Balanced 0, ±1 Matrices Part II. Recognition Algorithm

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Published In
Journal of Combinatorial Theory, Series B, 81, 2, 257-305.
Balanced $0, \pm 1$ Matrices
Part II: Recognition Algorithm

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revised September 2000

Abstract

In this paper we give a polynomial time recognition algorithm for balanced $0, \pm 1$ matrices. This algorithm is based on a decomposition theorem proved in a companion paper.

Keywords: balanced matrix, decomposition, recognition algorithm, 2-join, 6-join, extended star cutset

Running head: Recognition of balanced $0, \pm 1$ matrices

1 Introduction

A $0, \pm 1$ matrix is balanced if, in every square submatrix with two nonzero entries per row and column, the sum of the entries is a multiple of four. In [3], Conforti, Cornuéjols and Rao prove a decomposition theorem for balanced $0, 1$ matrices and they use it to obtain a polynomial time recognition algorithm for these matrices. In this paper, using a similar approach, we give a polynomial time recognition algorithm for balanced $0, \pm 1$ matrices, using a decomposition result derived in the companion paper [1]. For a survey of results on balanced matrices, see [2].

A convenient setting for working with balanced $0, \pm 1$ matrices is to consider their signed bipartite graph representations. A signed graph $G$ is a graph together with an assignment of $+1$ or $-1$ weights to the edges. Given a $0, \pm 1$ matrix $A$, the signed bipartite graph representation of $A$ is a signed bipartite graph $G$, with the two sides of the bipartition $V^r$ and $V^c$.

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This work was supported in part by NSF grants DMI-9802773, DMS-9509581 and ONR grant N00014-97-1-0196.
representing respectively the rows and columns of $A$, and for each nonzero entry $a_{ij}$ of $A$, there is an edge between nodes $i \in V^r$ and $j \in V^c$ with weight $a_{ij}$.

A signed bipartite graph $G$ is \textit{balanced} if it is the signed bipartite graph representation of a balanced 0,±1 matrix. Thus a signed bipartite graph $G$ is balanced if and only if for every hole $H$ of $G$, the sum of the weights of the edges of $H$ is a multiple of 4. A \textit{hole} in a bipartite graph is a chordless cycle. A hole is \textit{balanced} if it is of weight 0 modulo 4, and it is \textit{unbalanced} if it is of weight 2 modulo 4. A graph $G$ \textit{contains} a graph $H$, if $H$ is an induced subgraph of $G$. So, a signed bipartite graph is balanced if and only if it does not contain an unbalanced hole.

In this paper we construct a recognition algorithm that takes as input a signed bipartite graph $G$, and outputs YES if $G$ is balanced, and NO otherwise. The algorithm runs in time polynomial in the size of $V^r$ and $V^c$. This algorithm can be used to obtain a polynomial time algorithm for finding an unbalanced hole in a graph that contains one, in the following way.

\textbf{If} Recognition$(G)$=YES, \textbf{return} "$G$ is balanced".

\textbf{Else} set $H = G$.

\textbf{While} there exists some node $v$ in $H$ such that Recognition$(H \setminus \{v\})$ = NO, set $H = H \setminus \{v\}$.

\textbf{Return} "$H$ is an unbalanced hole of $G$".

As mentioned above, the recognition algorithm is based on a decomposition theorem, which we state in Section 1.1. The organization of the paper is described in Section 1.2.

\section{1.1 Decomposition Theorem}

A set $S$ of nodes (respectively edges) of a connected graph $G$ is a \textit{node cutset} (respectively an \textit{edge cutset}) if the subgraph $G \setminus S$, obtained from $G$ by removing the nodes (respectively edges) in $S$, is disconnected.

A \textit{biclique} is a complete bipartite graph $K_{AB}$ where the two sides of the bipartition $A$ and $B$ are both nonempty.

\textbf{Extended Star Cutset}

For a node $x$, let $N(x)$ denote the set of all neighbors of $x$. In a bipartite graph $G$, an \textit{extended star} $(x;X;Y;R)$ consists of disjoint subsets $X,Y,R$ of $V(G)$ and a node $x \in X$ such that

(i) $Y \cup R \subseteq N(x)$,

(ii) the node set $X \cup Y$ induces a biclique (with node set $X$ on one side of the bipartition and node set $Y$ on the other),

(iii) if $|X| \geq 2$, then $|Y| \geq 2$. 

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In a connected bipartite graph, an extended star cutset is an extended star \((x; X; Y; R)\) where \(X \cup Y \cup R\) is a node cutset. When \(R = \emptyset\) the extended star is a biclique, and the cutset is called a biclique cutset. When \(|X| = 1\) then the extended star cutset is also called a star cutset.

2-Join

Let \(G\) be a connected bipartite graph with more than four nodes, containing bicliques \(K_{A_1,A_2}\) and \(K_{B_1,B_2}\), where \(A_1, A_2, B_1, B_2\) are disjoint nonempty node sets. The edge set \(E(K_{A_1,A_2}) \cup E(K_{B_1,B_2})\) is a 2-join if it satisfies the following properties:

(i) The graph \(G' = G \setminus (E(K_{A_1,A_2}) \cup E(K_{B_1,B_2}))\) is disconnected.

(ii) Every connected component of \(G'\) has a nonempty intersection with exactly two of the sets \(A_1, A_2, B_1, B_2\) and these two sets are either \(A_1\) and \(B_1\) or \(A_2\) and \(B_2\). For \(i = 1, 2\), let \(G'_i\) be the subgraph of \(G'\) containing all its connected components that have nonempty intersection with \(A_i\) and \(B_i\).

(iii) If \(|A_1| = |B_1| = 1\), then \(G'_1\) is not a chordless path or \(A_2 \cup B_2\) induces a biclique. If \(|A_2| = |B_2| = 1\), then \(G'_2\) is not a chordless path or \(A_1 \cup B_1\) induces a biclique.

The purpose of Property (iii) is to exclude "improper" 2-joins.

6-Join

In a connected bipartite graph \(G\), let \(A_i\), \(i = 1, \ldots, 6\) be disjoint, nonempty node sets such that, for each \(i\), every node in \(A_i\) is adjacent to every node in \(A_{i-1} \cup A_{i+1}\) (indices are taken modulo 6), and these are the only edges in the subgraph \(A\) induced by the node set \(\bigcup_{i=1}^{6} A_i\). (Note that, for convenience of notation, the modulo 6 function is assumed to return values between 1 and 6, instead of the usual 0 to 5). The edge set \(E(A)\) is a 6-join if

(i) The graph \(G' = G \setminus E(A)\) is disconnected.

(ii) The nodes of \(G\) can be partitioned into \(V_1\) and \(V_2\) so that \(A_1 \cup A_3 \cup A_5 \subseteq V_1\), \(A_2 \cup A_4 \cup A_6 \subseteq V_2\) and the only adjacencies between the nodes of \(V_1\) and \(V_2\) are the edges of \(E(A)\).

(iii) \(|V_i| \geq 4\) for \(i = 1, 2\).

When the graph \(G\) comprises more than one connected component, we say that \(G\) has a 2-join, a 6-join or an extended star cutset if at least one of its connected components does.

Basic Classes of Graphs

A signed bipartite graph is strongly balanced if it is balanced and contains no cycle with exactly one chord. The recognition problem for this class of graphs is polynomial (Conforti and Rao [5]). \(R_{10}\) is the bipartite graph defined by the cycle \(x_1, \ldots, x_{10}, x_1\) of length 10 with chords \(x_i; x_{i+5}\), \(1 \leq i \leq 5\) (indices taken modulo 10). \(R_{10}\) can be signed to be balanced, say with weight +1 on the edges of the cycle \(x_1, \ldots, x_{10}, x_1\) and −1 on the chords.

In [1] we prove the following decomposition theorem.

**Theorem 1.1** A signed bipartite graph that is balanced but not strongly balanced is either \(R_{10}\) with proper signing or it contains a 2-join, a 6-join or an extended star cutset.
1.2 Organization of the Paper

The general idea of our recognition algorithm for balanced signed bipartite graphs is as follows. Let \( G \) be a signed bipartite graph. If \( G \) is strongly balanced or the underlying graph is \( R_{10} \), then we are done. Else, we search for one of the three cutsets described above. If none exists, \( G \) is not balanced as a consequence of Theorem 1.1. If one exists, its removal disconnects \( G \) into several connected components. From these components, we construct blocks by adding some new nodes and edges with some signing. In other words, we decompose \( G \) into these blocks. Ideally, the blocks should be constructed so that \( G \) is balanced if and only if all the blocks are. Let \( B \) stand for the class of signed bipartite graphs that are balanced. We say that a decomposition is \( B \)-preserving if it satisfies the following: \( G \) belongs to \( B \) if and only if all the blocks of the decomposition belong to \( B \). The three decompositions are then applied recursively to the blocks until no cutset can be found. We show that only a polynomial number of such basic blocks are generated. For each, we check whether it is \( R_{10} \) or strongly balanced. \( G \) is balanced if and only if all basic blocks are balanced (assuming all decompositions are \( B \)-preserving).

In Section 2, we show how to construct blocks that are \( B \)-preserving for the 2-join and the 6-join decompositions. In Section 3, we deal with the node cutset decomposition. For the extended star cutset, we are not able to construct blocks to be \( B \)-preserving. Instead, in our recognition algorithm we first apply a certain cleaning procedure to the input graph \( G \), which transforms it into a graph \( G' \) with the property that \( G' \) is balanced if and only if \( G \) is and, if \( G \) contains an unbalanced hole then \( G' \) contains an unbalanced hole that will either never be broken by extended star cutset decompositions or it will be detected while performing the decomposition. To construct such a procedure we need to study signed bipartite graphs that do contain unbalanced holes. In Section 3.2, we obtain certain properties of a smallest unbalanced hole which allow us to construct the cleaning procedure in Section 4.1. In Section 4, we present the recognition algorithm for signed bipartite graphs that are balanced, and prove its validity and polynomiality.

2 Edge Cutset Decompositions

Throughout the rest of the paper, we assume that \( G \) is a signed bipartite graph.

By scaling \( G \) at node \( u \), we mean changing the sign of the weights on all the edges incident with \( u \).

Remark 2.1 Let \( G' \) be a signed bipartite graph obtained from \( G \) by scaling at node \( u \). A hole is balanced in \( G' \) if and only if it is balanced in \( G \).

Let \( u, v \) be two nonadjacent nodes of \( G \) in opposite sides of the bipartition. A 3-path configuration connecting \( u \) and \( v \), denoted by \( 3PC(u, v) \), is defined by three chordless paths \( P_1, P_2 \) and \( P_3 \) with endnodes \( u \) and \( v \), such that the node set \( V(P_i) \cup V(P_j) \), \( i, j \in \{1, 2, 3\} \), \( i \neq j \), induces a hole. Since paths \( P_1, P_2 \) and \( P_3 \) of a 3-path configuration are of length 1 or 3 modulo 4, the sum of the weights of the edges in each path is also 1 or 3 modulo 4. It follows that two of the three paths induce a hole of weight 2 modulo 4. So a signed bipartite graph that contains a 3-path configuration is not balanced.
A wheel, denoted by \((H, x)\), is defined by a hole \(H\) and a node \(x \not\in V(H)\) which has at least three neighbors in \(H\), say \(x_1, \ldots, x_n\). The wheel \((H, x)\) is even if \(n\) is even and it is odd otherwise. An edge \(xx_i\) is a spoke. A subpath of \(H\) connecting \(x_i\) and \(x_j\) is called a sector if it contains no intermediate node \(x_k\), \(1 \leq i < j \leq n\). Consider a wheel \((H, x)\) which is signed to be balanced. By Remark 2.1, we can assume that all spokes of the wheel are signed \(+1\). This implies that the sum of the weights of the edges in each sector is \(2\) modulo \(4\). Hence if \((H, x)\) is an odd wheel, the hole \(H\) has weight \(2\) modulo \(4\). So a signed bipartite graph that contains an odd wheel is not balanced.

2.1 2-Join Decomposition

A 2-join \(E(K_{A_1, A_2}) \cup E(K_{B_1, B_2})\) is rigid if \(A_1 \cup B_1\) or \(A_2 \cup B_2\) induces a biclique. The following easy result was proved in [3].

**Lemma 2.2** Let \(G\) be a bipartite graph that has no extended star cutset. Then \(G\) has no rigid 2-join.

Let \(K_{A_1, A_2}\) and \(K_{B_1, B_2}\) define a 2-join of \(G\) that is not rigid. The blocks \(G_1\) and \(G_2\) of the 2-join decomposition are defined as follows. For \(i = 1, 2\), let \(G_i^t\) be the subgraph of \(G \setminus (E(K_{A_1, A_2}) \cup E(K_{B_1, B_2}))\) containing all its connected components that have nonempty intersection with \(A_i\) and \(B_i\). To obtain \(G_i\), we first add to \(G_i^t\) a node \(a_i\), adjacent to all the nodes in \(A_i\) and to no other node of \(G_i^t\) and a node \(b_i\), adjacent to all the nodes in \(B_i\) and to no other node of \(G_i^t\). Let \(Q_1\) be a path in \(G_2^t\) with smallest number of edges connecting a node in \(A_2\) to a node in \(B_2\), and let \(Q_2\) be a path in \(G_1^t\) with smallest number of edges connecting a node in \(A_1\) to a node in \(B_1\). Note that the existence of \(Q_1, Q_2\) is guaranteed by (ii) in the definition of 2-joins. For \(i = 1, 2\), add to \(G_i\) a marker path \(M_i\) connecting \(a_i\) and \(b_i\) with length \(4 \leq |E(M_i)| \leq 5\) and edge weights \(+1\) or \(-1\) chosen so that the weight of \(M_i\) is congruent to the weight of \(Q_i\) modulo \(4\).

**Theorem 2.3** Let \(G_1\) and \(G_2\) be the blocks of the decomposition of the signed bipartite graph \(G\) by a 2-join \(E(K_{A_1, A_2}) \cup E(K_{B_1, B_2})\) that is not rigid. If \(G\) does not contain an unbalanced hole of length \(4\), then \(G\) is balanced if and only if both \(G_1\) and \(G_2\) are balanced.

The following lemma is used in the proof of Theorem 2.3.

**Lemma 2.4** Let \(G\) be a signed bipartite graph with no unbalanced hole of length four. For every biclique \(K_{BD}\) in \(G\), we can scale \(G\) on the nodes in \(B \cup D\) so that every edge in \(E(K_{BD})\) has weight \(+1\).

**Proof:** If \(|B| = 1\) then we can scale on nodes in \(D\) to obtain the result. Similarly, for \(|D| = 1\).

We can assume \(|B| \geq 2\) and \(|D| \geq 2\). Let \(b \in B\) and \(d \in D\). Scale at nodes \(d' \in D\) so that all edges \(bd'\) have weight \(+1\). Scale at nodes \(b' \in B\) so that all edges \(b'd\) have weight \(+1\). Every \(d' \in D \setminus \{d\}\) and \(b' \in B \setminus \{b\}\) induce a hole \(b, d, b', d', b\) of length four. By assumption this hole is balanced. Hence \(b'd'\) must have weight \(+1\). \(\Box\)
Remark 2.5 Let $G$ be a signed bipartite graph with no unbalanced hole of length 4. By Lemma 2.4 there exists a signed graph $G'$, which is obtained from $G$ by a sequence of scalings, such that all the edges in $E(K_{A_1,A_2}) \cup E(K_{B_1,B_2})$ have weight +1, since $K_{A_1,A_2}$ and $K_{B_1,B_2}$ are node disjoint.

Proof of Theorem 2.3: By Remark 2.5 we can assume that all the edges in $E(K_{A_1,A_2})$ and $E(K_{B_1,B_2})$ have weight +1. First we show that $G_1$ and $G_2$ are balanced if $G$ is balanced. Every hole $H$ in $G_1$ corresponds to a hole $H'$ in $G$, except for the case where $H$ contains nodes $\alpha_1$ and $\beta_1$ and no other nodes of $M_1$, and $A_2 \cup B_2$ is a biclique in $G$. The existence of such a biclique would contradict our assumption that $E(K_{A_1,A_2}) \cup E(K_{B_1,B_2})$ is a 2-join that is not rigid. The hole $H'$ has the same weight as $H$, since all the edges of $E(K_{A_1,A_2}) \cup E(K_{B_1,B_2})$ are signed positive. Thus $G_1$ is balanced if $G$ is balanced. Similarly for $G_2$.

Now assume that $G_1$ and $G_2$ are balanced, but $G$ is not. Let $H$ be an unbalanced hole of $G$. It contains no edge of $G'_2$, there exists a hole in $G_1$ which is unbalanced. The same argument holds for $G_1'$. So $H$ must contain both an edge of $G_1'$ and an edge of $G_2'$. Hence $H$ must contain an edge of $E(K_{A_1,A_2}) \cup E(K_{B_1,B_2})$, say an edge $a_1a_2$ where $a_1 \in A_1$ and $a_2 \in A_2$. Since $H$ is a hole it cannot contain any node of $K_{A_1,A_2} \setminus \{a_1,a_2\}$. So $H$ must also contain an edge $b_1b_2$ where $b_1 \in B_1$ and $b_2 \in B_2$, and similarly $H$ cannot contain any node of $K_{B_1,B_2} \setminus \{b_1,b_2\}$. So $H = a_1, a_2, P_2, b_2, b_1, P_1, a_1$ where $P_2$ is a path in $G_2'$ from $a_2$ to $b_2$ having no intermediate nodes in $A_2 \cup B_2$, and $P_1$ is a path in $G_1'$ from $b_1$ to $a_1$ having no intermediate nodes in $A_1 \cup B_1$. Since the hole $a_1, a_1, M_1, b_1, b_1, a_1$ is balanced in $G_1$, $w(P_2)$ and $w(M_1)$ are not congruent modulo 4. But by definition of a block, there exists a path $Q_2$ in $G_2'$ from $d'_2 \in A_2$ to $b_2 \in B_2$, such that $w(Q_2)$ is congruent to $w(M_1)$ modulo 4. The holes $H_1 = d'_2, Q_2, b'_2, b_2, M_2, a_2, d'_2$ and $H_2 = a_2, P_2, b_2, b_2, M_2, a_2, a_2$ in $G_2$ have distinct weights modulo 4. Hence one of them must be unbalanced, contradicting our assumption. $\square$

2.2 6-Join Decomposition

Let $G$ be a signed bipartite graph that has a 6-join $E(A)$. Blocks $G_1$ and $G_2$ of a 6-join decomposition are constructed as follows. For $i = 1,\ldots, 6$ let $a_i$ be any node of $A_i$, $G_1$ is a subgraph of $G$ induced by the node set $V_1 \cup \{a_2, a_4, a_6\}$ and $G_2$ is a subgraph of $G$ induced by the node set $V_2 \cup \{a_1, a_3, a_5\}$.

Theorem 2.6 Let $G_1$ and $G_2$ be the blocks of the decomposition of the signed bipartite graph $G$ by a 6-join $E(A)$. If $G$ does not contain an unbalanced hole of length 4 or 6, then $G$ is balanced if and only if both $G_1$ and $G_2$ are balanced.

We first prove the following lemma.

Lemma 2.7 If $A$ does not contain an unbalanced hole of length 4 or 6, then there exists a signing of $G$ which is obtained by a sequence of scalings on the nodes of $A$, such that for every biclique $K_{A_{i_1}A_{i_2}}$, $i_1, i_2 \in \{1,\ldots, 6\}$ (where indices are taken modulo 6) the edges in the biclique are all signed +1 or they are all signed -1.

Proof: By Lemma 2.4 we can sign all the edges in $E(K_{A_1,A_2})$, $E(K_{A_3,A_4})$ and $E(K_{A_5,A_6})$ to be +1. W.l.o.g. let $E(K_{A_2,A_3})$ contain an edge signed +1 and another signed -1. Now there
exist in \( A \) two holes of length 6 which differ in weight by 2. Clearly one of these must be unbalanced contradicting our assumption that \( A \) contains no unbalanced hole of length 6. \( \Box \)

Proof of Theorem 2.6: It follows from the definition of the blocks that \( G_1 \) and \( G_2 \) are induced subgraphs of \( G \) and so are balanced if \( G \) is balanced.

To prove the converse assume that \( G_1 \) and \( G_2 \) are balanced, but \( G \) contains an unbalanced hole \( H \). By Lemma 2.7 we may assume that for every biclique \( K_{A_i,A_{i+1}}, i \in \{1, \ldots, 6\} \), the edges of the biclique are all signed +1 or they are all signed −1. So \( H \) must contain an edge with both ends in \( V_2 \), since otherwise there exists a hole in \( G_1 \) which is unbalanced. Similarly \( H \) must also contain an edge with both ends in \( V_1 \). Since \( H \) is a hole it must have exactly 4 nodes in common with \( V(A) \). Then w.l.o.g. \( H = a_1^0, P_1, a_5^0, a_1^1, P_2, a_5^1, a_1^2 \) where \( a_1^0 \in A_1, a_5^0 \in A_2, a_4^0 \in A_3, a_4^1 \in A_4, a_5^1 \in A_5 \), \( P_1 \) is a path with nodes in \( V_1 \) that connects \( a_1^0 \) to \( a_5^0 \), and \( P_2 \) is a path with nodes in \( V_2 \) that connects \( a_1^1 \) to \( a_1^2 \). The hole \( H_1 = a_1^0, P_1, a_5^1, a_5^0, a_1^2 \) is a hole of \( G_1 \) and \( H_2 = P_2, a_4^0, a_3^1, a_4^1 \) is a hole of \( G_2 \). Since \( G_1 \) and \( G_2 \) are balanced, both \( H_1 \) and \( H_2 \) are balanced. Also \( H' = a_1^0, a_5^0, a_3^1, a_4^1, a_4^0, a_5^1, a_5^2 \) is a hole of \( G \) (in particular) and by the construction of blocks the edges \( a_1^0a_5^0 \) and \( a_5^0a_3^1 \) (resp. \( a_4^0a_3^1 \) and \( a_5^0a_3^1 \)) are signed in \( G \) the same as the corresponding edges \( a_1^0a_5^0 \) and \( a_5^0a_5^2 \) (resp. \( a_4^0a_3^1 \) and \( a_5^0a_3^1 \)) are signed in \( G_1 \) (resp. \( G_2 \)). So \( w(H') \equiv (w(H) + w(H_1) + w(H_2)) \pmod{4} \). Since \( H \) is unbalanced and \( H_1 \) and \( H_2 \) are balanced, this implies that \( w(H') \equiv 2 \pmod{4} \), and hence \( H' \) is an unbalanced hole of \( A \), contradicting the assumption that \( G \) does not contain an unbalanced hole of length 6. \( \Box \)

3 Node Cutset Decompositions

Let \( S \) be a node cutset in a signed bipartite graph \( G \), and let \( C_1, \ldots, C_k \) be the connected components of \( G \setminus S \). We define the blocks of decomposition to be signed bipartite graphs \( G_1, \ldots, G_k \), where each \( G_i \) is a subgraph of \( G \) induced by the node set \( V(C_i) \cup S \).

With this definition of blocks, the decomposition by an extended star cutset is not \( B \)-preserving. For example, consider an odd wheel \( (H, x) \) in which all the spokes have weight +1, and the sectors are of weight 2 modulo 4. Then the wheel is not balanced, since \( H \) is an unbalanced hole, but all the blocks of decomposition by a star cutset \( N(x) \cup \{x\} \) are balanced.

In the next section we define a notion of a clean unbalanced hole and show that either some such hole is not broken by the node cutset decompositions we use in the recognition algorithm, or an unbalanced hole is detected while performing the decomposition.

To ensure that we end up with a polynomial number of blocks, instead of using extended star cutset decompositions, we use the removal of dominated nodes together with double star cutset decompositions. A node \( u \) is said to be dominated if there exists a node \( v \), distinct from \( u \), such that \( N(u) \subseteq N(v) \). A graph is said to be undominated if it does not contain any dominated nodes. A double star cutset in a graph \( G \) is a node cutset \( S = N(u) \cup N(v) \), where \( uv \) is an edge of \( G \).

Lemma 3.1 [3] If a bipartite graph contains an extended star cutset, then it contains a dominated node or a double star cutset.
3.1 Decompositions in Clean Graphs

Definition 3.2 A node $u$ is strongly adjacent to a hole $H$ in the graph $G$, if $u$ is not a node of $H$ and it has at least two neighbors in $H$. It is odd-strongly adjacent if it has an odd number of neighbors in $H$ and it is even-strongly adjacent if it has an even number of neighbors in $H$.

Definition 3.3 A tent $τ(H,u,v)$ is a subgraph of $G$ induced by node set $V(H) \cup \{u,v\}$, where $H$ is a hole of $G$ and $u \in V^r$ and $v \in V^s$ are adjacent nodes which are even-strongly adjacent to $H$ with the following property: the nodes of $H$ can be partitioned into two subpaths $P_u$ and $P_v$ containing the nodes in $N(u) \cap H$ and $N(v) \cap H$ respectively. A tent $τ(H,u,v)$ is referred to as a tent containing $H$.

Definition 3.4 A hole $H$ is said to be clean in $G$ if the following three conditions hold:

(i) No node is odd-strongly adjacent to $H$.

(ii) Every even-strongly adjacent node to $H$ has exactly two neighbors in $H$ and these two neighbors are at distance two in $H$.

(iii) There is no tent containing $H$.

Definition 3.5 Let $G$ be a signed bipartite graph containing a hole $H$. Then $C_G(H) = \{H_i \mid H_i$ is obtained from $H$ by a sequence of holes $H = H_0, H_1, \ldots, H_i$, where $H_j$ and $H_{j-1}$, for $j = 1,2,\ldots, i$, differ in one node $\}$. 

Lemma 3.6 Let $G$ be a signed bipartite graph which contains no unbalanced holes of length 4. Let $H$ be an unbalanced hole in $G$. If $H'$ and $H$ differ in at most one node, then $H'$ is unbalanced.

Proof: Let $H'$ be obtained from $H$ by replacing node $u$ by node $v$. Let $x$ and $y$ be the common neighbors of $u$ and $v$ in $H$. Since $G$ contains no unbalanced of length four, the paths $x,u,y$ and $x,v,y$ have the same weight modulo 4. Thus, $H'$ is unbalanced. □

An unbalanced hole $H^*$ of $G$ is smallest if its number of edges is smallest.

Lemma 3.7 If $H^*$ is a smallest unbalanced hole in $G$, then every even-strongly adjacent node to $H^*$ has exactly two neighbors in $H^*$ and these two neighbors are at distance two in $H^*$.

Proof: Suppose $u$ has an even number of neighbors, $u_1,u_2,\ldots,u_{2k}$, $k \geq 2$ in $H^*$. Let $S_i$, $i = 1,2,\ldots,2k$ be the sectors of $(H^*,u)$ having nodes $u_i,u_{i+1}$ as endnodes (where indices are taken modulo $2k$).

By scaling of the graph at every node $u_i$ for which the edge $uu_i$ has weight $-1$, we can obtain a graph in which all the spokes of $(H^*,u)$ have weight $+1$. Now since $H^*$ is unbalanced, there is a sector, say $S_i$, of weight 0 mod 4. Then the hole $u,u_i,S_i,u_{i+1},u$ is unbalanced and has smaller length than $H^*$. Hence if $u$ is an even-strongly adjacent node in $H^*$ it must have exactly two neighbors, say $u_1$ and $u_2$. W.l.o.g the edges $uu_1$ and $uu_2$ have weight $+1$. Clearly the two $u_1u_2$-subpaths of $H^*$ say $P_1$ and $P_2$, are such that one of them is of weight 0 mod 4.
and the other is of weight 2 mod 4. Suppose $P_2$ is of weight 2 mod 4. Then $P_2$ must have length two for otherwise $u, v_1, P_1, v_2, u$ would be an unbalanced hole of smaller length than $H^*$. Hence $u_1$ and $u_2$ are at distance 2 in $H^*$. □

When referring to a tent $\tau(H^*, u, v)$ we assume that $H^*$ is a smallest unbalanced hole. By Lemma 3.7, $u$ has two neighbors in $H^*$ say $u_1, u_2$, both adjacent to $u_0$ in $H^*$. Similarly the neighbors of $v$ in $H^*$ are $v_1, v_2$, both adjacent to $v_0$ in $H^*$. We assume that nodes $u_1, u_0, u_2, v_1, v_0, v_2$ are encountered in this order, when traversing $H^*$.

**Definition 3.8** A wheel with three spokes and at least two sectors of length 2 is said to be a short 3-wheel.

**Lemma 3.9** Let $G$ be a signed bipartite graph containing a smallest unbalanced hole $H^*$, but not containing a short 3-wheel and not containing an unbalanced hole of length 4. If $H^*$ is clean in $G$, then every hole $H^*_i$ in $\mathcal{C}_G(H^*)$ is clean in $G$.

**Proof:** It suffices to show that, if $H^*_i$ is a hole that differs from $H^*$ in only one node, then $H^*_i$ is clean in $G$.

By Lemma 3.6, $H^*_i$ is an unbalanced hole of smallest length. By Lemma 3.7, condition (ii) of Definition 3.4 is satisfied. Hence, if the lemma is false, condition (i) or (iii) of Definition 3.4 is not satisfied. Therefore we consider the following two cases.

**Case 1:** Condition (i) of Definition 3.4 is not satisfied.

Now a node $w$ must be odd-strongly adjacent to $H^*_1$. Since no node is odd-strongly adjacent to $H^*$, it follows that $w$ has three neighbors, say $w_1, w_2, w_3$ in $H^*_1$. Two of these neighbors, say $w_1$ and $w_2$ must be in $H^*$ and, by Lemma 3.7, they have a common neighbor, say $w_0$ in $H^*$. Since $w_3$ is in $H^*_1$ but not in $H^*$, it follows that $H^*_1$ is obtained from $H^*$ by replacing some node $u \neq w_1, w_2$ in $H^*$ with $w_3$. Let $u_1$ and $u_2$ be the neighbors of $u$ in $H^*$. Note that $w_3$ is adjacent to $u_1$ and $u_2$ and $u$ does not coincide with $w_1$ or $w_2$. Hence $u_1$ and $u_2$ do not coincide with $w_0$. Now $\tau(H^*, w_3, w)$ is a tent, contradicting the assumption that $H^*$ is clean in $G$.

**Case 2:** Condition (iii) of Definition 3.4 is not satisfied.

There must be a tent $\tau(H^*_1, u, v)$. We first show the following claim:

**Claim:** At least one of the nodes $u_1, u_2, v_1, v_2$ does not belong to the hole $H^*$.

**Proof of Claim:** Assume not. Since $u$ and $v$ are not in $H^*_1$, it follows that at most one of them is in $H^*$. If $u$ is in $H^*$, then $u_0$ is not in $H^*$ and $v$ is odd-strongly adjacent to $H^*$, contradicting (i) of Definition 3.4. So $u$ is not in $H^*$ and, by symmetry, node $v$ is not in $H^*$.

Let $w \neq u_1, u_2, v_1, v_2$ be a node in $H^*$ but not in $H^*_1$. Nodes $w$ and $v$ are not adjacent, otherwise node $w$ is odd-strongly adjacent to $H^*$, contradicting the assumption that $H^*$ is clean. By symmetry, it follows that nodes $w$ and $v$ are not adjacent. Now $\tau(H^*, w, v)$ is a tent, contradicting the assumption that $H^*$ is clean and the proof of the claim is complete.

By the above claim, one of the nodes $u_1, u_2, v_1, v_2$ is not in $H^*$. Assume w.l.o.g. that $u_2$ is not in $H^*$. Clearly, node $u$ is not in $H^*$. Node $v$ is not in $H^*$, otherwise node $v_0$ is not in $H^*$, node $u_2$ coincides with $v_0$ and $\tau(H^*_1, u, v)$ is not a tent.
Thus the hole $H^*_1$ is obtained from $H^*$ by replacing a node $w$ with $u_2$, where $w$ is adjacent to $u_0$. Let $u_3$ in $H^*$ be the other neighbor of $u_2$. It follows that $u_3$ is adjacent to $w$. Let $Q$ denote the $v_1u_3$-subpath of $H^*$ not containing $v_2$. Consider the hole $C = u, v, v_1, Q, u_3, w, u_3, u_1, u$. Now the wheel $(C, u_2)$ is a short 3-wheel, contradicting the fact that $G$ does not contain a short 3-wheel. □

**Definition 3.10** A signed bipartite graph $G$ is clean if either $G$ is balanced or $G$ contains a smallest unbalanced hole $H^*$ such that all the holes in $C_G(H^*)$ are clean.

In the next section we show how to construct, from a signed bipartite graph $G$, a clean graph $G'$ that has the property that $G$ is balanced if and only if $G'$ is.

**Lemma 3.11** Let $G$ be a clean graph with family $C_G(H^*)$ of clean smallest unbalanced holes. Let $u$ be a dominated node of $G$ and let $G' = G \setminus \{u\}$. Then some hole in $C_G(H^*)$ is contained in $G'$.

**Proof:** If $u$ is not in $H^*$, then $H^*$ belongs to $G'$. So assume that $u \in V(H^*)$ and that it is dominated by node $v$. Let $u_1$ and $u_2$ be the neighbors of $u$ in $H^*$. Then $v$ is adjacent to $u_1$ and $u_2$, and since $H^*$ is clean, these are the only neighbors of $v$ in $H^*$. The hole induced by the node set $(V(H^*) \setminus \{u\}) \cup \{v\}$ is in $C_G(H^*)$ and is contained in $G'$. □

**Definition 3.12** A 3PC($x, y$), with the three paths $P_1$, $P_2$ and $P_3$, is decomposition detectable w.r.t. the double star cutset $S = N(u) \cup N(v)$ if $P_1 = x, u, v, y$ and the intermediate nodes of $P_2$ and $P_3$ are in different components of $G \setminus S$.

**Lemma 3.13** Let $G$ be a clean graph with family $C_G(H^*)$ of clean smallest unbalanced holes. Furthermore assume that $G$ does not contain an unbalanced hole of length 4. When decomposing $G$ with a double star cutset $S$, then either some hole in $C_G(H^*)$ is contained in one of the blocks of the decomposition or there exists a decomposition detectable 3PC($x, y$) w.r.t. $S$.

**Proof:** Let $S = N(u) \cup N(v)$ be a double star cutset of $G$. Let $C_1, \ldots, C_k$ be the connected components of $G \setminus S$ and $G_1, \ldots, G_k$ be the corresponding blocks of decomposition. We consider the following three cases.

**Case 1:** Both nodes $u$ and $v$ belong to $H^*$.

Let $u_1$ (resp. $v_1$) be the neighbor of $u$ (resp. $v$) in $H^*$ that is distinct from $v$ (resp. $u$). The nodes of $V(H^*) \setminus \{u, v, u_1, v_1\}$ are in some connected component $C_i$ and hence $H^*$ is contained in $G_i$.

**Case 2:** Exactly one of the nodes $u$ or $v$ is in $H^*$.

Assume w.l.o.g. that $u$ is in $H^*$ and $v$ is not. Let $u_1$ and $u_2$ be the neighbors of $u$ in $H^*$. Note that, since $H^*$ is clean, $v$ can have at most one neighbor distinct from $u$ in $H^*$. First suppose that $v$ does not have any neighbor other than $u$ in $H^*$. Then the node set $V(H^*) \setminus \{u, u_1, u_2\}$ is contained in some connected component $C_i$ and hence $G_i$ contains $H^*$. Now suppose that $v$ has a neighbor $v_1$, distinct from $u$, in $H^*$. Nodes $v_1$ and $u$ must have a common neighbor in $H^*$, say $u_1$. Then the node set $V(H^*) \setminus \{v_1, u, u_1, u_2\}$ is contained in some connected component $C_i$ and hence $G_i$ contains $H^*$.

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**Case 3:** Neither $u$ nor $v$ is in $H^*$.  

Assume w.l.o.g. that $|N(u) \cap V(H^*)| \leq |N(v) \cap V(H^*)|$. We consider the following three subcases.

**Case 3.1:** $N(u) \cap V(H^*) = \emptyset$

If $|N(v) \cap V(H^*)| = 0$ or 1, then $H^*$ is contained in some block $G_i$. Suppose that $N(v) \cap V(H^*) = \{v_1, v_2\}$. Let $v_0$ be the common neighbor of $v_1$ and $v_2$ in $H^*$. The node set $V(H^*) \setminus \{v_0, v_1, v_2\}$ is contained in some connected component $C_i$. Let $H$ be the hole obtained from $H^*$ by replacing $v_0$ with $v$. Then $H$ belongs to $C_G(H^*)$ and the block $G_i$ contains $H$.

**Case 3.2:** $N(u) \cap V(H^*) = \{u_1\}$

Then $|N(v) \cap V(H^*)| = 1$ or 2. First suppose that $N(v) \cap V(H^*) = \{v_1\}$. If $u_1$ and $v_1$ are adjacent in $H^*$, then $H^*$ is contained in some block $G_i$. Suppose that $u_1$ and $v_1$ are not adjacent. Let $P$ and $Q$ be the two $u_1v_1$-subpaths of $H^*$. The nodes of $V(P) \setminus \{u_1, v_1\}$ are contained in some connected component $C_i$ and the nodes in $V(Q) \setminus \{u_1, v_1\}$ are contained in some connected component $C_j$. If $i = j$ then $H^*$ is contained in the block $G_i$. If $i \neq j$ then the node set $V(H^*) \cup \{u, v\}$ induces a decomposition detectable $3PC(u_1, v_1)$ w.r.t. $S$.

Now suppose that $N(v) \cap V(H^*) = \{v_1, v_2\}$. Let $v_0$ be the common neighbor of $v_1$ and $v_2$ in $H^*$. If $u_1 = v_0$ then $H^*$ is contained in some block $G_i$. So suppose that $u_1 \neq v_0$. Scale at $v_1$ and $v_2$ to get the edges $v_1v_2$ and $vv_2$ to have weight +1. Since $G$ does not contain an unbalanced hole of length 4, the weight of the path $v_1, v_0, v_2$ is congruent to 2 mod 4. Scale at $u$ and $u_1$ to get the edges $uv$ and $u_1v_1$ to have weight +1. Let $P$ be the $u_1v_1$-subpath of $H^*$ that does not contain $v_2$, and let $Q$ be the $u_1v_2$-subpath of $H^*$ that does not contain $v_1$. Then $w(P)$ and $w(Q)$ are congruent to 1 or 3 mod 4. Since the weight of the path $v_1, v_0, v_2$ is congruent to 2 mod 4, $w(P) \neq w(Q)$ mod 4. If $u_1$ is not adjacent to $v_1$ or $v_2$, then either $v, u, u_1, P, v_1, v$ or $v, u, u_1, Q, v_2, v$ is an unbalanced hole of length smaller than $H^*$. So suppose w.l.o.g. that $u_1$ is adjacent to $v_1$. Then the nodes of $V(H^*) \setminus \{u_1, v_1, u_0, v_2\}$ are contained in some connected component $C_i$. Let $H$ be the hole obtained from $H^*$ by replacing $v_0$ with $v$. Then $H$ belongs to $C_G(H^*)$ and the block $G_i$ contains $H$.

**Case 3.3:** $N(u) \cap V(H^*) = \{u_1, u_2\}$

Then $N(v) \cap V(H^*) = \{v_1, v_2\}$. Let $v_0$ be the common neighbor of $u_1$ and $v_1$ in $H^*$ and let $v_0$ be the common neighbor of $v_1$ and $v_2$ in $H^*$. Since there is no tent containing $H^*$ and $N(u) \cap V(H^*) = \{u_1, u_2\}$ and $N(v) \cap V(H^*) = \{v_1, v_2\}$, we have that $v_0$ is adjacent to $v$ and $v_0$ is adjacent to $u$. Therefore $H^*$ is contained in some block $G_i$. □

### 3.2 Properties of Smallest Unbalanced Holes

Let $H$ be a hole of $G$. By $A_r(H)$ ($A_s(H)$) we denote the set of all odd-strongly adjacent nodes to $H$ which belong to $V^c$ (resp. $V^e$).

**Theorem 3.14** Let $G$ be a signed bipartite graph which does not contain an unbalanced hole of length 4. Let $H^*$ be a smallest unbalanced hole of $G$. Then $H^*$ contains two edges $x_1x_2$ and $y_1y_2$ such that

1. $A_r(H^*) \subseteq N(x_1) \cup N(y_1)$

2. $A_s(H^*) \subseteq N(x_2) \cup N(y_2)$
(ii) $A_i(H^*) \subseteq N(x_2) \cup N(y_2)$

(iii) for every tent $\tau(H^*, u, v)$, either $u \in N(x_1) \cup N(y_1)$ or $v \in N(x_2) \cup N(y_2)$.

This section is devoted to the proof of the above theorem. We assume that $G$ is a signed bipartite graph that is not balanced but does not contain an unbalanced hole of length 4. We denote by $H^*$ a smallest unbalanced hole of $G$.

**Lemma 3.15** If $u, v \in A_i(H^*)$, then they have at least one common neighbor in $H^*$. Moreover in any sector of $(H^*, v)$, node $u$ has either an even number of neighbors, or exactly one neighbor adjacent to $v$.

**Proof:** First we show that $u$ cannot have an odd number, greater than one, of neighbors in any one sector of $(H^*, v)$. Suppose not. Let $u$ have an odd number of neighbors, greater than one in sector $S_k$ of $(H^*, v)$. Let $H = v, S_k, v$. Now $(H, u)$ is an odd wheel, therefore this wheel contains an unbalanced hole which must be of smaller length than $H^*$. Hence $u$ must have either an even number or exactly one neighbor in any sector of $(H^*, v)$.

Next we show that if node $u$ has exactly one neighbor in some sector then this node is also adjacent to $v$. This in turn implies that at least one node in $H^*$ is a neighbor of both $u$ and $v$ since node $u$ has an odd number of neighbors in $H^*$.

Suppose in sector $S_k$ node $u$ has a unique neighbor $u_k$ which is not a neighbor of $v$. Let $v_{k-1}$ and $v_k$ be the end nodes of $S_k$, $P_1$ and $P_2$ be the $v_{k-1}u_k$ and $v_k u_k$-subpaths of $S_k$ respectively. Since $u$ is strongly adjacent to $H^*$, it has a neighbor in another sector, say $S_l$ having one endnode $v_l$ distinct from $v_{k-1}$ and $v_k$. Let $u_l$ be the neighbor of $u$ closest to $v_l$ in sector $S_l$. (Note that since $u, v \in V^c$, then $v_{k-1}, v_k, u_l \in V^c$ and hence $u_l$ cannot be adjacent to $v_{k-1}$ or $v_k$). Now there is a $3PC(u_k, v)$ using paths $P_1, P_2$ and nodes $u_l$ and $v_l$. This 3-path configuration must contain an unbalanced hole which must be of smaller length than $H^*$, which contradicts our choice of $H^*$. □

**Lemma 3.16** Every three nodes in $A_i(H^*)$ have a common neighbor in $H^*$.

**Proof:** Let $U = \{u_1, u_2, u_3\} \subseteq A_i(H^*)$. Note that by Lemma 3.15 every pair of nodes in $A_i(H^*)$ has a common neighbor in $H^*$. Assume that there is no node of $H^*$ that is adjacent to all three nodes of $U$.

Let $A_{12}$ be the set of nodes of $H^*$ adjacent to $u_1$ and $u_2$. $A_{13}$ and $A_{23}$ are analogously defined.

By our assumption $A_{12} \cap A_{23} = \emptyset$. Consider the wheel $(H^*, u_1)$ and the strongly adjacent node $u_3$. For any $j, k \in \{1, 2, 3\}$ with $j \neq k$, define $A^o_{jk} = \{v \in A_{jk} \mid \text{in the two adjacent sectors of } (H^*, u_j) \text{ with the common node } v, \text{there are in total an odd number of neighbors of } u_j\}$. (Note that this definition is not symmetric, i.e. $A^o_{jk}$ is not necessarily equal to $A^o_{kj}$). Now we prove two claims.

**Claim 1:** $A^o_{jk}$ contains an odd number of elements.

**Proof of Claim 1:** We prove that $|A^o_{13}|$ is odd. Consider the wheel $(H^*, u_1)$ and let $S_1, \ldots, S_n$ be the sectors of this wheel, with $S_i$ having endnodes $s_i$ and $s_{i+1}$ (where indices are taken modulo $n$). For every $i = 1, \ldots, n$ let $x_i$ denote the number of neighbors of $u_3$ in
sector $S_i$. By Lemma 3.15 every sector of $(H^*, u_1)$ either has an even number of neighbors of $u_3$ or exactly one neighbor, in which case the neighbor is in $A_{13}$. This and the definition of $A_{13}$ leads to the following properties:

(a) If $s_i \in A_{13}^o$ then either $x_{i-1} = x_i = 1$, or both $x_{i-1}$ and $x_i$ are even.

(b) If $s_i \in A_{13} \setminus A_{13}^o$ then either $x_{i-1} = 1$ and $x_i$ is even, or $x_{i-1}$ is even and $x_i = 1$.

(c) If $s_i$ and $s_{i+1}$ are not in $A_{13}$ then $x_i$ is even.

Now we show that

$$\sum_{i=1}^{n} x_i \equiv |A_{13} \setminus A_{13}^o| \mod 2 \quad (1)$$

Clearly the parity of $\sum_{i=1}^{n} x_i$ is the parity of the number of sectors with an odd number of neighbors of $u_3$. We refer to these sectors as odd sectors. By Properties (a), (b) and (c), if $S_i$ is an odd sector, then it has exactly one neighbor of $u_3$ (i.e. $x_i = 1$), and either $s_i$ or $s_{i+1}$ is an element of $A_{13}$. Each element in $A_{13}$ belongs to 0, 1 or 2 odd sectors. Clearly the parity of the number of odd sectors is equal to the parity of the number of elements in $A_{13}$ which belong to exactly one odd sector. By Properties (a) and (b), $A_{13} \setminus A_{13}^o$ is the set of elements of $A_{13}$ that belong to exactly one odd sector. Thus the parity of $\sum_{i=1}^{n} x_i$ is the same as the parity of $|A_{13} \setminus A_{13}^o|$.

In the summation $\sum_{i=1}^{n} x_i$, every neighbor of $u_3$ which is in $A_{13}$ is counted twice, so the total number of neighbors of $u_3$ on $H^*$ is

$$|N(u_3) \cap V(H^*)| = \sum_{i=1}^{n} x_i - |A_{13}| \quad (2)$$

Now by (1) and (2) we have

$$|N(u_3) \cap V(H^*)| \equiv (|A_{13} \setminus A_{13}^o| - |A_{13}|) \mod 2$$

$$\equiv -|A_{13}^o| \mod 2$$

Since $u_3$ is an odd-strongly adjacent node to $H^*$, we have that $|A_{13}^o|$ is odd. This completes the proof of Claim 1.

**Claim 2:** Let $v_1, v_2 \in V(H^*) \setminus A_{12}$ be neighbors of $u_1$ and $u_2$ respectively. If $P$ is a $v_1 v_2$-subpath of $H^*$, such that $u_1$ and $u_2$ have no neighbors in $V(P) \setminus \{v_1, v_2\}$, then $u_3$ has an even number of neighbors on $P$.

**Proof of Claim 2:** Suppose that $u_3$ has an odd number of neighbors on $P$.

Assume first that $u_3$ has exactly one neighbor $v_3$ on $P$.

W.l.o.g. $v_3 \neq v_i$. By Lemma 3.15, any two nodes of $A_d(H^*)$ have a common neighbor on $H^*$. Let $v_1 \in V(H^*)$ be a common neighbor of $u_1$ and $u_2$, and let $v_{13} \in V(H^*)$ be a common neighbor of $u_1$ and $u_3$. By our assumption $A_{12} \cap A_{13} = \emptyset$, so $v_{12} \neq v_{13}$. Now there
is a $3\text{PC}(v_3, u_1)$ where nodes $v_1, v_{12}, v_{13}$ belong to distinct paths of the 3-path configuration, which must contain an unbalanced hole of length smaller than $H^*$. This contradicts our choice of $H^*$.

Assume now that $u_3$ has an odd number of neighbors, greater than one, on $P$.

Let $v_{12}$ be defined as above. Now there is an odd wheel $(C, u_3)$, where $C = u_1, v_1, P, v_2, u_2, v_{12}, u_1$. Since $u_1$ is an odd-strongly adjacent node either the $v_{12}v_1$-subpath of $H^*$ that does not contain $v_2$ or the $v_{12}v_1$-subpath of $H^*$ that does not contain $v_1$, is of length greater than two. Therefore the wheel contains an unbalanced hole of length smaller than $H^*$, which contradicts our choice of $H^*$. This completes the proof of Claim 2.

Now let $s_1, \ldots, s_n$ be the neighbors of $u_1$ on $H^*$, and $t_1, \ldots, t_m$ be the neighbors of $u_2$ on $H^*$. Let $P_1, \ldots, P_i$ be all the subpaths of $H^*$, whose endnodes belong to $\{s_1, \ldots, s_n, t_1, \ldots, t_m\}$ but have no intermediate node in this set. For every $i = 1, \ldots, l$, let $x_i$ denote the number of neighbors of $u_3$ in $P_i$. Let the endnodes of $P_i$ be denoted by $p_i$ and $p_i+1$ (where the indices are taken modulo $l$). By Lemma 3.15 and Claim 2, if $x_i$ is odd, then $x_i = 1$. Furthermore, by property (c) in Claim 1, if $x_i = 1$ then exactly one of $p_i$ or $p_i+1$ is in $A_{13} \cup A_{23}$.

The $P_i$'s with exactly one neighbor of $u_3$ are characterized as follows:

(i) If $x_i = 1$ and $p_i \in A_{13}^0$, then by Claim 2, $p_{i+1}$ is a neighbor of $u_1$. Now by Property (a) in Claim 1 $x_{i+1} = 1$ and hence by Claim 2, $p_{i-1}$ is a neighbor of $u_1$. Similarly if $x_i = 1$ and $p_i \in A_{23}^0$, then $x_{i-1} = 1$ and both $p_{i-1}$ and $p_{i+1}$ are neighbors of $u_2$.

(ii) If $x_i = 1$ and $p_i \in A_{13} \setminus A_{13}^0$, then by Claim 2, $p_{i+1}$ is a neighbor of $u_1$. Also either by Property (b) in Claim 1 or by Claim 2, $x_{i+1}$ is even. Similarly if $x_i = 1$ and $p_i \in A_{23} \setminus A_{23}^0$, then $p_{i+1}$ is a neighbor of $u_2$ and $x_{i-1}$ is even.

In the summation $\sum_{i=1}^{n} x_i$, every neighbor of $u_3$ which is in $A_{13} \cup A_{23}$ is counted twice, so the total number of neighbors of $u_3$ on $H^*$ is

$$|N(u_3) \cap V(H^*)| = \sum_{i=1}^{n} x_i - |A_{13}| - |A_{23}|$$

(3)

Further we will show that

$$\sum_{i=1}^{n} x_i \equiv (|A_{13} \setminus A_{13}^0| + |A_{23} \setminus A_{23}^0|) \mod 2$$

(4)

Now by (3) and (4) we have

$$|N(u_3) \cap V(H^*)| \equiv (|A_{13} \setminus A_{13}^0| - |A_{13}| + |A_{23} \setminus A_{23}^0| - |A_{23}|) \mod 2$$

$$\equiv -(|A_{13}^0| + |A_{23}^0|) \mod 2$$

By Claim 1 $|A_{13}^0| + |A_{23}^0|$ is even, which contradicts our choice of $u_3$. Thus $A_{13}$ and $A_{23}$ cannot be disjoint.

Now we prove (4). Clearly the parity of $\sum_{i=1}^{n} x_i$ is the same as the parity of the number of sectors with an odd number of neighbors of $u_3$. Recall that if $P_i$ has an odd number of
neighbors of \( w_3 \), then it has exactly one neighbor (i.e., \( x_i = 1 \)) and exactly one of \( p_i \) or \( p_{i+1} \) is an element of \( A_{13} \cup A_{23} \). W.l.o.g. let \( p_i \in A_{13} \cup A_{23} \). Pair off \( P_{i-1} \) and \( P_i \) if the only neighbor of \( w_3 \) in these paths is the node common to \( P_{i-1} \) and \( P_i \), namely \( p_i \). By Property (i) and (ii) this is possible if and only if \( p_i \in A_{13} \cup A_{23} \). Notice that in this case \( x_{i-1} + x_i = 2 \) and the sectors together provide an even count in the sum \( \sum_{i=1}^{n} x_i \). Hence the parity of \( \sum_{i=1}^{n} x_i \) is the same as the parity of \( |A_{13} \setminus A_{13}| + |A_{23} \setminus A_{23}| \), and so (4) holds.

This completes the proof that \( A_{13} \) and \( A_{23} \) are not disjoint. Hence we have proved the lemma. \( \square \)

**Lemma 3.17** \( H^* \) contains a node adjacent to all the nodes in \( A_r(H^*) \) and a node adjacent to all the nodes in \( A_r(H^*) \).

**Proof:** By symmetry, it suffices to prove the first statement. If \( H^* \) is of length 6 or less then the property clearly holds. Suppose now that \( H^* \) has length greater than 6. Suppose \( W \subseteq A_r(H^*) \) is such that for every proper subset \( W' \) of \( W \) there exists a node of \( H^* \) which is adjacent to all nodes in \( W' \), but there exists no node of \( H^* \) adjacent to all nodes in \( W \). By Lemma 3.15 and Lemma 3.16, \( |W| > 3 \). Let \( W = \{w_i|i = 1, \ldots, p\} \) and let \( W_i = \{w_i|i = 1, \ldots, p, i \neq l\} \). Now for \( l = 1, 2, \ldots, p \), all the nodes in \( W_i \) have a common neighbor say \( t_i \) in \( H^* \). Hence for \( i = 1, \ldots, p \), node \( t_i \) is adjacent to \( w_j \), for \( j = 1, \ldots, p, j \neq i \), but \( t_i \) is not adjacent to \( w_i \). Now there exists an odd wheel, \( w_1, t_2, w_3, t_1, w_2, t_3, w_1 \) with center \( t_4 \), hence it must contain an unbalanced hole smaller than \( H^* \). This contradicts the choice of \( H^* \). \( \square \)

**Lemma 3.18** For a tent \( \tau(H^*, u, v) \) the following hold:

- \( A_r(H^*) \subseteq N(v_0) \cup N(u_1) \) or \( A_r(H^*) \subseteq N(v_0) \cup N(u_2) \).
- \( A_r(H^*) \subseteq N(u_0) \cup N(v_1) \) or \( A_r(H^*) \subseteq N(u_0) \cup N(v_2) \).

**Proof:** We prove the first part. Suppose \( w \in A_r(H^*) \) is not adjacent to \( v_0 \). Consider the hole \( H^*_1 \) obtained from \( H^* \) by replacing \( v_0 \) with node \( v \) of \( \tau(H^*, u, v) \). By Lemma 3.6, \( H^*_1 \) is unbalanced, and since it is of the same length as \( H^* \), it also is a smallest unbalanced hole. Now \( w \) cannot be adjacent to \( v \), for otherwise \( w \) is even-strongly adjacent to \( H^*_1 \), which violates Lemma 3.7. Node \( u \) is in \( A_r(H^*_1) \) and has neighbors \( u_1, u_2 \) and \( v \) in \( H^*_1 \). Since \( w \) is not adjacent to \( v \), by Lemma 3.17 it follows that \( w \) is adjacent to \( u_1 \) or \( u_2 \). Furthermore, by Lemma 3.17 the nodes in \( A_r(H^*) \) which are not adjacent to \( v_0 \) are either all adjacent to \( u_1 \) or they are all adjacent to \( u_2 \). Therefore \( A_r(H^*) \subseteq N(v_0) \cup N(u_1) \) or \( A_r(H^*) \subseteq N(v_0) \cup N(u_2) \).

The second part of the lemma can be proved similarly. \( \square \)

**Lemma 3.19** Let \( \tau(H^*, u, v) \) and \( \tau(H^*, w, y) \) be two tents, where \( w_1, w_2 \) are the neighbors of \( w \) and \( y_1, y_2 \) are the neighbors of \( y \) in \( H^* \). Let \( w_0 \) and \( y_0 \) be the common neighbors in \( H^* \) of \( w_1, w_2 \) and \( y_1, y_2 \) respectively. Then at least one of the following properties holds:

- Nodes \( u_1 \) and \( u_2 \) coincide with \( w_1 \) and \( w_2 \).
- Nodes \( v_1 \) and \( v_2 \) coincide with \( y_1 \) and \( y_2 \).
• Node $v_0$ coincides with $y_1$ or $y_2$.

• Node $v_0$ coincides with $w_1$ or $w_2$.

Proof: Suppose the contrary. Then node $u$ does not coincide with $w$, node $v$ does not coincide with $y$, nodes $v_0$ and $y$ are not adjacent and nodes $v_0$ and $w$ are not adjacent. Let $P$ denote the $v_2v_1$-subpath of $H^*$ not containing any other neighbor of $u$ or $v$. Similarly, let $Q$ denote the $v_2v_1$-subpath of $H^*$ not containing any other neighbors of $u$ and $v$. Now it follows that $y_1$ and $y_2$ are contained in $P$ or $Q$ since they are at distance two by Lemma 3.7, and $v_1$ and $w_2$ are contained in $P$ or $Q$. Assume w.l.o.g. that $y_1$ and $y_2$ are contained in $P$. We now prove the following two claims.

Claim 1: Node $y$ is not adjacent to $u$ and node $w$ is not adjacent to $v$.

Proof of Claim 1: Suppose that $y$ and $u$ are adjacent. Now there is an odd wheel $u_2, P, v_1, v, u, u_2$ with center $y$. This wheel contains an unbalanced hole, which is by construction, of smaller length than $H^*$, which contradicts our choice of $H^*$. Hence $y$ is not adjacent to $u$. By symmetry, it follows that $w$ is not adjacent to $v$. This completes the proof of Claim 1.

Claim 2: Nodes $w_1$ and $w_2$ belong to $Q$.

Proof of Claim 2: Suppose not. Then $w_1$ and $w_2$ belong to $P$. By assumption, $y_1$ and $y_2$ belong to $P$. Let $P'$ be the path obtained from $P$ by substituting $y$ for $y_0$. Now by Claim 1, there is an odd wheel $u_2, P', v_1, v, u, u_2$ with center $w$. This wheel contains an unbalanced hole, which is by construction, of smaller length than $H^*$. This contradicts our choice of $H^*$. Hence $w_1$ and $w_2$ belong to $Q$. This completes the proof of Claim 2.

Now by Claim 1 and Claim 2, there is a $3PC(u, y)$ that uses at most as many edges as there are in $H^*$. This 3-path configuration contains an unbalanced hole, of smaller length than $H^*$, which contradicts our choice of $H^*$. □

Proof of Theorem 3.14: First assume that there is no tent in $G$ that contains $H^*$. By Lemma 3.17 $H^*$ contains a node $x_2$ that is adjacent to all nodes in $A_r(H^*)$. By Lemma 3.17 $H^*$ contains a node $y_1$ that is adjacent to all nodes in $A_r(H^*)$. Let $x_1$ be a neighbor of $x_2$ in $H^*$, and let $y_2$ be a neighbor of $y_1$ in $H^*$. Then the edges $x_1x_2$ and $y_1y_2$ satisfy (i), (ii) and (iii).

Now assume that $G$ contains a tent $\tau(H^*, u, v)$. By Lemma 3.18 $A_r(H^*) \subseteq N(v_0) \cup N(u_1)$ or $A_r(H^*) \subseteq N(u_0) \cup N(v_2)$, and $A_r(H^*) \subseteq N(v_0) \cup N(u_1)$ or $A_r(H^*) \subseteq N(u_0) \cup N(v_2)$. Assume that $A_r(H^*) \subseteq N(v_0) \cup N(u_1)$ and $A_r(H^*) \subseteq N(u_0) \cup N(v_1)$. By Lemma 3.19, for every tent $\tau(H^*, w, y)$ in $G$, either $w \in N(v_0) \cup N(u_1)$ or $y \in N(u_0) \cup N(v_1)$. Hence the edges $u_0u_1$ and $v_0v_1$ satisfy (i), (ii) and (iii). The other cases follow similarly. □

4 Recognition Algorithm and its Validity

In this section we present the algorithm that recognizes whether a signed bipartite graph is balanced.
4.1 Cleaning Procedure

CLEANING PROCEDURE

**Input:** A signed bipartite graph $G$ which does not contain an unbalanced hole of length 4.

**Output:** A family $\mathcal{L}$ of induced subgraphs of $G$ such that if $G$ is not balanced, then some $G'$ in $\mathcal{L}$ contains a smallest unbalanced hole that is clean in $G'$.

**Step 1** Let $\mathcal{L} = \{G\}$. Let $U$ be the set of all $(P_1; P_2)$ where $P_1$ and $P_2$ are chordless paths in $G$ of length 3.

**Step 2** For every $(P_1 = x_0, x_1, x_2, x_3; P_2 = y_0, y_1, y_2, y_3) \in U$, add to $\mathcal{L}$ the graph obtained from $G$ by removing the node set $(N(x_1) \cup N(x_2) \cup N(y_1) \cup N(y_2)) \setminus (V(P_1) \cup V(P_2))$.

**Remark 4.1** The number of graphs in list $\mathcal{L}$ produced by the Cleaning Procedure is bounded by $|V^e|^4|V^x|^4$.

**Lemma 4.2** The Cleaning Procedure produces the desired output.

**Proof:** Assume that $G$ is not balanced and let $H^*$ be a smallest unbalanced hole in $G$. By Theorem 3.14 $H^*$ contains edges $x_1x_2$ and $y_1y_2$ that satisfy (i), (ii) and (iii) of Theorem 3.14. Let $P_1 = x_0, x_1, x_2, x_3$ and $P_2 = y_0, y_1, y_2, y_3$ be the two subpaths of $H^*$ with middle edges $x_1x_2$ and $y_1y_2$. Let $G'$ be the graph obtained from $G$ by removing the node set $(N(x_1) \cup N(x_2) \cup N(y_1) \cup N(y_2)) \setminus (V(P_1) \cup V(P_2))$. $G'$ is one of the graphs in $\mathcal{L}$ and it contains $H^*$. By Lemma 3.7 and Theorem 3.14, $H^*$ is clean in $G'$. □

4.2 Short 3-Wheels

SHORT 3-WHEEL PROCEDURE

**Input:** A signed bipartite graph $G$.

**Output:** A short 3-wheel of $G$ or the fact that $G$ does not contain such a node induced subgraph.

**Step 1:** Enumerate all distinct subsets of six nodes with three nodes in $V^x$ and three nodes in $V^e$ and declare them as unscanned. Go to Step 2.

**Step 2:** If all subsets are scanned, $G$ does not contain a short 3-wheel, stop. Otherwise choose an unscanned subset $U$. If $U$ induces a 6-cycle $C = a_1, a_2, a_3, a_4, a_5, a_6, a_1$, having unique chord $a_2a_5$, go to Step 3. Otherwise declare $U$ as scanned and repeat Step 2.

**Step 3:** Remove the nodes in $N(a_2) \cup N(a_4) \cup N(a_5) \cup N(a_6) \setminus \{a_1, a_3\}$. If $a_1$ and $a_3$ are in the same connected component, then a short 3-wheel with spokes $a_2a_1, a_2a_3, a_2a_5$ is identified, stop. If not, remove the nodes in $N(a_1) \cup N(a_2) \cup N(a_3) \cup N(a_5) \setminus \{a_4, a_6\}$. If $a_4$ and $a_6$ are in the same connected component, then a short 3-wheel with spokes $a_5a_2, a_5a_4, a_5a_6$ is identified, stop. Otherwise declare $U$ as scanned return to Step 2.
4.3 6-Join Decomposition

We now give an algorithm that finds a 6-join in a connected undominated graph $G$ or shows that $G$ does not have one.

Note that, if a connected undominated graph has a 6-join, then (using the notation given in the introduction) there exists a node in $V_1 \setminus (A_1 \cup A_3 \cup A_5)$ that is adjacent to a node of $A_1 \cup A_3 \cup A_5$ (otherwise some node in $A_1 \cup A_3 \cup A_5$ would be dominated) and there exists a node in $V_2 \setminus (A_2 \cup A_4 \cup A_6)$ that is adjacent to a node of $A_2 \cup A_4 \cup A_6$. Let $a_1, \ldots, a_6, u_1, u_2$ be 8 distinct nodes of $G$ such that $\{a_1, \ldots, a_6\}$ induces a hole of length 6, $u_1$ is adjacent to at least one node in $\{a_1, a_3, a_5\}$, and $u_2$ is adjacent to at least one node in $\{a_2, a_4, a_6\}$ but not to $u_1$. The following rules yield a 6-join $E(A)$ with $\{a_1, a_3, a_5, u_1\} \subseteq V_1$ and $\{a_2, a_4, a_6, u_2\} \subseteq V_2$, or show that $G$ does not have such a 6-join. (Note that if such a 6-join is found then, for $i = 1, \ldots, 6$, $a_i \in A_i$, $u_1 \in V_1 \setminus (A_1 \cup A_3 \cup A_5)$ and $u_2 \in V_2 \setminus (A_2 \cup A_4 \cup A_6)$).

Initially $V_1 = \{a_1, a_3, a_5, u_1\}$ and $V_2 = V(G) \setminus V_1$. Then forcing rules will be applied to move nodes from $V_2$ to $V_1$.

During the algorithm the nodes $u$ in $V_1$ are partitioned into four sets:

- $u \in A_1$ if it is adjacent to $a_2$ and $a_6$ but not to $a_4$,
- $u \in A_3$ if it is adjacent to $a_2$ and $a_4$ but not to $a_6$,
- $u \in A_5$ if it is adjacent to $a_4$ and $a_6$ but not to $a_2$,
- $u \in V_1 \setminus (A_1 \cup A_3 \cup A_5)$ if it is not adjacent to any node $a_2, a_4, a_6$.

The case where some node $u$ in $V_1$ is adjacent to exactly one of the nodes $a_2, a_4, a_6$ or to all three of them will not be permitted.

Forcing rules that move nodes from $V_2$ to $V_1$ are as follows.

- If $u \in V_2 \setminus \{a_2, a_4, a_6, u_2\}$ is adjacent to at least one node in $V_1 \setminus (A_1 \cup A_3 \cup A_5)$ then remove $u$ from $V_2$ and add it to $V_1$.
- If $u \in V_2 \setminus \{a_2, a_4, a_6, u_2\}$ is adjacent to at least one node in $A_1 \cup A_3 \cup A_5$ and $N(u) \cap (A_1 \cup A_3 \cup A_5) \neq A_1 \cup A_3 \cup A_5$ or $A_1 \cup A_5$, then remove $u$ from $V_2$ and add it to $V_1$.

Clearly, if there exists a 6-join $E(A)$ with $\{a_1, a_3, a_5, u_1\} \subseteq V_1$ and $\{a_2, a_4, a_6, u_2\} \subseteq V_2$ and $u$ satisfies one of the above rules, then $u$ must be in $V_1$.

If some node $u$ which is moved from $V_2$ to $V_1$ does not satisfy the following: $N(u) \cap \{a_2, a_4, a_6\} = \emptyset$, $\{a_2, a_4\}$, $\{a_2, a_6\}$ or $\{a_4, a_6\}$, and $N(u) \cap \{a_2\} = \emptyset$, then the algorithm terminates since no 6-join $E(A)$ with $\{a_1, a_3, a_5, u_1\} \subseteq V_1$ and $\{a_2, a_4, a_6, u_2\} \subseteq V_2$ exists. If this situation never occurs, we continue moving nodes from $V_2$ to $V_1$ until no forcing rule applies.

At this stage the nodes of $V_2$ satisfy the following: no node of $V_2$ is adjacent to a node of $V_1 \setminus (A_1 \cup A_3 \cup A_5)$ and if a node $u \in V_2$ is adjacent to a node of $A_1 \cup A_3 \cup A_5$ then $N(u) \cap (A_1 \cup A_3 \cup A_5) = A_1 \cup A_3 \cup A_5$ or $A_1 \cup A_5$. Denote by $A_2$ the nodes of $V_2$ that are adjacent to all nodes in $A_1 \cup A_3$, by $A_4$ the nodes of $V_2$ that are adjacent to all nodes in $A_3 \cup A_5$ and by $A_6$ the nodes of $V_2$ that are adjacent to all nodes in $A_1 \cup A_5$. Let $A$ be the graph induced by the node set $\bigcup_{i=1}^{6} A_i$. Then $E(A)$ is a 6-join of $G$ with partition $V_1$ and $V_2$. 

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To determine whether a graph $G$ has a 6-join one would apply the above algorithm to all 8-tuples $(a_1, \ldots, a_6, u_1, u_2)$ of nodes of $G$ for which $\{a_1, \ldots, a_6\}$ induces a hole of length 6, $u_1$ is adjacent to at least one node in $\{a_1, a_3, a_5\}$, and $u_2$ is adjacent to at least one node in $\{a_2, a_4, a_6\}$ but not $u_1$. Clearly all of this can be implemented to run in polynomial time.

Let $S$ be a double star cutset in a graph $G$, and let $G_1, \ldots, G_k$ be the blocks of decomposition. The refined blocks of decomposition are graphs $G_1^*, \ldots, G_k^*$, where $G_i^*$ is obtained from $G_i$ by removing all dominated nodes.

When we say "remove all dominated nodes from a graph $F"$, we mean to apply the following procedure:

Step 1: If $F$ contains a dominated node $u$, then go to Step 2. Otherwise, stop and output $F$.

Step 2: Let $F = F \setminus \{u\}$ and go to Step 1.

DOUBLE STAR CUTSET AND 6-JOIN DECOMPOSITION ALGORITHM

Input: A signed bipartite graph $G$ that does not contain a short 3-wheel or an unbalanced hole of length 4 or 6.

Output: Either $G$ is identified as not being balanced, or a list $L$ of induced subgraphs of $G$ with the following properties:

- The graphs in $L$ do not contain a 6-join, a double star cutset or any dominated nodes.
- If the input graph $G$ contains a family $C_G(H^*)$ of clean smallest unbalanced holes, then one of the graphs $G'$ in $L$ contains a hole $H'$ of $C_G(H^*)$, and $C_G(H')$ is a family of clean smallest unbalanced holes in $G'$.

Step 1: Remove all dominated nodes from $G$ and initialize $M = \{G\}$ and $L = \emptyset$.

Step 2: If $M$ is empty, return $L$ and stop. Otherwise, remove a graph $F$ from $M$.

Step 3: If $F$ contains a double star cutset $S$ go to Step 4 and otherwise go to Step 5. (Note that checking whether $F$ contains a double star cutset involves checking for every pair of adjacent nodes $u$ and $v$ whether $S = N(u) \cup N(v)$ is a cutset).

Step 4: Check whether there exists a decomposition detectable 3PC($x, y$) w.r.t. $S$. If it does, identify $G$ as not balanced and stop. Otherwise, construct the refined blocks of the decomposition by $S$, add them to $M$ and go to Step 2.

Step 5: Check whether $F$ contains a 6-join. If it does, construct the blocks of the 6-join decomposition, remove all dominated nodes from the blocks, add these graphs to $M$ and go to Step 2. Otherwise, add $F$ to $L$ and go to Step 2.

Theorem 4.3 The Double Star Cutset and 6-Join Decomposition Algorithm produces the desired output.

Proof: Let $G$ be a signed bipartite graph that does not contain an unbalanced hole of length 4 or 6, or a short 3-wheel. If the algorithm terminates in Step 4, then $G$ is correctly identified as not being balanced. So suppose that the algorithm does not terminate in Step 4. By the construction of the algorithm, the graphs in $L$ do not contain a 6-join, a double star cutset or any dominated nodes. Suppose that $G$ contains a family $C_G(H^*)$ of clean smallest unbalanced holes. To prove the theorem it is enough to show the following.
(1) If $G'$ is the graph obtained from $G$ by removing dominated nodes, then $G'$ contains a hole in $C_G(H^*)$.

(2) If $G_1^*, \ldots, G_k^*$ are the refined blocks of decomposition of $G$ by a double star cutset, then for some $i$, $G_i^*$ contains a hole in $C_G(H^*)$.

(3) If $G_1$ and $G_2$ are the blocks of decomposition of $G$ by a 6-join $E(A)$, then for some $i$, $G_i$ contains a hole in $C_G(H^*)$.

(4) If $G'$ contains a hole $H'$ of $C_G(H^*)$, then $C_G(H')$ is a family of clean smallest unbalanced holes in $G'$.

(1) and (2) follow from Lemma 3.11 and Lemma 3.13. (4) follows from the fact that if a hole $H'$ is clean in $G$, then it is also clean in any induced subgraph $G'$. To prove (3) suppose that $H^*$ is contained in neither $G_1$ nor $G_2$. Then $H^*$ must contain an edge of $E(A)$. Since $G$ does not contain an unbalanced hole of length 6, not all of the edges of $H^*$ can be in $E(A)$. Hence w.l.o.g. we may assume that either (i) $H^* = a_1', a_2', a_3', P_1, a_4'$ where $a_i' \in A_i$ for $i = 1, 2$ and 3, and $P_1$ is a path with nodes in $V_1$ from $a_1'$ to $a_3'$, or (ii) $H^* = a_1', a_2', a_3', a_4', a_5', P_1, a_1'$ where $a_i' \in A_i$ for $i = 1, 2, 4$ and 5, $P_1$ is a path with nodes in $V_1$ from $a_1'$ to $a_5'$, and $P_2$ is a path with nodes in $V_2$ from $a_2'$ to $a_4'$. If (i) holds, then the hole obtained from $H^*$ by substituting $a_2$ for $a_2'$ is a hole of $C_G(H^*)$ and is contained in $G_1$. So assume (ii) holds. Since node $a_3$ has neighbors $a_2'$ and $a_4'$ in $H^*$, and $H^*$ is clean, the path $P_2$ must be of length 2. Similarly path $P_1$ must be of length 2. Hence $H^*$ is of length 6, contradicting our assumption.

\[\square\]

**Lemma 4.4** The number of graphs in the list $L$ produced by the Double Star Cutset and 6-Join Decomposition Algorithm is bounded by $|V^r|^{|V^r|}(|V^r| + |V^r|)$.

**Proof:** Let $G$ be a signed bipartite graph that does not contain a short 3-wheel, and let $L$ be the list of induced subgraphs of $G$ produced by the algorithm. Note that we are assuming that the algorithm does not terminate in Step 4, with identifying a decomposition detectable 3PC($x$, $y$). We prove the lemma by showing that the number of decompositions used to decompose $G$ by the algorithm is bounded by the number of chordless paths of length 5 in $G$. (So in the decomposition tree the number of parents of the leaves, i.e. graphs added to $L$, is bounded by $|V^r|^{|V^r|}$, and hence the number of graphs in $L$ is bounded by $|V^r|^{|V^r|}$). This will be shown by proving that if $F$ is a graph decomposed in Step 4 or Step 5 of the algorithm, $F$ has the property that it contains a chordless path of length 5 that is not contained in any of the blocks of decomposition that are added to list $M$, and that no two blocks of decomposition contain the same chordless path of length 5. So the lemma follows from the following four claims.

First suppose that $F$ is decomposed in Step 5 by a 6-join $E(A)$. Let $F_1$ and $F_2$ be the blocks of decomposition.

**Claim 1:** $F_1$ and $F_2$ do not contain the same chordless path of length 5.

**Proof of Claim 1:** Any chordless path of length 5 in $F_1$ must contain a node of $V_1 \setminus (A_1 \cup A_3 \cup A_5)$ and hence cannot be a path of $F_2$. This completes the proof of Claim 1.
Claim 2: $F$ contains at least one chordless path of length 5 that is contained neither in $F_1$ nor in $F_2$.

Proof of Claim 2: By (iii) of the definition of a 6-join, $|V_1| \geq 4$. Let $U_1 = V_1 \setminus (A_1 \cup A_3 \cup A_5)$. We must have $U_1 \neq \emptyset$, otherwise some node of $A_1 \cup A_3 \cup A_5$ is dominated in $F$, a contradiction. No node $v \in U_1$ can have neighbors in each of the sets $A_1$, $A_3$ and $A_5$ since, otherwise, $v$ would be the center of a short 3-wheel. So, w.l.o.g., there exists a node $u_1$ with no neighbor in $A_5$, but at least one neighbor in $A_1$. In $F$, node $u_1$ is not dominated by a node of $A_2$. This implies that $u_1$ is adjacent to a node $v_1 \in U_1$ that is at distance two from $A_1 \cup A_3 \cup A_5$, i.e., $v_1, u_1, a'_1$ is a chordless path where $a'_1 \in A_1$. Similarly, let $v_2$ be a node of $F_2$ that is at distance two from $A_2 \cup A_4 \cup A_6$. Let $v_2, a_1, a_2$ be a chordless path with $a_2 \in V_2 \setminus (A_2 \cup A_4 \cup A_6)$ and $a'_2 \in A_1$, $i = 2, 4$ or 6. If $i = 2$ or 6, then $v_1, u_1, a'_1, a'_2, u_2, v_2$ is the desired path. So assume that $i = 4$ and $u_2$ is not adjacent to any node of $A_2 \cup A_6$. Then $u_1, a'_1, a_6, a_4, a_2$ is the desired path. This completes the proof of Claim 2.

Now assume that $F$ is decomposed in Step 4 by a double star cutset $S = N(u) \cup N(v)$. Let $C_1, \ldots, C_k$ be the connected components of $F \setminus S$. Let $F_1, \ldots, F_k$ be the blocks of decomposition and $F^*_1, \ldots, F^*_k$ the refined blocks of decomposition. Note that $F$ is an undominated graph, and by the definition of refined blocks so are $F^*_1, \ldots, F^*_k$. Also, w.l.o.g., we assume that $F$ is a connected graph.

Claim 3: No two graphs $F^*_i, \ldots, F^*_k$ contain the same chordless path of length 5.

Proof of Claim 3: We actually prove a stronger statement that no two graphs $F^*_i, \ldots, F^*_k$ contain the same chordless path of length 3. Assume otherwise and let $P = a, b, c, d$ be a chordless path that is contained in both $F^*_i$ and $F^*_j$, $i \neq j$. Then $\{a, b, c, d\} \subseteq S$ Since $a, b, c, d$ must alternate between $N(u)$ and $N(v)$, and $P$ is a chordless path, $u$ and $v$ cannot coincide with $a$ or $d$, and similarly for $v$. So w.l.o.g., $a \in N(u) \setminus \{v\}$ and $d \in N(v) \setminus \{u\}$. If $a$ does not have a neighbor in $C_i$, then $a$ is dominated by $v$ in $F$, and hence $a$ is dominated by some node in $F^*_i$, contradicting the assumption that $F^*_i$ is an undominated graph. So $a$ must have a neighbor in $C_i$, and similarly it must have a neighbor in $C_j$. By the same argument $d$ has a neighbor in both $C_i$ and $C_j$. But then there is a decomposition detectable $3PC(a, d)$ w.r.t. $S$, contradicting our assumption. This completes the proof of Claim 3.

Claim 4: $F$ contains at least one chordless path of length 5 that is not contained in any of the graphs $F^*_1, \ldots, F^*_k$.

Proof of Claim 4: Each of the connected components $C_1, \ldots, C_k$ must contain at least two nodes, since $F$ is an undominated graph. Since $F$ is connected, a node of $C_i$, $i = 1, \ldots, k$, must have a neighbor in $S$.

First assume that there exist nodes $p_1 \in V(C_1)$ and $p_2 \in V(C_2)$ such that they have a common neighbor $a_1 \in N(u)$. Since $|V(C_1)| \geq 2$, $C_1$ contains a node $q_1$ adjacent to $p_1$. Similarly, $C_2$ contains a node $q_2$ adjacent to $p_2$. Since $q_2$ is not dominated by $a_1$, $q_2$ must have a neighbor $t_2$ that $a_1$ is not adjacent to. If $t_2$ is adjacent to $q_1$, then $t_2 \in N(v)$ and hence there is a decomposition detectable $3PC(a_1, t_2)$ w.r.t. $S$, contradicting our assumption. So $t_2$ is not adjacent to $q_1$, and hence $P = q_1, p_1, a_1, p_2, q_2, t_2$ is the desired path.

Now assume that no two nodes, one from $C_1$ and one from $C_2$, have a common neighbor in $S$. Let $p_1$ (resp. $p_2$) be a node of $C_1$ (resp. $C_2$) that is adjacent to $a_1 \in S$ (resp. $a_2 \in S$). Let $q_1$ (resp. $q_2$) be a neighbor of $p_1$ (resp. $p_2$) in $C_1$ (resp. $C_2$). If $a_1, a_2 \in N(u)$,
then $P = q_1, p_1, a_1, u, a_2, p_2$ is the desired path. So we may assume that $a_1 \in N(u)$ and $a_2 \in N(v)$. If $a_1 a_2$ is not an edge then $P = p_1, a_1, u, v, a_2, p_2$ is the desired path. Otherwise, $P = q_1, p_1, a_1, a_2, p_2, q_2$ is the desired path. This completes the proof of Claim 4. □

4.4 2-Join Decomposition

In [6] an algorithm that either finds a 2-join in a graph $G$ or concludes that $G$ does not have one is given. We outline here this algorithm for the sake of completeness, in the case where $G$ contains no extended star cutset.

Lemma 4.5 Let $G$ be a bipartite graph that has no extended star cutset. Then, in every 2-join, $|V(G_i^2)| \geq 4$, for $i = 1, 2$.

Proof: By Lemma 2.2, the 2-join is not rigid. Suppose $|V(G_i^2)| \leq 3$, for $i = 1$ or 2.

If there exists a node $u \in V(G_i^2) \setminus (A_i \cup B_i)$, then $|A_i| = |B_i| = 1$ and, since the 2-join is not rigid, (ii) of the definition of 2-join implies that $u$ is adjacent to both these nodes. This contradicts (iii) of the definition of 2-join.

So $V(G_i^2) \setminus (A_i \cup B_i) = \emptyset$. By (iii) of the definition of 2-join, every node of $A_i$ has a neighbor in $B_i$ and vice versa, every node in $B_i$ has a neighbor in $A_i$. Since the 2-join is not rigid, this implies that $|A_i| \geq 2$ and $|B_i| \geq 2$. □

Let $a_1, b_1, a_2, b_2$ be 4 distinct nodes of a bipartite graph $G$, such that $a_1 a_2$ and $b_1 b_2$ are edges, but $a_1 b_2$, $a_2 b_1$ are not. The following procedure yields a 2-join $E(K_{A_1 A_2}) \cup E(K_{B_1 B_2})$ with $a_1 \in A_1$, $b_1 \in B_1$, $a_2 \in A_2$ and $b_2 \in B_2$, or shows that no such 2-join exists.

For every 2 distinct nodes $u_1, v_1 \in V(G) \setminus \{a_1, a_2, b_1, b_2\}$, each adjacent to at most one node in $\{a_2, b_2\}$, the following rules identify a partition of $V(G)$ into $V_1$ and $V_2$, where $a_1, b_1, u_1, v_1 \in V_1$ and $a_2, b_2 \in V_2$, such that the edges with one endnode in $V_1$ and the other in $V_2$ induce two disjoint bicliques $K_{A_1 A_2}$ and $K_{B_1 B_2}$ satisfying Properties (i) and (ii) in the definition of 2-join, or show no such partition exists.

Initially we let $V_1 = \{a_1, b_1, u_1, v_1\}$ and $V_2 = V(G) \setminus \{a_1, b_1, u_1, v_1\}$. Then forcing rules will be applied that will move nodes from $V_2$ to $V_1$.

During the algorithm, the nodes $u$ in $V_1$ are partitioned into three sets:

- $u \in A_1$ if $ua_2$ is an edge but $ub_2$ is not,
- $u \in B_1$ if $ub_2$ is an edge but $ua_2$ is not,
- $u \in V_1 \setminus (A_1 \cup B_1)$ if neither $ua_2$ nor $ub_2$ is an edge.

The case where some node $u$ in $V_1$ is adjacent to both $a_2$ and $b_2$ will not be permitted.

The forcing rules that move nodes from $V_2$ to $V_1$ are as follows.

- If $u \in V_2 \setminus \{a_2, b_2\}$ is adjacent to at least one node in $V_1 \setminus (A_1 \cup B_1)$, add $u$ to $V_1$ and remove it from $V_2$.
- If $u \in V_2 \setminus \{a_2, b_2\}$ is adjacent to some node in $A_1 \cup B_1$ and $N(u) \cap (A_1 \cup B_1) \neq A_1$ or $B_1$, then add $u$ to $V_1$ and delete it from $V_2$. 
Note that if there is a 2-join \( E(K_{A_1,A_2}) \cup E(K_{B_1,B_2}) \) with \( a_1, b_1, u_1, v_1 \in V_1 \) and \( a_2, b_2 \in V_2 \), and \( u \) satisfies one of the above rules, then \( u \) would have to be in \( V_1 \).

If some node \( u \) which is moved from \( V_2 \) to \( V_1 \) is adjacent to both \( a_2 \) and \( b_2 \), then the algorithm terminates since no 2-join \( E(K_{A_1,A_2}) \cup E(K_{B_1,B_2}) \) with \( a_1, b_1, u_1, v_1 \in V_1 \) and \( a_2, b_2 \in V_2 \) exists. If this situation never occurs, we continue moving nodes from \( V_2 \) to \( V_1 \) until no forcing rule applies.

At this stage the nodes of \( V_2 \) satisfy the following: no node of \( V_2 \) is adjacent to a node of \( V_1 \setminus (A_1 \cup B_1) \), and if a node \( u \) of \( V_2 \) is adjacent to a node of \( A_1 \cup B_1 \), then \( N(u) \cap (A_1 \cup B_1) = A_1 \) or \( B_1 \). Denote by \( A_2 \) the nodes of \( V_2 \) that are adjacent to all nodes in \( A_1 \), and by \( B_2 \) the nodes of \( V_2 \) that are adjacent to all nodes in \( B_1 \). The edge set \( E(K_{A_1,A_2}) \cup E(K_{B_1,B_2}) \) satisfies (i) of the definition of 2-join. By our assumption that \( G \) has no extended star cutset, (ii) of the definition of 2-join holds as well.

Now, if (iii) also holds, we have a 2-join with \( a_1, b_1, u_1, v_1 \in V_1 \) and \( a_2, b_2 \in V_2 \). On the other hand, if no choice of \( u_1, v_1 \) yields an edge set \( E(K_{A_1,A_2}) \cup E(K_{B_1,B_2}) \) satisfying (iii), then no 2-join with \( a_1, b_1 \in V_1 \) and \( a_2, b_2 \in V_2 \) exists. Indeed, the only way in which a choice \( u_1, v_1 \) can fail to yield a 2-join with \( a_1, b_1, u_1, v_1 \in V_1 \) and \( a_2, b_2 \in V_2 \) when such a 2-join exists is if, at termination, \( |A_1| = |B_1| = 1 \) and \( V_1 \) induces a chordless path \( P \). Furthermore, any 2-join with \( a_1, b_1, u_1, v_1 \in V_1 \) and \( a_2, b_2 \in V_2 \) satisfies \( V_1 \subseteq V_2 \). Therefore, the choice \( u_1, v_1 \), where \( v'_2 \in V_2 \setminus V(P) \) yields the desired 2-join.

To determine whether a bipartite graph \( G \) without extended star cutsets has a 2-join, one would apply the above algorithm to all 4-tuples \( (a_1, b_1, a_2, b_2) \) of nodes of \( G \) for which \( a_1a_2 \) and \( b_1b_2 \) are edges, but \( a_1b_2, a_2b_1 \) are not. Clearly all of this can be implemented to run in polynomial time.

2-JOIN DECOMPOSITION ALGORITHM

Input: A signed bipartite graph \( G \) that does not contain a short 3-wheel, an unbalanced hole of length 4, an extended star cutset or a 6-join.

Output: A list of signed bipartite graphs \( L \) with the following properties:

- The graphs in \( L \) do not contain an extended star cutset, a 6-join or a 2-join.
- \( G \) is balanced if and only if all the graphs in \( L \) are balanced.

Step 1: Let \( M = \{G\} \) and \( L = \emptyset \).

Step 2: If \( M \) is empty, return \( L \) and stop. Otherwise remove a graph \( M \) from \( M \).

Step 3: Check whether \( M \) has a 2-join. If it does not, then add \( M \) to \( L \) and go to Step 2. Otherwise, the 2-join is not rigid (we justify this in Theorem 4.6). Construct the blocks of the 2-join decomposition, add them to \( M \) and go to Step 2.

Theorem 4.6 The 2-Join Decomposition Algorithm produces the desired output.

Proof: Let \( G \) be a signed bipartite graph that does not contain a short 3-wheel, an unbalanced hole of length 4, an extended star cutset or a 6-join. Let \( E(K_{A_1,A_2}) \cup E(K_{B_1,B_2}) \) be a 2-join of \( G \). Let \( G_1 \) and \( G_2 \) be the blocks of the decomposition. To prove the validity of the algorithm it is enough to show that (i) \( G_1 \) and \( G_2 \) do not contain a short 3-wheel, an
unbalanced hole of length 4, an extended star cutset or a 6-join, and (ii) $G$ is balanced if and only if $G_1$ and $G_2$ are balanced.

By Lemma 2.2, the 2-join is not rigid. So by Theorem 2.3, (ii) holds. By the construction of the blocks, there is no hole of length less than 7 in the blocks that uses the marker paths. Hence $G_1$ and $G_2$ do not contain an unbalanced hole of length 4, a short 3-wheel or a 6-join.

We now show that $G_1$ and $G_2$ do not contain an extended star cutset. Suppose w.l.o.g. that $G_1$ contains an extended star cutset $S = (x; X; Y; R)$. Recall that the marker path $M_1$ of $G_1$ is of length 4 or 5. Let $G'_1 = G_i \setminus V(M_i)$.

**Case 1:** Node $x$ coincides with $\alpha_1$ or $\beta_1$.

Assume w.l.o.g. that $x$ coincides with $\alpha_1$. Since $|E(M_1)| \geq 4$, $\beta_1$ is not in $S$. So, $S$ is a cutset that separates $\beta_1$ from a node in $G'_1 \setminus S$. We can assume w.l.o.g. that the neighbor of $\alpha_1$ in $M_1$ is not in $S$, since the set obtained by removing that neighbor from $S$ would also be an extended star cutset of $G_1$. So $Y \cup R \subseteq A_1$. If $S$ is a star cutset, i.e. $X = \{x\}$ and $R = \emptyset$, then $S^* = Y \cup A_2$ is a biclique cutset of $G$, separating $B_2$ from a node in $G'_1 \setminus S$. So assume that $|X| \geq 2$. Then at least two nodes of $A_1$ are contained in $Y$. Let $x^*$ be any node of $A_2$. Then $S^* = (x^*; (X \cup A_2) \setminus \{x\}; Y; R)$ is an extended star cutset of $G$ separating $B_2$ from a node in $G'_1 \setminus S$.

**Case 2:** Node $x$ is an intermediate node of $M_1$.

Since $M_1$ has length at least 4, we must have $|X| = 1$, i.e. $S$ is a star cutset. W.l.o.g. assume $\beta_1 \notin S$. Then $S$ separates $\beta_1$ from a node in $G'_1 \setminus S$. But then $S' = \{\alpha_1\}$ is also a star cutset of $G_1$. So, by Case 1, we are done.

**Case 3:** Node $x$ is in $A_1$ or $B_1$.

W.l.o.g. assume that $x$ is in $A_1$. If $\beta_1 \notin S$, then $S$ separates $\beta_1$ from a node in $G'_1 \setminus S$. If $\alpha_1 \notin Y \cup R$, let $S^* = S$. If $\alpha_1 \in R$, let $S^* = (x; X; Y; \{\alpha_1\}) \cup A_2$ and if $\alpha_1 \in Y$, let $S^* = (x; X; Y \setminus \{\alpha_1\}) \cup A_2; R)$. Then $S^*$ is an extended star cutset of $G$ separating $B_2$ from a node in $G'_1 \setminus S$. So $\beta_1 \in S$ and hence $\beta_1 \in X$. Thus $Y \subseteq B_1$. Now $S^* = (x; (X \setminus \{\beta_1\}) \cup B_2; Y; \{\alpha_1\}) \cup A_2$ is an extended star cutset of $G$ separating a node of $G'_1 \setminus S$ from a node of $G'_2 \setminus (A_2 \cup B_2)$. Indeed, this graph is nonempty by the following claim.

**Claim:** $V(G'_2 \setminus (A_2 \cup B_2)) \neq \emptyset$.

**Proof:** Assume otherwise, namely $V(G'_2 \setminus (A_2 \cup B_2)) = \emptyset$. By (ii) in the definition of a 2-join, every node of $A_2$ has a neighbor in $B_2$ and, vice versa, every node in $B_2$ has a neighbor in $A_2$. Since the 2-join is not rigid, this implies that $|A_2| \geq 2$ and $|B_2| \geq 2$. Furthermore, every node in $A_2$ has a node in $B_2$ that it is not adjacent to (otherwise, there is a star cutset) and every node in $B_2$ has a node in $A_2$ that it is not adjacent to. Let $u$ be a node of largest degree in the graph induced by $A_2 \cup B_2$. W.l.o.g. assume $u \in A_2$. Let $Q$ be the set of neighbors of $u$ in $B_2$ and let $v \in B_2 \setminus Q$. Let $w \in A_2$ be a neighbor of $v$. Then $w$ is not adjacent to some node $q \in Q$, by our choice of $u$. Since the 2-join is not rigid, $A_1 \cup B_1$ is not a biclique, i.e. there exist $a_1 \in A_1$ and $b_1 \in B_1$ which are not adjacent. So $a_1 uvb_1qu$ is a 6-hole. Now, if $x$ is adjacent to $b_1$, it induces a short 3-wheel with this 6-hole, a contradiction. Therefore $x$ is not adjacent to $b_1$ and $uxvub_1qu$ is a 6-hole. But then, any $y \in Y$ induces a short 3-wheel with this 6-hole, a contradiction. This completes the proof of the claim.

**Case 4:** Node $x$ is in $G'_1 \setminus (A_1 \cup B_1)$. 

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Not both $\alpha_1$ and $\beta_1$ can be in $S$. Assume w.l.o.g. that $\beta_1 \notin S$. Then $S$ is a cutset separating $\beta_1$ from a node in $G'_1 \setminus S$. If $\alpha_1 \notin S$, then $S$ is a cutset of $G$ separating $B_2$ from a node in $G'_1 \setminus S$. So $\alpha_1 \in S$. Then $\alpha_1 \in X$, $Y \subseteq A_1$ and hence $S^* = (x; (X \setminus \{\alpha_1\}) \cup A_2; Y; R)$ is an extended star cutset of $G$ separating $B_2$ from a node in $G'_1 \setminus S$. □

**Lemma 4.7** The number of graphs in the list $\mathcal{L}$ produced by the $2$-Join Decomposition Algorithm is linear in the size of the input graph $G$.

**Proof**: For a graph $G$, let $\Phi(G) = |E(G)| - |V(G)| - 1$.

First, we show that, if a connected graph $G$ has a $2$-join with blocks $G_1$, $G_2$, then $\Phi(G_1) + \Phi(G_2) < \Phi(G)$. Consider a $2$-join of $G$, say $E(K_{A_1,A_2}) \cup E(K_{B_1,B_2})$, and let $G'_1$, $G'_2$ be the graphs described in the definition of a $2$-join. Then

$$\Phi(G) = |E(G'_1)| + |E(G'_2)| + |A_1| \times |A_2| + |B_1| \times |B_2| - |V(G'_1)| - |V(G'_2)| - 1$$

and

$$\Phi(G_i) = |E(G'_i)| + |A_i| + |B_i| - |V(G'_i)| - 2.$$ 

Now $\Phi(G_1) + \Phi(G_2) < \Phi(G)$ follows by observing that any positive integers $p$, $q$ satisfy $p + q \leq p \times q + 1$.

Now we show that, if $G$ has a $2$-join but no extended star cutset, then $\Phi(G) > 0$, $\Phi(G_1) \geq 0$ and $\Phi(G_2) \geq 0$. Since $G$ has a $2$-join, it has more than four nodes and therefore it is $2$-connected. Thus, for $i = 1, 2$, $G_i$ is $2$-connected as well and its number of edges is at least $|V(G_i)|$, i.e. $\Phi(G_i) \geq -1$. If $\Phi(G_i) = -1$, then $G_i$ is a hole, but this is impossible by Property (iii) in the definition of a $2$-join. Therefore $\Phi(G_i) \geq 0$. Since $\Phi(G_1) + \Phi(G_2) < \Phi(G)$, it follows that $\Phi(G) > 0$.

This implies that the total number of blocks created in the $2$-join decomposition algorithm is at most $2\Phi(G)$, i.e. it is linear in the size of the input graph. □

### 4.5 Recognition Algorithm

We now give the recognition algorithm, prove its validity and polynomial time bound.

**RECOGNITION ALGORITHM**

**Input**: A signed bipartite graph $G$.

**Output**: YES if $G$ is balanced and NO otherwise.

**Step 1**: Check whether $G$ contains an unbalanced hole of length 4 or 6. If it does output NO.

**Step 2**: Apply the Short 3-Wheel Procedure to check whether $G$ contains a short 3-wheel. If it does, output NO.

**Step 3**: Apply the Cleaning Procedure to $G$ and let $\mathcal{L}_1$ be the output family of graphs.

**Step 4**: For each $L \in \mathcal{L}_1$, apply the Double Star Cutset and 6-Join Decomposition Algorithm. If $L$ is identified as not being balanced output NO, and otherwise union the output with $\mathcal{L}_2$.

**Step 5**: For each $L \in \mathcal{L}_2$, apply the 2-Join Decomposition Algorithm and union the output with $\mathcal{L}_3$. 

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Step 6: For each $L \in \mathcal{L}_3$, check whether $L$ is strongly balanced. If some $L \in \mathcal{L}_3$ is not strongly balanced, then output NO. If every $L \in \mathcal{L}_3$ is strongly balanced, output YES.

Remark 4.8 An algorithm that tests whether a signed bipartite graph is strongly balanced is given in [5]. Hence the details of Step 6 are omitted in this paper.

Theorem 4.9 The Recognition Algorithm produces the desired output and it can be implemented to run in time polynomial in the size of the input graph $G$.

Proof: If $G$ contains an unbalanced hole of length 4 or 6, a short 3-wheel or a 3-path configuration, then the algorithm correctly identifies $G$ as not being balanced. So suppose that the algorithm does not terminate in Step 1, 2 or 4.

Claim 1: No $L \in \mathcal{L}_3$ contains an extended star cutset, a 6-join or a 2-join.

Proof of Claim 1: The graphs in $\mathcal{L}_2$ do not contain a 6-join, a double star cutset or any dominated nodes. By Lemma 3.1, they do not contain an extended star cutset. So by the 2-Join Decomposition Algorithm, graphs in $\mathcal{L}_3$ do not contain an extended star cutset, a 6-join or a 2-join. This completes the proof of Claim 1.

Claim 2: $G$ is balanced if and only if all the graphs in $\mathcal{L}_3$ are balanced.

Proof of Claim 2: If $G$ is balanced, then all the induced subgraphs of $G$ are balanced, and hence all the graphs in $\mathcal{L}_3$ are balanced. Suppose that $G$ is not balanced. Then $G$ contains a smallest unbalanced hole $H^*$. By the Cleaning Procedure, some graph $G' \in \mathcal{L}_1$ contains $H^*$ and $H^*$ is clean in $G'$. By Lemma 3.9 all the holes in $\mathcal{C}_{G'}(H^*)$ are clean in $G'$. By the Double Star Cutset and 6-Join Decomposition Algorithm, some graph $G'' \in \mathcal{L}_2$ contains an unbalanced hole in $\mathcal{C}_{G'}(H^*)$. So $G$ is balanced if and only if all the graphs in $\mathcal{L}_2$ are balanced. Then, by the 2-Join Decomposition Algorithm, $G$ is balanced if and only if all the graphs in $\mathcal{L}_3$ are balanced. This completes the proof of Claim 2.

So by Claim 1, Claim 2 and Theorem 1.1, $G$ is balanced if and only if every $L \in \mathcal{L}_3$ is strongly balanced. Hence the algorithm correctly identifies $G$ as balanced or not balanced.

Now we show that the Recognition Algorithm can be implemented to run in time polynomial in the size of the input graph $G$. Steps 1 and 2 can clearly be implemented to run in polynomial time. By Remark 4.1, Lemma 4.4 and Lemma 4.7, the Cleaning Procedure, the Double Star Cutset and 6-Join Decomposition Algorithm and the 2-Join Decomposition Algorithm can be implemented to run in polynomial time. Furthermore, the number of graphs in $\mathcal{L}_3$ is polynomial in the size of $G$. So by Remark 4.8, Step 6 can also be implemented to run in polynomial time. □

References


