On a class of non-linear elliptic boundary value problems

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BOUNDARY VALUE PROBLEMS

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1. Introduction. In [5], a theorem is proved which asserts the existence of a non-trivial solution to the problem

\[ Ay + yF(y^2, x) = 0 \quad \text{in } \Omega, \quad y|_{\partial \Omega} = 0, \]

where \( A \) is the Laplace operator, \( \Omega \) is a bounded region in \( \mathbb{R}^N \) for which the Dirichlet problem is solvable, and \( F \) is a function locally Hölder continuous on \( \mathbb{R}^N \times \mathbb{R} \) satisfying, for some \( \epsilon > 0 \) and all \( x \in \Omega, \)

\[ 0 < n_1 F(r_1 x) \leq F(r_2 x), \quad \text{for } 0 < r_1 \leq r_2 < \infty, \]

and also, for all \( x \in \Omega, \)

\[ F(\eta x) \leq c \eta^y + \text{or}, \quad 0 < \eta < \infty, \]

where \( c, a \) and \( y \) are positive constants, \((N-2)y < 2\). This result is the analogue of a result of [8] concerning a boundary value problem for a non-linear ordinary differential equation. The result of [5] concerning (1) was obtained by treating the integral equation equivalent to (1) by methods similar to those used in [9].

In this note we shall derive from the results of [5] an existence theorem for a boundary value problem of the form

\[ TU = uF(u^2, x) \quad \text{in } \Omega, \quad D^\alpha |^\alpha = 0, \quad |\alpha| < m-1, \]
where $r$ is an elliptic operator of order $2m$, \[ D = \frac{\partial^2 |\alpha|}{\partial \alpha_1 \ldots \partial \alpha_N}, \]

$|a| = a_1 + \ldots + o^\infty$; $m \geq 1$, $N \geq 1$. The result obtained here was suggested by the main theorem of Berger's study, [4], of a non-linear elliptic eigenvalue problem.
2. The differential operator. We shall assume throughout this section that $Q$, is a region of class $C^{2m}$ (see def. 9.2, [3]) and that the differential operator $r$, given in the form

$$\tau = \sum |\alpha| |\beta| \leq m \alpha \beta \alpha \beta (x) D^\alpha \beta,$$

has real coefficients satisfying

$$a_{\alpha \beta} (x) \in C^{m} (Q), \quad \text{all } \alpha, \beta.$$

In addition we assume that $T$ is uniformly strongly elliptic in $C_1$ and that there exists a positive constant $c_0$ such that

$$B [\varphi, \varphi] \geq c_0 \| \varphi \|^2, \quad \text{all } \varphi \in C^{2m} (Q),$$

where

$$B [\varphi, \varphi] = \int_0^1 \sum |\alpha| |\beta| \leq m \alpha \beta \alpha \beta \varphi (x) D^\alpha \beta \varphi (x) dx.$$

We shall use standard notation. For $r > 1$, $\| \ldots \|_m, r$ is the Sobolev norm defined as follows,

$$\| \phi \|_{m, r} = \left( \int_\Omega |\phi (x)|^r dx \right)^{1/r}$$

for $\phi$ having strong $L^r$-derivatives of order up to $m$ in $Q$; $W^{m, r} (Q)$ is the space of all such functions (because of the smoothness assumption concerning $C_1$ this is equivalent to the more usual definition of $W^{3n+1} (a)$); $W^{m, r} (\Omega)$ is normed by $\| \ldots \|_{m, r}$.

Finally, $W^{m, r} (Q)$ is the closure of $C_0^0 (Q)$ in $W^{m, r} (\Omega)$.

Without exception the function space considered here will be
understood to consist of real valued functions.

By a standard result, see Theorem 8.2, [3], the generalized Dirichlet problem with zero boundary data

\[ B[p,u] = (p,f), \quad \text{all } p \in C^0(0), \]

has a unique solution \( u \in V^q(0) \) for each \( f \in L^q(0) \). Actually the same is true for \( f \in L^q(0) \) provided

\[ q > q_0 = \max(1,2N/(N+2m)), \quad q < 2. \]

This follows from the fact that, because of the Sobolev imbedding theorem, \( W^{1,2} \) is stronger than \( L^p(0) \) when \( \frac{1}{p} + \frac{1}{q} = 1 \) and (8) holds. Thus for \( f \in L^q(0) \), \( p \in V^{q^2}(0) \),

\[ (9) \quad |(p,f)| \leq \|f\|_{L^q(0)} \|f\|_{W^{1,2}(0)} \leq \text{const}. \]

The same proof as in the case where \( f \in L \) then works for \( f \in L^q \).

We define an operator \( A \), whose domain is \( U > L^q(0) \) and whose range is contained in \( V^q(0) \), by

\[ B[p,Af] = ((0,f), \quad \text{all } p \in V^{q^2}(0), \]

upon taking \( p = Af \) in (10) it follows from (7) and (9) that \( A \) acts as a bounded operator from \( L^q(0) \) to \( W^{1,2}(0) \) for each \( q > q_0 \).

We shall require the following results from [1] and [2].

(*) If \( f \in L^r(0), \ r > q_0 \) then \( A f \in W^{2m,r}(0) \), and there exists a constant \( k^r \) such that

\[ (11) \quad \|Af\|_{2m,r} \leq k^r \|f\|_r + H^{1,f}_r \|f\|_r. \]
(**) II. \( f \in L^0 (Q) \) and \( u = Af \), then (after modification on a set of measure zero) \( u \in C^{2m} (O) \) and \( u \) satisfies the boundary conditions

\[
D^a u = 0, \quad \text{on } \partial Q, \quad |a| < m.
\]

(***) if \( f \in C^\infty (n) \) and if

\[
\text{for some } \eta: 0 < \eta < 1, \text{ then } u = A f \in C^m (O) \text{ and } u \text{ satisfies } (12).
\]

\( C^{\infty} (f2) \) is the space consisting of those functions in \( C^k (O) \) whose \( k \)-th order derivatives are uniformly Holder continuous of order \( \eta \) in \( Q \).

The first assertion^\(^(*)\), follows from Theorem 8.2, [2]; the proof of (**) is also in [2]; (***) follows from Theorem A5.1, Appendix 5, [1]. Although we do not use the fact here it is interesting to note that the assertion (**) is equivalent to the assertion that for \( r > q \) \( 0 \), a function \( u \) belongs to \( W^{2m, r} (O) \) if and only if it is the limit in \( W^{2m, r} (O) \) of a sequence \( (u_n) \), where, for each \( n \), \( u_n \in C^{2m, 1} (O) \) and \( u = u_n \) satisfies (12).

If \( r > 0 \) then \( TU \) can be defined for \( u \in W^{2m, r} (O) \), if \( r > q \) \( 0 \) then \( W^{2m, r} \) can be imbedded in \( W^{m, \infty} \), we shall denote by \( A \) for \( r > q \) \( 0 \) the operator in \( L^2 (C) \) whose domain is \( \mathcal{M}^r = W^{2m, r} (O) \) and which sends \( u \) - \( TU \). Since

\[
B(\varphi, \psi) = (\omega, \tau \psi) = \langle \varphi, \psi \rangle, \quad \text{for } \varphi \in C^\infty (O), \psi \in \mathcal{M}^r,
\]

1 Actually this is not so unless (6) is strengthened. An adaptation of the arguments in [2] to the case where the operator is given in divergence form gives a result implying (*) under condition (6).
it follows that

\[ A^r u = u, \text{ for } u \in M. \]  

On the other hand, by (*) , \( A \) maps \( L^r(Cl) \) into \( M^r \) so

\[ B[p, A^r] = ((p, *^r Af), \text{ for } p \in C^0_{(d)}, f \in L^r(Q), \]

thus

\[ *^r Af = f, \text{ for } f \in L^r(Q). \]

We put \( A_r = A|L^r(O) \).

**Lemma 1.** For each \( r > q \),

\[ A_r : L^r(O) \rightarrow M^r \]

is a bijection. Moreover there are positive constants \( c_r \) such that for \( f \in L^r(O) \)

\[ \|f\|_{L^r} \leq c_r \|f\|_{W^{2mr}(a)} \]  

**Proof.** We have shown that \( A_r \) and \( A_r \) are inverses of one another. It readily follows from (6) that

\[ \|f\|_{L^r} \leq \text{const}, \|u\|_{L^s} \leq c_r \|u\|_{L^r} \]

which implies the second inequality of (16). \( M \) is a subspace of \( W^{2mr}(a) \) so the first inequality of (16) follows from the open mapping theorem.

**Lemma 2.** Let \( r > q \). If \( 2mr < N \) then \( A \) maps \( L^r(O) \) compactly into \( L^s(Q) \) for

\[ 1 < s < \frac{Nr}{N-2mr}; \]

\[ \text{OrK'112}. \]
if $2mr > N$ then $A$ can be regarded as a compact mapping of $L^r(0)$ into $C(JT)$.

Proof. Let $q_0 < r$, then by Lemma 1 $A$ maps $L^r(Cl)$ into $W^{2m,r}(0)$. If $2mr < N$ then, by Sobolev's theorem, $W^{2m,r}(Q)$ can be imbedded compactly in $L^s(Cl)$ for any $s$ satisfying (17). If $2mr > N$ then $W^{2m,r}(f2)$ can be imbedded compactly in $C(\Omega)$.

The following lemma will simplify matters by making the results of [5] applicable, as they stand, to the problem considered here.

Lemma 3. Let $a = N/(N-2m)$ or let $a^* = OD$ according as $N > 2m$ or $N < 2m$. There exists a measurable function $G(x,t)$ on $0 \times 0$ such that the mapping

$$ x - G(x, \cdot) $$

is uniformly continuous from $0$ to $L^a(0)$ for $1 < a < a_0$, and

$$ \text{ess sup} \int_{x \in \Omega} G(x,t) \frac{a}{s} dt < \infty, \quad \text{ess sup} \int_{t \in \Omega} |G(x,t)| \frac{a}{r} dx < \infty, $$

for $1 < a < a_0$. For $f \in L^r(0)$, $r > q_0$,

$$ [Af](x) = \int_{\Omega} (G(K,t)f(t))dt, \quad \text{a.e. in } 0. $$

Proof. Let $2mr > N$, so that $A$ can be regarded as a map of $L^r(Q)$ into $C(C2)$. By a well known representation theorem, (Theorem VI. 7.1, [6]) there is a continuous map $x - \cdot G(x, \cdot)$ of $SI$ into $L^a(Q)$, where $\frac{1}{r} + \frac{1}{a} = 1$, such that (19) holds everywhere.
in \( Q \) for \( f \in L^r(f) \). It is easily seen that the function \( G(x,t) \) is independent of the particular choice of \( r \). Let

\[
(20) \quad r^* = \sum |\beta| \cdot D^\beta a \cdot g(x) \cdot D^\alpha,
\]

(notice the symmetry of (6)), and let \( A^* \) be defined by

\[
(21) \quad B[A^*f,0] = (\xi, \psi), \quad \psi \in \mathcal{E}'(\Omega), \quad \xi \in L^q(\Omega), \quad q > q_0.
\]

It follows that for \( f \in L^q(f) \), \( q > q_0 \), we have

\[
(22) \quad (A^*g,f) = (\xi, \Phi)\cdot (\xi, A_f).
\]

Because of the symmetry of (6), \( A^* \) has an integral representation analogous to (19) for \( f \in L^r(f) \), \( 2mr > N \); let \( G^*(x,t) \) denote the corresponding kernel. It follows then from (22) that

\[
\int_{\Omega} G^*(t,x) f(x) g(t) \, dx \, dt = \int_{\Omega} G(x,t) f(x) g(t) \, dx \, dt,
\]

for \( f, g \in L^r(\Omega) \), \( 2mr > N \). Thus we have

\[
G^*(t,x) = G(x,t), \quad \text{a.e. in } Q \times O,
\]

and from the uniform continuity of \( x \mapsto G(x,*) \) and \( t \mapsto G^*(t,*) \) as mappings from \( Q \) to \( L^a(Q) \), \( 1 < a < a_0 \), follow the inequalities (18).

It remains to show that (19) is valid for \( f \in L^r(f) \) when \( r > q_0 \), and \( 2mr \leq N \). In this case however it follows from (18) and Theorem 9.5.6, [7], that the right hand side of (19) defines a compact mapping from \( L^r(f_2) \) to \( L^{q>(Q)} \) for

\[
1 < s < \frac{Nr}{N-2mr}.
\]
Thus since (19) is valid for $f$ in a dense subset of $L^r(Q)$, (namely for $f \in L^1(Q), 2m \geq N$), it follows from Lemma 2 that it is valid for $f \in L^r(Q)$. 

3, The non-linear problem. Let $O$ and $r$ be as in Section 2. The main result of this note is the following.

**Theorem.** Suppose that (13) holds, for some positive $\mu$ less than 1 and that $F$ is uniformly Hölder continuous on $\mathbb{R} \times Q$. Suppose also that $F$ satisfies (2) for some $e > 0$ and that (3) holds, for all $x \in I$, with positive constants $C, \alpha$ and $\gamma$ where

\begin{equation}
0 < \gamma, \quad \gamma(N-2m) < 2m.
\end{equation}

Finally assume that $r$ is formally self-adjoint. Then there exists a function $u \in C^{2}(O; \mathbb{R})$ which is not identically zero and satisfies (4) (in the ordinary sense).

**Proof.** We consider the operator equation

\begin{equation}
u = AuF(u^2, x),
\end{equation}

where $A$ has the same meaning as in Section 2. By Lemma 3 this is equivalent to an integral equation

\begin{equation}
u(x) = \int_{O} G(x, t) u(t) F(u^2(t), t) dt,
\end{equation}

where, since $r$ is formally self-adjoint, $G(x, t)$ is symmetric; (18) holds for $1 < a$, $a(N-2m) < N$. It readily follows from (7) that $A$, regarded as an operator in $L^p(O)$, is positive definite; the range of $A$ contains $C^\infty_0(O)$ and is therefore dense in $L^p(O)$ for any $p > 1$. Now from Theorems 1 and 3 of [5] it follows that (25) has a non-trivial essentially bounded solution $u$; see also the remarks following the statement of Theorem 2 of [5]. From the equivalence of (24) and (25) it follows that $u$
satisfies (24). By (3), \( u(x)F(u^2(x), x) \) is essentially bounded and thus, by (**) \( ueC^{2,1/1}([0]) \) and \( u \) satisfies (12). From the differentiability of \( u \) and the hypothesis concerning \( F \) it follows that \( u(x)F(u^2(x), x) \) is uniformly Hölder continuous in \( Q \). Finally by (***) we conclude that \( ueC^2(0) \) and is an ordinary solution of (4). This completes the proof.
References


