w \in \mathcal{NF})(w)_2 = y$	$\forall_{\in}$I: 24
26	*	$\lfloor (X)_{112} \rfloor \in \mathbb{S} \,\&\, \lfloor (X)_{1122} \rfloor \in \mathbb{F} \,\&$	
	*	$(\forall y \in \lfloor (X)_{1112} \rfloor)(\exists w \in \mathcal{NF})(w)_2 = y$	&I: 12,13,25
27	*	$\langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor \rangle,$	
	*	$\langle\langle \lfloor \forall \rfloor, \lfloor (X)_{112122} \rfloor \rangle, \lfloor (X)_{1122} \rfloor \rangle\rangle \in z$	→E: 11,26
28	*	$\lfloor [[(X)_{11}, (X)_{1112}], [[\forall, (X)_{112122}], (X)_{1122}]] \rfloor \in z$	DefI(Code): 27
29	*	$\lfloor X \rfloor \in z$	RepE(=): 28,1
30	*	$(\forall z \in p)\lfloor X \rfloor \in z$	$\forall_{\in}$I: 29
31	*	$\lfloor X \rfloor \in \bigcap p$	DefI(∩): 30
32	*	$\lfloor X \rfloor \in \mathbb{P}$	DefI($\mathbb{P}$): 31
33		$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 32

Inductive Case 15: Universal Elimination

$$\textsc{Proof}((X)_{11}) \longrightarrow ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{P}$$

1	$X = [[(X)_{11}, (X)_{1112}], (X)_2]$	Premise
2	$\textsc{Proof}((X)_{11})$	Premise
3	$\textsc{Form}((X)_{112})$	Premise
4	$(X)_{11211} = \forall$	Premise
5	$\textsc{Subst}((X)_2)$	Premise
6	$(X)_{2122} = (X)_{11212}$	Premise
7	$\textsc{FreeFor}((X)_{2121})$	Premise
8	$(X)_{21212} = (X)_{1122}$	Premise
9	$(X)_{212112} = (X)_{11212}$	Premise
10	$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{P}$	IH: 2
11	* $\lfloor (X)_{11} \rfloor \in \mathbb{P}$	ProvE: 10
12	* $z \in p$	Assume
13	* $(\forall x \in z)(\forall y \in \mathbb{S})(((x)_2 \in \mathbb{F} \,\&\, (x)_{211} = \lfloor \forall \rfloor \,\&$	
	* $(y)_{121} \in \mathcal{FF} \,\&\, (y)_{1212} = (x)_{22} \,\&\, (y)_{12112} = (x)_{212})$	
	* $\longrightarrow \langle \langle x, (x)_{12} \rangle, y \rangle \in z)$	DefE(p): 12
14	* $\lfloor (X)_{11} \rfloor \in z$	Lemma(6): 11
15	* $\lfloor (X)_2 \rfloor \in \mathbb{S}$	RepE(Subst): 5
16	* $(\lfloor (X)_{112} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11211} \rfloor = \lfloor \forall \rfloor \,\&\, \lfloor (X)_{2121} \rfloor \in \mathcal{FF} \,\&$	
	* $\lfloor (X)_{21212} \rfloor = \lfloor (X)_{1122} \rfloor \,\&\, \lfloor (X)_{212112} \rfloor = \lfloor (X)_{11212} \rfloor)$	
	* $\longrightarrow \langle \langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor \rangle, \lfloor (X)_2 \rfloor \rangle \in z$	$\forall_{\in}$E: 13,14,15

(cont'd)

1	*	$\lfloor (X)_{112} \rfloor \in \mathbb{F}$	RepE(Form): 3
18	*	$\lfloor (X)_{11211} \rfloor = \lfloor \forall \rfloor$	RepE(=): 4
19	*	$\lfloor (X)_{2121} \rfloor \in \mathcal{F}\mathcal{F}$	RepE(FreeFor): 7
20	*	$\lfloor (X)_{21212} \rfloor = \lfloor (X)_{1122} \rfloor$	RepE(=): 8
21	*	$\lfloor (X)_{212112} \rfloor = \lfloor (X)_{11212} \rfloor$	RepE(=): 9
22	*	$\lfloor (X)_{112} \rfloor \in \mathbb{F} \,\&\, \lfloor (X)_{11211} \rfloor = \lfloor \forall \rfloor \,\&\, \lfloor (X)_{2121} \rfloor \in \mathcal{F}\mathcal{F} \,\&$	
	*	$\quad \lfloor (X)_{21212} \rfloor = \lfloor (X)_{1122} \rfloor \,\&\, \lfloor (X)_{212112} \rfloor = \lfloor (X)_{11212} \rfloor$	&I: 17-21
23	*	$\langle \langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor \rangle, \lfloor (X)_2 \rfloor \rangle \in z$	→E: 16,22
24	*	$\lfloor [[(X)_{11}, (X)_{1112}], (X)_2] \rfloor \in z$	DefI(Code): 23
25	*	$\lfloor X \rfloor \in z$	RepE(=): 24,1
26	*	$(\forall z \in p) \lfloor X \rfloor \in z$	$\forall_{\in}$I: 25
27	*	$\lfloor X \rfloor \in \bigcap p$	DefI(∩): 26
28	*	$\lfloor X \rfloor \in \mathbb{P}$	DefI(ℙ): 27
29		$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 28

Inductive Case 16: Existential Introduction

Induction Hypothesis: For arbitrary X, assume

$$\textsc{Proof}((X)_{11}) \longrightarrow ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{P}$$

1	$X = [[(X)_{11}, (X)_{1112}], [[\exists, (X)_{212}], (X)_{1122}]]$	Premise
2	$\textsc{Proof}((X)_{11})$	Premise
3	$\textsc{Subst}((X)_{112})$	Premise
4	$\textsc{Form}((X)_{1122})$	Premise
5	$\textsc{Var}((X)_{212})$	Premise
6	$ZF \vdash \lfloor (X)_{11} \rfloor \in \mathbb{P}$	IH: 2
7	$* \quad \lfloor (X)_{11} \rfloor \in \mathbb{P}$	ProvE: 6
8	$* \quad z \in p$	Assume
9	$* \quad (\forall x \in z)(\forall y \in \mathbb{V})([(x)_2 \in \mathbb{S} \,\&\, (x)_{22} \in \mathbb{F}] \longrightarrow$	
10	$* \quad \langle\langle x, (x)_{12}\rangle, \langle\langle \lfloor \exists \rfloor, y\rangle, (x)_{22}\rangle\rangle \in z)$	DefE(p): 8
11	$* \quad \lfloor (X)_{11} \rfloor \in z$	Lemma(6): 7
12	$* \quad \lfloor (X)_{212} \rfloor \in \mathbb{V}$	RepE(Var): 5
13	$* \quad [\lfloor (X)_{112} \rfloor \in \mathbb{S} \,\&\, \lfloor (X)_{1122} \rfloor \in \mathbb{F}] \longrightarrow$	
14	$* \quad \langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \langle\langle \lfloor \exists \rfloor, \lfloor (X)_{212} \rfloor\rangle, \lfloor (X)_{1122} \rfloor\rangle\rangle \in z$	$\forall_\in$E: 9,11,12
15	$* \quad \lfloor (X)_{112} \rfloor \in \mathbb{S}$	RepE(Subst): 3
16	$* \quad \lfloor (X)_{1122} \rfloor \in \mathbb{F}$	RepE(Form): 4
17	$* \quad \lfloor (X)_{112} \rfloor \in \mathbb{S} \,\&\, \lfloor (X)_{1122} \rfloor \in \mathbb{F}$	&I: 15,16
18	$* \quad \langle\langle \lfloor (X)_{11} \rfloor, \lfloor (X)_{1112} \rfloor\rangle, \langle\langle \lfloor \exists \rfloor, \lfloor (X)_{212} \rfloor\rangle, \lfloor (X)_{1122} \rfloor\rangle\rangle \in z$	→E: 13,17
19	$* \quad \lfloor [[(X)_{11}, (X)_{1112}], [[\exists, (X)_{212}], (X)_{1122}]] \rfloor \in z$	DefI(Code): 18
20	$* \quad \lfloor X \rfloor \in z$	RepE(=): 19,1
21	$* \quad (\forall z \in p) \lfloor X \rfloor \in z$	$\forall_\in$I: 20
22	$* \quad \lfloor X \rfloor \in \bigcap p$	DefI(∩): 21
23	$* \quad \lfloor X \rfloor \in \mathbb{P}$	DefI($\mathbb{P}$): 22
24	$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 23

Inductive Case 17: Existential Elimination

Induction Hypothesis: Assume for arbitrary X that

$$\textsc{Proof}((X)_{11i}) \longrightarrow ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathbb{P}$$

1	$X = [[[(X)_{111}, (X)_{112}], (X)_{11112}], (X)_{1122}]$	Premise
2	$\textsc{Proof}((X)_{11i})$	Premise
3	$\textsc{Form}((X)_{11i2})$	Premise
4	$(X)_{111211} = \exists$	Premise
5	$(X)_{11212} = (X)_{11112} \cup (X)_{112122}$	Premise
6	$\textsc{Subst}((X)_{112122})$	Premise
7	$(X)_{112122122} = (X)_{111212}$	Premise
8	$\textsc{Var}((X)_{112122121})$	Premise
9	$(\exists Y)(\textsc{NotFree}(Y) \,\&\, (Y)_2 = (X)_{1122})$	Premise
10	$(\forall W_1 \in (X)_{11112})(\exists W_2)(\textsc{NotFree}(W) \,\&\, (W)_2 = W_1)$	Premise
11	$ZF \vdash \lfloor (X)_{11i} \rfloor \in \mathbb{P}$	IH: 2
12	$*$ $\lfloor (X)_{11i} \rfloor \in \mathbb{P}$	ProvE: 11
13	$*$ $z \in p$	Assume
14	$*$ $(\forall x, y \in z)(((x)_2, (y)_2 \in \mathbb{F} \,\&\, (x)_{211} = \lfloor \exists \rfloor \,\&$	
	$*$ $(y)_{12} = (x)_{12} \cup (y)_{122} \,\&\, (y)_{122} \in \mathbb{S} \,\&$	
	$*$ $(y)_{122122} = (x)_{212} \,\&\, (y)_{122121} \in \mathbb{V} \,\&$	
	$*$ $(\exists w \in \mathcal{NF})(w)_2 = (y)_2 \,\&$	
	$*$ $(\forall v_1 \in (x)_{12})(\exists v_2 \in \mathcal{NF})(v)_2 = v_1) \longrightarrow$	
	$*$ $\langle\langle\langle x, y\rangle, (x)_{12}\rangle, (y)_2\rangle \in z)$	DefE(p): 13
15	$*$ $\lfloor (X)_{11i} \rfloor \in z$	Lemma(6): 12

(cont'd)

1	$*$	$(\lfloor(X)_{1112}\rfloor, \lfloor(X)_{1122}\rfloor \in \mathbb{F} \,\&\, \lfloor(X)_{111211}\rfloor = \lfloor\exists\rfloor \,\&$	
	$*$	$\lfloor(X)_{11212}\rfloor = \lfloor(X)_{11112}\rfloor \cup \lfloor(X)_{112122}\rfloor$	
	$*$	$\&\, \lfloor(X)_{112122}\rfloor \in \mathbb{S} \,\&$	
	$*$	$\lfloor(X)_{112122122}\rfloor = \lfloor(X)_{111212}\rfloor \,\&\, \lfloor(X)_{112122121}\rfloor \in \mathbb{V} \,\&$	
	$*$	$(\exists w \in \mathcal{NF})(w)_2 = \lfloor(X)_{1122}\rfloor \,\&$	
	$*$	$(\forall v_1 \in \lfloor(X)_{11112}\rfloor)(\exists v_2 \in \mathcal{NF})(v)_2 = v_1) \longrightarrow$	
	$*$	$\langle\langle\langle\lfloor(X)_{111}\rfloor, \lfloor(X)_{112}\rfloor\rangle, \lfloor(X)_{11112}\rfloor\rangle, \lfloor(X)_{1122}\rfloor\rangle \in z$	$\forall_{\in}$E: 14,15
17	$*$	$\lfloor(X)_{1112}\rfloor, \lfloor(X)_{1122}\rfloor \in \mathbb{F}$	RepE(Form): 3
18	$*$	$\lfloor(X)_{111211}\rfloor = \lfloor\exists\rfloor$	RepE(=): 4
19	$*$	$\lfloor(X)_{11212}\rfloor = \lfloor(X)_{11112}\rfloor \cup \lfloor(X)_{112122}\rfloor$	RepE(=): 5
20	$*$	$\lfloor(X)_{112122}\rfloor \in \mathbb{S}$	RepE(Subst): 6
21	$*$	$\lfloor(X)_{112122122}\rfloor = \lfloor(X)_{111212}\rfloor$	RepE(=): 7
22	$*$	$\lfloor(X)_{112122121}\rfloor \in \mathbb{V}$	RepE(Var): 8
23		NotFree(Y)	Assume
24		$(Y)_2 = (X)_{1122}$	Assume
25	$*$	$\lfloor Y\rfloor \in \mathcal{NF}$	RepE(NotFree): 23
26	$*$	$\lfloor(Y)_2\rfloor = \lfloor(X)_{1122}\rfloor$	RepE(=): 24
27	$*$	$(\exists w \in \mathcal{NF})(w)_2 = \lfloor(X)_{1122}\rfloor$	$\exists_{\in}$I: 25,26
28		$ZF \vdash (\exists w \in \mathcal{NF})(w)_2 = \lfloor(X)_{1122}\rfloor$	ProvI: 27
29		$ZF \vdash (\exists w \in \mathcal{NF})(w)_2 = \lfloor(X)_{1122}\rfloor$	$\exists_{\in}$E: 9,28
30	$*$	$(\exists w \in \mathcal{NF})(w)_2 = \lfloor(X)_{1122}\rfloor$	ProvE: 29
31	$*$	$\lfloor W_1\rfloor \in \lfloor(X)_{11112}\rfloor$	Assume
32		$W_1 \in (X)_{11112}$	RepI(∈): 31

(cont'd)

1		$(\exists W_2)(\textsc{NotFree}(W)\,\&\,(W)_2 = W_1)$	$\forall_{\in}$E: 10,32
34		$\textsc{NotFree}(W)$	Assume
35		$(W)_2 = W_1$	Assume
36	*	$\lfloor W \rfloor \in \mathcal{NF}$	RepE($\in$): 34
37	*	$\lfloor (W)_2 \rfloor = \lfloor W_1 \rfloor$	RepE($=$): 35
38	*	$(\exists v_2 \in \mathcal{NF})(v)_2 = \lfloor W_1 \rfloor$	$\exists_{\in}$I: 36,37
39		$ZF \vdash (\exists v_2 \in \mathcal{NF})(v)_2 = \lfloor W_1 \rfloor$	ProvI: 38
40		$ZF \vdash (\exists v_2 \in \mathcal{NF})(v)_2 = \lfloor W_1 \rfloor$	$\exists$E: 33,39
41	*	$(\exists v_2 \in \mathcal{NF})(v)_2 = \lfloor W_1 \rfloor$	ProvE: 40
42	*	$(\forall v_1 \in \lfloor (X)_{11112} \rfloor)(\exists v_2 \in \mathcal{NF})(v)_2 = v_1$	$\forall_{\in}$I: 41
43	*	$\lfloor (X)_{1112} \rfloor, \lfloor (X)_{1122} \rfloor \in \mathbb{F}\,\&\,\lfloor (X)_{111211} \rfloor = \lfloor \exists \rfloor\,\&$	
	*	$\lfloor (X)_{11212} \rfloor = \lfloor (X)_{11112} \rfloor \cup \lfloor (X)_{112122} \rfloor$	
	*	$\&\,\lfloor (X)_{112122} \rfloor \in \mathbb{S}\,\&$	
	*	$\lfloor (X)_{112122122} \rfloor = \lfloor (X)_{111212} \rfloor\,\&\,\lfloor (X)_{112122121} \rfloor \in \mathbb{V}\,\&$	
	*	$(\exists w \in \mathcal{NF})(w)_2 = \lfloor (X)_{1122} \rfloor\,\&$	
	*	$(\forall v_1 \in \lfloor (X)_{11112} \rfloor)(\exists v_2 \in \mathcal{NF})(v)_2 = v_1$	&I: 17-22,30,42
44	*	$\langle\langle\langle \lfloor (X)_{111} \rfloor, \lfloor (X)_{112} \rfloor \rangle, \lfloor (X)_{11112} \rfloor \rangle, \lfloor (X)_{1122} \rfloor \rangle \in z$	$\rightarrow$E: 16,43
45	*	$\lfloor [[[(X)_{111}, (X)_{112}], (X)_{11112}], (X)_{1122}] \rfloor \in z$	DefI(Code): 44
46	*	$\lfloor X \rfloor \in z$	RepE($=$): 45,1
47	*	$(\forall z \in p)\lfloor X \rfloor \in z$	$\forall_{\in}$I: 46
48	*	$\lfloor X \rfloor \in \bigcap p$	DefI($\cap$): 47
49	*	$\lfloor X \rfloor \in \mathbb{P}$	DefI($\mathbb{P}$): 48
50		$ZF \vdash \lfloor X \rfloor \in \mathbb{P}$	ProvI: 49

Thus by the principle of induction for proofs,

$$(\forall X)[\textsc{Proof}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \in \mathbb{P}]$$

□

It is also necessary to show

$$(\forall X)[\textsc{NotProof}(X) \longrightarrow ZF \vdash \lfloor X \rfloor \notin \mathbb{P}]$$

but the proof will not be given here. The proof has the same structure as the proofs for the reprentabilty of non-variables and non-formulae, but as there are so many sub-cases to consider here, the proof is just not practical to do by hand. We will define non-proofs and give the basic outline of the proof so that we can at least take it to be provable in principle.

To show the complexity the inductive definitions can reach, we will explicitly provide the metatheoretic definition for non-proofs. It should be clear why proofs involving such definitions are just not practical for human use and would be better deferred to the machine. The object-theoretic definition is an immediate result from the metatheoretic definition and will thus not be given.

$$
\begin{aligned}
&(\forall X)((\textsc{NotAxiom}(X) \\
&\quad \vee [X = [[[(X)_{111}, (X)_{112}], (X)_{12}], [\&, [(X)_{221}, (X)_{222}]]] \,\&\, (\textsc{NotProof}((X)_{111}) \vee \\
&\quad\quad \textsc{NotProof}((X)_{112}) \vee \textsc{NotForm}((X)_{1112}) \vee \textsc{NotForm}((X)_{1122}) \vee \\
&\quad\quad (X)_{12} \neq (X)_{11112} \cup (X)_{11212} \vee (X)_{221} \neq (X)_{1112} \vee (X)_{222} \neq (X)_{1122})] \\
&\quad \vee [X = [[[(X)_{111}, [\&, (X)_{1122}]], (X)_{12}], (X)_{2}] \,\&\, (\textsc{NotProof}((X)_{11}) \vee \\
&\quad\quad \textsc{NotForm}((X)_{112}) \vee (X)_{12} \neq (X)_{1112} \vee (X)_{2} \neq (X)_{1122i})] \\
&\quad \vee [X = [[(X)_{11}, (X)_{12}], [\rightarrow, [(X)_{221}, (X)_{222}]]] \,\&\, (\textsc{NotProof}((X)_{11}) \vee \\
&\quad\quad \textsc{NotForm}((X)_{112}) \vee \textsc{NotForm}((X)_{221}) \vee (X)_{12} \neq (X)_{1112} \setminus (X)_{221} \vee \\
&\quad\quad (X)_{222} \neq (X)_{112})] \\
&\quad \vee [X = [[[[(X)_{1111}, [\rightarrow, (X)_{11122}]], (X)_{112}], (X)_{12}], (X)_{2}] \,\&\, (\textsc{NotProof}((X)_{111}) \vee \\
&\quad\quad \textsc{NotProof}((X)_{112}) \vee \textsc{NotForm}((X)_{1112}) \vee \textsc{NotForm}((X)_{1122}) \vee \\
&\quad\quad (X)_{12} \neq (X)_{11112} \cup (X)_{11212} \vee (X)_{11121} \neq (X)_{1122} \vee (X)_{11122} \neq (X)_{2})] \\
&\quad \vee [X = [[(X)_{11}, (X)_{12}], [\vee, [(X)_{221}, (X)_{222}]]] \,\&\, (\textsc{NotProof}((X)_{11}) \vee \\
&\quad\quad \textsc{NotForm}((X)_{112}) \vee \textsc{NotForm}((X)_{221}) \vee (X)_{12} \neq (X)_{1112} \vee \\
&\quad\quad (X)_{221} \neq (X)_{112} \vee (X)_{221} \neq (X)_{112})] \\
&\quad \vee [X = [[[[(X)_{1111}, [\vee, (X)_{11122}]], [(X)_{1121}, (X)_{1122}]], (X)_{12}], (X)_{2}] \,\& \\
&\quad\quad (\textsc{NotProof}((X)_{111}) \vee \textsc{NotProof}((X)_{1121}) \vee \textsc{NotProof}((X)_{1122}) \\
&\quad\quad \vee \textsc{NotForm}((X)_{1112}) \vee \textsc{NotForm}((X)_{11212}) \vee \textsc{NotForm}((X)_{11222}) \\
&\quad\quad \vee (X)_{12} \neq (X)_{11112} \vee (X)_{112112} \neq (X)_{11112} \cup (X)_{111221} \vee \\
&\quad\quad (X)_{112212} \neq (X)_{11112} \cup (X)_{111222} \vee (X)_{11212} \neq (X)_{2} \vee (X)_{11222} \neq (X)_{2})] \\
&\quad \vee [X = [[[(X)_{111}, (X)_{112}], (X)_{12}], [\leftrightarrow, [(X)_{221}, (X)_{222}]]] \,\&\, (\textsc{NotProof}((X)_{111}) \\
&\quad\quad \vee \textsc{NotProof}((X)_{112}) \vee \textsc{NotForm}((X)_{1112}) \vee \textsc{NotForm}((X)_{1122}) \vee \\
&\quad\quad (X)_{12} \neq (X)_{11112} \setminus (X)_{1122} \vee (X)_{12} \neq (X)_{11112} \setminus (X)_{1112} \vee \\
&\quad\quad (X)_{221} \neq (X)_{1122} \vee (X)_{222} \neq (X)_{1112})] \\
&\quad \vee [X = [[[[(X)_{1111}, [\leftrightarrow, (X)_{11122}]], (X)_{112}], (X)_{12}], (X)_{2}] \,\&\, (\textsc{NotProof}((X)_{111}) \vee \\
&\quad\quad \textsc{NotProof}((X)_{112}) \vee \textsc{NotForm}((X)_{1112}) \vee \textsc{NotForm}((X)_{1122}) \vee \\
&\quad\quad (X)_{12} \neq (X)_{11112} \cup (X)_{11212} \vee (X)_{11121} \neq (X)_{1122} \vee (X)_{2} \neq (X)_{11122})] \\
&\quad \vee [X = [[[[(X)_{1111}, [\leftrightarrow, (X)_{11122}]], (X)_{112}], (X)_{12}], (X)_{2}] \,\&\, (\textsc{NotProof}((X)_{111}) \vee \\
&\quad\quad \textsc{NotProof}((X)_{112}) \vee \textsc{NotForm}((X)_{1112}) \vee \textsc{NotForm}((X)_{1122}) \vee \\
&\quad\quad (X)_{12} \neq (X)_{11112} \cup (X)_{11212} \vee (X)_{11122} \neq (X)_{1122} \vee (X)_{2} \neq (X)_{11121})] \\
&\quad \vee [X = [[[(X)_{111}, [(X)_{1121}, [\neg, (X)_{11222}]]], (X)_{12}], \bot] \,\&\, (\textsc{NotProof}((X)_{111}) \vee \\
&\quad\quad \textsc{NotProof}((X)_{112}) \vee \textsc{NotForm}((X)_{1112}) \vee \textsc{NotForm}((X)_{1122}) \vee \\
&\quad\quad (X)_{12} \neq (X)_{11112} \cup (X)_{11212} \vee (X)_{112} \neq (X)_{11222})]
\end{aligned}
$$

$\vee [X = [[[(X)_{111}, \bot], (X)_{12}], [\neg, (X)_{22}]] \,\&\, (\textsc{NotProof}((X)_{11}) \vee$
$\textsc{NotForm}((X)_{112}) \vee (X)_{12} \neq (X)_{1112} \setminus (X)_{22})]$
$\vee [X = [[[(X)_{111}, \bot], (X)_{12}], (X)_2] \,\&\, (\textsc{NotProof}((X)_{11}) \vee$
$\textsc{NotForm}((X)_{112}) \vee (X)_{12} \neq (X)_{1112} \setminus [\neg, (X)_2])]$
$\vee [X = [[(X)_{11}, (X)_{12}], [[\forall, (X)_{212}], (X)_{22}]] \,\&\, (\textsc{NotProof}((X)_{11}) \vee$
$\textsc{NotSubst}((X)_{112}) \vee \textsc{NotForm}((X)_{1122}) \vee (X)_{12} \neq (X)_{1112} \vee$
$(X)_{212} \neq (X)_{112122} \vee (X)_{22} \neq (X)_{1122} \vee$
$(\exists Y \in (X)_{12})(\forall W)(\textsc{Free}(W) \vee (W)_2 \neq Y))]$
$\vee [X = [[[(X)_{111}, [\forall, (X)_{1122}]], (X)_{12}], (X)_2] \,\&\, (\textsc{NotProof}((X)_{11}) \vee$
$\textsc{NotForm}((X)_{112}) \vee \textsc{NotSubst}((X)_2) \vee (X)_{2122} \neq (X)_{11212} \vee$
$\textsc{NotFreeFor}((X)_{2121}) \vee (X)_{21212} \neq (X)_{1122} \vee$
$(X)_{212112} \neq (X)_{11212} \vee (X)_{12} \neq (X)_{1112})]$
$\vee [X = [[(X)_{11}, (X)_{12}], [[\exists, (X)_{212}], (X)_{22}]] \,\&\, (\textsc{NotProof}((X)_{11}) \vee$
$\textsc{NotSubst}((X)_{112}) \vee \textsc{NotForm}((X)_{1122}) \vee \textsc{NotVar}((X)_{212}) \vee$
$(X)_{12} \neq (X)_{1112} \vee (X)_{22} \neq (X)_{1122})]$
$\vee [X = [[[[(X)_{1111}, [\exists, (X)_{11122}]], (X)_{112}], (X)_{12}], (X)_2] \,\&\, (\textsc{NotProof}((X)_{111}) \vee$
$\textsc{NotProof}((X)_{112}) \vee \textsc{NotForm}((X)_{1112}) \vee \textsc{NotForm}((X)_{1122}) \vee$
$(X)_{11212} \neq (X)_{11112} \cup (X)_{112122} \vee \textsc{NotSubst}((X)_{112122}) \vee$
$(X)_{112122122} \neq (X)_{111212} \vee \textsc{NotVar}((X)_{112122121}) \vee$
$(\exists Y)(\textsc{Free}(Y) \,\&\, (Y)_2 = (X)_{1122}) \vee$
$(\exists W_1 \in (X)_{11112})(\exists W_2)(\textsc{Free}(W_2) \,\&\, (W)_2 = W_1) \vee (X)_{12} \neq (X)_{11112} \vee$
$(X)_2 \neq (X)_{1122})]) \longrightarrow \textsc{NotProof}(X))$

We would then define the set of non-proofs $\mathbb{NP}$ in ZF using the same idea of enumerating all the possible cases in which a set-theoretic object could fail to be a coded proof. The representability argument would then proceed in the same way that the proof for showing the representability of formulae did. We would need to show:

1. $(\forall X)(\textsc{Proof}(X) \longrightarrow \lfloor X \rfloor \in \mathbb{P})$
2. $(\forall X)(\textsc{NotProof}(X) \longrightarrow \lfloor X \rfloor \in \mathbb{NP})$
3. $ZF \vdash (\forall x)(x \in \mathbb{P} \longrightarrow x \notin \mathbb{NP})$
4. $ZF \vdash (\forall x)(x \in \mathbb{NP} \longrightarrow x \notin \mathbb{P})$

(1) is the first representability condition and was shown previously by the metatheoretic principle of induction for proofs. (2) would utilize the metatheoretic principle of induction for non-proofs as defined above. (3) and (4) together show that $\mathbb{P}$ and $\mathbb{NP}$ are disjoint and each are shown by using the induction principles justified by the definitions of the respective sets. (1), together with (3)-(4) establish the desired representability conditions, i.e. they show R_1 and R_2 for the metatheoretic notion of proofs:

- $(\forall X)(\textsc{Proof}(X) \longrightarrow \lfloor X \rfloor \in \mathbb{P})$
- $(\forall X)(\textsc{NotProof}(X) \longrightarrow \lfloor X \rfloor \notin \mathbb{P})$

2.12 Representability of Theorems

The representability arguments shown previously have all been pointing to a single goal, namely to show the representability of the theorem predicate inside of ZF. The undecidability of ZF precludes a complete representation of theorems, so the theorem predicate is only semi-representable, in the sense that we can define what it means to be a theorem and show this notion to be representable inside of ZF, but we cannot define what it means to *not* be a theorem and also show the collection of non-theorems to be reprsentable in ZF.

Theorems are taken to be precisely the statements that are provable in ZF, that is, statements for which there exists a proof. The previous argument established the representability of proofs, where proofs in the metatheory are represented by binary trees whose right sub-tree is the formula that the proof is of. Thus, we can define theorems to be the collection of objects for which there exists a Y in the collection of proofs whose right sub-tree is the formula to be proved. More formally, the metatheoretic definition of theorems is:

$$(\forall X)(\textsc{Theo}(X) \longleftrightarrow (\textsc{Form}(X) \,\&\, (\exists Y)(\textsc{Proof}(Y) \,\&\, (Y)_2 = X)))$$

In the object theory we can then define the set of theorems to be

$$\mathbb{TH} = \{z \in \mathbb{F} \,|\, (\exists y \in \mathbb{P})(y)_2 = z\}$$

The representability is clear. Non-theorems would then be defined using the negation of the existential (but, as indicated above, cannot be shown to be representable in ZF as the collection of theorems is only recursively enumerable).

The theorem predicate is then used in the self-reference lemma to construct an undecidable sentence of ZF. The sentence is of the form

$$G \longleftrightarrow \lfloor G \rfloor \notin \mathbb{TH}$$

We can informally verify that such a sentence would cause some trouble in ZF. If $ZF \vdash G$, then $\lfloor G \rfloor \notin \mathbb{TH}$, therefore $\neg(\exists y \in \mathbb{P})(y)_2 = \lfloor G \rfloor$. But since $ZF \vdash G$, it must be that $(\exists y \in \mathbb{P})(y)_2 = \lfloor G \rfloor$, a contradiction. So consider instead the case in which it is not the case that $ZF \vdash G$. Then $\neg(\exists y \in \mathbb{P})(y)_2 = \lfloor G \rfloor$, so $\lfloor G \rfloor \notin \mathbb{TH}$. From this and the construction of G, we are able to derive G. Thus $ZF \vdash G$, also a contradiction. Thus G is undecidable, and ZF is incomplete.

3 Conclusion

The goal of this project was to provide foundational support to the automated proofs given by AProS. Though the verification was carried out by hand here, we would eventually like the proofs to be verified using AProS. The way in which the proofs were done in this work almost requires the help of a machine, as the more complex metatheoretic definitions generate a huge number of cases to verify and the proofs get very long and tedious, losing all intelligible structure.

Though it is good to see the task of representability in full, the reader should have noticed a certain sense of redundancy in it all. Each of the metatheoretic and object-theoretic definitions had the same basic structure, and each of the representability proofs proceeded in exactly the same way. Thus a more elegant approach would be to formulate a general specification for inductive definitions in the metatheory and provide an argument that all metatheoretic definitions adhering to the general specification are representable. Once this is done, the representability proof itself can be represented in the object-theory, which is the first key step to verifying the derivability conditions of Gödel's second incompleteness theorem. This is an area for further research and is planned to be carried out over the summer of 2011.

AProS is already equipped to give metamathematical proofs, but some additional features will need to be added in order to attempt the representability proofs. There will need to be a slight expansion of syntax to handle a representation of binary tree notation (the $[X, Y]$ notation used here would be most reasonable). It would also need to be able to work with a more robust formal metatheory that includes a series of inductive definitions and induction principles for each. The proof search would then need to be expanded to consider proofs by induction.

The next step is thus to try to formulate a more general description of inductive definitions, and the particular proofs that were carried out here for formulae, proofs, etc. will motivate the requirements that the general specification must meet. Once this is sufficiently developed, AProS will be expanded accordingly so that the representability proof can be done mechanically. The abstract axiomatic-level proofs have already been given in AProS, and with an automated proof of representability, it would then only remain to verify the derivability conditions and the self-reference lemma (that allows the construction of an undecidable sentence). The proof of representability is a large piece of the verification project and brings the prospect of automatically verifying Gödel's incompleteness theorems in AProS closer to realization.

A Index of Lemmata

1. $(\forall Z)(Z = S \vee (\exists X, Y)Z = [X, Y])$: follows directly from the axioms for binary trees.

2. $(\forall x, y)(\langle x, y\rangle = \emptyset \longrightarrow \perp)$: immediate from the definitions of ordered pairs and the empty set.

3. $(\forall Z \neq S)(\exists X, Y)Z = [X, Y]$: consequence of Lemma 1.

4. $(\forall X_1, X_2, Y_1, Y_2)([X_1, Y_1] \neq [X_2, Y_2] \longrightarrow (X_1 \neq X_2 \vee Y_1 \neq Y_2))$: follows from axioms for binary trees.

5. $(\forall x_1, x_2, y_1, y_2)(x_1 \neq x_2 \longrightarrow \langle x_1, y_1\rangle \neq \langle x_2, y_2\rangle)$: follows from FTOP.

6. $(\forall x, y, z, w)((x = \bigcap y \,\&\, z \in x \,\&\, w \in y) \longrightarrow z \in w)$: trivial definitional reasoning.

7. $(\forall x)(x \in \mathbb{NV} \longleftrightarrow (x = \emptyset \vee (\exists y \neq \emptyset)(\exists z)x = \langle y, z\rangle \vee (\exists z \in \mathbb{NV})(z \neq \lfloor\exists\rfloor \,\&\, x = \langle\emptyset, z\rangle)))$: follows from the definition of $\mathbb{NV}$

8.
$$\begin{aligned}
&(\forall w)(w \in \mathbb{NF} \longleftrightarrow w = \emptyset \vee w = \langle\lfloor=\rfloor, \emptyset\rangle \vee w = \langle\lfloor\Box\rfloor, \emptyset\rangle \\
&\quad \vee (\exists x)(\exists y \in \mathbb{NT})(w = \langle\lfloor=\rfloor, \langle x, y\rangle\rangle \vee w = \langle\lfloor=\rfloor, \langle y, x\rangle\rangle) \\
&\quad \vee (\exists x_1, \ldots, x_n)(\exists x \in pred)(\exists y \in \mathbb{NT})(w = \langle x, \langle y, \ldots, \langle x_1, x_n\rangle \cdots\rangle\rangle \\
&\qquad \vee \cdots \vee w = \langle x, \langle x_1, \ldots, \langle x_n, y\rangle \cdots\rangle\rangle) \\
&\quad \vee (\exists x)(\exists y \in \mathbb{NV})w = \langle\langle\lfloor Q\rfloor, y\rangle, x\rangle \\
&\quad \vee (\exists x)(\exists y \in npred)(y \neq \lfloor\neg\rfloor \& y \neq \lfloor\Box\rfloor \& y \neq \lfloor=\rfloor \& (y)_1 \neq \lfloor Q\rfloor \& w = \langle y, x\rangle) \\
&\quad \vee (\exists x \in z)w = \langle\lfloor\neg\rfloor, x\rangle \vee (\exists x)(\exists y \in z)(w = \langle\lfloor\Box\rfloor, \langle x, y\rangle\rangle \\
&\quad \vee w = \langle\lfloor\Box\rfloor, \langle y, x\rangle\rangle) \vee (\exists x)(\exists y \in z)w = \langle\langle\lfloor Q\rfloor, x\rangle, y)
\end{aligned}$$

   : follows immedately from the definition of $\mathbb{NF}$.

9. $(\forall x)(\lfloor\neg\rfloor = x \longrightarrow (\exists y_1)\emptyset = \langle\emptyset, y_1\rangle)$: immediate from coding of negation.

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