Decomposition of Balanced Matrices

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Abstract

A 0,1 matrix is balanced if it does not contain a square submatrix of odd order with two ones per row and per column. We show that a balanced 0,1 matrix is either totally unimodular or its bipartite representation has a cutset consisting of two adjacent nodes and some of their neighbors. This result yields a polytime recognition algorithm for balancedness. To prove the result, we first prove a decomposition theorem for balanced 0,1 matrices that are not strongly balanced.

to the memory of Ray Fulkerson, who planted the seeds of this research twenty five years ago, in a graduate course at Cornell University.

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1 Introduction

1.1 Decomposition theorem

In an undirected graph $G$, a cycle is balanced if its length is a multiple of 4. The graph $G$ is balanced if all its chordless cycles are balanced. Clearly, a balanced graph is simple and bipartite.

We prove a decomposition theorem for balanced graphs and we give a polytime recognition algorithm based on this decomposition theorem. The theorem states that every balanced graph is either "strongly balanced" or contains a cutset, namely an "extended star cutset" or a "2-join". These three concepts are defined next.

A biclique is a complete bipartite graph $K_{AB}$ where the two sides of the bipartition $A$ and $B$ are both nonempty.

Extended star cutsets

In a connected bipartite graph $G$, an extended star $(x; T; A; R)$ is defined by disjoint node sets $T$, $A$, $R$ and a node $x \in T$ such that all nodes in $A \cup R$ are neighbors of $x$ and the node set $T \cup A$ induces a biclique. Furthermore, if $|T| \geq 2$, then $|A| \geq 2$. The set $R$ may be empty. See Figure 1. The node set $S = T \cup A \cup R$ is an extended star cutset if $G \setminus S$ is a disconnected graph.

Since the nodes in $T \cup A$ induce a biclique, an extended star cutset with $R = \emptyset$ is called a biclique cutset. An extended star cutset having $T = \{x\}$ is called a star cutset, since it is composed by a node $x$ and a subset of its neighbors. Note that a star cutset is a special case of a biclique cutset.

2-join

Let $G$ be a connected bipartite graph with more than four nodes, containing bicliques $K_{A_1,A_2}$ and $K_{B_1,B_2}$, where $A_1$, $A_2$, $B_1$, $B_2$ are disjoint nonempty node sets. The edge set $E(K_{A_1,A_2}) \cup E(K_{B_1,B_2})$ is a 2-join if it satisfies the following properties (see Figure 1):

(i) The graph $G' = G \setminus (E(K_{A_1,A_2}) \cup E(K_{B_1,B_2}))$ is disconnected.

(ii) Every connected component of $G'$ has a nonempty intersection with exactly two of the sets $A_1$, $A_2$, $B_1$, $B_2$ and these two sets are either $A_1$ and $B_1$ or $A_2$ and $B_2$. For $i = 1, 2$, let $G'_i$ be the subgraph of $G'$ containing all its connected components that have nonempty intersection with $A_i$ and $B_i$.

(iii) If $|A_1| = |B_1| = 1$, then $G'_1$ is not a chordless path or $A_2 \cup B_2$ induces a biclique. If $|A_2| = |B_2| = 1$, then $G'_2$ is not a chordless path or $A_1 \cup B_1$ induces a biclique.

The purpose of Property (iii) is to exclude "improper" 2-joins, as we discuss later, in Section 2.

When the graph $G$ comprises more than one connected component, we say that $G$ has an extended star cutset or a 2-join if at least one of its connected components does.
Figure 1: An extended star and a 2-join
Strongly balanced graphs

A graph is strongly balanced if it is balanced and contains no cycle with exactly one chord.

**Theorem 1.1** If $G$ is a balanced graph that is not strongly balanced, then $G$ has an extended star cutset or a 2-join.

Balanced matrices

Given a 0,1 matrix $A$, the bipartite representation of $A$ is the bipartite graph $G = (V^r \cup V^c, E)$ having a node in $V^r$ for every row of $A$, a node in $V^c$ for every column of $A$ and an edge $ij$ joining nodes $i \in V^r$ and $j \in V^c$ if and only if the entry $a_{ij}$ of $A$ equals 1. Conversely, let $G = (V^r \cup V^c, E)$ be a bipartite graph with no parallel edges. Up to permutations of rows and columns and taking transpose, there is a unique 0,1 matrix $A$ having $G$ as bipartite representation.

A 0,1 matrix $A$ is balanced if its bipartite representation is a balanced graph. Equivalently, $A$ is balanced if and only if $A$ does not contain a square submatrix of odd order with exactly two ones per row and per column. Balanced matrices were first introduced by Berge [2] and we summarize here their relevance in combinatorial optimization.

Berge [3] showed that if $A$ is balanced, the polyhedra $P(A) = \{ x \geq 0 \mid Ax \leq 1 \}$ and $Q(A) = \{ y \geq 0 \mid yA \geq 1 \}$ have only vertices with 0,1 components. Berge and Las Vergnas [8] showed that if $A$ is balanced, then the maximum number of 1’s in a 0,1 vector $x \in P(A)$ is equal to the minimum number of 1’s in a 0,1 vector $y \in Q(A)$. More generally, Fulkerson, Hoffman and Oppenheim [23] showed that the inequalities defining $P(A)$ and $Q(A)$ are totally dual integral systems. A hypergraph is balanced if its node-edge incidence matrix is a balanced matrix. Balanced hypergraphs can be viewed as a natural generalization of bipartite graphs. This is the motivation that led Berge to introduce the notion of balancedness. Indeed, a hypergraph with exactly two nodes in each edge is balanced if and only if it is a bipartite graph. Balanced hypergraphs can be characterized by a bicoloring theorem [2]: The nodes of a balanced hypergraph can be colored either red or blue in such a way that every edge with at least two nodes contains both a red node and a blue node. Hall’s theorem [26] about the existence of a perfect matching in a bipartite graph extends to balanced hypergraphs [14]. Further results on balanced matrices are surveyed in [12].

Totally unimodular matrices

A matrix is totally unimodular if every square submatrix has a determinant equal to 0, +1 or -1. A consequence of Theorem 1.1 is the following result, proved in Section 2 of this paper.

**Theorem 1.2** If a 0,1 matrix is balanced but not totally unimodular, then its bipartite representation has an extended star cutset.

Let $(P)$ be a property of 0,1 matrices that is invariant upon permutation of rows, permutation of columns and taking transpose (such as total unimodularity, balancedness, etc). It will be convenient to refer to a “bipartite graph with Property $(P)$” to mean that it is the bipartite representation of a 0,1 matrix with Property $(P)$.
In Section 3, we use Theorem 1.2 to recognize in polytime whether a 0,1 matrix is balanced. The algorithm is presented in terms of bipartite representations. Given a connected bipartite graph $G$, let $S$ be a node set such that $G \setminus S$ is a disconnected graph. Let $G_1', \ldots, G_k'$ denote the connected components of $G \setminus S$. The blocks of the decomposition of $G$ by the node cutset $S$ are the graphs $G_i'$ induced by $V(G_i') \cup S$. The idea of the algorithm is to use extended star cutsets to decompose $G$, and then its blocks, recursively, until no block contains an extended star cutset. This yields a polytime algorithm to recognize whether $G$ is balanced provided that:

- total unimodularity can be recognized in polytime,
- extended star cutsets can be found in polytime,
- the total number of blocks generated is polynomial,
- $G$ is balanced if and only if the blocks are balanced.

Total unimodularity can be recognized in polytime, see [28]. In Section 3, we show how to find extended star cutsets in polytime, and we prove that the number of blocks is polynomial. In general, when $G$ has an extended star cutset, it is false that $G$ is balanced if and only if its blocks are balanced. The major task of Section 3 is to show how to overcome this difficulty.

The rest of the paper (Sections 4 to 8) is devoted to the proof of Theorem 1.1. In fact, we prove that this theorem holds for a more general class of graphs.

Balanceable graphs

A signed graph is a graph whose edges are labelled with +1 or −1. In a signed graph, a cycle is balanced if the sum of its labels is a multiple of 4. A signed graph $G$ is balanced if all its chordless cycles are balanced. Note that every balanced signed graph is bipartite and that, given a balanced undirected graph, we obtain a balanced signed graph by labelling +1 all the edges.

A graph is balanceable if there exists an assignment of ±1 labels to its edges so that the resulting signed graph is balanced. Balanced graphs are obviously balanceable. Two examples of bipartite graphs that are not balanceable are odd wheels and 3-path configurations, which we define next.

Odd wheels, 3-path configurations and a theorem of Truemper

A hole of a bipartite graph is a chordless cycle. A wheel $(H, v)$ is a bipartite graph consisting of a hole $H$ and a node $v$ having at least three neighbors in $H$. The wheel $(H, v)$ is odd if $v$ has an odd number of neighbors in $H$.

A 3-path configuration is a bipartite graph consisting of three internally node-disjoint chordless paths, connecting two nonadjacent nodes $u$ and $v$ in opposite sides of the bipartition, and containing no edge other than those of the paths (see Figure 2). In all figures of this paper, solid lines represent edges and dotted lines represent paths with at least one edge.

Both a 3-path configuration and an odd wheel have the following properties: They contain an odd number of edges and each edge belongs to exactly two holes. Therefore in any signing,
Figure 2: An odd wheel and a 3-path configuration

the sum of the labels of all holes is equal to 2 mod 4. This implies that at least one of the holes is not balanced, showing that neither 3-path configurations nor odd wheels are balanceable. These are in fact the only minimal bipartite graphs that are not balanceable, as a consequence of a theorem of Truemper [30].

**Theorem 1.3** A bipartite graph is balanceable if and only if it does not contain an odd wheel or a 3-path configuration as an induced subgraph.

An easy proof of Truemper's theorem can be found in [15], as well as a discussion of its consequences.

**Weakly balanced graphs**

Two examples of graphs that are balanceable but not balanced are connected 6-holes and $R_{10}$, to be defined next.

A *triad* is a bipartite graph consisting of three internally node-disjoint paths $t, \ldots, u; t, \ldots, v$ and $t, \ldots, w$, where nodes $t, u, v, w$ are distinct and belong to the same side of the bipartition. Furthermore, the graph induced by the nodes of the triad contains no other edges than those of the three paths. The nodes $u, v, w$ are the *attachments* of the triad.

A *fan* consists of a chordless path $P = x, \ldots, y$ together with a node $z$ not in $P$, adjacent to a positive even number of nodes in $P$, where nodes $x, y, z$ belong to the same side of the bipartition and are the *attachments* of the fan.

A *connected 6-hole* is a bipartite graph induced by two disjoint node sets $A$ and $B$ such that each induces either a triad or a fan, their attachments induce a 6-hole and there are no other adjacencies between the nodes of $A$ and $B$.

$R_{10}$ is the graph defined by a cycle $C = x_1, x_2, \ldots, x_{10}, x_1$ with chords $x_i x_{i+5}, 1 \leq i \leq 5$.

A bipartite graph $G$ is *weakly balanced* if $G$ contains no odd wheel, no 3-path configuration, no connected 6-hole and no $R_{10}$ as induced subgraphs. Equivalently, $G$ is weakly balanced if $G$ is balanceable and contains no connected 6-hole and no $R_{10}$ as induced subgraphs. Clearly, every balanced graph is weakly balanced and every weakly balanced graph is balanceable.
A graph is strongly balanceable if it is balanceable and contains no cycle with exactly one chord. Since $R_{10}$ and connected 6-holes contain cycles with a unique chord, every strongly balanceable graph is weakly balanced.

In this paper, we prove the following stronger version of Theorem 1.1.

**Theorem 1.4** If $G$ is a weakly balanced graph that is not strongly balanceable, then $G$ has an extended star cutset or a 2-join.

In [13], a decomposition theorem for the class of balanceable graphs is presented. This result builds on Theorem 1.4: A third decomposition is introduced to deal with connected 6-holes, and the class of strongly balanceable graphs, which is used as "basic class" in Theorem 1.4, is enlarged to contain $R_{10}$.

### 1.2 Additional Definitions and Notation

**Adjacency**

Let $G$ be a graph. Given a node set $S \subseteq V(G)$, node $u \not\in S$ is strongly adjacent to $S$ if $u$ has at least two neighbors in $S$. (Throughout this paper $\subset$ denotes strict inclusion while $\subseteq$ denotes inclusion.) We denote by $N(u)$ the set of neighbors of $u$ in $G$, and by $N_H(u)$ the set of neighbors of $u$ in $H$, for any subgraph $H$ of $G$. For a node $v \in V(H)$, we say that node $u \not\in V(H)$ is a twin of $v$ relative to $H$, if $N_H(u) = N_H(v)$.

**Paths and Cycles**

A path $P$ may be denoted by the sequence of its distinct nodes $x_1, x_2, \ldots, x_n$, $n \geq 1$. A path having $x_1$ and $x_n$ as endnodes is an $x_1x_n$-path. Let $x_i$ and $x_l$ be two nodes of $P$, where $l \geq i$. The path $x_i, x_{i+1}, \ldots, x_l$ is called the $x_i x_l$-subpath of $P$ and is denoted by $P_{x_i x_l}$. We write $P = x_1, x_2, x_3, x_4, x_5, x_6$. When $n \geq 3$, we denote by $P$ the $x_2 x_n$-subpath of $P$.

For a path $P$ (a cycle $C$), the edges connecting consecutive nodes of $P$ (of $C$) are called the edges of $P$ (edges of $C$) and this edge set is denoted by $E(P)$ ($E(C)$ respectively). The length of $P$ or $C$ is equal to the cardinality of $E(P)$ or $E(C)$. 

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**Figure 3:** $R_{10}$ and a connected 6-hole
Direct Connections

Let \( A, B, C \) be three disjoint node sets such that no node of \( A \) is adjacent to a node of \( B \). A path \( P = x_1, x_2, \ldots, x_n \) connects \( A \) and \( B \) if \( x_1 \) is adjacent to at least one node in \( A \) and \( x_n \) is adjacent to at least one node in \( B \). The path \( P \), connecting \( A \) and \( B \), is a direct connection between \( A \) and \( B \) if, in the subgraph induced by the nodes \( V(P) \cup A \cup B \), no path connecting \( A \) and \( B \) is shorter than \( P \). A direct connection between \( A \) and \( B \) avoids \( C \) if \( V(P) \cap C = \emptyset \).

1.3 Classes of balanced graphs and decomposition theorems

The decomposition approach adopted in this paper was already used in the literature for several classes of balanced graphs and for related classes of matrices. The result of Yannakakis [31] for restricted unimodular matrices, the results of Anstee and Farber [1], Hoffman, Kolen and Sakanovich [27], Golumbic and Goss [24] for totally balanced matrices, the results of Conforti and Rao for strongly balanced matrices [16] and linear balanced matrices [17] and Seymour’s [29] characterization of totally unimodular matrices are all in this spirit.

Restricted balanced graphs

A graph is restricted balanced if every cycle is balanced. Restricted balanced graphs were introduced by Commoner [10]. A graph is basic if it is bipartite and all the nodes in one side of the bipartition have degree at most two. Testing whether a graph is basic is trivial and testing whether a basic graph is restricted balanced amounts to testing whether a graph is bipartite. Yannakakis [31] proved the following decomposition theorem:

**Theorem 1.5** Let \( G \) be a biconnected restricted balanced graph that is not basic. Then \( G \) has a 2-join consisting of two edges.

An algorithm for checking whether a graph is restricted balanced follows from this theorem, see [31]. See [16] for a different algorithm.

1-Joins and strongly balanced graphs

Let \( K_{AB} \) be a bidice of a connected graph \( G \) with the property that \( G \setminus E(K_{AB}) \) is disconnected. Let \( V_A \) be the node set of the connected components of \( G \setminus E(K_{AB}) \) with at least one node in \( A \). Similarly, let \( V_B \) be the node set of the connected components with at least one node in \( B \). The set \( E(K_{AB}) \) forms a 1-join if \( |V_A| \geq 2 \) and \( |V_B| \geq 2 \). This concept was introduced by Cunningham and Edmonds [22]. Conforti and Rao [16] prove the following decomposition theorem for strongly balanced graphs:

**Theorem 1.6** Let \( G \) be a strongly balanced graph that is not restricted balanced. Then \( G \) has a 1-join.

An algorithm for checking whether a graph is strongly balanced follows from this theorem, see [16].
Figure 4: Classes of balanced graphs
Totally balanced graphs

A graph is totally balanced if it is bipartite and every hole has length 4. Totally balanced graphs arise in location theory and were the first balanced graphs to be the object of an extensive study. Several authors (Golumbic and Goss [24], Golumbic [25], Anstee and Farber [1] and Hoffman, Kolen and Sakarovitch [27]) gave properties of these graphs. An edge \( uv \) of a bipartite graph is bisimplicial if the node set \( N(u) \cup N(v) \) induces a biclique. Note that if \( uv \) is a bisimplicial edge and nodes \( u \) and \( v \) have degree at least \( 2 \), then \( G \) has a 2-join formed by the edges connecting \( u \) and \( v \) to nodes in \( G \setminus \{u, v\} \). The following theorem of Golumbic and Goss characterizes totally balanced graphs and can be used in a recognition algorithm, see [24].

**Theorem 1.7** A totally balanced graph has a bisimplicial edge.

Linear balanced graphs

A bipartite graph is linear if it does not contain a cycle of length 4. Note that an extended star cutset in a linear bipartite graph is always a star cutset. Conforti and Rao [17] prove the following decomposition theorem for linear balanced graphs. This can be used to check whether a linear bipartite graph is balanced, see [19], [20].

**Theorem 1.8** Let \( G \) be a linear balanced graph that is not strongly balanced. Then \( G \) has a star cutset.

Figure 4 shows the Venn diagram for the classes of balanced graphs defined above.

1.4 Even wheels, parachutes, connected squares, goggles

Here we define four weakly balanced graphs that play an important role in the proof of Theorem 1.4.

**Even wheels**

A wheel \( (H, v) \) is even if \( v \) has an even number \( (\geq 4) \) of neighbors in \( H \).

**Parachutes**

A parachute is defined by four paths of positive length \( P_1 = v_1, \ldots, z, P_2 = v_2, \ldots, z, M = v_3, \ldots, z \) and \( T = v_1, \ldots, v_2 \), where nodes \( v \) and \( z \) are in the same side of the bipartition, nodes \( v_1 \) and \( v_2 \) are adjacent to \( v \) and \( |E(P_1)| + |E(P_2)| \geq 3 \). No other adjacency exists in a parachute. See Figure 5.

**Connected Squares**

Connected squares are defined by four chordless paths of positive lengths \( P_1 = s_1, \ldots, t_1; P_2 = s_2, \ldots, t_2; P_1' = s_1', \ldots, t_1'; P_2' = s_2', \ldots, t_2' \), where nodes \( s_1 \) and \( s_2 \) are adjacent to both \( s_1' \) and \( s_2' \) and nodes \( t_1 \) and \( t_2 \) are adjacent to both \( t_1' \) and \( t_2' \), as in Figure 6(a). No other adjacency exists in connected squares. The nodes \( s_1, s_2, t_1, t_2 \) are in one side of the bipartition and \( s_1', s_2', t_1', t_2' \) are in the other.
Figure 5: Parachute

Figure 6: Connected squares and goggles
Goggles

Goggles are defined by a chordless path $T = h, \ldots, v$ and a cycle $C = h, P, x, a, Q, v, R, b, u, S, h$, with chords $ua$ and $xb$ (and $hv$ when $T$ has length 1), where $P, Q, R, S$ denote chordless paths of length greater than one, such that no intermediate node of $T$ belongs to $C$. No other edge with both endnodes in $V(C) \cup V(T)$ exists in goggles. The nodes $a, b, h$ are on one side of the bipartition and $x, u, v$ are in the other, see Figure 6(b).

1.5 Outline of the proof of the main theorem

In this subsection, we state the results used in the proof of Theorem 1.4. We first introduce the following classes of graphs:

- A bipartite graph is wheel-free if it contains no wheel.
- A bipartite graph is wheel-and-parachute-free if it contains no wheel and no parachute.

Let $G$ be a weakly balanced graph. In Section 4, we study the case where $G$ is wheel-and-parachute-free and we prove the following theorem:

**Theorem 1.9** If $G$ is a wheel-and-parachute-free weakly balanced graph that is not strongly balanceable, then $G$ has a 2-join.

Section 5 proves a decomposition result when $G$ contains a wheel:

**Theorem 1.10** If $G$ is a weakly balanced graph that contains a wheel, then $G$ has an extended star cutset.

Sections 6, 7 and 8 deal with the remaining case, namely the case when $G$ contains a parachute but no wheel. Section 6 proves the following result:

**Theorem 1.11** Let $G$ be a wheel-free weakly balanced graph containing a parachute. Then $G$ has an extended star cutset, or $G$ contains connected squares or goggles.

Section 7 studies connected squares:

**Theorem 1.12** Let $G$ be a wheel-free weakly balanced graph containing connected squares. Then $G$ has a biclique cutset or a 2-join.

Section 8 studies goggles:

**Theorem 1.13** Let $G$ be a wheel-free weakly balanced graph containing goggles but no connected squares. Then $G$ has an extended star cutset or a 2-join.

Clearly, Theorem 1.4 follows from these five results.
1.6 Some conjectures and open questions

The following conjecture has been formulated in [17]:

**Conjecture 1.14** Every balanced graph $G$ has an edge $e$ with the property that $G \setminus e$ remains balanced.

In other words, if a 0,1 matrix is balanced, the 1’s can be turned into 0’s sequentially so that all intermediate matrices are balanced. Conjecture 1.14 is obviously equivalent to the following:

**Conjecture 1.15** Every balanced graph contains an edge that is not the unique chord of a cycle.

This property holds for totally balanced graphs, as a consequence of Theorem 1.7. Note that every edge of the graph $R_{10}$ is the unique chord of a cycle of length 8, hence the above conjectures cannot be extended to the class of balanceable graphs.

A biclique cutset is a special case of an extended star cutset, hence the question arises whether Theorem 1.1 can be strengthened, by showing that every balanced graph that is not strongly balanced has a biclique cutset or a 2-join.

**Conjecture 1.16** If $G$ is a wheel-free balanced graph that is not strongly balanced, then $G$ has a biclique cutset or a 2-join.

Note that Theorem 1.12 proves this conjecture when $G$ contains connected squares. Such a result would be interesting since biclique cutsets preserve balancedness in wheel-free graphs.

**Theorem 1.17** If $G$ is a wheel-free graph that contains a biclique cutset, then $G$ is balanced if and only if all the blocks are.

*Proof:* The "only if" part is obvious, since the blocks are induced subgraphs of $G$.

To prove the "if" part, assume $G$ has biclique cutset $A \cup B$, $G$ is not balanced but all the blocks are. Let $H$ be an unbalanced hole of $G$. At least two nonconsecutive nodes of $H$, say $v_i$ and $v_j$, belong to $A \cup B$, else $H$ is contained in some block. Furthermore, nodes $v_i$ and $v_j$ belong to the same set, else $H$ has a chord. Assume w.l.o.g. that $v_i, v_j \in A$. Let $w$ be a node of $B$. If $w \in V(H)$, then $wv_1$ and $wv_2$ are edges of $H$ and $H$ contains no other node of $A \cup B$, else $H$ has a chord. Now, it follows that $H$ is contained in some block, a contradiction. So $w \notin V(H)$. Assume that $w$ has no neighbor in $V(H)$ other than nodes $v_i, v_j$, and let $P_1, P_2$ be the two subpaths of $H$ connecting $v_i$ and $v_j$. Then the holes $H_1 = v_i, w, v_j, P_1, v_i$ and $H_2 = v_i, w, v_j, P_2, v_i$ have distinct length mod 4, and each one belongs to a block, a contradiction to the assumption that all blocks are balanced. Hence $w$ has at least three neighbors in $H$, and $(H, w)$ is a wheel. □

The graph in Figure 7 shows that Conjecture 1.16 cannot be extended to all balanced graphs. More generally, we define an infinite family of graphs as follows. Let $H$ be a hole where nodes $u_1, \ldots, u_p, v_1, \ldots, v_q, w_1, \ldots, w_p, x_1, \ldots, x_q$ appear in this order when traversing $H$, but are not necessarily adjacent. Let $Y = \{y_1, \ldots, y_p\}$ and $Z = \{z_1, \ldots, z_q\}$ be two node
sets having empty intersection with $V(H)$ and inducing a biclique $K_{YZ}$. Node $y_i$ is adjacent to $u_i$ and $w_i$ for $1 \leq i \leq p$. Node $z_i$ is adjacent to $v_i$ and $x_i$ for $1 \leq i \leq q$. Any balanced graph of this form for $p, q \geq 2$ is called a $W_{pq}$. For all values of $p, q \geq 2$, the graph $W_{pq}$ has no 2-join and no biclique cutset.

Since the graphs $W_{pq}$ contain a wheel, a stronger form of the above conjecture is the following.

**Conjecture 1.18** If $G$ is a balanced graph that is not strongly balanced, then $G$ is either a $W_{pq}$ or has a biclique cutset or a 2-join.
2 2-Joins, Balancedness and Total Unimodularity

2.1 Introduction

In this section, we define the blocks $G_1$ and $G_2$ of a 2-join decomposition and we show that $G$ is balanced if and only if $G_1$ and $G_2$ are balanced.

A 2-join $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$ is rigid if $A_1 \cup B_1$ or $A_2 \cup B_2$ induces a biclique. For a 2-join that is not rigid, we show that $G$ is totally unimodular if and only if the blocks $G_1$ and $G_2$ are totally unimodular.

These results are then used to deduce Theorem 1.2 from Theorem 1.1.

2.2 2-Join decomposition

Let $K_{A_1A_2}$ and $K_{B_1B_2}$ define a 2-join of $G$. The blocks $G_1$ and $G_2$ of the 2-join decomposition are defined as follows. For $i = 1, 2$, let $G'_i$ be the subgraph of $G \setminus (E(K_{A_1A_2}) \cup E(K_{B_1B_2}))$ containing all its connected components that have nonempty intersection with $A_i$ and $B_i$. To obtain $G_i$, we first add to $G'_i$ a node $\alpha_i$, adjacent to all the nodes in $A_i$ and to no other node of $G'_i$ and a node $\beta_i$, adjacent to all the nodes in $B_i$ and to no other node of $G'_i$.

If neither $A_1 \cup B_1$ nor $A_2 \cup B_2$ induces a biclique, let $Q_1$ be a shortest path in $G'_2$ connecting a node in $A_1$ to a node in $B_2$, and let $Q_2$ be a shortest path in $G'_1$ connecting a node in $A_1$ to a node in $B_1$. Note that the existence of $Q_1$, $Q_2$ is guaranteed by (ii) in the definition of 2-joins. For $i = 1, 2$, add to $G_i$ a marker path $M_i$ connecting $\alpha_i$ and $\beta_i$ with length $3 \leq |E(M_i)| \leq 6$ and $|E(M_i)|$ congruent to $|E(Q_i)|$ modulo 4.

If exactly one set $A_1 \cup B_1$ or $A_2 \cup B_2$ induces a biclique, say $A_1 \cup B_1$, then no marker path is added in $G_1$ and a marker path $M_2$ consisting of a single edge, connecting $\alpha_2$ and $\beta_2$, is added to $G_2$.

If both $A_1 \cup B_1$ and $A_2 \cup B_2$ induce bicliques, then no marker path is added in $G_1$ and $G_2$.

The graphs $G_1$ and $G_2$ are the blocks of the 2-join decomposition of $G$. It follows from (iii) in the definition of a 2-join that the blocks $G_1$ and $G_2$ are both distinct from $G$. Furthermore, some graph invariant decreases. In that sense, the 2-join decomposition is "proper". For example, let $\Phi(G) = |E(G)| - |V(G)| - 1$.

**Lemma 2.1** If a connected graph $G$ has a 2-join with blocks $G_1$, $G_2$, then $\Phi(G_1) + \Phi(G_2) < \Phi(G)$. Furthermore, if $G$ has no extended star cutset, then $\Phi(G_1) \geq 0$ and $\Phi(G_2) \geq 0$.

**Proof:** Consider a 2-join of $G$, say $E(K_{A_1A_2}) \cup E(K_{B_1B_2})$, and let $G'_1$, $G'_2$ be defined as above. Then

$$\Phi(G) = |E(G'_1)| + |E(G'_2)| + |A_1| \times |A_2| + |B_1| \times |B_2| - |V(G'_1)| - |V(G'_2)| - 1$$

and

$$\Phi(G_i) \leq |E(G'_i)| + |A_i| + |B_i| - |V(G'_i)| - 2.$$

Now $\Phi(G_1) + \Phi(G_2) < \Phi(G)$ follows by observing that any positive integers $p$, $q$ satisfy $p + q \leq p \times q + 1$.

Now assume that $G$ has no extended star cutset. Since $G$ has a 2-join, it has more than four nodes and therefore it is 2-connected. Thus, for $i = 1, 2$, $G_i$ is 2-connected as well and
its number of edges is at least $|V(G_1)|$, i.e. $\Phi(G_i) \geq -1$. If $\Phi(G_i) = -1$, then $G_i$ is a hole, but this is impossible by Property (iii) in the definition of a 2-join. Therefore $\Phi(G_i) \geq 0$. □

**Theorem 2.2** Let $G_1$, $G_2$ be the blocks of a 2-join decomposition of a connected bipartite graph $G$. Then $G$ is a balanced graph if and only if both $G_1$ and $G_2$ are balanced graphs.

**Proof:** We first prove that, if $G$ is balanced, then $G_1$ and $G_2$ are balanced. Assume not and let $H$ be an unbalanced hole of $G_1$ or $G_2$.

**Case 1** Neither $A_1 \cup B_1$ nor $A_2 \cup B_2$ induces a biclique.

Assume w.l.o.g. that $H$ is in $G_1$. Then $H$ must contain nodes $a_1$ and $b_1$, but not the marker path connecting them, otherwise $G$ would contain a hole with the same length mod 4 as $H$. Since $G$ contains nonadjacent nodes $a_2 \in A_2$ and $b_2 \in B_2$, the hole induced by $(V(H) \setminus \{a_1, b_1\}) \cup \{a_2, b_2\}$ is an unbalanced hole of $G$.

**Case 2** Exactly one of $A_1 \cup B_1$ or $A_2 \cup B_2$ induces a biclique.

Then $G_1$ and $G_2$ are induced subgraph of $G$ and therefore they are balanced.

**Case 3** Both $A_1 \cup B_1$ and $A_2 \cup B_2$ induce bicliques.

W.l.o.g. $H$ is in $G_1$. $H$ must contain nodes $a_1$ and $b_1$, otherwise $G$ would contain a hole with the same length as $H$. But then $H$ has chords, a contradiction.

We now prove that, if $G_1$ and $G_2$ are balanced, then $G$ is balanced. Assume not and let $H$ be an unbalanced hole of $G$. Then $H$ must contain at least one node in each of the set $A_1$, $A_2$, $B_1$, $B_2$, otherwise $H$ is also a hole contained in $G_1$ or $G_2$. If $H$ contains no edge of $G_i'$, then $H = a_1', a_2', a_1'', \ldots, b_1', b_2', b_1', \ldots, a_1'$ where $a_1', a_1'' \in A_1$ and $b_1', b_2', b_1' \in B_1$, $a_2' \in A_2$ and $b_2' \in B_2$. Then neither $A_1 \cup B_1$ nor $A_2 \cup B_2$ induces a biclique and $H' = a_1', a_1'', \ldots, b_1', b_2', b_1', \ldots, a_1'$ is an unbalanced hole of $G_1$, a contradiction. So $H = a_1', a_2', P_2, b_2', b_1', P_1, a_1'$, where $a_1' \in A_1$ and $b_1' \in B_1$, $a_2' \in A_2$ and $b_2' \in B_2$. If both sets $A_1 \cup B_1$ and $A_2 \cup B_2$ induce bicliques, then $H$ has length 4, contradicting the choice of $H$. If exactly one set $A_1 \cup B_1$ or $A_2 \cup B_2$ induces a biclique, say $A_1 \cup B_1$, then $G_2$ contains a hole of the same length as $H$. Now consider the case where neither $A_1 \cup B_1$ nor $A_2 \cup B_2$ induces a biclique. Since $G_1$ contains no unbalanced hole, the length of its marker path $M_1$ is not congruent to the length of $P_2$ mod 4. It follows that $G_2'$ contains a chordless path $Q_2'$ connecting a node $a_2'' \in A_2$ to a node $b_2'' \in B_2$ whose length is not congruent to the length of $P_2$ mod 4. The holes $a_2''', P_2, b_2'', \beta_2, M_2, a_2', a_2'''$ and $a_2''', Q_2, b_2'', \beta_2, M_2, a_2', a_2'''$ have distinct lengths mod 4. Hence one of them is unbalanced, contradicting the assumption that $G_2$ is balanced. □

Next, we prove a lemma which is a graphical analog of the fact that 3-sums preserve regularity in matroid theory.

**Lemma 2.3** Let $G$ be a connected bipartite graph with a 2-join that is not rigid, and let $G_1$ and $G_2$ be the blocks of the corresponding 2-join decomposition of $G$. Then $G$ is totally unimodular if and only if both $G_1$ and $G_2$ are totally unimodular.
Proof: A graph is Eulerian if all its nodes have even degree. By Camion's theorem [9], a bipartite graph is totally unimodular if and only if it contains no Eulerian induced subgraph with 2 (mod 4) edges. If \( G_1 \) or \( G_2 \) contains an Eulerian induced subgraph with 2 (mod 4) edges, then so does \( G \) since the length of a marker path is the same (mod 4) as that of a path in \( G \).

So, to prove the lemma, it suffices to show that, if \( G \) contains an Eulerian induced subgraph with 2 (mod 4) edges, then so does \( G_1 \) or \( G_2 \). Let \( H \) be an Eulerian induced subgraph of \( G \) with 2 (mod 4) edges. Let \( E(\Gamma_{A_1,A_2}) \cup E(\Gamma_{B_1,B_2}) \) be the 2-join of \( G \) and let \( A_i = V(H) \cap A_i^* \), \( B_i = V(H) \cap B_i^* \), for \( i = 1, 2 \). We remove from \( H \) subsets of edges that form 4-cycles in \( E(\Gamma_{A_1,A_2}) \) or in \( E(\Gamma_{B_1,B_2}) \), as follows (the resulting graph \( H' \) is Eulerian with 2 (mod 4) edges, but may not be induced in \( G \):

(i) If \( |A_1| \) or \( |A_2| \) are both even, then all edges of \( K_{A_1,A_2} \) are removed.

(ii) If \( |A_1| \) is even and \( |A_2| \) is odd, then choose \( a_2 \in A_2 \) and remove all edges of \( K_{A_1,A_2 \setminus \{a_2\}} \).

(iii) If both \( |A_1| \) and \( |A_2| \) are odd, then choose \( a_1 \in A_1 \) and \( a_2 \in A_2 \), and remove all edges of \( K_{A_1 \setminus \{a_1\}, A_2 \setminus \{a_2\}} \).

Edges with endnodes in \( B_1 \cup B_2 \) are removed in the same way, based on the parity of \( |B_1| \) and \( |B_2| \). Note that (iii) occurs for \( A_1, A_2 \) if and only if (iii) occurs for \( B_1, B_2 \), since \( H \) is Eulerian and therefore the number of edges in \( E(K_{A_1,A_2}) \cup E(K_{B_1,B_2}) \) is even.

Case 1 (i) or (ii) occurs.

Then the edges of \( H' \) partition into two Eulerian graphs \( H_1 \) and \( H_2 \) that are induced subgraphs of \( G_1 \) and \( G_2 \). One of these subgraphs has 2 (mod 4) edges.

Case 2 (iii) occurs.

For \( i = 1, 2 \), define \( H'_i \) to be the subgraph of \( H' \setminus (E(K_{A_1,A_2}) \cup E(K_{B_1,B_2})) \) containing the connected components that have nonempty intersection with \( A_i \) and \( B_i \). Define \( H_i \) to be the graph induced by \( V(H'_i) \) and the nodes of the marker path \( M_i \) of \( G_i \). Then \( H_1 \) and \( H_2 \) are Eulerian induced subgraphs of \( G_1 \) and \( G_2 \) respectively. Furthermore, \( |E(H_1)| + |E(H_2)| \) is equal to \( |E(H')| + |E(M_1)| + |E(M_2)| + 2 \), since the edges \( a_1a_2 \) and \( b_1b_2 \) of \( H' \) correspond to edges in both \( H_1 \) and \( H_2 \) and all the other edges appear exactly once.

If \( |E(M_1)| + |E(M_2)| \) equals 2 (mod 4), it follows that \( |E(H_1)| \) or \( |E(H_2)| \) equals 2 (mod 4) and we are done.

Otherwise, \( |E(M_1)| + |E(M_2)| \) equals 0 (mod 4), since this sum must be even. Now \( M_2 \) has the same length (mod 4) as a shortest path \( Q_2 \) in \( G_1' \) connecting a node in \( A_1 \) to a node in \( B_1 \). Therefore the nodes of \( Q_2 \) and \( M_1 \) induce a hole of length 2 (mod 4) in \( G_1 \). This is the required Eulerian induced subgraph.

\( \square \)
2.3 Bipartite graphs without extended star cutsets

**Lemma 2.4** Let $G$ be a bipartite graph that has no extended star cutset. Then $G$ has no rigid 2-join.

*Proof:* Assume $G$ has a rigid 2-join $E(K_{A_1,A_2}) \cup E(K_{B_1,B_2})$ and let $A_1 \cup B_1$ induce a biclique. If $|A_1| > 1$ or $|B_1| > 1$ or $V(G'_i) \setminus (A_1 \cup B_1) \neq \emptyset$, then $G$ has a star cutset with center in $A_1$ or $B_1$. So $|A_1| = |B_1| = 1$ and $V(G'_i) \setminus (A_1 \cup B_1) = \emptyset$. Now, $A_2 \cup B_2$ must be a biclique by (iii) of the definition of a 2-join. So, as above, it follows that $|A_2| = |B_2| = 1$ and $V(G'_2) \setminus (A_2 \cup B_2) = \emptyset$. But then $G$ has only four nodes, contradicting the definition of 2-join. \hfill \Box

**Lemma 2.5** Let $G$ be a balanced graph that has no extended star cutset. Then, in every 2-join, $V(G'_i) \setminus (A_i \cup B_i) \neq \emptyset$, for $i = 1, 2$.

*Proof:* Assume otherwise, say $V(G'_i) \setminus (A_1 \cup B_1) = \emptyset$. By (ii) in the definition of a 2-join, every node of $A_1$ has a neighbor in $B_1$ and, vice versa, every node in $B_1$ has a neighbor in $A_1$. By Lemma 2.4, the 2-join is not rigid. These two facts imply that $|A_1| \geq 2$ and $|B_1| \geq 2$. Furthermore, every node in $A_1$ has a node in $B_1$ that it is not adjacent to (otherwise, there is a star cutset) and every node in $B_1$ has a node in $A_1$ that it is not adjacent to. Let $u$ be a node of largest degree in the graph induced by $A_1 \cup B_1$. W.l.o.g. assume $u \in A_1$. Let $Y$ be the set of neighbors of $u$ in $B_1$ and let $v \in B_1 \setminus Y$. Let $w \in A_1$ be a neighbor of $v$. Then $w$ is not adjacent to some node $y \in Y$, by our choice of $u$. Since the 2-join is not rigid, $A_2 \cup B_2$ is not a biclique, i.e. there exist $a_2 \in A_2$ and $b_2 \in B_2$ that are not adjacent. Now $ua_2wvb_2yu$ is a $6$-hole, a contradiction. \hfill \Box

**Lemma 2.6** Let $G$ be a balanced graph that has no extended star cutset. If $G$ has a 2-join, then the blocks $G_1, G_2$ of the 2-join decomposition do not have an extended star cutset.

*Proof:* Assume otherwise, i.e. one of the blocks, say $G_1$, has an extended star cutset $S = (x; T; Q; R)$. By Lemma 2.4, the 2-join is not rigid. So, for $i = 1, 2$, $G_i$ contains a marker path $M_i = \alpha_i, \ldots, \beta_i$ of length $|E(M_i)| \geq 3$. Let $G'_i = G_i \setminus V(M_i)$.

**Case 1** Node $x$ coincides with $\alpha_1$ or $\beta_1$.

Assume w.l.o.g. that $x$ coincides with $\alpha_1$. Since $|E(M_1)| \geq 3$, $\beta_1$ is not in $S$. So, $S$ is a cutset that separates $\beta_1$ from a node in $G'_1 \setminus S$. We can assume w.l.o.g. that the neighbor of $\alpha_1$ in $M_1$ is not in $S$, since the set obtained by removing that neighbor from $S$ would also be an extended star cutset of $G_1$. So $Q \cup R \subseteq A_1$. If $S$ is a star cutset, i.e. $T = \emptyset$ and $Q = \emptyset$, then $S^* = R \cup A_2$ is a biclique cutset of $G$, separating $B_2$ from a node in $G'_1 \setminus S$. So assume that $|T| \geq 2$. Then at least two nodes of $A_1$ are contained in $Q$. Let $x^*$ be any node of $A_2$. Then $S^* = (x^*; (T \cup A_2) \setminus \{x\}; Q; R)$ is an extended star cutset of $G$ separating $B_2$ from a node in $G'_1 \setminus S$.

**Case 2** Node $x$ is an intermediate node of $M_1$.

Since $M_1$ has length at least 3, we must have $|T| = 1$, i.e. $S$ is a star cutset. W.l.o.g. assume $\beta_1 \notin S$. Then $S$ separates $\beta_1$ from a node in $G'_1 \setminus S$. But then $S' = \{\alpha_1\}$ is also a star cutset of $G_1$. So, by Case 1, we are done.
Case 3 Node $x$ is in $A_1$ or $B_1$.

W.l.o.g. assume that $x$ is in $A_1$. If $\beta_1 \notin S$, then $S$ separates $\beta_1$ from a node in $G' \setminus S$. If $\alpha_1 \notin Q \cup R$, let $S^* = S$. If $\alpha_1 \in R$, let $S^* = (x; T; Q; (R \setminus \{\alpha_1\}) \cup A_2)$ and if $\alpha_1 \in Q$, let $S^* = (x; T; (Q \setminus \{\alpha_1\}) \cup A_2; R)$. Then $S^*$ is an extended star cutset of $G$ separating $B_2$ from a node in $G' \setminus S$. So $\beta_1 \in S$ and hence $\beta_1 \in T$. Thus $Q \subseteq B_1$. Now $S^* = (x; (T \setminus \{\beta_1\}) \cup B_2; Q; (R \setminus \{\alpha_1\}) \cup A_2)$ is an extended star cutset of $G$ separating a node of $G' \setminus S$ from a node of $G' \setminus (A_2 \cup B_2)$. (Note that this graph is nonempty by Lemma 2.5.)

Case 4 Node $x$ is in $G' \setminus (A_1 \cup B_1)$.

Not both $\alpha_1$ and $\beta_1$ can be in $S$. Assume w.l.o.g. that $\beta_1 \notin S$. Then $S$ is a cutset separating $\beta_1$ from a node in $G' \setminus S$. If $\alpha_1 \notin S$, then $S$ is a cutset of $G$ separating $B_2$ from a node in $G' \setminus S$. So $\alpha_1 \in S$. Then $\alpha_1 \in T$, $Q \subseteq A_1$ and hence $S^* = (x; (T \setminus \{\alpha_1\}) \cup A_2; Q; R)$ is an extended star cutset of $G$ separating $B_2$ from a node in $G' \setminus S$.

Now we prove that Theorem 1.2 follows from Theorem 1.1 and the above results.

Proof of Theorem 1.2: Assume that $G$ is a balanced graph that has no extended star cutset. Decompose $G$ into blocks using 2-join decompositions, recursively, until no 2-joins exist. This process is finite, by Lemma 2.1. All the blocks are balanced by Lemma 2.2. By Lemma 2.6, the blocks have no extended star cutset. So, by Theorem 1.1, all the blocks are strongly balanced. Strongly balanced graphs are totally unimodular [16]. By Lemma 2.4, the 2-joins used in the decomposition are not rigid, and, by Lemma 2.3, 2-joins that are not rigid preserve total unimodularity. It follows that $G$ is totally unimodular. □
3 Polytime Recognition Algorithm

In this section, we use Theorem 1.2 to recognize in polytime whether a bipartite graph $G$ is balanced. Since it is not true that the blocks of an extended star cutset decomposition are balanced if and only if $G$ is balanced, we decompose a family of subgraphs of $G$, say $G_1, \ldots, G_p$, instead of just $G$. Then, by applying extended star cutset decompositions to all the $G_i$'s, we can show that the desired property holds for this larger family of blocks, namely, all blocks in this family are balanced if and only if $G$ is balanced. To describe the appropriate family of graphs $G_i$, we need to study smallest unbalanced holes in bipartite graphs $G$ that are not balanced.

3.1 Undominated Graphs

A node $u$ is said to be dominated if there exists a node $v$, distinct from $u$, such that $N(u) \subseteq N(v)$. The graph $G$ is said to be undominated if it contains no dominated nodes. A double star is a node set $S = N(u) \cup N(v)$ where $u, v$ are adjacent nodes. $S$ is a double star cutset if $G \setminus S$ is nonempty and contains more connected components than $G$. The lemma below shows a relation between double star cutsets and extended star cutsets for undominated bipartite graphs.

Lemma 3.1 If an undominated bipartite graph $G$ has an extended star cutset, then it has a double star cutset.

Proof: Let $S = (x; T; A; R)$ be an extended star cutset of $G$ and let $G'_1, G'_2, \ldots, G'_k$ be the connected components of $G \setminus S$. By definition, $A \neq \emptyset$. Define $S^* = N(x) \cup N(v)$, where $v$ is a node in $A$. Clearly, $S \subseteq S^*$. Suppose $S^*$ is not a double star cutset of $G$. Then all the nodes in one of the connected components of $G \setminus S$, say $G'_i$, belong to $S^* \setminus S$. Hence $V(G'_i) \subseteq N(x) \cup N(v)$, i.e. each node in $G'_i$ is dominated by $x$ or by $v$. \square

Lemma 3.1 and Theorem 1.2 imply:

Theorem 3.2 If $G$ is an undominated balanced graph that is not totally unimodular, then $G$ has a double star cutset.

3.2 Smallest unbalanced holes

Let $G$ be a bipartite graph that is not balanced and let $H^*$ be a smallest unbalanced hole in $G$. In this subsection, we study properties of strongly adjacent nodes to $H^*$.

A strongly adjacent node $u$ to a hole $H$ in $G$ is odd-strongly adjacent if $u$ has an odd number of neighbors in $H$, and it is even-strongly adjacent if it has an even number of neighbors in $H$. The sets $A^r(H)$ and $A^e(H)$ contain the odd-strongly adjacent nodes to $H$ that belong to $V^r$ and $V^e$ respectively.

The following properties of the sets $A^r(H^*)$ and $A^e(H^*)$, associated with a smallest unbalanced hole $H^*$ were proven by Conforti and Rao in [19].

Property 3.3 There exists a node $x^r \in V^r \cap V(H^*)$ that is adjacent to all the nodes in $A^r(H^*)$. There exists a node $x^e \in V^e \cap V(H^*)$ that is adjacent to all the nodes in $A^e(H^*)$. 

Property 3.4 Every even strongly adjacent node to $H^*$ is a twin of a node in $H^*$

The above properties were used in [20] to design a polytime algorithm to test whether a linear bipartite graph is balanced. To test balancedness of a bipartite graph, we need the following additional properties of strongly adjacent nodes.

Definition 3.5 A tent $\tau(H, u, v)$ is a subgraph of $G$ induced by a hole $H$ and two adjacent nodes $u$ and $v$ that are even strongly adjacent to $H$ with the following property:

The nodes of $H$ can be partitioned into two subpaths containing the nodes in $N(u) \cap V(H)$ and $N(v) \cap V(H)$ respectively.

A tent $\tau(H, u, v)$ is referred to as a tent containing $H$. We now study properties of a tent $\tau(H^*, u, v)$ containing a smallest unbalanced hole $H^*$. By Property 3.4, both $u$ and $v$ are twins of nodes of $H$. We assume throughout that the first node, say $u$ in the definition of a tent $\tau(H, u, v)$ belongs to $V^r$ and that the second node, say $v$, belongs to $V^c$. We use the notation of Figure 8, where nodes $u_1, u_0, u_2, v_1, v_0, v_2$ are encountered in this order, when traversing $H^*$.

\[\begin{tikzpicture}
  \node (u) at (0,0) [circle, fill, inner sep=2pt] {}; \node (v) at (0,-2) [circle, fill, inner sep=2pt] {v}; \node (u0) at (1,1) [circle, fill, inner sep=2pt] {}; \node (u1) at (2,1) [circle, fill, inner sep=2pt] {}; \node (v0) at (1,-1) [circle, fill, inner sep=2pt] {}; \node (v1) at (2,-1) [circle, fill, inner sep=2pt] {}; \node (u2) at (3,0) [circle, fill, inner sep=2pt] {}; \node (v2) at (3,-2) [circle, fill, inner sep=2pt] {};
  \draw (u) -- (v) -- (u0) -- (u1) -- (v0) -- (v1) -- (v2) -- (u2) -- (u) -- (v);
\end{tikzpicture}\]

Figure 8: Tent

Lemma 3.6 Let $H^*$ be a smallest unbalanced hole and $\tau(H^*, u, v)$ be a tent containing it. At least one of the two sets $N(v_0) \cup N(u_1)$, $N(v_0) \cup N(u_2)$ contains $A^r(H^*)$. At least one of the two sets $N(u_0) \cup N(v_1)$, $N(u_0) \cup N(v_2)$ contains $A^c(H^*)$.

Proof: By symmetry, we only need to prove the first statement. Suppose $v_0$ is not adjacent to a node $w \in A^r(H^*)$. Consider the hole $H^*_1$ obtained from $H^*$ by replacing $v_0$ with node $v$. Now $w$ in not adjacent to $v$, for otherwise $w$ is even strongly adjacent to $H^*_1$, violating Property 3.4. Therefore, $w$ is in $A^r(H^*_1)$. Node $u$ is in $A^r(H^*_1)$ and has neighbors $u_1, u_2$ and $v$ in $H^*_1$. By Property 3.3, all nodes in $A^r(H^*_1)$ have a common neighbor in $H^*_1$. So it follows that this common neighbor must be $u_1$ or $u_2$. \qed
Lemma 3.7 Let $H^*$ be a smallest unbalanced hole and $\tau(H^*, u, v)$, $\tau(H^*, w, y)$ be two tents containing $H^*$, where $w_1$, $w_2$ are the neighbors of $w$ and $y_1$, $y_2$ are the neighbors of $y$ in $H^*$. Let $w_0$ and $y_0$ be the common neighbors in $H^*$ of $w_1$, $w_2$ and $y_1$, $y_2$ respectively. Then at least one of the following properties holds:

- Nodes $u_1$ and $u_2$ coincide with $w_1$ and $w_2$.
- Nodes $v_1$ and $v_2$ coincide with $y_1$ and $y_2$.
- Nodes $u_0$ and $y$ are adjacent.
- Nodes $v_0$ and $w$ are adjacent.

Proof: Suppose the contrary. Then $u$, $v$, $w$, $y$ are all distinct nodes and one of the following two cases occurs. The edges of $H^*$ can be partitioned in two paths $P_1$, $P_2$ with common endnodes so that either (Case 1:) $P_1$ contains $u_1$, $u_2$, $v_1$, $v_2$ and $P_2$ contains $w_1$, $w_2$, $y_1$, $y_2$ or (Case 2:) $P_1$ contains $u_1$, $u_2$, $y_1$, $y_2$ and $P_2$ contains $v_1$, $v_2$, $w_1$, $w_2$.

Assume $u$ and $y$ are adjacent and consider the hole $H^*_{wy}$ contained in $V(H^*) \cup \{w, y\}$, containing $w$, $u_1$, $u_2$. Then $(H^*_{wy}, u)$ is an odd wheel and all the holes of $(H^*_{wy}, u)$ are smaller than $H^*$. Since one of them is unbalanced, we have a contradiction to the minimality of $H^*$. By symmetry, $w$ and $v$ are nonadjacent as well.

In Case 1, consider the hole $H^*_{vwy}$ contained in $V(H^*) \cup \{v, w, y\}$, containing $v$, $w$, $y$, $u_1$, $u_2$. Then $(H^*_{vwy}, u)$ is an odd wheel and all the holes of $(H^*_{vwy}, u)$ are smaller than $H^*$, a contradiction. In Case 2, nodes $u$ and $y$ are connected by a $3PC(u, y)$. The three holes of this 3-path configuration are smaller than $H^*$ and at least one of them is unbalanced. □

3.3 A Recognition Algorithm

In this subsection, we give an algorithm to test whether a bipartite graph is balanced.

Definition 3.8 A hole $H$ is said to be clean in $G$ if the following three conditions hold:

(i) No node is odd-strongly adjacent to $H$.

(ii) Every even-strongly adjacent node is a twin of a node in $H$.

(iii) There is no tent containing $H$.

In a wheel $(W, v)$, a subpath of $W$ having two nodes of $N_W(v)$ as endnodes and only nodes of $V(W) \setminus N_W(v)$ as intermediate nodes is called a sector of $(W, v)$. A short 3-wheel is a wheel with three sectors, at least two of which have length 2.

RECOGNITION ALGORITHM

Input: A bipartite graph $G$.

Output: $G$ is identified as balanced or not balanced.

Step 1 Apply Procedure 1 to check whether $G$ contains a short 3-wheel. If so, $G$ is not balanced, otherwise go to Step 2.
Step 2 Apply Procedure 2 to create at most \( |V^r|^4 |V^c|^4 \) induced subgraphs \( G_1, \ldots, G_p \) such that, if \( G \) is not balanced, one of the induced subgraphs created, say \( G_i \), contains an unbalanced hole \( H^* \) that is clean in \( G_i \).

Step 3 Apply Procedure 3 to each of the graphs \( G_1, \ldots, G_p \) to decompose them into undominated induced subgraphs \( B_1, \ldots, B_s \) that do not contain a double star cutset. While decomposing a graph with a double star cutset \( N(u) \cup N(v) \), Procedure 3 also checks for the existence of a 3-path configuration containing nodes \( u \) and \( v \) and nodes in two distinct connected components resulting from the decomposition. If such a configuration is found, then \( G \) is not balanced, otherwise go to Step 4.

Step 4 Test whether all the blocks \( B_1, \ldots, B_s \) are totally unimodular. If so, \( G \) is balanced, otherwise \( G \) is not balanced.

An algorithm to test whether a bipartite graph is totally unimodular can be found in [28]. Hence the details of Step 4 are omitted in this paper.

### 3.4 Short 3-Wheels

PROCEDURE 1, for identifying whether \( G \) contains a short 3-wheel, can be described as follows: Let \( C = a_1, a_2, a_3, a_4, a_5, a_6, a_1 \) be a 6-cycle of \( G \) having unique chord \( a_2a_5 \). If \( a_1 \) and \( a_3 \) are in the same connected component of \( G \setminus (N(a_2) \cup N(a_4) \cup N(a_5) \cup N(a_6) \setminus \{a_1, a_3\}) \), or if \( a_4 \) and \( a_6 \) are in the same connected component of \( G \setminus (N(a_1) \cup N(a_2) \cup N(a_3) \cup N(a_5) \setminus \{a_4, a_6\}) \), then a short 3-wheel containing \( C \) is identified. Otherwise \( G \) has no short 3-wheel containing \( C \). Perform such a test for all 6-cycles of \( G \) with a unique chord.

The complexity of this procedure is of order \( O(|V^r|^4 |V^c|^4) \).

### 3.5 Clean Unbalanced Holes

In this subsection, we show how to create at most \( |V^r|^4 |V^c|^4 \) induced subgraphs of \( G \) such that, if \( G \) is not balanced, one of these subgraphs, say \( G_i \), contains an unbalanced hole that is smallest in \( G \) and is clean in \( G_i \).

Given a graph \( F \) and nodes \( i, j, k, l \) of \( F \) that induce the chordless path \( i, j, k, l \), we define \( F_{ijkl} \) to be the induced subgraph obtained from \( F \) by removing the nodes in \( N_F(j) \cup N_F(k) \setminus \{i, j, k, l\} \).

PROCEDURE 2

Input: A bipartite graph \( G \).

Output: A family \( \mathcal{L} = G_1, \ldots, G_p \), where \( p \leq |V^r|^4 |V^c|^4 \), of induced subgraphs of \( G \) such that if \( G \) is not balanced, one of the subgraphs in \( \mathcal{L} \), say \( G_i \), contains an unbalanced hole that is smallest in \( G \) and is clean in \( G_i \).

Step 1 Let \( \mathcal{L}^* = \{G_{ijkl} : \text{nodes } i, j, k, l \text{ of } G \text{ induce the chordless path } i, j, k, l\} \)

Step 2 Let \( \mathcal{L} = \{Q_{ijkl} : \text{the graph } Q \text{ is in } \mathcal{L}^* \text{ and nodes } i, j, k, l \text{ of } Q \text{ induce the chordless path } i, j, k, l\} \).

We now prove the validity of Procedure 2.

**Lemma 3.9** If \( G \) is not balanced, one of the graphs in \( \mathcal{L} \), say \( G_i \), contains an unbalanced hole \( H^* \) that is smallest in \( G \) and is clean in \( G_i \).
Proof: Let $H^*$ be a smallest unbalanced hole in $G$. Choose two induced paths $u_1, u_0, u_2, u_3$ and $v_1, v_0, v_2, v_3$ on $H^*$ as follows:

- If no tent contains $H^*$: $A^r(H^*) \subseteq N(u_0)$ and $A^r(H^*) \subseteq N(v_0)$. (This choice is possible by Property 3.3.)

- If some tent $\tau(H^*, u, v)$ contains $H^*$: $\{u_1, u_2\} = N(u) \cap V(H^*)$ and $\{v_1, v_2\} = N(v) \cap V(H^*)$. By Lemma 3.6, we can index $u$, $i = 1, 2$, so that $A^r(H^*) \subseteq N(u_0) \cup N(u_2)$ and we can index $v$, $i = 1, 2$, so that $A^r(H^*) \subseteq N(v_0) \cup N(v_2)$. By Lemma 3.7, for every tent $\tau(H^*, w, y)$ that contains $H^*$, $w$ or $y$ is adjacent to one of the nodes in $\{u_0, u_2, v_0, v_2\}$.

So $(G_{u_1 u_0 u_2 u_3})_{v_1 v_0 v_2 v_3}$ belongs to $\mathcal{L}$ and contains $H^*$ but in $(G_{u_1 u_0 u_2 u_3})_{v_1 v_0 v_2 v_3}$, no tent contains $H^*$ and $H^*$ has no strongly adjacent nodes other than twins.  

\[\square\]

3.6 Double Star Cuset Decompositions

We describe a procedure to decompose a bipartite graph $G$ into blocks that are induced subgraphs and do not have a double star cutset.

Definition 3.10 Let $H$ be a hole in a graph. Then $\mathcal{C}(H) = \{H_i \mid H_i$ is a hole that can be obtained from $H$ by a sequence of holes $H = H_0, H_1, \ldots, H_i$; where $|V(H_j) \setminus V(H_{j-1})| = 1$, for $j = 1, 2, \ldots, i\}$. 

Lemma 3.11 Let $G$ be a bipartite graph that is not balanced and contains no short 3-wheel. If $H$ is a smallest unbalanced hole and is clean in $G$, then every hole $H_i$ in $\mathcal{C}(H)$ is clean in $G$, $|H_i| = |H|$ and $\mathcal{C}(H_i) = \mathcal{C}(H)$.

Proof: Let $H = H_0, H_1, \ldots, H_i$ be a sequence of holes as in Definition 3.10. It suffices to show the lemma for $H_1$. Since $A^r(H) \cup A^r(H) = \emptyset$, by Property 3.4, $H_1$ has been obtained from $H = x_1, x_2, x_3, \ldots, x_n, x_1$ by substituting one node, say $x_3$, with its twin $y_3$. We assume w.l.o.g. that $x_3, y_3 \in V^r$. So $|H_1| = |H|$ and $\mathcal{C}(H_1) = \mathcal{C}(H)$.

Assume $A^r(H_1) \cup A^r(H_1) \neq \emptyset$. Since $H$ is clean, then $H$ must contain a twin $y$ of $x_1$ in $V^c$, where $y$ is adjacent to $y_3$ but not to $x_3$. Now $\tau(H, y_3, y)$ is a tent, a contradiction to the assumption that $H$ is clean, and this proves that $H_1$ satisfies (i) of Definition 3.8.

The fact that $H$ is clean shows that $H_1$ satisfies (ii) of Definition 3.8.

Finally, assume $H_1$ contains a tent $\tau(H_1, x, y)$, where $x \in V^r$. Then $x_3 \neq x$, else $y \in A^r(H)$. So $x$ is a twin of a node $x_3$ in $G$ and $y$ is adjacent to $y_3$. We can assume that $x_1$ is the unique neighbor of $y$ in $H$. Now let $x_{i}$ be the neighbor of $x$ with lowest index and let $C = x_1, x_2, x_3, \ldots, x_i, x, y_3, x_1$. The neighbors of $y_3$ in $C$ are $y$, $x_2, x_4$ and $(C, y_3)$ is a short 3-wheel. This proves that $H_1$ satisfies (iii) of Definition 3.8.  

\[\square\]

PROCEDURE 3

Input: A bipartite graph $F$ not containing a short 3-wheel.

Output: Either a 3-path configuration of $F$, or a list of undominated induced subgraphs $F^1, \ldots, F^q$ of $F$ each containing an induced path of length 3, where $q \leq |V^r(F)|^2|V^r(F)|^2$ with the following properties:

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• The graphs \( F_1^*, \ldots, F_q^* \) do not contain a double star cutset.

• If the input graph \( F \) is not balanced and contains a clean unbalanced hole that is smallest in \( F \), then one of the graphs in the list, say \( F_i^* \), contains an unbalanced hole \( H^* \) in \( \mathcal{C}(H) \), that is smallest and clean in both \( F \) and \( F_i^* \).

**Step 1** Delete dominated nodes in \( F \) until \( F \) becomes undominated. Let \( M = \{F\}, T = \phi \).

**Step 2** If \( M \) is empty, stop. Otherwise remove a graph \( R \) from \( M \). If \( R \) has no double star cutset, add \( R \) to \( T \) and repeat Step 2. Otherwise, let \( S = N_R(u) \cup N_R(v) \) be a double star cutset of \( R \). Let \( R_1^*, \ldots, R_q^* \) be the connected components of \( R \setminus S \) and let \( R_1, \ldots, R_i \) be the corresponding blocks, that is \( R_i \) is the graph induced by \( V(R_i^*) \cup S \). Go to Step 3.

**Step 3** Consider every pair of nonadjacent nodes \( u_p \) and \( v_q \) adjacent to \( u \) and \( v \) respectively and distinct from \( u \) and \( v \). If there exist two distinct connected components of \( R \setminus S \) that each contain neighbors of \( u_p \) and neighbors of \( v_q \), there is a \( 3PC(u_p, v_q) \) and \( F \) is not balanced. Otherwise go to Step 4.

**Step 4** From each block \( R_i \), remove dominated nodes until the resulting graph \( R_i^* \) becomes undominated. Add to \( M \) all the graphs \( R_i^* \) that contain at least one chordless path of length 3. Go to Step 2.

**Lemma 3.12** Let \( F \) be a bipartite graph that does not contain a short 3-wheel and let \( H^* \) be a smallest unbalanced hole that is clean in \( F \).

If Procedure 3, when applied to \( F \), does not detect a 3-path configuration in Step 3, then one of the graphs \( F_i^* \), obtained as output of Procedure 3, contains an unbalanced hole in \( \mathcal{C}(H^*) \).

**Proof:** Let \( N(u) \cup N(v) \) be a double star cutset of \( F \), used in Procedure 3. Let \( F_1^*, \ldots, F_q^* \) be the connected components of \( F \setminus (N(u) \cup N(v)) \) and \( F_1, \ldots, F_i \) be the corresponding blocks. We first show that if no 3-path configuration is detected in Step 3, an unbalanced hole \( H' \in \mathcal{C}(H^*) \) is contained in some block \( F_i \) obtained at the end of Step 3.

Choose \( H' \in \mathcal{C}(H^*) \) such that \( V(H') \cap \{u, v\} \) is maximal. By Lemma 3.11, \( H' \) is clean in \( F \), so \( u \) is either in \( H' \) or has at most one neighbor in \( H' \) and the same holds for \( v \).

Let \( W \) be the subgraph induced by \( V(H') \setminus (N(u) \cup N(v)) \). We have three possibilities for \( W \):

(i) If \( H' \) contains no neighbor of \( u \) and \( v \), then \( W = H' \).

(ii) If both \( u \) and \( v \) have a single neighbor \( u_1 \) and \( v_1 \) in \( H' \) and \( u_1, v_1 \) are nonadjacent, then \( W \) consists of two paths.

(iii) In all the remaining cases, it is easy to check that \( W \) consists of a single path.

If \( H' \) does not belong to any of the blocks \( F_1, \ldots, F_i \), the graph \( W \) must be disconnected and have a component in, say, \( F_i^* \) and another in, say, \( F_j^* \). So (ii) holds. Let \( u_1 \) and \( v_1 \) be the neighbors of \( u \) and \( v \) in \( H' \). Then \( V(W) \cup \{u, v\} \) induces a \( 3PC(u_1, v_1) \) which is detected in Step 3 of the algorithm.

So, at the end of Step 3, one block \( F_i \) contains \( H' \) and, by Lemma 3.11, \( H' \in \mathcal{C}(H^*) \) is clean in \( F_i \). Since \( H' \) is clean, the graph \( F_i^* \), obtained from \( F_i \) by removing dominated nodes, contains a hole \( H'' \in \mathcal{C}(H') = \mathcal{C}(H^*) \), where possibly \( H' = H'' \). \[\square\]
Lemma 3.13 The number of graphs $F_1^*, \ldots, F_q^*$ produced by Procedure 3 applied to $F$ is bounded by $|V^*(F)|^2|V^+(F)|^2$. So is the number of double star cutsets used by Procedure 3.

Proof: Let $N(u) \cup N(v)$ be a double star cutset of $F$. Let $F_1', \ldots, F_t'$ be the connected components of $F \setminus (N(u) \cup N(v))$ and let $F_1^*, \ldots, F_t^*$ be the corresponding undominated blocks.

Claim 1 No two distinct undominated blocks contain the same chordless path of length 3.

Proof of Claim 1: Suppose by contradiction that a chordless path $P = a, b, c, d$ belongs to two distinct undominated blocks $F_i^*$ and $F_j^*$. Then $\{a, b, c, d\} \subseteq N_F(u) \cup N_F(v)$.

Node $u$ is distinct from $a$ and $d$ for otherwise $a$ and $d$ are adjacent and $P$ is not a chordless path. By symmetry, $v$ is also distinct from $a$ and $d$. Since both $F_i^*$ and $F_j^*$ are undominated, both nodes $a$ and $d$ have at least one neighbor in both the connected components $F_i'$ and $F_j'$. Now Step 3 of Procedure 3 detects a 3-path configuration. This completes the proof of Claim 1.

Claim 2 The graph $F$ contains at least one chordless path of length 3 that is not contained in any of the undominated blocks $F_i^*$.

Proof of Claim 2: Each of the connected components $F_1', \ldots, F_t'$ must contain at least two nodes, since $F$ is an undominated graph. At least one node in $F_i'$ must be adjacent to a node in $N_F(u) \cup N_F(v)$. Assume w.l.o.g. that node $p_i$ in $F_i'$ is adjacent to a neighbor of $v$, say $v_i$.

Suppose now no node in $F_i'$ is adjacent to a node in $N(u)$. Then the nodes in $N(u) \setminus \{v\}$ are dominated by $v$ in $F_i' \cup N(u) \cup N(v)$. Thus, the undominated block $F_i^*$ does not contain any neighbor of $u$ except $v$. This in turn implies that node $u$ is dominated by $v_i$. Thus $u$ would have been deleted from $F_i^*$. Now $P = p_i, v_i, v, u$ is a chordless path of length 3 in $F$ but $P$ is not in any of the undominated blocks $F_1^*, \ldots, F_t^*$.

So a node in $F_i'$ must be adjacent to a node, say $u$, that is a neighbor of $u$. Repeating the same argument for $j = 1, \ldots, t$, it follows that each connected component $F_j'$ contains a node, say $w_j$, that is adjacent to a node, say $u_j \in N_F(u)$. Suppose now $u_j$ has a neighbor, say $g$ in a connected component $F_k'$, distinct from $F_j'$. Let $q$ be a neighbor of $g$ in $F_k'$. Then $P = q, g, u_j, w_j$ is a chordless path of length 3 contained in $F$ but not in any of the undominated blocks $F_1^*, \ldots, F_t^*$. Suppose now that $u_j$ does not have any neighbor in $F_k'$, $k \neq j$. Then, in Step 4 of Procedure 3, node $u_j$ is deleted from the undominated block $F_k^*$. Now the path $w_k, u_k, u, u_j$ is a chordless path of length 3 contained in $F$ but not in any of the undominated blocks $F_1^*, \ldots, F_t^*$. This completes the proof of Claim 2.

Every undominated block that is added to the list $M$ in Step 4 of Procedure 3 contains a chordless path of length 3. Hence every undominated block that is added to the list $T$ in Step 2 contains a chordless path of length 3. By Claim 1, the same chordless path of length 3 is not in any other undominated block that is added to the list $T$. So the number of graphs in the list $F_1^*, \ldots, F_q^*$ is at most $|V^*(F)|^2|V^+(F)|^2$. By Claim 2, it follows that the number of double star cutsets used to decompose the graph $F$ with Procedure 3 is at most $|V^*(F)|^2|V^+(F)|^2$.

3.7 Validity of the Algorithm

We now prove the validity of the recognition algorithm given in Subsection 3.3.
Theorem 3.14 The running time of the recognition algorithm is polynomial in the size of the input graph $G$, and the algorithm correctly identifies $G$ as balanced or not.

Proof: The recognition algorithm described in Subsection 3.3 applies first Procedures 1, 2 and 3. The running time of each of these procedures has been shown to be polytime in its respective subsection. Finally, in Step 4, the algorithm checks whether each of the (polynomially many) blocks is totally unimodular. Total unimodularity can be checked in polytime [28]. Hence the running time of the recognition algorithm described in Subsection 3.3 is polynomial.

Suppose $G$ is balanced. Then $G$ does not contain a short 3-wheel or a 3-path configuration. All the induced subgraphs of $G$ are balanced, so the graphs produced by Procedures 2 and 3 are balanced. Now, by Theorem 1.2, every graph in the list $B_1,\ldots,B_s$ is totally unimodular. Then Step 4 of the algorithm identifies $G$ as balanced.

Suppose $G$ is not balanced. If $G$ contains a short 3-wheel, Step 1 of the algorithm identifies $G$ as not balanced. Suppose $G$ does not contain a short 3-wheel. Clearly $G$ contains an unbalanced hole of smallest cardinality. Now, by Lemma 3.9, one of the induced subgraphs, say $G_i$, of $G$, in the list produced by Procedure 2 contains an unbalanced hole $H^*$, of smallest cardinality, that is clean in $G_i$. Now $G_i$ is one of the graphs considered for double star cutset decompositions by Procedure 3. By Lemma 3.12, Procedure 3 either detects a 3-path configuration or one of the undominated blocks, say $B_j$, in the final list produced by Procedure 3 contains an unbalanced hole in the family $C(H^*)$. In the former case $G$ is correctly identified as not balanced. In the latter case, $B_j$ is not totally unimodular and Step 4 of the algorithm identifies $G$ as not balanced. \qed

Remark 3.15 A polytime recognition algorithm for balancedness that does not use total unimodularity testing can be obtained as follows. Instead of stopping the recognition algorithm in Step 4, where the blocks are checked for total unimodularity, continue the decomposition process, using 2-join decompositions for each $B_j$. A polytime algorithm for identifying 2-joins is given in [21] for a slightly different definition. This algorithm can be adapted to the 2-join decomposition used here. By pursuing the decomposition process until the blocks contain no 2-join, one can show, using Theorem 1.1, that $G$ is balanced if and only if all the blocks are strongly balanced. In fact, by Theorems 1.6 and 1.5, one can show that $G$ is balanced if and only if all the blocks are basic. This property can be checked in polytime and the number of blocks is polynomial by Lemma 2.1.
4 Wheel-and-Parachute-Free Graphs

4.1 Introduction

In this section, we consider weakly balanced graphs that are wheel-and-parachute-free.

Remark 4.1 The class of wheel-and-parachute-free weakly balanced graphs properly contains totally balanced graphs and strongly balanceable graphs.

Proof: The cycle $H$ of a wheel $(H,v)$ and the cycle induced by the paths $T,P_1,P_2$ in a parachute $Par(T,P_1,P_2,M)$ are holes of length strictly greater than 4. Hence totally balanced graphs are wheel-and-parachute-free.

In a wheel $(H,v)$, a cycle with a unique chord is induced by $v$ and an appropriate subpath of $H$. In a parachute, assume w.l.o.g. that $P_1$ has length greater than 1. Then the graph obtained from the parachute by removing the intermediate nodes of $P_1$ is a cycle with a unique chord. Hence strongly balanceable graphs are wheel-and-parachute-free.

To see that the inclusion is proper, note that a cycle $C$ with a unique chord having length 10 or more, is neither strongly balanceable nor totally balanced. Yet, the cycle $C$ is a wheel-and-parachute-free weakly balanced graph.

A 2-join $E(K_{A_1,A_2}) \cup E(K_{B_1,B_2})$ is said to be rigid if $A_1 \cup B_1$ or $A_2 \cup B_2$ induces a biclique. The main result of this section is the following.

Theorem 4.2 If $G$ is a wheel-and-parachute-free weakly balanced graph that is not strongly balanceable, then $G$ has a rigid 2-join.

Assume $uv$ is a bisimplicial edge of a bipartite graph $G$ and both $A = N(u) \setminus \{v\}$ and $B = N(v) \setminus \{u\}$ are nonempty. Then $G$ contains a rigid 2-join $E(K_{A,\{u\}}) \cup E(K_{B,\{v\}})$ (note that Property (iii) of the definition of a 2-join is satisfied since $A \cup B$ induces a biclique). So, Theorem 4.2 can be viewed as an extension of Theorem 1.7 about totally balanced graphs.

The next result describes the different types of possible strongly adjacent nodes to a cycle with a unique chord.

Theorem 4.3 Let $C$ be a cycle with a unique chord $uv$ in a weakly balanced graph and let $C_1$ and $C_2$ be the two holes of the graph induced by $V(C)$. If $x$ is a strongly adjacent node to $C$, then $x$ is of one of the following types:

Type 1 The set $N_C(x)$ is contained in $V(C_1)$ or in $V(C_2)$.

Then $|N_C(x)|$ is even.

Type 2 The set $N_C(x)$ is not contained in $V(C_1)$ or in $V(C_2)$ and $N(x) \cap \{u,v\} \neq \emptyset$.

Then $|N_{C_1}(x)|$ and $|N_{C_2}(x)|$ are both even.

Type 3 The set $N_C(x)$ is not contained in $V(C_1)$ or in $V(C_2)$ and $N(x) \cap \{u,v\} = \emptyset$.

Then either $|N_{C_1}(x)|$ is even and $|N_{C_2}(x)| = 1$ or $|N_{C_2}(x)|$ is even and $|N_{C_1}(x)| = 1$. Furthermore the unique neighbor of $x$ in $C_1$ or $C_2$ is adjacent to $u$ or $v$. 

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Proof: If, for $i = 1$ or $2$, $N_{C}(x)$ is contained in $V(C_{i})$, then $|N_{C}(x)|$ is even, else $(C_{i}, x)$ is an odd wheel. Thus $x$ of Type 1.

If $N_{C}(x)$ is not contained in $V(C_{1})$ or $V(C_{2})$ and $x$ is adjacent to $u$ or $v$, then $|N_{C_{i}}(x)|$ is even for $i = 1, 2$, else $(C_{i}, x)$ is an odd wheel. Thus $x$ of Type 2.

If $N_{C}(x)$ is not contained in $V(C_{1})$ or $V(C_{2})$ and $x$ is not adjacent to $u$ or $v$, then assume w.l.o.g. that $u$ and $x$ are in opposite sides of the bipartition.

Assume that $|N_{C_{1}}(x)|$ or $|N_{C_{2}}(x)|$ is even. Then there is a $3PC(u, x)$ unless $x$ has a unique neighbor, adjacent to $v$, in $V(C_{1})$ or in $V(C_{2})$. Thus $x$ of Type 3.

Assume that $|N_{C_{1}}(x)| = 1$ for $i = 1, 2$. If both neighbors of $x$ in $V(C)$ are adjacent to $v$, then there is an odd wheel with center $v$. If one neighbor of $x$ in $V(C)$, say $y$, is not adjacent to $v$, then there is a $3PC(y, v)$.

\[\square\]

4.2 Decomposition

Let $G$ be a wheel-and-parachute-free weakly balanced graph. In this subsection we show that, for every edge $uv$ that is the unique chord of at least one cycle, the graph $G$ has a biclique cutset $K_{AB}$ with $u \in A$ and $v \in B$. This result will then be used to prove Theorem 4.2.

For a cycle $C$ with unique chord $uv$, we use the notation of Figure 9. It will be convenient to write $C = (C_{1}, C_{2})$, where $C_{1}$ and $C_{2}$ are the two holes of the graph induced by $V(C)$. We assume w.l.o.g. that $u$ is in $V^{e}$ and that $v$ is in $V^{c}$.

![Figure 9: Cycle with unique chord $C = (C_{1}, C_{2})$](image)

Lemma 4.4 Every node $x$ that is strongly adjacent to $C$ is either of Type 1 $[4,3]$ and has two neighbors in $C_{1}$ or in $C_{2}$, or is a twin of $u$ or $v$ relative to $C$.

Proof: Every strongly adjacent node $x$ is of Type 1, 2 or 3$[4,3]$ and has at most two neighbors in $C_{1}$ and in $C_{2}$, since $G$ contains no wheel.

If $x$ is of Type 2 $[4,3]$, assume w.l.o.g. that $x$ is adjacent to $u$. Then $x$ has exactly two other neighbors in $C$, one in $C_{1}$ and one in $C_{2}$, say $x_{1}$ and $x_{2}$ respectively. If $x_{1}$ is distinct from $c$ (see Figure 9), then there is a parachute with paths $T = u, a, \ldots, x_{1}, P_{1} = u, v, P_{2} = x_{1}, \ldots, c, v$ and $M = x, x_{2}, \ldots, d, v$. So $x_{1} = c$. By symmetry, it follows that $x_{2} = d$.

If $x$ is of Type 3 $[4,3]$, assume w.l.o.g. that $x$ is adjacent to $b$. Then $x$ has exactly two neighbors in $V(C_{1}) \setminus \{u, v\}$, say $x_{1}$ and $x_{2}$. The nodes of $V(C_{1}) \cup \{b, x\}$ induce a parachute, a contradiction.

\[\square\]

A node $x$ fits $C$ if it is adjacent to $u$ or $v$ and has no neighbor in $V(C) \setminus \{u, v, a, b, c, d\}$.

Lemma 4.5 Each node in a direct connection from $V(C_{1}) \setminus \{u, v\}$ to $V(C_{2}) \setminus \{u, v\}$ fits $C$. 

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Proof. We prove the lemma by induction on the length of the direct connection. In case it has only one node, then, by Lemma 4.4, if consists of $u$, $v$ or one of their twins; clearly, such nodes fit $C$. Let $P = x_1, \ldots, x_n$ be a direct connection from $V(C_1) \setminus \{u, v\}$ to $V(C_2) \setminus \{u, v\}$.

Claim If a node of $P$ fits $C$ or is strongly adjacent to $C$, all other nodes of $P$ fit $C$.

Proof of Claim: Let $x_i$ fit $C$ or be strongly adjacent to $C$. By symmetry between $C_1$ and $C_2$, it suffices to show that all $x_j$ with $j > i$ fit $C$. We may as well assume that we have chosen $i$ as small as possible. By symmetry between $u$ and $v$ we may assume that $x_i$ is not adjacent to $v$. So if $i \neq 1$, $x_i$ is adjacent to $u$, and if $i = 1$, $x_1 = x_i$ has two neighbors in $C_1$.

Let $y_1$ be the neighbor of $x_1$ on $C_1$ closest to $v$. If $i \neq 1$, let $y_2 = u$, and if $i = 1$, let $y_2$ be the neighbor of $x_1$ in $C_1$ distinct from $y_1$. If $y_1$ and $y_2$ are adjacent, then $y_2 = u$ and $y_1 = a$.

But, in that case, the subgraph induced by $V(P) \cup V(C) \setminus \{u\}$ contains a hole with at least three neighbors of $a$, namely $a$, $x_i$, and $v$, which contradicts the fact that $G$ is wheel-free. So $y_1$ and $y_2$ are nonadjacent. Consequently, the cycle $C^*$, obtained from $C$ by replacing the $y_1y_2$-subpath of $C_1$ not using $uv$ by $y_1, x_1, P_{x_1x_2}, \ldots, y_2$, has a unique chord. Let $C_1^*$ and $C_2^*$ be the two holes in $C^*$. Then $P_{x_i+1}x_n$ is a direct connection from $V(C_1^*) \setminus \{u, v\}$ to $V(C_2^*) \setminus \{u, v\}$. As it is shorter than $P$, all its nodes fit $C^*$, by the induction hypothesis. Hence, as $C_2^* = C_2$, the nodes of $P_{x_i+1}x_n$ fit $C$. This completes the proof of the claim. □

Let $S(C_1, C_2)$ denote the set of all nodes $x$ in $G$ such that there exists a direct connection from $V(C_1) \setminus \{u, v\}$ to $V(C_2) \setminus \{u, v\}$ starting with $x$ (i.e., with $x$ adjacent to $V(C_1) \setminus \{u, v\}$). Note that $S(C_1, C_2)$ contains $u$, $v$ and their twins relative to $C$ and that, by Lemma 4.5, all the other nodes in $S(C_1, C_2)$ are adjacent to $u$ and $c$ or to $v$ and $a$.

Theorem 4.6 Let $G$ be a wheel-and-parachute-free weakly balanced graph. Let $C$ be a cycle with a unique chord $uv$ and let $C_1$ and $C_2$ be the two holes induced by $C$. Then $S(C_1, C_2)$ is a biclique cutset of $G$ separating $V(C_1) \setminus \{u, v\}$ from $V(C_2) \setminus \{u, v\}$.

Proof: Clearly, $S(C_1, C_2)$ separates $V(C_1) \setminus \{u, v\}$ from $V(C_2) \setminus \{u, v\}$ as it intersects all direct connections. It remains to prove that $S(C_1, C_2)$ induces a biclique. Assume not; then there exist two nonadjacent nodes $x \in V^c \cap S(C_1, C_2)$ and $y \in V^c \cap S(C_1, C_2)$. By Lemma 4.5, $x$ is adjacent to both $u$ and $v$, and $y$ is adjacent to both $c$ and $u$. Let $P$ and $Q$ be direct connections from $V(C_1) \setminus \{u, v\}$ to $V(C_2) \setminus \{u, v\}$ starting with $x$ and $y$ respectively. Let $L$ denote the graph induced by the nodes in $P$, $Q$ and $V(C_2) \setminus \{u, v\}$. Let $R$ be the shortest path connecting $x$ and $y$ in $L$. Let $w$ be the node of $R$ adjacent to $x$. As $x$ and $y$ are not adjacent, $w \neq y$. If $w \in V(C_2) \setminus \{u, v\}$, it is $b$ (as $x$ fits $C$); otherwise it lies on $(V(P) \cup V(Q)) \setminus \{x, y\}$. So, by Lemma 4.5, $w$ is adjacent to $c$. Now, let $T$ be the ac-subpath of $C_1$ not using $uw$. Then $H = x, R, y, c, T, a, x$ is a hole containing at least three neighbors of $u$ (namely $a$, $y$ and $w$), contradicting the fact that $G$ is wheel-free. □

Let $G(C_1, C_2)$ be the block (as defined in Subsection 1.1) containing $C_1$ in the decomposition of $G$ by $S(C_1, C_2)$. Let $W(G(C_1, C_2)) = V(G(C_1, C_2)) \setminus S(C_1, C_2)$. 29
Proof of Theorem 4.2: Let $C = (C_1, C_2)$ be a cycle with a unique chord $uv$ such that $G(C_1, C_2)$ has the smallest possible number of nodes. Let $R$ be the set of nodes in $W(C_1, C_2)$ that are adjacent to all the nodes in $S(C_1, C_2) \cap V^c$ or to all the nodes in $S(C_1, C_2) \cap V^c$. Moreover, let $Z$ be the set of nodes in $W(C_1, C_2) \setminus R$ that have at least one neighbor in $S(C_1, C_2)$. If $R \cup Z$ and $S(C_1, C_2)$ do not induce a rigid 2-join, $Z$ is not empty. So we may assume that $Z \cap V^c$ is nonempty. Let $Q$ be a shortest path in $W(C_1, C_2)$ connecting a node $z \in Z \cap V^c$ with a node $r \in R \cap V^c$. (Note that, by Theorem 4.6, $R \cap V^c \neq \emptyset$ as it contains $u$.) Let $x \in S(C_1, C_2)$ be adjacent to $z$ and let $y \in S(C_1, C_2) \cap V^c$ be nonadjacent to $z$. Clearly, as $Q$ was chosen as short as possible, no intermediate node of $Q$ is adjacent to $x$ or to $y$. Hence the cycle $H_1 = x, z, Q, r, x$ is a hole. Let $\Gamma$ be the component of $G \setminus S(C_1, C_2)$ containing $C_2$. By the definition of $S(C_1, C_2)$, each member of $S(C_1, C_2)$ is adjacent to a node in $\Gamma$. Let $T$ be the shortest $xy$-path in $\Gamma \cup \{x, y\}$ and $H_2 = r, y, T, x, r$. Then $(H_1, H_2)$ is a cycle with a unique chord, namely $xr$. As $H_1 \setminus \{x\}$ is in $W(C_1, C_2)$, we have that $S(H_1, H_2)$ is contained in $G(C_1, C_2)$. On the other hand, $H_2 \setminus \{x, y, r\}$ is in $\Gamma$, and so, as each node of $S(C_1, C_2)$ has a neighbor in $\Gamma$, we see that $S(C_1, C_2) \cap W(H_1, H_2) = \emptyset$. Hence, $G(H_1, H_2)$ in contained in $G(C_1, C_2)$. As $y$ lies in $G(C_1, C_2)$ and not in $G(H_1, H_2)$, that containment is proper, contradicting the minimality of $G(C_1, C_2)$. \qed
5 Even Wheels

5.1 Introduction

In this section, we prove Theorem 1.10, which states that if a graph is weakly balanced and contains an even wheel, then it has an extended star cutset. The proof is in two parts, treated in Subsections 5.2 and 5.3 respectively. In Subsection 5.2, we give properties of the strongly adjacent nodes to even wheels. In Subsection 5.3 we prove Theorem 1.10. In fact, we prove a stronger result. In order to state it, we first introduce some notation.

Given an even wheel \((H, v)\), a subpath of \(H\) having two nodes of \(N_H(v)\) as endnodes and only nodes of \(V(H) \setminus N_H(v)\) as intermediate nodes is called a sector of \((H, v)\). Two sectors are adjacent if they have a common endnode and two nodes of \(N_H(v)\) are consecutive if they are the endnodes of some sector. We paint the nodes of \(V(H) \setminus N_H(v)\) with two colors, say blue and green, in such a way that nodes of \(V(H) \setminus N_H(v)\) have the same color if they are in the same sector, and have distinct colors if they are in adjacent sectors. The nodes of \(N_H(v)\) are left unpainted.

In a bipartite graph \(G\), an even wheel is small if no even wheel of \(G\) contains strictly fewer nodes.

**Theorem 5.1** Let \(G\) be a weakly balanced graph that contains a wheel. Then \(G\) contains an extended star cutset \((x; T; A; R)\) and a small even wheel \((H, x)\) such that \(|N_H(x) \cap A| \geq 2\) and the extended star cutset separates the blue nodes of \(H\) from the green nodes.

5.2 Strongly Adjacent Nodes to an Even Wheel

The goal of this subsection is to prove the following two theorems.

**Theorem 5.2** Let \((H, v)\) be an even wheel of a weakly balanced graph \(G\). Assume that \(v \in V^r\). Let \(u \in V \setminus N(v)\) be a node with neighbors in at least two distinct sectors of \(H\). Then \(u\) has exactly two neighbors in \(H\), say \(u_j\) and \(u_k\), belonging to two distinct sectors of the same color.

For an even wheel \((H, v)\), define the set of nodes:

\[ T(H, v) = \{ u \in V(G) \mid \text{No sector of } (H, v) \text{ entirely contains } N_H(u) \text{ and } |N_H(v) \cap N_H(u)| \geq 2 \}. \]

**Theorem 5.3** If a weakly balanced graph contains an even wheel, then it contains a small even wheel \((H, v)\) such that

\[ | \bigcap_{u \in T(H, v)} N_H(u) | \geq 2. \]

**Proof of Theorem 5.2:** Assume that node \(u\) has neighbors in at least three different sectors, say \(S_1, S_j, S_k\). If none of these sectors is adjacent to the other two, then there exist three unpainted nodes \(v_i, v_j, v_k\), such that \(v_i \in V(S_1) \setminus (V(S_j) \cup V(S_k))\), \(v_j \in V(S_j) \setminus (V(S_i) \cup V(S_k))\), \(v_k \in V(S_k) \setminus (V(S_i) \cup V(S_j))\). This implies the existence of a 3 PC \((u, v)\), where each of the nodes \(v_i, v_j, v_k\) belongs to a distinct path of the 3-path configuration. If \(H\) has four sectors
each containing neighbors of $u$, then each sector has exactly one neighbor of $u$ (otherwise there is a $3PC(u,v)$). But now Theorem 4.3 is contradicted in the cycle formed by two adjacent sectors. So $u$ has neighbors in exactly three sectors and one of them is adjacent to the other two, say $S_j$ is adjacent to both $S_i$ and $S_k$. Let $v_i$ be the unpainted node in $V(S_i) \cap V(S_j)$ and $v_k$ the unpainted node in $V(S_j) \cap V(S_k)$. Then, there is a $3PC(u,v)$ unless node $u$ has a unique neighbor $u_i$ in $S_i$ which is adjacent to $v_i$ and a unique neighbor $u_k$ in $S_k$ which is adjacent to $v_k$. When this is the case, node $u$ has an even number of neighbors in $S_j$ (else $(H,u)$ is an odd wheel) and therefore the nodes in $(H,v)$ together with $u$ induce a connected 6-hole with fan sides and 6-hole $u,u_i,v_i,v_k$.

So $u$ has neighbors in at most two sectors of the wheel, say $S_j$ and $S_k$. If these two sectors are adjacent, let $v_i$ be their common endnode and $v_j, v_k$ the other endnodes of $S_j$ and $S_k$ respectively. Let $H'$ be the hole obtained from $H$ by replacing $V(S_j) \cup V(S_k)$ by the shortest path in $V(S_j) \cup V(S_k) \cup \{u\} \setminus \{v_i\}$. The wheel $(H',v)$ is an odd wheel. So the sectors $S_j$ and $S_k$ are not adjacent.

If $u$ has three neighbors or more on $H$, say two or more in $S_j$ and at least one in $S_k$, then denote by $v_j$ and $v_{j-1}$ the endnodes of $S_j$ and by $v_k$ one of the endnodes of $S_k$. There exists a $3PC(u,v)$ where each of the nodes $v_j$, $v_{j-1}$, and $v_k$ belongs to a different path. Therefore $u$ has only two neighbors in $H$, say $u_j \in V(S_j)$ and $u_k \in V(S_k)$. Let $C_1$ and $C_2$ be the holes formed by the node $u$ and the two $u_ju_k$-subpaths of $H$, respectively. In order for both $(C_1,v)$ and $(C_2,v)$ to be even wheels, the sectors $S_j$ and $S_k$ must be of the same color. □

Let $(H,v)$ be an even wheel of a weakly balanced graph $G$. In the remainder of this subsection, we assume that $v \in V^r$. Before proving Theorem 5.3, we need to establish results
about nodes $u \in V^r$ that are strongly adjacent to $H$. Node $u$ can be of four types:

**Type 1** There exists a sector of $(H, v)$ containing all the nodes of $N_H(u)$.

**Type 2** Node $u$ is not of Type 1 and all its neighbors in $H$ are unpainted. Note that, in particular, the center $v$ of the wheel is of Type 2.

**Type 3** Node $u$ is not of Types 1 or 2 and all its painted neighbors in $H$ have the same color.

**Type 4** Node $u$ has painted neighbors of both colors.

**Lemma 5.4** If $u \in V^r$ is strongly adjacent to $H$ and has a unique neighbor $w$ in some sector of $(H, v)$, then node $w$ is unpainted.

**Proof:** Assume that node $u$ has a unique neighbor $w$ in sector $S_i$ and that $w$ is painted. Let $v_i$ and $v_{i-1}$ be the endnodes of $S_i$. Since node $u$ is strongly adjacent to $H$, it has at least one neighbor in the path induced by $V(H) \setminus V(S_i)$. Choose $u^*$ among the nodes of $N_H(u) \setminus V(S_i)$ and choose $v^*$ among the nodes of $N_H(v) \setminus V(S_i)$ in such a way that the $u^*v^*$-subpath of $H$ not containing $S_i$ is shortest. Note that $u^* \in V^r$, hence $u^*$ cannot be adjacent to $v_i$ or $v_{i-1}$. This implies a $3PC(w, v)$, where each of the nodes $v_i, v_{i-1}$ and $v^*$ belongs to a different path. □

**Lemma 5.5** Let $u \in V^r$ be a Type 4 strongly adjacent node to an even wheel $(H, v)$. Let $s$ and $t$ be a green and a blue neighbor of $u$, respectively. Each of the $st$-subpaths of $H$ contains at least one unpainted neighbor of $u$. Hence $u \in T(H, v)$.

**Proof:** Assume that one of the two $st$-subpaths of $H$ contains no unpainted neighbor of $u$. Let $Q$ be this subpath. Let $P$ be a $s't'$-subpath of $Q$ such that $s'$ is a green neighbor of $u$, $t'$ is a blue neighbor of $u$, and $P$ contains no other painted neighbor of $u$. $P$ contains an odd number of unpainted nodes, none of which are adjacent to $u$. If this number is three or more, then $v$ is the center of an odd wheel with hole induced by the nodes of $P$ and $u$.

So $P$ contains exactly one neighbor of $v$, say $x$. Consider the cycle $C$ with unique chord $ux$ induced by $v$ and the two sectors of $H$ having $x$ as an endnode. Node $u$ is a strongly adjacent node relative to $C$ and therefore must be of one of the three types of Theorem 4.3. It is not of Type 1[4.3] since $s'$ and $t'$ are in different sectors. It is not of Type 2[4.3] either since $u$ is not adjacent to $v$ or $x$. Since $s'$ and $t'$ are painted, they are not adjacent to $v$ and since $s', t', x \in V^c$, the nodes $s'$ and $t'$ are not adjacent to $x$ either. So the node $u$ is not of Type 3[4.3] relative to $C$, a contradiction to Theorem 4.3. Therefore $Q$ must contain an unpainted node adjacent to $u$. □

Now assume the even wheel $(H, v)$ is small.

**Remark 5.6** Every strongly adjacent node to $H$ has at most two neighbors in each sector of $(H, v)$.

**Lemma 5.7** If $(H, v)$ is a small even wheel, $x$ is an unpainted node of $(H, v)$ and $u \in V^r$ is a Type 3 or 4 node not adjacent to $x$, then $u$ has at least one painted neighbor in one of the two sectors of $(H, v)$ adjacent to $x$ and no painted neighbor in the other.
Proof: If \(u\) has no painted neighbor in the two sectors of \((H, v)\) adjacent to \(x\), then \(v\) has at least three neighbors in a sector of \((H, u)\), a contradiction to \((H, v)\) being small. Since \(u\) is not adjacent to \(x\), by Lemma 5.5, \(u\) cannot have painted neighbors in both sectors of \((H, v)\) adjacent to \(x\). \(\square\)

**Lemma 5.8** Let \((H, v)\) be a small even wheel and let \(u\) be a Type 2, 3 or 4 node. Then \(|N_H(u)| = |N_H(v)|\) and therefore \((H, u)\) is a small even wheel.

**Proof:** When \(u\) is a Type 2 node, the result is immediate. When \(u\) is a Type 3 node, it follows from Remark 5.6, Lemma 5.4 and Lemma 5.7.

Now consider the case when \(u\) is a Type 4 node. By Lemma 5.5, we have that \(|N_H(u) \cap N_H(v)| \geq 2\). Let \(P\) be a subpath of \(H\) with endnodes in \(N_H(u) \cap N_H(v)\), say \(x\) and \(y\) but no intermediate node in \(N_H(u) \cap N_H(v)\). The nodes \(x\) and \(y\) are said to be *consecutive nodes* of \(N_H(u) \cap N_H(v)\) in \(H\). Now Lemma 5.5 implies that \(V(P) \cap N(u)\) does not contain nodes of distinct colors. Assume w.l.o.g. that \(V(P) \cap N(u)\) contains no green node. Then Lemmas 5.4, 5.7 and the fact that \((H, v)\) is a small wheel imply that \(u\) has exactly two neighbors in every blue sector of \(P\). This shows that \(|N_H(u) \cap V(P)| = |N_H(v) \cap V(P)|\). As this holds for each pair of consecutive nodes of \(N_H(u) \cap N_H(v)\) in \(H\), we get the equality claimed in the lemma. \(\square\)

**Lemma 5.9** Let \((H, v)\) be a small even wheel and let \(u\) be a Type 4 node having painted neighbors \(u_i\) and \(u_{i+1}\) in two adjacent sectors, say \(S_i, S_{i+1}\). Then every Type 2, 3 or 4 node is adjacent to the common endnode \(v_i\) of \(S_i, S_{i+1}\).

**Proof:** Node \(v_i\) belongs to \(N(u)\), as a consequence of Lemma 5.5. Assume by contradiction that there exists a node \(w\) of Type 2, 3 or 4, that is not adjacent to \(v_i\). By Lemma 5.7, node \(w\) has a painted neighbor in \(S_i\) or \(S_{i+1}\). If \(w\) has a painted neighbor in both \(S_i\) and \(S_{i+1}\), then Lemma 5.5 implies that \(w\) is adjacent to \(v_i\). Therefore we assume w.l.o.g. that \(w\) has a painted neighbor in \(S_i\) but no painted neighbor in \(S_{i+1}\). Remark 5.6 applied to \((H, w)\) and \(v\) implies that \(w\) has a neighbor in \(S_{i+2}\). Let \(w_i\) be a painted neighbor of \(w\) in \(S_i\) that is closest to \(v_i\) and let \(w_{i+2}\) be the neighbor of \(w\) in \(S_{i+2}\) that is closest to the common endnode \(v_{i+1}\) of \(S_i, S_{i+2}\). (Possibly \(w_i = u_i\) or \(w_{i+2} = v_i+1\)). Since \(u\) is adjacent to \(v_i\) and has a painted neighbor in \(S_{i+1}\), it follows from Remark 5.6 that \(u\) is not adjacent to \(v_{i+1}\). Thus, \(u\) has no painted neighbor in \(S_{i+2}\) by Lemma 5.5. Now there is a \(3PC(v_{i+1}, u)\): \(P_1 = v_{i+1}, v, u; P_2 = v_{i+1}, (S_i)_{i+3}^{i+4}, w_i+2, w, v; P_3 = v_{i+1}, (S_{i+1})_{i+3}^{i+4}, u_{i+1}, u\). \(\square\)

**Lemma 5.10** Let \((H, v)\) be a small even wheel and assume that a Type 4 node exists. Then \(T(H, v)\) contains all Type 2, 3 and 4 nodes and

\[
|\bigcap_{u \in T(H, v)} N_H(u)| \geq 2.
\]

**Proof:** Let \(z\) be a Type 4 node, having painted neighbors \(u_i\) and \(u_j\) of distinct colors in \((H, v)\). We show that each of the two \(u_i; u_j\)-subpaths of \(H\) contains a node in \(\bigcap_{u \in T(H, v)} N_H(u)\).

Let \(P\) be one of the two \(u_i; u_j\)-subpaths of \(H\) and let \(X\) be the set of Type 2, 3, 4, nodes in \((H, v)\). Pick a pair of nodes \(x, y \in X\) such that \(y\) is of Type 4 relative to \((H, x)\) and \(y\) has
two painted neighbors \( y_l, y_m \) in \( P \) of distinct colors in \((H, x)\). Furthermore nodes \( x \) and \( y \) are chosen such that the \( y_l y_m \)-subpath \( P_{y_l y_m} \) of \( P \) contains the smallest number of unpainted nodes. (Note that \((H, x)\) and \((H, v)\) have the same set of nodes of Type 2, 3, 4.) If \( P_{y_l y_m} \) contains exactly one unpainted node of \((H, x)\), then the proof follows from Lemma 5.9. Now consider the case where \( P_{y_l y_m} \) contains more than one unpainted neighbor. This number is odd, say \( 2k + 1 \), and let \( x^* \) be the \( k + 1 \)st unpainted node in \( P_{y_l y_m} \), starting from either end. We show that every Type 2, 3, 4 node with respect to \((H, x)\) is adjacent to \( x^* \).

Assume not. Then there exists a node \( w \) of Type 3 or 4 that is not adjacent to \( x^* \). Since \( y \) has no painted neighbors in the two sectors adjacent to \( x^* \), Lemma 5.7 shows that \( y \) is adjacent to \( x^* \). So \( w \neq y \). Let \( S_l \) and \( S_m \) be the sectors of \((H, x)\) containing \( y_l \) and \( y_m \) respectively and having \( x_l, x_{l+1} \) and \( x_m, x_{m+1} \) as endnodes, see Figure 11.

![Figure 11:](image)

Let \( P_{x_l x_{m+1}} \) be the \( x_l x_{m+1} \)-subpath of \( H \) containing \( x^* \). Then in \( P_{x_l x_{m+1}} \), node \( w \) has either two neighbors in every green sector of \((H, x)\) or two neighbors in every blue sector. For, if not, let \( w_1 \) and \( w_2 \) be neighbors of \( w \) that are painted with distinct colors and are closest in \( P_{x_l x_{m+1}} \). Since \( w \) is not adjacent to \( x^* \), \( w_1 \) and \( w_2 \) cannot belong to \( S_l \) and \( S_m \) by Lemma 5.7. But this contradicts our assumption on \( P_{y_l y_m} \).

Assume w.l.o.g. that node \( w \) has two neighbors in every blue sector of \((H, x)\) and that \( S_l \) is painted blue. Let \( x_k \) be the other endnode of the blue sector having \( x^* \) as endnode. In this sector, let \( w^* \) be the neighbor of \( w \) closest to \( x^* \) and \( w_k \) the one closest to \( x_k \).
If \( w \) has at least one painted neighbor in \( S_i \), then let \( w_i \) be such a neighbor and assume w.l.o.g. that the \( y_iw_i \)-subpath \( P_{y_iw_i} \) of \( S_i \) does not contain another neighbor of \( w \) or \( y \). Then the following three paths induce a 3PC(\( x^*, w \): \( P_1 = x^*, y, y_1, P_{y_iw_i}, w, w_i; \ P_2 = x^*, x, x_1, \ldots, w_k, w; \ P_3 = x^*, \ldots, w^*, w \).

Hence \( w \) is adjacent to \( x_i \) and \( x_{i+1} \). In the wheel \( (H, w) \), node \( y \) is of Type 4, having neighbors \( x^* \) and \( y_1 \) in sectors of \( (H, w) \) of distinct colors. But now the number of neighbors of \( w \) in the \( x^*y_i \)-subpath of \( P \) is smaller than the number of neighbors of \( x \) in \( P_{y_iw_i} \), a contradiction to the choice of the pair \( x, y \).

Proof: By Lemma 5.8, \((H, u)\) and \((H, w)\) are small even wheels. Assume that neither \( w \triangleright u \) nor \( u \triangleright w \) relative to \((H, v)\). There are three cases.

**Case 1** In some sector, the neighbors of \( u, w \), say \( u_1, u_2 \) and \( w_1, w_2 \) respectively, appear in the order \( u_1, w_1, u_2, w_2 \) where \( u_1 \neq w_1 \) and \( u_2 \neq w_2 \).

Then, in the wheel \((H, u)\), node \( w \) has a unique neighbor, namely \( w_1 \), in one sector and \( w_1 \) is painted in \((H, u)\). This contradicts Lemma 5.4.

**Case 2** In some sector \( S_i \), the neighbors of \( u, w \), say \( u_1, u_2 \) and \( w_1, w_2 \) respectively, appear in the order \( u_1, u_2, w_1, w_2 \) (possibly \( u_2 = w_1 \)).

Let \( S_i \) have endnodes \( v_i, v_{i+1} \) where w.l.o.g. \( v_i, u_1, u_2, w_1, w_2, v_{i+1} \) appear in this order in \( H \). Let \( S_{i+1} \) be the sector adjacent to \( S_i \) with endnodes \( v_{i+1}, v_{i+2} \) and \( S_{i+2} \) the sector adjacent to \( S_{i+1} \) distinct from \( S_i \). Let \( u_3 \) be the neighbor of \( u \) in \( S_{i+2} \) that is closest to \( v_{i+2} \) in \( S_{i+2} \). Similarly, let \( w_3 \) be the neighbor of \( w \) in \( S_{i+2} \) closest to \( v_{i+2} \). The nodes \( u_3 \) and \( w_3 \) exist by Lemma 5.8. By Remark 5.6, node \( w \) can have at most two neighbors in a sector of \((H, u)\). This implies that \( u_3 \) is closer to \( v_{i+2} \) than \( w_3 \), in \( S_{i+2} \). Define the hole \( H' \) as follows: \( w, w_1, (S_i)_{v_i, v_{i+1}}, v_i, v_{i+2}, (S_{i+2})_{v_{i+2}, w_3}, w \). \( H' \) is shorter than \( H \) and node \( u \) has at least three neighbors in \( H \), contradicting the assumption that \((H, v)\) is small.

**Case 3** There exist two sectors \( S_i \) and \( S_j \) in which the nodes \( u \) and \( w \) have their neighbors as in Figure 12 where, possibly, either \( u_{i_1} = w_{i_1} \) or \( u_{i_2} = w_{i_2} \), but not both, and possibly, either \( u_{j_1} = w_{j_1} \) or \( u_{j_2} = w_{j_2} \), but not both.

In other words, at least one of \( u_{i_1} \) and \( u_{i_2} \) is painted in \((H, w)\) and lies in a sector \( W \) adjacent to the \( w_{i_1}, w_{i_2} \)-sector \( W_i \) of \((H, w)\). Similarly, at least one of \( u_{j_1} \) and \( u_{j_2} \) is painted in \((H, w)\) and lies in the \( w_{j_1}, w_{j_2} \)-sector \( W_j \) of \((H, w)\). However, as \( w \) is a Type 3 node with respect to \((H, v)\), sector \( W_i \) of \((H, w)\) has the same color in \((H, w)\) as sector \( W_j \). Hence, \( W \) and \( W_j \) are painted differently in \((H, w)\). So \( u \) is of Type 4 with respect to \((H, w)\), a contradiction.

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We are now ready to prove the main result of this subsection.

**Proof of Theorem 5.3:** If there exists a small even wheel having at least one Type 4 node, the result holds as a consequence of Lemma 5.10. Moreover, by Lemma 5.8, the theorem follows also if there is a small even wheel \((H, v)\) with no Type 3 node in \(T(H, v)\). So we assume that no small wheel has Type 4 nodes and that each small wheel has a Type 3 node in \(T(H, v)\).

The key observation of this proof is the fact that if \(u\) and \(w\) are Type 3 nodes of a small even wheel \((H, v)\) such that the painted neighbors of \(u\) and \(w\) have the same color, then \[ w > u \implies N_H(w) \cap N_H(v) \supseteq N_H(u) \cap N_H(v). \]

**Claim 1** There exists a small even wheel \((H, v^*)\) and a blue-green painting of its sectors such that all Type 3 nodes of \((H, v^*)\) have blue neighbors.

**Proof of Claim 1:** Let \((H, v)\) be a small even wheel, then the relation \(>\) is transitive in the family \(B(H, v)\) of all the Type 3 nodes with blue neighbors on \((H, v)\). Hence, by Lemma 5.11, there is a Type 3 node \(v^*\) in \(B(H, v)\) such that each Type 3 node \(w\) in \(B(H, v)\) satisfies \(w > v^*\). Assume the wheel \((H, v^*)\) is painted such that \(v \in B(H, v^*)\). Then, from the choice of \(v^*\), it is easy to see that all Type 3 nodes in \((H, v^*)\) lie in \(B(H, v^*)\). So Claim 1 follows.

**Claim 2** There exists a Type 3 node \(u^*\) in \(T(H, v^*)\) such that each Type 3 node \(w\) in \(T(H, v^*)\) satisfies \(w > u^*\) relative to \((H, v^*)\).
Proof of Claim 2: Observe that \((H, v^*)\) has no Type 4 nodes and at least one node in \(B(H, v^*) \cap T(H, v^*)\). Hence by Lemma 5.11, \(\geq\) is a linear order on the family \(B(H, v^*) \cap T(H, v^*)\). So Claim 2 follows.

Claims 1 and 2 and the earlier observation show that \((H, v^*)\) has the property that all nodes in \(T(H, v^*)\) are adjacent to all the nodes in \(N_H(u^*) \cap N_H(v^*)\).

\[\square\]

5.3 An Extended Star Cutset Theorem for Small Even Wheels

In this subsection, we prove the following key result concerning the decomposition of weakly balanced graphs that contain an even wheel.

**Theorem 5.12** Let \((H, v)\) be a small even wheel in a weakly balanced graph. Then every path connecting a blue node to a green node of \((H, v)\) contains a node in \(N(v) \cup T(H, v)\).

Before proving this result, we observe that Theorem 5.1 follows as a corollary.

**Proof of Theorem 5.1:** There exists a small even wheel \((H, v)\) such that \(|\cap_{u \in T(H, v)} N_H(u)| \geq 2\), as a consequence of Theorem 5.3. Now, for any \(a_1, a_2 \in \cap_{u \in T(H, v)} N_H(u)\), Theorem 5.12 implies that \(N(v) \cup (N(a_1) \cap N(a_2))\) is an extended star cutset of \(G\) separating the blue sectors of \((H, v)\) from the green sectors.

The remainder of this subsection is devoted to the proof of Theorem 5.12. We make use of the following lemma.

**Lemma 5.13** Let \((H, v)\) be an even wheel in a weakly balanced graph \(G\) and let \(P\) be a chordless path with nodes in \(V(G) \setminus (V(H) \cup N(v))\) such that any \(x \in V(P)\) is adjacent to at most one node in \(N_H(v)\) and to no painted node of \(H\). Then at most two nodes of \(N_H(v)\) have a neighbor in \(P\).

**Proof:** Assume the lemma is not true and let \(P' = y_1, y_2, \ldots, y_n\) be a shortest subpath of \(P\) with the property that three distinct nodes of \(N_H(v)\) have a neighbor in \(P'\). Denote by \(v_1, v_2, v_3\) the three nodes of \(N_H(v)\) with a neighbor in \(P'\). We can assume w.l.o.g. that \(v_1\) is adjacent to \(y_1\) and no other node of \(P'\), \(v_2\) is adjacent to \(y_2\) and no other node of \(P'\), and \(v_2\) is adjacent to some intermediate nodes of \(P'\). Let \(y_s\) and \(y_t\) be such nodes, such that the \(y_s y_t\)-subpath \(P'_{y_s y_t}\) of \(P'\) and the \(y_t y_s\)-subpath \(P'_{y_t y_s}\) of \(P'\) are as short as possible.

Let \(P_{i;j}\) be the \(v_i; v_j\)-subpath of \(H\) not containing \(v_k\), for \(i, j, k \in \{1, 2, 3\}\) and \(i \neq j \neq k\). Let \(H_{12}\) be the hole induced by the nodes in \(P_{12}\) and in \(P'_{y_t y_s}\), \(H_{13}\) the hole induced by the nodes in \(P_{13}\) and in \(P'\) and let \(H_{23}\) be the hole induced by the nodes in \(P_{23}\) and in \(P'_{y_t y_s}\). Since \((H, v)\) is an even wheel, at least one of the paths \(P_{i;j}\) contains an odd number of intermediate nodes in \(N(v)\). Hence there exists an odd wheel.

**Proof of Theorem 5.12:** If the theorem does not hold for \((H, v)\) let \(P = s^*, s, \ldots, t, t^*\) be a shortest path connecting nodes of \(H\) with distinct colors, and containing no node of \(N(v) \cup T(H, v)\). W.l.o.g. assume that \(v \in V^*\), \(s^*\) is green and \(t^*\) is blue. The following possibilities can occur for nodes \(s\) and \(t\).

(a) Node \(s\) (or \(t\)) has only one neighbor in \(H\), namely \(s^*\) (\(t^*\) respectively),
(b) Node \( s \) (or \( t \)) belongs to \( V^c \setminus N(v) \), is strongly adjacent to \( H \), but all its neighbors are in the same sector of \((H, v)\).

(c) Node \( s \) (or \( t \)) belongs to \( V^c \setminus N(v) \) and has exactly two neighbors in \((H, v)\), one in sector \( S_i \) and one in sector \( S_j \), \( i \neq j \), where \( S_i \) and \( S_j \) have the same color.

(d) Node \( s \) (or \( t \)) belongs to \( V^r \setminus T(H, v) \) and is a Type 1 node.

(e) Node \( s \) (or \( t \)) belongs to \( V^r \setminus T(H, v) \) and is a Type 3 node with at most one neighbor in \( N_H(v) \).

It follows from Theorem 5.2, Lemma 5.5 and the definition of \( T(H, v) \) that no other possibility can occur for the node \( s \) (or \( t \)).

Next, we show that we can dispose of the possibilities (b) and (d) by modifying the wheel \((H, v)\) and the path \( P \).

**Claim 1** There exists a small even wheel \((H', v)\) and a path \( P' = s^{s'}, s', \ldots, t', t^s \) connecting nodes of distinct colors in \((H', v)\), containing no node of \( N(v) \cup T(H', v) \), such that the nodes \( s' \) and \( t' \) satisfy one of the properties (a), (c) or (e) above and, furthermore, the nodes of \( V(P') \setminus \{s'', s', t', t^s\} \) have at most one neighbor in \( H' \).

**Proof of Claim 1:** First, assume that some node \( u \) of \( V(P) \setminus \{s, s, t, t^s\} \) has at least two neighbors in \( H \). These neighbors are unpainted, otherwise a shorter path \( P \) would exist. All Type 2 nodes are in \( T(H, v) \), so \( u \) must be of Type 1. Denote by \( v_i \) and \( v_{i-1} \) the nodes of \( H \) adjacent to \( u \) and by \( S_i \) the \( v_i v_{i-1} \)-sector of \((H, v)\). Assume w.l.o.g. that \( S_i \) is a blue sector. Construct \( H' \) from \( H \) by replacing the sector \( S_i \) by the sector \( v_{i-1}, u, v_i \) and let \( P' \) be the \( s'^{s'}-u \)-subpath of \( P \). Note that \((H', v)\) is small and \( T(H', v) = T(H, v) \). Therefore, \( P' \) connects sectors of distinct colors in \((H', v)\) and contains no node of \( N(v) \cup T(H', v) \). In \( P' \), the node \( t' \) adjacent to \( u \) is different from \( s \) (if \( s = t' \), then this node violates Theorem 5.2 with respect to \((H', v)\)). Note also that \( P' \) is shorter than \( P \). So by repeating the above procedure, we can dispose of all the nodes of \( V(P) \setminus \{s, s, t, t^s\} \) with at least two neighbors in \( H \). In the remainder, we assume w.l.o.g. that the nodes of \( V(P) \setminus \{s, s, t, t^s\} \) have at most one neighbor in \( H \) and, if this neighbor exists, it is unpainted.

Assume that \( s \) satisfies property (b) or (d) and let \( S_i \) be the sector containing \( s^s \). Denote by \( v_i \) and \( v_{i-1} \) the endnodes of \( S_i \) and by \( s_i \) and \( s_{i-1} \) the neighbors of \( s \) in \( S_i \) that are closest to \( v_i \) and \( v_{i-1} \) respectively. Let \((H', v)\) be the wheel obtained from \((H, v)\) by substituting the \( s_{i-1} s_i \)-subpath of \( S_i \) with \( s_i, s, s_i \) and let \( P' \) be the subpath obtained from \( P \) by removing the node \( s^s \), namely \( P' = s, s', \ldots, t, t^s \). The wheel \((H', v)\) is small and, since \( T(H', v) = T(H, v) \), the path \( P' \) connects two sectors of \((H', v)\) with distinct colors and contains no node of \( N(v) \cup T(H', v) \). Note that \( s' = t \) cannot occur, since this node would either violate Theorem 5.2 with respect to \((H', v)\) or would be Type 4 relative to \((H', v)\), a contradiction to the fact that \( P' \) contains no node of \( T(H', v) \). Therefore, \( s' \) has at most two neighbors in \( H' \). If \( s' \) does have two neighbors, it must be of Type 1 relative to \((H', v)\), i.e. Property (d) holds. In this case the above procedure can be repeated and \( P' \) can be shortened again. The proof of Claim 1 is now complete.

As a consequence of this claim, we can assume w.l.o.g. that \((H, v)\) and \( P = s^s, s, \ldots, t, t^s \) have the following properties, in addition to those already stated at the beginning of the
proof: $s$ and $t$ satisfy Properties (a), (c) or (e) and the nodes of $V(P) \setminus \{s^*, s, t, t^*\}$ have at most one neighbor in $H$.

**Claim 2** Assume $s$ is a Type 3 node.

(i) If $N_H(s) \cap N_H(v) = \emptyset$, then no node of $V(P) \setminus \{s^*, s, t, t^*\}$ is adjacent to a node of $H$.

(ii) If $N_H(s) \cap N_H(v) = \{a\}$, then no node of $V(P) \setminus \{s^*, s, t, t^*\}$ is adjacent to a node of $H \setminus \{a\}$.

**Proof of Claim 2:** Assume not and let $u$ be the node of $V(P) \setminus \{s^*, s, t, t^*\}$ that is closest to $s$ in $P$ and adjacent to a node of $H$ (Case (i)) or of $H \setminus \{a\}$ (Case (ii)). By Claim 1, node $u$ can only be adjacent to one node of $H$ and this node is unpainted. Let $x \in N_H(v)$ be this node. By Lemma 5.8, node $s$ has exactly two neighbors in each green sector of $(H, v)$. Let $S_i$ be the green sector having $x$ as endnode and let $s_i$ be the neighbor of $s$ closest to $x$ in $S_i$. Let $S_j$ be a green sector distinct from $S_i$, say with endnodes $v_j$ and $v_{j-1}$ and let $s_j$ and $s_{j-1}$ be the neighbors of $s$ in $S_j$, closest to $v_j$ and $v_{j-1}$ respectively. Assume w.l.o.g. that $s_j$ is painted. Then $x \in V^c$ and $s \in V^c$ are connected by a $3PC(x, s)$: $P_1 = x, u, P_{a}, s; P_2 = x, (S_i)_{x_{S_i}}, s_i, s; P_3 = x, v_j, (S_j)_{v_j}, s_j, s$. This completes the proof of Claim 2.

A similar statement to Claim 2 holds when $t$ is of Type 3.

**Claim 3** Neither node $s$ nor node $t$ is of Type 3.

**Proof of Claim 3:** Assume $s$ is a Type 3 node. Let $S_i$ be the blue sector containing $t^*$ where we assume that $t^*$ is chosen such that $t$ has no neighbor in $N_H(v) \setminus V(S_i)$. (This choice is possible since $t$ satisfies Properties (a) or (c) or (e).)

Let $S_{i-1}$ and $S_{i+1}$ be the green sectors adjacent to $S_i$. Let $v_i$ be the common endnode of $S_i$ and $S_{i+1}$ and let $v_{i-2}$ be the endnode of $S_{i-1}$ not on $S_i$. As we have a symmetry between $S_{i-1}$ and $S_{i+1}$, Property (c) implies that we may choose the numbering so that $s$ is not adjacent to $v_i$ nor to $v_{i-2}$ and, by Claim 2, no intermediate node of $P_{i+s}$ is adjacent to $v_i$ or $v_{i-2}$. Let $s_{i-1}$ be the neighbor of $s$ on $S_{i-1}$ closest to $v_{i-2}$ and let $s_{i+1}$ be the neighbor of $s$ on $S_{i+1}$ closest to $v_i$. Moreover, let $t_i$ be the neighbor of $t$ on $S_i$ closest to $v_i$. (Note that possibly $t = v_i$.) Now the following three paths form a $3PC(v_i, s)$: $P_1 = v_i, (S_i)_{v_i}, t_i, t, P_{s}, s; P_2 = v_i, (S_{i+1})_{v_{i+1}}, s_{i+1}, s; P_3 = v_i, v_i, v_{i-2}, (S_{i-1})_{v_{i-2}}, s_{i-1}, s$. This completes the proof of Claim 3.

Let $U$ be the set of unpainted nodes of $H$ adjacent to at least one node of $V(P) \setminus \{s^*, s, t, t^*\}$.

**Claim 4** If $s$ satisfies Property (a), let $v_{i-1}$ and $v_i$ be the endnodes of the sector $S_i$ containing $s^*$. Then $U \subseteq \{v_{i-1}, v_i\}$.

**Proof of Claim 4:** Assume not and let $x$ be the node of $V(P) \setminus \{s^*, s, t, t^*\}$ adjacent to $v_i$, $k \neq i - 1$, $i$ such that $P_{sx}$ is shortest.

**Case 1** No intermediate node of $P_{sx}$ is adjacent to an unpainted node of $H$.

Let $P_1$ and $P_2$ be the $v_i s^*$-subpaths of $H$ and let $C_1 = v_k, P_1, s^*, s, P_{sx}, x, v_k, C_2 = v_k, P_2, s^*, s, P_{sx}, x, v_k$. Then either $(C_1, v)$ or $(C_2, v)$ is an odd wheel.
**Case 2** Some intermediate node of $P_{sx}$ is adjacent to an unpainted node of $H$.

By Lemma 5.13, only one unpainted node of $H$ is adjacent to some intermediate node of $P_{sx}$, and by the choice of node $x$, we can assume w.l.o.g. this node to be $v_i$. Let $y$ be the neighbor of $v_i$ in $P_{sx}$ such that $P_{sy}$ is shortest, let $P_1$ be the $vy$-subpath of $H$ not containing $v_{i-1}$ and $P_2$ be the $vs$-subpath of $H$ not containing $v_i$. Consider the cycles $C_1 = v_k, P_{sx}, v_{i-1}, v_y, P_{sy}, v_{i-1}, v_k$ and $C_2 = v_i, s', s_i, P_{sx}, x, v_k, P_2, s'$. An easy counting argument shows that either $(C_1, v)$ or $(C_2, v)$ is an odd wheel.

**Claim 5** If $s$ satisfies Property (c), let $v_{i-1}$ $v_i$ and $v_{j-1}$ $v_j$ be the endnodes of sectors $S_i$ and $S_j$ containing the two neighbors $s^* = s_i$ and $s_j$ of $s$ in $H$. Then $U$ is contained either in $\{v_{i-1}, v_i\}$ or in $\{v_{j-1}, v_j\}$. Furthermore if $U \neq \emptyset$, at least one node of $U$ is adjacent to $s_i$ or $s_j$.

**Proof of Claim 5:** We first show $U \subseteq \{v_{i-1}, v_i, v_{j-1}, v_j\}$.

Assume not and let $v_k$, $k \neq i - 1, i, j - 1, j$ be the node of $U$ adjacent to the node $x \in V(P) \setminus \{s^*, s, t, t^*\}$ such that the $xs$-subpath $P_{sx}$ of $P$ is shortest. Lemma 5.13 shows $|U| \leq 2$. Hence by symmetry, we assume w.l.o.g. that, among $v_{i-1}, v_i, v_{j-1}$ and $v_j$, only node $v_i$ can be adjacent to an intermediate node of $P_{sx}$. Now the following three paths induce a $3PC(v, s)$:

- $P_1 = v, v_{i-1}, (S_i)_{v_{i-1}v_i}, s_i, s$;  
- $P_2 = v, v_{j-1}, (S_j)_{v_{j-1}v_j}, s, s_j$;  
- $P_3 = v, v_k, x, P_{sx}, s$.

This shows $U \subseteq \{v_{i-1}, v_i, v_{j-1}, v_j\}$.

Assume that nodes $v_{i-1}, v_i, v_{j-1}, v_j$ appear in this order when traversing $H$ clockwise starting in $v_{i-1}$. We now show $U \neq \{v_i, v_j\}$.

Assume not and let $x$ and $y$ be nodes of $V(P) \setminus \{s^*, s, t, t^*\}$ adjacent to $v_i, v_j$ respectively such that the $xy$-subpath $P_{xy}$ of $P$ is shortest and let $P_1$ be a $vy$-subpath of $H$. Then the wheel $(C, v)$, induced by the cycle $C = v_j, P_1, v_i, x, P_{xy}, y, v_j$ is odd since $S_i$ and $S_j$ are sectors with the same color.

We now show $U \neq \{v_{i-1}, v_j\}$.

Assume w.l.o.g. that when traversing $P$ starting from $s$ we first encounter nodes that are adjacent to $v_{i-1}$. Let $y$ be the first encountered neighbor of $v_j$. Then $s_i$ is adjacent to $v_{i-1}$, else the following three paths induce a $3PC(v, s)$:

- $P_1 = v, v_i, (S_i)_{v_{i-1}v_i}, s_i, s$;  
- $P_2 = v, v_{j-1}, (S_j)_{v_{j-1}v_j}, s, s_j$;  
- $P_3 = v, v_j, P_{xy}, s$.

A similar argument shows that $s_i$ is adjacent to $v_{i-1}$. Now the nodes of $P_{sx}$ together with the nodes in $(H, v)$ induce a connected 6-hole with fan sides and 6-hole $s_i, s, s_j, v_j, v, v_{i-1}, s_i$.

By symmetry, the above two arguments show that $U \neq \{v_{i-1}, v_j\}$ and $U \neq \{v_i, v_{j-1}\}$.

Hence $U$ is contained either in $\{v_{i-1}, v_i\}$ or in $\{v_{j-1}, v_j\}$.

Finally, if $U$ is nonempty, let $x$ be the node of $V(P) \setminus \{s^*, s, t, t^*\}$ adjacent to a node of $U$, such that the $sx$-subpath of $P$ is shortest. Assume w.l.o.g. that $v_i$ is adjacent to $x$. Then $s_i$ and $v_i$ are adjacent, else there exists a $3PC(v, s)$, which completes the proof of Claim 5.

**Claim 6** $U = \emptyset$.

**Proof of Claim 6:** Lemma 5.13 shows $|U| \leq 2$. Since $s$ and $t$ satisfy Property (a) or (c) the above two claims, applied to both $s$ and $t$ rule out the possibility that $|U| = 2$. So $|U| = 1$. Assume $U = \{v_i\}$, let $S_i$ and $S_{i+1}$ be the sectors of $(H, v)$ having $v_{i-1}, v_i$ and $v_i, v_{i+1}$ as endnodes, where $S_i$ is green and consider the following cases:
Case 1 Both $s$ and $t$ satisfy Property (a).

Claim 4, applied to both $s$ and $t$ shows that $s^*$ belongs to $S_i$ and $t^*$ belongs to $S_{i+1}$. Let $P_1$ be the $t^*s^*$-subpath of $H$ not containing $v_i$ and $C = t^*, P_1, s^*, P, t^*$. Then $(C, v)$ is an odd wheel.

Case 2 Node $s$ satisfies Property (a) and node $t$ satisfies Property (c).

Claim 4 applied to $s$ and Claim 5 applied to $t$ shows that $s^*$ belongs to $S_i$ and $t^*$ has neighbor $t^* = t_{i+1}$ in $S_{i+1}$ which is adjacent to $v_i$. Let $t_j$ be the second neighbor of $t$ in $H$ and let $v_j$ be an endnode of the sector $S_j$ containing $t_j$, distinct from $v_{i-1}$. If $t_j$ is not adjacent to $v_{i-1}$, the following three paths induce a $3PC(v, t)$: 
\[ P_1 = v, v_1, (S_{i+1})v_{i+1}, t_{i+1}, t; \]
\[ P_2 = v, v_j, (S_j)v_j, t_j, t; \]
\[ P_3 = v, v_{i-1}, (S_i)v_{i-1}, s^*, s, P_{st}, t. \]

If $t_j$ is adjacent to $v_{i-1}$ (i.e. $S_j = S_{i-1}$), then the nodes in $P$ together with the ones in $(H, v)$, induce a connected 6-hole with fan sides and 6-hole $v, v_i, t_{i+1}, t, t_j, v_{i-1}, v$.

Case 3 Both $s$ and $t$ satisfy Property (c).

Claim 5 applied to both $s$ and $t$ shows that one neighbor of $s$ in $H$, say $s^*$, belongs to $S_i$ and one neighbor of $t$ in $H$, say $t^*$, belongs to $S_{i+1}$ and both $s^*$ and $t^*$ are adjacent to $v_i$. Let $S_j$ and $S_k$ be the sectors containing the other neighbors $s_j$ and $t_k$ of $s$ and $t$ respectively. Let $P_1$ be the $t_js_j$-subpath of $H$ not containing $v_i$. Consider the cycle $C_1 = s_j, s, P_{st}, t, t_k, P_1, s_j$. Since $s_j$ and $t_k$ belong to sectors with distinct colors, $P_1$ contains an odd number of neighbors of $v$. In fact, $P_1$ contains a unique neighbor of $v$, say $z_1$, otherwise $(C_1, v)$ is an odd wheel. Consider now the cycle $C_2 = s, P_{st}, t, t^*, \ldots, v_{i+1}, v, v_{i-1}, \ldots, s^*, s$. Then $P_{st}$ has an odd number of neighbors of $v_i$, else the wheel $(C_2, v_i)$ is odd. In fact, $P_{st}$ contains a unique neighbor of $v_i$, say $z_2$, otherwise $(C_1, v_i)$ is an odd wheel. Now the two $z_1z_2$-subpaths of $C_1$, together with $z_1, v, v_i, z_2$, induce a $3PC(z_1, z_2)$.

By symmetry, this completes the proof of Claim 6.

Claim 7 The path $P$ cannot exist.

Proof of Claim 7: Assume $P$ exists. Then Claim 6 shows that $U = \emptyset$.

Case 1 Both $s$ and $t$ satisfy Property (a).

Let $P_1$ be a $t^*s^*$-subpath of $H$ containing more that one unpainted node and let $C = s^*, P, t^*, P_1, s^*$. Then since $s^*$ and $t^*$ belong to sectors with distinct colors, $(C, v)$ is an odd wheel.

Case 2 Node $s$ satisfies Property (a) and node $t$ satisfies Property (c).

Let $v_{i-1}$ and $v_i$ be the endnodes of the sector $S_i$ containing $s^*$ and let $t^*$ and $t_k$ be the neighbors of $t$ in $H$. Then $t^*$ is adjacent to $v_{i-1}$ and $t_k$ is adjacent to $v_i$ (or vice versa), else there is a $3PC(v, t)$. Note that $s^* \notin V^c$, as otherwise there is a $3PC(s^*, t)$. Now the nodes in $P$ together with the ones in $(H, v)$, induce a connected 6-hole with a fan side and a triad side and 6-hole $v, v_{i-1}, t^*, t, t_k, v_i, v$. 

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Case 3 Both \( s \) and \( t \) satisfy Property (c).

Let \( S_i, S_j \) be the sectors containing the neighbors \( s^* = s_i \) and \( s_j \) of \( s \) in \( H \) and \( S_k, S_l \) be the sectors containing the neighbors \( t_k \) and \( t_l \) of \( t \) in \( H \). Then \( S_k \) is adjacent to both \( S_i \) and \( S_j \), else each of these three sectors has one endnode that is not the endnode of any of the other two and this implies the existence of a \( 3PC(v,s) \). By symmetry, \( S_l \) is also adjacent to both \( S_i \) and \( S_j \). So \( (H,v) \) is a wheel with four spokes. Furthermore \( t_k \) is adjacent to both endnodes of \( S_k \), else there is a \( 3PC(v,t) \). By symmetry, \( s_i \) is adjacent to both endnodes of \( S_i \). Now the common neighbor of \( t_k \) and \( s_i \) is the center of a wheel with three spokes.

By symmetry, this completes the proof of Claim 7 and of the theorem. \( \square \)
6 Parachutes

6.1 Introduction

In this section, we consider a wheel-free weakly balanced graph $G$ that contains a parachute. We use the notation introduced in Section 1. Let $II = Par(P_1, P_2, M, T)$ be a parachute of $G$. The node $z$ is called bottom node, $v_1$ and $v_2$ are called side nodes and $v$ is called center node. We assume w.l.o.g. that $v, z \in V^c$ and $v_1, v_2 \in V^r$. Let $m$ be the neighbor of $v$ in the path $M$. The nodes of $V(II) \setminus \{v, v_1, v_2, m\}$ induce two connected components called the top of $II$, induced by $V(T) \setminus \{v_1, v_2\}$, and the bottom of $II$, induced by $V(P_1) \cup V(P_2) \cup V(M) \setminus \{v, v_1, v_2, m\}$.

Recall that, for a path $P = x_1, x_2, \ldots, x_{n-1}, x_n$ where $n \geq 3$, we denote by $\hat{P}$ the $x_2x_{n-1}$-subpath of $P$. With this notation, the top of $II$ is $\hat{T}$ and the bottom of $II$ is induced by $V(\hat{P}_1) \cup V(\hat{P}_2) \cup (V(M) \setminus \{m\}) \cup \{z\}$.

When $|E(T)| = 2$, the parachute $II$ is said to have a short top; the top is long when $|E(T)| \geq 4$. Similarly, the parachute $II$ is said to have a short middle when $|E(M)| = 2$, and long middle otherwise. Finally, when $|E(P_1)| = 1$ or $|E(P_2)| = 1$, the parachute $II$ is said to have one short side; otherwise, we say that $II$ has long sides.

This section is organized as follows. In Subsection 6.2, we list all possible strongly adjacent nodes to $II$. In Subsections 6.3 and 6.4, we list all possible direct connections from the bottom of $II$ to the top of $II$, avoiding $N(v) \cup ((N(v_1) \cap N(v_2)) \setminus V(T))$. When no such path exists, the graph $G$ has an extended star cutset disconnecting the top of $II$ from the bottom. When there is such a path, at least one of the following possibilities arises.

- The graph $G$ contains no parachute with long sides. This case is treated in Subsection 6.5 where we prove the existence of an extended star cutset in $G$ (Theorem 6.6).

- The graph $G$ contains a stabilized parachute. This concept is defined in Subsection 6.6 and $G$ is shown to have an extended star cutset (Theorem 6.10).

- The graph $G$ contains a parachute with short middle path and long sides, but $G$ contains no stabilized parachute and no connected squares. This case is treated in Subsection 6.8 where we prove the existence of an extended star cutset (Theorem 6.16).

- The graph $G$ contains connected squares. This case is treated in Section 7.

- The graph $G$ contains goggles. This case is treated in Section 8.

6.2 Strongly Adjacent Nodes

Theorem 6.1 Let $II = Par(P_1, P_2, M, T)$ be a parachute in a wheel-free weakly balanced graph $G$. Let $w \in V(G) \setminus V(II)$ be a strongly adjacent node to $II$. Then $w$ satisfies one of the following properties:

Node $w$ has exactly two neighbors in $II$ and both belong to one of the paths $P_1$, $P_2$, $M$ or $T$. Node $w$ is of one of the following types, see Figure 13.

- Type a Node $w \in V^c$ is adjacent to the neighbors of $z$ in $\hat{P}_1$ and $\hat{P}_2$ respectively and to no other node of $II$.
• **Type b** Node \( w \in V^r \) is adjacent to one node in \( \tilde{P}_1 \), to one node in \( \tilde{P}_2 \) and to no other node of \( \Pi \).

• **Type c** Node \( w \in V^r \) is adjacent to exactly two nodes of \( \Pi \), one of which is the neighbor of \( z \) in \( \tilde{M} \) and the other is the neighbor of \( z \) in \( P_1 \) or in \( P_2 \).

• **Type d** Node \( w \in V^r \) is adjacent to one node in \( V(M) \setminus \{z\} \), to one node in either \( \tilde{P}_1 \) or \( \tilde{P}_2 \) (but not both) and to no other node of \( \Pi \).

• **Type e** Node \( w \in V^r \) is adjacent to \( v \), to one node in \( \tilde{T} \) and to no other node of \( \Pi \).

• **Type f** Node \( w \in V^r \) is adjacent to one node in \( P_1 \), to one node in \( P_2 \), to one node of \( M \) and to no other node of \( \Pi \).

• **Type g** Node \( w \in V^r \) is adjacent to \( m \), to two nodes in \( T \) at least one of which is in \( \tilde{T} \), and to no other node of \( \Pi \).

• **Type h** Node \( w \in V^r \) is adjacent to \( v \), to one node in \( \bar{T} \), to one node in \( V(M) \setminus \{v\} \) and to no other node of \( \Pi \).

• **Type i** Node \( w \in V^r \) is adjacent to \( v \), to one node in \( \bar{T} \), to one node in either \( \tilde{P}_1 \) or \( \tilde{P}_2 \) (but not both) and to no other node of \( \Pi \).

When \( \Pi \) has a short side, say \( P_2 \), the following additional types of strongly adjacent nodes can occur.

• **Type j** Node \( w \in V^r \) is adjacent to \( v_2 \), to one node in \( \tilde{M} \) distinct from the neighbor of \( z \), and to no other node of \( \Pi \).

• **Type k** Node \( w \in V^r \) is adjacent to one node in \( V(T) \setminus \{v_1\} \), to one node in \( \tilde{P}_1 \) and to no other node of \( \Pi \).

• **Type l** Node \( w \in V^r \) is adjacent to the neighbors of \( v_1 \) in \( \tilde{T} \) and \( \tilde{P}_1 \) respectively and to no other node of \( \Pi \).

• **Type m** Node \( w \in V^r \) is adjacent to two nodes of \( V(M) \setminus \{v\} \), to the neighbor of \( v_2 \) on \( \tilde{T} \) and to no other node of \( \Pi \).

• **Type n** Node \( w \in V^r \) is adjacent to \( v_2 \), to one node in \( \tilde{T} \), to one node in \( V(M) \setminus \{m\} \) and to no other node of \( \Pi \).

When \( \Pi \) has a short top and long sides, one additional type of strongly adjacent node can occur.

• **Type o** Node \( w \in V^r \) is adjacent to two nodes of \( V(M) \setminus \{v\} \), to the unique node of \( V(\bar{T}) \) and to no other node of \( \Pi \).
Figure 13: Strongly adjacent nodes
Claim 6. \( w \neq w' \) and \( w \notin P \). Then if \( w \notin M \) and \( w \notin M' \), then \( w \notin P \) or \( w \notin P' \).

Let \( w_2 \) be the third neighbor of \( w \) in \( M \).

Claim 5. \( w \) has exactly three neighbors in \( M \).

Proof of Claim 5. If \( w \) has at least four neighbors in \( M \), then by Claim 1, \( w \) is the center of a wheel in \( M \) and \( w \) has exactly two neighbors in \( M \).

If \( w \) is a neighbor of \( w \) not in \( P \), then \( w \) is not of Type 1 or 2. But then, there is a 3PC of \( w \).

Let \( w \) be the neighbor of \( w \) in \( M \) closest to \( v \).

Claim 4. \( w \in M \cap V(P) \).

Proof of Claim 4. Suppose the claim is false. By symmetry, we may assume that \( w \) is the only neighbor of \( w \) in \( M \). Then \( w \) is not of Type 1 or 2. But then, there is a 3PC of \( w \).

But this implies the existence of a 3PC of \( w \).

Claim 3. \( w \in V(P) \).

Proof of Claim 3. If the claim is false, then by Claim 1, \( w \) has exactly two neighbors in \( M \). Then \( w \) is the center of a wheel in \( M \) and \( w \) has exactly two neighbors in \( M \).

If \( w \) is a neighbor of \( w \) not in \( P \), then \( w \) is not of Type 1 or 2. But then, there is a 3PC of \( w \).

But this implies the existence of a 3PC of \( w \).

Claim 2. \( w \neq V(P) \) and \( M' \) at most one neighbor in \( P \) at most two in each of \( P \) and \( P' \).

Proof of Claim 2. If the claim is false, then by Claim 1, \( w \) has exactly two neighbors in \( M \). Then \( w \) is the center of a wheel in \( M \) and \( w \) has exactly two neighbors in \( M \).

If \( w \) is a neighbor of \( w \) not in \( P \), then \( w \) is not of Type 1 or 2. But then, there is a 3PC of \( w \).

But this implies the existence of a 3PC of \( w \).

Claim 1. \( w \) is a neighbor of \( w \) not in \( P \), \( P' \), or \( T \) in \( P \).
II \setminus \bar{P}_1$, because if it were, then $w_1 = v$ and $w_3 \in V(\bar{P}_1)$, so $w$ would be of Type $i$ with respect to II. Suppose $w$ is of Type 2[4.3]. Since $w$ is not of Type $h$ or $i$, $w_1 \neq v$. Therefore $w_3 = v_2$ (as $w_3 \neq v$, by the choice of $w_1$), $w_1 \neq m$ since $w$ is not of Type $g$, and $P_2$ is long since $w$ is not of Type $h$ or $i$. Now there is a $3PC(z, v_2)$. So $w$ is of Type 3[4.3]. As $w$ is not of Type $g$ and $w_1 \neq z$, it follows that $w_3 \notin V(T)$. Since $w_2 \neq v_1$, we have that $w_2 \in N(v_2)$ and $w_3 \in M \cup \bar{P}_2$. As $w$ is not of Type $h$ or $i$, $w_1 \neq m$.

We may assume that $P_1$ is long. Hence, by Claim 6, so is $P_2$, as otherwise $w$ would be of Type $m$. Now Claim 6 again implies that $T$ is short and $w_3 \in M$. But then $w$ is of Type $o$, which yields a contradiction.

6.3 Parachute Modifications

Let $\Pi = Par(P_1, P_2, M, T)$ be a parachute with center node $v \in V^c$ and side nodes $v_1, v_2$.

Let $S(\Pi) = N(v) \cup ((N(v_1) \cap N(v_2)) \setminus V(T))$. In this subsection and the next, we enumerate all possible direct connections from the bottom of II to the top of II avoiding $S(\Pi)$, in two cases:

- II has long sides,
- II has a short side and $G$ contains no parachute with long sides.

Let $Q = x_1, \ldots, x_n$ denote a direct connection from bottom to top avoiding $S(\Pi)$, where $x_1$ is adjacent to a node in the bottom of II and $x_n$ is adjacent to a node in the top of II. It follows from the definition of a direct connection that no node of $\bar{Q}$ is adjacent to a node of $V(\Pi) \setminus \{v_1, v_2, m\}$. Furthermore, since $Q$ avoids $S(\Pi)$, a node of $\bar{Q}$ is adjacent to at most one of $v_1, v_2$. To reduce the number of possible path types that need to be enumerated in the main theorem of this subsection (Theorem 6.4), we introduce the concept of parachute modification.

**Definition 6.2** Assume $y \in V(G) \setminus V(\Pi)$ has exactly two neighbors in II, both are in $T$ and at least one is in $T$. A parachute modification at the top consists of replacing II by the unique parachute II$'$ that is induced by a subset of $V(\Pi) \cup \{y\}$ and is distinct from II.

Assume $y \in V(G) \setminus V(\Pi)$ has exactly two neighbors in II that are both in $P_1$, or both in $P_2$ or both in $V(M) \setminus \{v\}$, A parachute modification at the bottom consists of replacing II by the unique parachute II$'$ that is induced by a subset of $V(\Pi) \cup \{y\}$ and is distinct from II.

**Remark 6.3** (i) If II$'$ is obtained from II by a parachute modification at the top, then both II and II$'$ have long top. If II$'$ is obtained from II by parachute modification at the bottom, then II$'$ has long sides if and only if II has long sides.

(ii) Let II$'$ be obtained from II by a parachute modification using a node of a direct connection $Q = x_1, \ldots, x_n$ from bottom to top avoiding $S(\Pi)$. Then $n > 1$ and if the modification is at the top, it involves node $x_n$. If the modification is at the bottom, it involves node $x_1$. Furthermore $S(\Pi) = S(\Pi')$.

**Theorem 6.4** Let $G$ be a wheel-free weakly balanced graph. Let $\Pi = Par(P_1, P_2, M, T)$ be a parachute in $G$ and $Q = x_1, \ldots, x_n$ a direct connection from bottom to top avoiding $S(\Pi)$ such that no parachute modification exists relative to a node of $Q$.
(i) If $\Pi$ has long top and long sides, then $n \geq 2$ and, up to symmetry between $P_1$ and $P_2$, $Q$ is of one of the following types, see Figure 14.

- **Type a** Node $x_1$ is a strongly adjacent node of Type $f[6.1]$, adjacent to $v_1, m$ and one node of $P_2$. Node $x_n$ is not strongly adjacent and its unique neighbor is adjacent to $v_1$ in $T$. Exactly one node of $\bar{Q}$ is adjacent to $m$ and none is adjacent to $v_1, v_2$.

- **Type b** Node $x_1$ is not strongly adjacent and its unique neighbor is the node of $\bar{P}_2$ adjacent to $v_2$. Node $x_n$ is of Type $g[6.1]$ adjacent to $v_2, m$ and one node of $\bar{T}$. No node of $\bar{Q}$ has a neighbor in $\Pi$. Furthermore, $\Pi$ has a short middle path.

- **Type c** Node $x_1 \in V^c$ is not strongly adjacent and its unique neighbor is in $\bar{P}_2$. Node $x_n$ is a strongly adjacent node of Type $g[6.1]$, adjacent to $v_2, m$ and one node of $\bar{T}$. Exactly one node of $\bar{Q}$ is adjacent to $v_2$ and none is adjacent to $v_1, m$. Furthermore, $\Pi$ has a short middle path.

- **Type d** Node $x_1$ is a strongly adjacent node of Type $c[6.1]$, with neighbors in $\bar{P}_1$ and $M$. Node $x_n$ is a strongly adjacent node of Type $g[6.1]$, adjacent to $v_2, m$ and one node of $\bar{T}$. Exactly one node of $\bar{Q}$ is adjacent to $v_2$ and none is adjacent to $v_1, m$. Furthermore, $\Pi$ has a short middle path.

- **Type e** Node $x_1$ is a strongly adjacent node of Type $f[6.1]$, adjacent to $v_2, m$ and one node of $\bar{P}_1$. Node $x_n$ is a strongly adjacent node of Type $g[6.1]$, adjacent to $v_2, m$ and one node of $\bar{T}$. No node of $\bar{Q}$ has a neighbor in $\Pi$.

(ii) If $\Pi$ has short top and long sides, then either $n = 1$ and $x_1$ is a strongly adjacent node of Type $c[6.1]$, or $n \geq 2$ and, up to symmetry between $P_1$ and $P_2$, $Q$ is a direct connection of Type a above or is of the types described below, see Figures 14 and 15.

- **Type f** Node $x_1 \in V^c$ is not strongly adjacent and its unique neighbor is adjacent to $m$ in $M$. Node $x_n$ is not strongly adjacent. Exactly one node of $\bar{Q}$ is adjacent to $m$ and none is adjacent to $v_1$ or $v_2$.

- **Type g** Node $x_1 \in V^c$ is not strongly adjacent and its unique neighbor is in $V(M) \setminus \{m\}$. Node $x_n$ is not strongly adjacent. Exactly two nodes of $\bar{Q}$ are adjacent to $m$ and none is adjacent to $v_1$ or $v_2$.

- **Type h** Node $x_1$ is a strongly adjacent node of Type $f[6.1]$ adjacent to $m$, one node in $\bar{P}_1$ and one node in $\bar{P}_2$. Node $x_n$ is not strongly adjacent. Exactly one node of $\bar{Q}$ is adjacent to $m$ and none is adjacent to $v_1$ or $v_2$.

- **Type i** Node $x_1$ is a strongly adjacent node of Type $a[6.1]$. Node $x_n$ is not strongly adjacent. Exactly two nodes of $\bar{Q}$ are adjacent to $m$ and none is adjacent to $v_1$ or $v_2$.

- **Type j** Node $x_1 \in V^c$ is not strongly adjacent and its unique neighbor belongs to $V(M) \setminus \{m\}$. Node $x_n$ is not strongly adjacent. No node of $\bar{Q}$ has a neighbor in $\Pi$.

- **Type k** Node $x_1$ is a strongly adjacent node of Type $a[6.1]$. Node $x_n$ is not strongly adjacent. No node of $\bar{Q}$ has a neighbor in $\Pi$. 

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(iii) If $\Pi$ has a short side, say $P_2$, and $G$ contains no parachute with long sides, then $\Pi$ has short top and either $n = 1$ and $x_1$ is a strongly adjacent node of Type $l[6.1]$, or $n \geq 2$ and $Q$ is of Type $l$ described below, see Figure 15.

- **Type 1** Nodes $x_1, x_n \in V^r$ are not strongly adjacent and their respective neighbors $b \in V(\tilde{P}_1)$ and $t \in V(\tilde{T})$ are adjacent to $v_1$. No node of $Q$ has a neighbor in $\Pi$.

**Proof:** Throughout this proof, we assume that either $\Pi$ has long sides, or $\Pi$ has a short side and $G$ contains no parachute with long sides.

First, we consider the case $n = 1$, i.e. $Q$ consists of a single node $x_1$ that is strongly adjacent to $\Pi$. Then it follows from Theorem 6.1 that $\Pi$ has a short side and $x_1$ is of Type $k, l, m, n[6.1]$ or that $\Pi$ has a short top and long sides and $x_1$ is of Type $o[6.1]$. When $x_1$ is of Type $k, m$ or $n[6.1], \Pi$ has a short side and the graph induced by $V(\Pi) \cup \{x_1\}$ contains a parachute with long sides, a contradiction. (For example, if $x_1$ is of Type $n[6.1]$, assume w.l.o.g. that $x_1$ is adjacent to $v_2$. Then $P_2$ is short, $x_1$ is not adjacent to $v_1$ and the parachute with long sides has center node $v_2$, bottom node $v_1$ and side nodes $x_1$ and $z$. When $x_1$ is of Type $l[6.1], \Pi$ has a short side and the graph induced by $V(\Pi) \cup \{x_1\}$ contains a parachute with long sides unless $\Pi$ has short top. This proves the theorem when $n = 1$.

Now consider the case $n \geq 2$. By Theorem 6.1, either $x_n$ is not strongly adjacent to $\Pi$ or $x_n$ is a strongly adjacent node of Type $g[6.1]$. Similarly, either $x_1$ is not strongly adjacent to $\Pi$ or it is a strongly adjacent node of Type $a, b, c, d, f, j$ or $k[6.1]$. When $x_1$ is of Type $j[6.1], \Pi$ has a short side, say $P_2$, and $\Pi \cup \{x_1\}$ contains a parachute with long sides with center node $v_2$, sides nodes $v, z$ and top path induced by $V(\tilde{P}_1) \cup \{v\}$, since the neighbor of $x_1$ in $M$ is distinct from $m$ (as $Q$ connects bottom to top) and from the neighbor of $z$ (by the definition of Type $j[6.1]$). So the case where $x_1$ is of Type $j[6.1]$ does not occur. When $x_1$ is of Type $k[6.1], \Pi$ has a short side, say $P_2$, and $x_1$ is adjacent to $v_2$ and the neighbor of $z$ in $\tilde{P}_1$ (since, otherwise, there is a parachute with long sides).

We will divide the proof into two parts, depending on whether $x_n$ is of Type $g[6.1]$ or is not strongly adjacent to $\Pi$. Then, in each of the two parts, the proof will be broken down further based on the adjacencies between the intermediate nodes of $Q$ and $\{v_1, v_2, m\}$. Finally, subcases will occur depending on the node type of $x_1$. The two following claims reduce the number of cases that have to be considered.

We say that node $x_i \in V(\bar{Q})$ adjacent to $m$ and node $x_j \in V(\bar{Q})$ adjacent to $v_1$ or $v_2$, say $v_2$, are consecutive in $Q$ if $N(v_1) \cap V(Q_{x,v_j}) = \emptyset, N(v_2) \cap V(Q_{x,v_j}) = \{x_j\}$ and $N(m) \cap V(Q_{x,v_j}) = \{x_i\}$. We allow $x_i = x_j$, and $Q_{x,x_j} = x_i$ in this case.

**Claim 1** If $x_i, x_j \in V(\bar{Q})$ are consecutive in $Q$, where $x_i$ is adjacent to $m$ and $x_j$ is adjacent to $v_2$, then $P_2$ is short.

**Proof of Claim 1:** $V(Q_{x,v_j}) \cup V(\bar{T}) \cup V(P_1) \cup V(M)$ induces an odd wheel with center $v$ unless $P_2$ is short. This proves Claim 1.

**Claim 2** At most two of the nodes $v_1, v_2, m$ are adjacent to a node of $\bar{Q}$.

**Proof of Claim 2:** For any pair of nodes $x_i, x_j \in V(\bar{Q})$ such that node $x_i$ is adjacent to $v_1$, node $x_j$ is adjacent to $v_2$ and the subpath $Q_{x,v_j}$ of $Q$ connecting them contains no other node adjacent to $v_1$ or $v_2$, the following property holds: $x_i \neq x_j$, no intermediate node of $Q_{x,v_j}$ is adjacent to $m$, and at most one of $x_i, x_j$ is adjacent to $m$. (The first statement follows from
Figure 14: Direct connections from bottom to top
Figure 15: Direct connections from bottom to top (continued)
the fact that $Q$ contains no node of $N(v_1) \cap N(v_2)$, the second and third follow from Claim 1
and the fact that $P_1$ and $P_2$ cannot both be short).

Therefore, if each of the nodes $v_1, v_2, m$ has a neighbor in $\bar{Q}$, then by symmetry, we can
assume that $\bar{Q}$ contains a node $x_1$ adjacent to $v_1$ but not $m$ and a node $x_2$ adjacent to $m$ but
not $v_1$ with the property that $Q_{x_1, x_2, v_1}$ contains at least one node (possibly $x_2$ itself) adjacent
to $v_2$ but no intermediate node of $Q_{x_1, x_2, v_1}$ is adjacent to $m$ or $v_1$. Now $P_2$ is short, by Claim 1.
In fact, $Q_{x_1, x_2, v_1}$ contains at exactly one node, say $x^*$, adjacent to $v_2$, else there is a wheel with
center $v_2$. But then $V(Q_{x_1, x_2, v_1}) \cup V(\bar{Q}) \setminus \{v\}$ induces a parachute with long sides: the center
node is $v_2$, the side nodes are $x^*$ and $z$ and the bottom node is $v_1$. This proves Claim 2.

Part 1 Node $x_n$ is of Type g[6.1]

By symmetry, we may assume $x_n$ is not adjacent to $v_1$.

Claim 3 If $P$ is any chordless $x_n, v_1$-path in $(V(Q) \cup V(P_1) \cup V(P_2) \cup V(\bar{M})) \setminus \{m, v_2\}$, then
$v_2$ and $m$ both have a neighbor in $P$. Moreover, no node of $\bar{Q}$ is adjacent to $v_1$.
Proof of Claim 3: Let $T_1$ and $T_2$ be the chordless paths from $x_n$ to $v_1$ and from $x_n$ to $v_2$ that only use nodes of $V(\bar{Q}) \cup \{x_n\}$. Since the paths $P$, $T_1$ and $x_n, T_2, v_2, v, v_1$ do not form a 3PC$(x_n, v_1)$, node $v_2$ is adjacent to at least one node of $\bar{Q}$. Since the paths $P$, $T_1$
and $x_n, m, v, v_1$ do not form a 3PC$(x_n, v_1)$, node $m$ is adjacent to at least one node of $\bar{Q}$. Finally, no node of $\bar{Q}$ is adjacent to $v_1$. For otherwise, by construction of $P$, we have that
$P$ is a subpath of $\bar{Q}$ and by the above argument, $v_1, v_2$ and $m$ all have neighbors in $\bar{Q}$, a
contradiction to Claim 2. This proves Claim 3.

Claim 4 Node $x_n$ is adjacent to $v_2$.
Proof of Claim 4: Suppose that neither $v_1$ nor $v_2$ are neighbors of $x_n$. Then, by Claim 3
and symmetry, $Q$ contains no neighbor of $v_1$ and no neighbor of $v_2$. Also by Claim 3 and
symmetry, we see that $x_1$ is not adjacent to $v_1$ or $v_2$. So $Q$ contains no neighbor of $v_1$ or $v_2$. Again by Claim 3 and symmetry, one now deduces that both $P_1$ and $P_2$ are short, a
contradiction. This proves Claim 4.

Case 1 No intermediate node of $Q$ is adjacent to $v_1, v_2$ or $m$.

Assume first that $x_1$ is not strongly adjacent to $\bar{Q}$ and let $b$ be the node of $\bar{Q}$ adjacent
to $x_1$. By Claim 3, it follows that either $b$ is adjacent to $v_2$ and $m$ is adjacent to $z$, or $b$
is adjacent to $m$ and $v_2$ is adjacent to $z$. When the first possibility occurs, the path $Q$
is of Type b. When the second possibility occurs, $P_2$ is short and there is a parachute
with long sides and center node $m$, bottom node $v_1$ and side nodes $x_n, b$.

Assume now that $x_1$ is strongly adjacent to $\bar{Q}$. By Claim 3, it follows that $x_1$ is not of
Type a, b, c, d or k[6.1]. Assume $x_1$ is of Type f[6.1]. Then $x_1$ is adjacent to $v_2$ and to $m$,
by Claim 3. Hence the path $Q$ is of Type e.

Case 2 $N(V(\bar{Q})) \cap \{v_1, v_2, m\} = \{v_2\}$.

Since there is no wheel, $\bar{Q}$ contains exactly one neighbor of $v_2$.

Assume first that $x_1$ is not strongly adjacent to $\bar{Q}$ and let $b$ be the node of $\bar{Q}$ adjacent
to $x_1$. By Claim 3, it follows that either $b$ belongs to $P_2$ and $m$ is adjacent to $z$, or $b$ is
the neighbor of $m$ in $\bar{M}$. If $b$ belongs to $P_2$ and is adjacent to $v_2$, then there is a wheel with center $v_2$. If $b$ belongs to $P_2$ and is not adjacent to $v_2$, then there is a $3PC(b, v_2)$ when $b \in V'$ or $Q$ is of Type $d$ when $b \in V'$. If $b$ is the neighbor of $m$ in $\bar{M}$, then either there is a $3PC(z, v_2)$ when $P_2$ is long, or there is a wheel with center $v_2$ when $P_2$ is short.

Assume now that $x_1$ is strongly adjacent to II. By Claim 3, it follows that $x_1$ is not of Type $a$, $b$, $d$ or $k[6.1]$. If $x_1$ is of Type $c[6.1]$, the middle path $M$ must be short, i.e. $x_1$ is adjacent to $m$. If $x_1$ is adjacent to $m$ and to a node $b$ in $\bar{P}_2$, there is a $3PC(x_1, v_2)$.

If $x_1$ is adjacent to $m$ and to a node $b$ in $\bar{P}_1$, the path $Q$ is of Type $d$. If $x_1$ is of Type $f[6.1]$ and is adjacent to $v_2$, there is a wheel with center $v_2$. If $x_1$ is of Type $f[6.1]$ and is not adjacent to $v_2$, there is a $3PC(x_1, v_2)$.

**Case 3** $N(V(\bar{Q})) \cap \{v_1, v_2, m\} = \{m\}$.

Since there is no wheel, $\bar{Q}$ contains exactly one neighbor of $m$.

Assume first that $x_1$ is not strongly adjacent to II and let $b$ be the neighbor of $x_1$ in II. By Claim 3, we have that $b$ is adjacent to $v_2$. If $b$ is not adjacent to $m$, there is a $3PC(m, b)$; else there is a wheel with center $m$.

Assume now that $x_1$ is strongly adjacent to II. By Claim 3, it follows that $x_1$ is not of Type $a$, $b$, $d$ or $k[6.1]$. If $x_1$ is of Type $c[6.1]$, then it is adjacent to $v_2$. So, if $x_1$ is of Type $c[6.1]$, there is an odd wheel with center $m$ (if $x_1$ and $m$ are adjacent), or there is a $3PC(m, x_1)$ (if $x_1$ and $m$ are not adjacent). If $x_1$ is of Type $k[6.1]$, there exists a parachute with long sides with center $m$ and bottom node $v_1$.

**Case 4** $N(V(\bar{Q})) \cap \{v_1, v_2, m\} = \{v_2, m\}$.

$\bar{Q}$ has exactly one neighbor of $m$ (or else there is a wheel with center $m$). By Claim 1, $P_2$ is short.

Assume first that $x_1$ is not strongly adjacent to II and let $b$ be the neighbor of $x_1$ in II. If $b \in V(M) \setminus \{v, m\}$, there is a wheel with center $v_2$. So $b$ belongs to $\bar{P}_1$ and there exists a parachute with long sides and center node $m$.

Assume now that $x_1$ is strongly adjacent to II. If $x_1$ is of Type $c[6.1]$, with neighbors in $\bar{P}_1$ and $M$, then there is a wheel with center $v$. If $x_1$ is of Type $c[6.1]$ with neighbors $v_2$ and in $\bar{M}$, or if $x_1$ is of Type $f$ or $k[6.1]$, then there is a wheel with center $v_2$. If $x_1$ is of Type $d[6.1]$, with neighbors in $\bar{P}_1$ and $\bar{M}$, then there is a $3PC(x_1, z)$.

**Part 2** Node $x_n$ has a unique neighbor, say $t$ in II.

**Case 1** $N(V(\bar{Q})) \cap \{v_1, v_2, m\} = \emptyset$.

**Case 1.1** Node $x_1$ has a unique neighbor, say $b$ in II.

Assume first that $b$ is in $\bar{P}_1$. Then $z$ is adjacent to $v_2$, else there is a $3PC(z, v_2)$. The nodes $b$ and $t$ belong to the same side of the bipartition, else there is a $3PC(b, t)$. If $b, t \in V'$, then they are both adjacent to $v_1$, else there is a $3PC(b, v_1)$ or a $3PC(t, v_1)$. Furthermore $T$ is short, else there exists a parachute with long sides. Hence $Q$ is of Type $l$. If $b, t \in V'$, then there is a parachute with long sides.
Assume now that $b \in V(M) \setminus \{v, m\}$. Then $b \in V^r$, else there is a $3PC(b, v_1)$ or $3PC(b, v_2)$.

If the top path of II is long, assume w.l.o.g. that $t$ is not adjacent to $v_1$. Then $P_2$ is short, else there is a $3PC(z, v_2)$. Node $t$ is adjacent to $v_2$, else there is a $3PC(z, v_1)$. This yields a parachute with long sides induced by $V(Q) \cup V(II) \setminus V(P_1)$.

If the top path of II is short, then the path $Q$ is of Type j when the side paths are long and, when $P_2$ is short, there is a parachute with long sides induced by $V(Q) \cup V(II) \setminus V(P_1)$.

Case 1.2 Node $x_1$ is strongly adjacent to II.

Assume first that $x_1$ is of Type a[6.1]. If the top path $T$ is long, there exists a $3PC(x_1, v_1)$ or a $3PC(x_1, v_2)$. So $T$ is short and $Q$ is a path of Type k.

Assume that $x_1$ is of Type b[6.1]. Let $b_1, b_2$ be the neighbors of $x_1$ in $P_1$ and $P_2$ respectively. If $b_1$ is not adjacent to $v_1$ or $b_2$ is not adjacent to $v_2$, there exists a $3PC(z, x_1)$. If $b_1$ is adjacent to $v_1$ and $b_2$ is adjacent to $v_2$ then $t \in V^r$, else there is a $3PC(t, x_1)$. This yields a connected 6-hole, contradicting the assumption that $G$ is weakly balanced.

Assume $x_1$ is of Type c[6.1], say with neighbors in $M$ and $P_2$. If $P_2$ is long, then $x_1$ is not adjacent to $v_2$ and there is a $3PC(x_1, v_2)$. If $P_2$ is short, then $x_1$ is adjacent to $v_2$. If $t$ is adjacent to $v_2$, then $v_2$ is the center of a wheel and if $t$ and $v_2$ are nonadjacent then $t \in V^r$, else there is a $3PC(t, v_2)$. Now there is a parachute with long sides, center node $v_2$, bottom node $t$.

If $x_1$ is of Type d[6.1], there is a $3PC(x_1, z)$.

If $x_1$ is of Type f[6.1]. Then $x_1$ is not adjacent to both $v_1, v_2$. If $x_1$ is not adjacent to $v_1$, there is a $3PC(x_1, v_1)$, and if $x_1$ is not adjacent to $v_2$, there is a $3PC(x_1, v_2)$.

Finally, if $x_1$ is of Type k[6.1], there is a wheel with center $v_2$.

Case 2 \( N(V(Q)) \cap \{v_1, v_2, m\} = \{m\} \).

Case 2.1 Node $x_1$ has a unique neighbor, say $b$ in II.

If $b$ is in $\tilde{P}_1$, then $b \in V^r$, else there is a $3PC(m, b)$ and $t \in V^r$, else there is a $3PC(v, t)$. This implies the existence of a $3PC(b, t)$.

If $b \in V(M) \setminus \{v, m\}$, then $V(Q) \cup \{b\}$ contains at most two nodes adjacent to $m$, otherwise there is a wheel. If it contains only one neighbor of $m$, say $x_1$, there is a $3PC(x_1, v_1)$. So, $V(Q) \cup \{b\}$ contains exactly two neighbors of $m$. If one side of II is short, there exists a parachute with long sides with center $m$ and bottom $v_1$. So II has long sides. Now II has short top, else there is a wheel with center $m$. If $b \in V^r$, $Q$ is of Type g. If $b \in V^r$ and $b$ is not adjacent to $m$, there is a $3PC(b, m)$. If $b$ is adjacent to $m$, then $Q$ is of Type f.

Case 2.2 Node $x_1$ is strongly adjacent to II.

Assume first that $x_1$ is of Type a[6.1]. Then $Q$ has at most two nodes adjacent to $m$, otherwise there is a wheel. If $Q$ has only one neighbor of $m$, say $x_1$, there is a
$3PC(x_i, v_1)$. So, $Q$ has exactly two neighbors of $m$. Now, if $II$ has long top, there is a wheel with center $m$. If $II$ has short top, $Q$ is of Type i.

If $x_1$ is of Type b or d[6.1], there is a $3PC(x_1, z)$.

If $x_1$ is of Type c [6.1], then $x_1$ is adjacent to $m$, else there is a $3PC(x_1, m)$. But now there is a wheel with center $m$, since $x_1$ is not adjacent to $v_1$ or $v_2$.

If $x_1$ is of Type f [6.1], then $x_1$ is adjacent to $m$, else there is a $3PC(x_1, m)$. Hence $\tilde{Q}$ contains exactly one neighbor of $m$, as otherwise there is a wheel with center $m$. Let $w_1$ and $w_2$ be the neighbors of $x_1$ in $P_1$ and $P_2$ respectively. Assume w.l.o.g. that $w_2 \neq v_2$. If $w_1 \neq v_1$, then either $T$ is long and there is a wheel with center $m$, or $T$ is short and $Q$ is of Type h. If $w_1 = v_1$, then the neighbor $t$ of $x_n$ is adjacent to $v_1$, otherwise there is a wheel with center $m$. Thus the path $Q$ is of Type a.

Finally, assume that $x_1$ is of Type k[6.1]. Then $Q$ has at most two nodes adjacent to $m$, otherwise there is a wheel. If $Q$ has only one neighbor of $m$, say $x_i$, there is a $3PC(x_i, v_1)$. So, $Q$ has exactly two neighbors of $m$. But now, there is a parachute with long sides, middle node $m$ and bottom node $v_1$.

**Case 3** $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \{v_1\}$ and $v_1$ is adjacent to $t$.

**Case 3.1** Node $x_1$ has a unique neighbor, say $b$, in $II$.

If $b \in V(P_1) \setminus \{v_1\}$, then $v_1$ has exactly one neighbor in $\tilde{Q}$ and $v_1$ is not adjacent to $b$, else there is a wheel with center $v_1$. Now $b \in V^r$, else there is a $3PC(b, v_1)$. $P_2$ is short, else there is a $3PC(v_2, z)$. Now the graph induced by $V(II) \cup V(Q)$ contains a parachute with long sides, center $v_1$ and bottom $b$.

If $b \in V(\tilde{P}_2) \cup V(M) \setminus \{m\}$, then there is a wheel with center $v_1$ when $b$ is not adjacent to $v_2$, and there is a $3PC(b, v_1)$ otherwise.

**Case 3.2** Node $x_1$ is strongly adjacent to $II$.

If $x_1$ is of Type a[6.1], there is a wheel with center $v_1$.

If $x_1$ is of Type b or d[6.1], then there is a wheel with center $v_1$.

Assume $x_1$ is of Type c[6.1]. If $x_1$ is not adjacent to $v_2$, then there is a wheel with center $v_1$. If $x_1$ is adjacent to $v_2$, then $x_1$ is not adjacent to $m$ and $P_2$ is short. Hence there is a parachute with long sides, center node $v_2$, bottom node $v_1$.

Assume that $x_1$ is of Type f [6.1]. Then, if $x_1$ is adjacent to $v_1$, there is a wheel with center $v_1$, and if $x_1$ is not adjacent to $v_1$, there is a $3PC(x_1, n_1)$.

Finally, if $x_1$ is of Type k[6.1], there is a wheel with center $v_2$.

**Case 4** $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \{v_1\}$ and $v_1$ is not adjacent to $t$.

Then $t \in V^r$, else there is a $3PC(t, v_1)$. Consider the parachute $II'$ obtained from $II$ by replacing the $tv_1$-subpath of $T$ by the path $x_i, \ldots, x_n, t$, where $x_i$ is the node of $\tilde{Q}$ of highest index adjacent to $v_1$. As $Q_{x_i, x_{i+1}}$ is shorter than $Q$, we may assume by induction that $Q_{x_i, x_{i+1}}$ and $II'$ satisfy Theorem 6.4. Since $II'$ has long top and $Q_{x_i, x_{i+1}}$ contains no neighbor of $m$, we get a contradiction. So Case 4 cannot occur.
Case 5 \(N(V(\bar{Q})) \cap \{v_1, v_2, m\} = \{v_2, m\}\).

As a consequence of Claim 1, \(P_2\) is short. Let \(x_j \in V(\bar{Q})\) be the neighbor of \(v_2\) closest to \(t\) in \(\bar{Q}\) and \(x_i \in V(\bar{Q})\) the neighbor of \(m\) closest to \(t\). Note that \(i = j\) is possible. Suppose that \(j < i\). If \(t \in V^r\), there is a \(3PC(v_2, t)\) or a wheel with center \(v_2\). So \(t \in V^r\). If \(v_2\) has more than one neighbor in \(Q_{x|x_n}\), there is a wheel with center \(v_2\). If \(x_j\) is the unique neighbor of \(v_2\) in \(Q_{x|x_n}\), there is parachute with long sides, center node \(v_2\), bottom node \(t\) and side nodes \(v, x_j\).

So \(j < i\). If there is no other neighbor of \(m\) on the \(tx_j\)-subpath of \(Q\), then there is a \(3PC(v_2, x_i)\) and if there are more than two neighbors, then there is a wheel with center \(m\). So there are exactly two neighbors of \(m\) on the \(tx_j\)-subpath of \(Q\), say \(x_i\) and \(x_k\).

Now there is a parachute with long sides, center node \(m\) and bottom node \(v_2\).

Case 6 \(N(V(\bar{Q})) \cap \{v_1, v_2, m\} = \{v_1, v_2\}\).

Let \(x_k\) be the node of smallest index in \(\bar{Q}\) adjacent to \(v_1\) for \(h = 1\) or 2 such that at least one of the nodes \(x_2, \ldots, x_{k-1}\) is adjacent to \(x_{h-1}\), and let \(x_i\) be such a node with highest index, \(l < k - 1\). W.l.o.g. assume \(x_j\) is adjacent to \(v_2\). Let \(\Pi'\) be parachute obtained from \(\Pi\) by replacing \(T\) by \(v_1, x_i, Q_{x|x_n}, x_k, v_2\). Let \(Q' = Q_{x|x_{i-1}}\). Then \(Q'\) is a direct connection from bottom to top of \(\Pi'\) avoiding \(S(\Pi')\). By induction, as \(Q'\) is shorter than \(Q\), we can assume that \(\Pi'\) and \(Q'\) satisfy Theorem 6.4. However, since \(\Pi'\) has long top and \(m\) has no neighbor on \(Q'\), we get a contradiction. So Case 6 cannot occur.

\[\square\]

6.4 Connections from Bottom to Top

In this subsection, we continue the study of direct connections \(Q\) from bottom to top of a parachute. These connections were considered in Theorem 6.4 under the assumption that parachute modifications relative to \(Q\) had been performed. Here we describe the possible direct connections before parachute modifications are performed.

Theorem 6.5 Let \(G\) be a wheel-free weakly balanced graph. Let \(\Pi = Par(P_1, P_2, M, T)\) be a parachute and let \(Q = x_1, \ldots, x_n\) be a direct connection from bottom to top avoiding \(S(\Pi)\).

(i) If \(\Pi\) has long top and long sides, then \(n \geq 2\) and, up to symmetry between \(P_1\) and \(P_2\), \(Q\) is of Type a, b, c, d or e[6.4] or one of the following types, see Figure 16.

- **Type a1** Node \(x_1\) is a strongly adjacent node of Type f[6.1], adjacent to \(v_1, m\) and one node in \(\bar{P_2}\). Node \(x_n\) is strongly adjacent to \(\Pi\), adjacent to \(v_1\) and to one node in \(T\).

- **Type b1** Node \(x_1\) is strongly adjacent to \(\Pi\), adjacent to \(v_2\) and to one node in \(\bar{P_2}\). Node \(x_n\) is a strongly adjacent node of Type g[6.1], adjacent to \(v_2, m\) and one node in \(T\). No node of \(Q\) has a neighbor in \(\Pi\). Furthermore, \(\Pi\) has a short middle path.

- **Type b2** Node \(x_1 \in V^r\) is strongly adjacent to \(\Pi\), with two neighbors in \(V(P_2) \setminus \{v_2\}\). Node \(x_n\) is adjacent to \(v_2\) and to no other node of \(\Pi\). Node \(x_n\) is a strongly adjacent
node of Type $g[6.1]$, adjacent to $v_2$, $m$ and one node in $\tilde{T}$. No node of $Q_{x_n x_{n-1}}$ has a neighbor in $\Pi$. Furthermore, $\Pi$ has a short middle path.

- **Type cl** Node $x_1 \in V^c$ is strongly adjacent to $\Pi$, adjacent to two nodes in $P_2$. Node $x_n$ is a strongly adjacent node of Type $g[6.1]$, adjacent to $v_2$, $m$ and one node in $\tilde{T}$. Exactly one node of $\tilde{Q}$ is adjacent to $v_2$ and none is adjacent to $v_1$, $m$. Furthermore, $\Pi$ has a short middle path.

(ii) If $\Pi$ has short top and long sides, then either $n = 1$ and the only node of $\tilde{Q}$ is of Type $a[6.1]$, or $n \geq 2$ and, up to symmetry between $P_1$ and $P_2$, $\tilde{Q}$ is of Types $f$, $g$, $h$, $i$, $j$ or $k[6.4]$ or of one of the following types. See Figure 16.

- **Type f1** Node $x_1 \in V^c$ is strongly adjacent to $\Pi$, adjacent to $m$ and a node in $\tilde{M}$. Node $x_n$ is not strongly adjacent to $\Pi$. Exactly one node of $\tilde{Q}$ is adjacent to $m$ and none is adjacent to $v_1$ or $v_2$.

- **Type g1** Node $x_1 \in V^c$ is strongly adjacent to $\Pi$ and its two neighbors both belong to $V(M) \setminus \{v, m\}$. Node $x_n$ is not strongly adjacent to $\Pi$. Exactly two nodes of $\tilde{Q}$ are adjacent to $m$ and none is adjacent to $v_1$ or $v_2$.

- **Type j1** Node $x_1 \in V^c$ is strongly adjacent to $\Pi$ and its two neighbors both belong to $V(M) \setminus \{v, m\}$. Node $x_n$ is not strongly adjacent to $\Pi$. No node of $\tilde{Q}$ has a neighbor in $\Pi$.

(iii) If $\Pi$ has a short side, say $P_2$, and $G$ contains no parachute with long sides, then $\Pi$ has short top and either $n = 1$ and the only node of $\tilde{Q}$ is of Type $l[6.1]$, or $n \geq 2$ and $\tilde{Q}$ is of Type $l[6.4]$ or as described below. See Figure 16.

- **Type l1** Node $x_1$ is strongly adjacent to $\Pi$ and is adjacent to $v_1$ and one node in $\tilde{P}_1$. Node $x_n$ is not strongly adjacent to $\Pi$. No node of $\tilde{Q}$ has a neighbor in $\Pi$.

- **Type l2** Node $x_1 \in V^c$ is strongly adjacent to $\Pi$ and has exactly two neighbors in $P_1$. Furthermore, one of these neighbors is adjacent to $v_1$. Node $x_n$ is adjacent to $v_1$ and to no other node of $\Pi$. Node $x_n$ is not strongly adjacent to $\Pi$. No node of $Q_{x_n x_{n-1}}$ has a neighbor in $\Pi$.

Proof: A parachute $\Pi'$ with direct connection $Q' = x'_1, \ldots, x'_n$ is called top-initial for $Q'$ if $\Pi'$ and $Q'$ cannot be obtained from a parachute $\Pi$ with direct connection $Q$ by one parachute modification at the top using one node of $Q$. It follows from Remark 6.3 that if $\Pi'$ is not top-initial for $Q'$, then $Q = x'_1, \ldots, x'_n, x'_{n+1}$, node $x'_{n+1}$ is in the top of $\Pi'$, and $V(\Pi') \setminus V(\Pi) = x'_{n+1}$.

Parachute $\Pi'$ with direct connection $Q' = x'_1, \ldots, x'_n$ is bottom-initial for $Q'$ if $\Pi'$ and $Q'$ cannot be obtained from a parachute $\Pi$ with direct connection $Q$ by one parachute modification at the bottom using one node of $Q$. Again, it follows from Remark 6.3 that if $\Pi'$ is not bottom-initial for $Q'$, then $Q = x'_0, x'_1, \ldots, x'_n$, node $x'_0$ is in the bottom of $\Pi'$, and $V(\Pi') \setminus V(\Pi) = x'_0$.

The proof of the theorem uses the following sufficient conditions for $\Pi'$ being top-initial or bottom-initial for $Q' = x'_1, \ldots, x'_n$.
Figure 16: Direct connections from bottom to top
(1) If the top of $\Pi'$ is short, then $\Pi'$ is top-initial for $Q'$.

(2) If $x'_1$ has more than one neighbor in the bottom of $\Pi'$, then $\Pi'$ is bottom-initial for $Q'$.

(3) If $\Pi'$ has long sides and $x'_n$ is adjacent to one of its side nodes $v_1$ or $v_2$ and to node $m$, then $\Pi'$ is top-initial for $Q'$.

(4) If $\Pi'$ has long sides and $x'_1$ is adjacent to one of its side nodes $v_1$ or $v_2$ and to node $m$, then $\Pi'$ is bottom-initial for $Q'$.

(1) follows from Remark 6.3. (2) follows from the definitions of direct connection and parachute modification. To prove (3): Assume that $\Pi'$ has long sides and that $x'_n$ is adjacent to $m$ and to one of $v_1$ and $v_2$. If $\Pi'$, $Q'$ come from $\Pi$, $Q$ by a parachute modification at the top, then $x'_n$ is a strongly adjacent node to II of Type j [6.1], but this contradicts Theorem 6.4 as $\Pi$ has long sides. Similarly, one proves (4).

Now, we prove the theorem as follows: Given a parachute $\Pi'$, we list all direct connections $Q'$ from bottom to top given by Theorem 6.4. For the ones that are not bottom and top-initial, we add to the list the ones that give them after one parachute modification at the top or at the bottom. We repeat this until all the direct connections added are both bottom and top-initial. Two rounds of this procedure will suffice. In the proof of (iii), the procedure also stops when a parachute with long sides is detected.

(ii) and (iii): $\Pi$ has long sides.

Assume $\Pi'$ with direct connection $Q'$ is derived from $\Pi$, $Q$ with one parachute modification at the top.

By (1) and (3) above, $Q'$ is of Type a[6.4]. It follows that $Q$ is of Type a1.

Assume now $\Pi'$ with direct connection $Q'$ is derived from $\Pi$, $Q$ with one parachute modification at the bottom. So by (2) and (4) above, $Q'$ is of Type b, c, d, f, g, j [6.4].

If $Q'$ is of Type b[6.4], then $Q$ is of Type b1.  
If $Q'$ is of Type c[6.4], then $Q$ is of Type c1.  
If $Q'$ is of Type d[6.4], then there is a wheel with center $v$.  
If $Q'$ is of Type f[6.4], then $Q$ is of Type f1.  
If $Q'$ is of Type g[6.4], then $Q$ is of Type g1.  
If $Q'$ is of Type j[6.4], then $Q$ is of Type j1.

We now examine all the newly added direct connections.

If $Q'$ is of Type a1, then $\Pi'$ is bottom-initial for $Q'$ by (4) and $\Pi'$ is top-initial for $Q'$, else there is a wheel with center $v$.

If $Q'$ is of Type b1, then $\Pi'$ is top-initial for $Q'$ by (2) and one parachute modification at the bottom gives Type b2.

If $Q'$ is of Type c1, then $\Pi'$ is top and bottom-initial for $Q'$ by (3) and (2).

If $Q'$ is of Type f1, then $\Pi'$ is top-initial for $Q'$ by (1) and one parachute modification at the bottom gives Type g1.

If $Q'$ is of Type g1, then $\Pi'$ is top and bottom-initial for $Q'$ by (1) and (2).

If $Q'$ is of Type j1, then $\Pi'$ is top and bottom-initial for $Q'$ by (1) and (2).

We now examine all the newly added direct connections.

If $Q'$ is of Type b2, then $\Pi'$ is top and bottom-initial for $Q'$ by (3) and (2).

(iii): $\Pi$ has a short side and $G$ contains no parachute with long sides.
Assume II' with direct connection $Q'$ is derived from II, $Q$ with one parachute modification at the top.

Then by Theorem 6.4, $Q'$ is of Type l[6.1] or l[6.4]. In both cases, II' is top-initial for $Q'$ by (1) and one parachute modification at the bottom gives Type l1.

Next, consider the case where $Q'$ is of Type l1. Then II' is top-initial for $Q'$ by (1). Moreover, if there has been one parachute modification at the bottom, then the first node $x_0$ of $Q$ has two neighbors in $P_1$. One of these neighbors is adjacent to $v_1$, as otherwise there is a parachute with long sides. Hence $Q$ is of Type l2.

Finally, if $Q'$ is of Type l2, then II' is top and bottom-initial for $Q'$ by (1) and (2).

\[\square\]

### 6.5 Parachutes with a Short Side

As in the earlier subsections, $G$ is a wheel-free weakly balanced graph. We show that, if $G$ contains a parachute with one short side but no parachute with long sides, then $G$ has an extended star cutset.

**Theorem 6.6** Let $G$ be a wheel-free weakly balanced graph containing no parachute with long sides. Let $\Pi = Par(P_1, P_2, M, T)$ be a parachute with a short side, say $P_2 = v_2, z$ and let its middle path be $M = v, m, \ldots, z$. Then $S(\Pi)$ or $N(v_2) \cup (N(z) \cap N(v)) \setminus \{m\}$ is an extended star cutset of $G$.

**Proof:** If $S(\Pi)$ is not an extended star cutset then, by Theorem 6.5(iii) and Remark 6.3, $\Pi$ has short top $T = v_1, t, v_2$, and we can assume that, after possibly parachute modifications at the bottom, there is a direct connection $Q = x_1, \ldots, x_n$ from bottom to top of Type l[6.4] or of Type l[6.1], where $x_1$ is adjacent to the neighbor, say $a$, of $v_1$ in $P_1$. Note that $\Pi$ induces another parachute with short side, namely the parachute with center node $v_2$, side nodes $v, z$, top path $M$, middle path $T$ and side paths $P_1$ and $v, v_1$. Denote by $\Pi'$ this parachute. Now, if $S(\Pi') = N(v_2) \cup (N(z) \cap N(v)) \setminus \{m\}$ is not an extended star cutset then, by Theorem 6.5(iii), $\Pi'$ has short top $v, m, z$ and, there is a direct connection $R = y_1, \ldots, y_r$ from the bottom of $\Pi'$ to the top $m$ of Type l[6.1], Type l[6.4], Type l1 or l2[6.5].

No node of $V(R) \setminus \{y_1\}$ is adjacent to or coincident with a node of $Q$, since otherwise $\Pi'$ would contain a direct connection from bottom to top violating Theorem 6.5. Now, if $R$ is of Type l1 or l2[6.5], there is a wheel with center $z$. So $R$ is of Type l[6.1] or Type l[6.4]. Node $y_1$ has a neighbor in $Q$ since, otherwise, there is a wheel with center $z$. Finally, $y_1$ is adjacent to $x_1$ but to no other node of $Q$ since, otherwise, $\Pi$ would contain a direct connection from bottom to top violating Theorem 6.5. Let $b$ be the neighbor of $z$ in $P_1$, see Figure 17.

Node $x_1$ is adjacent to $t$, else there is a $3PC(x_1, t)$. Similarly, $y_1$ is adjacent to $m$, else there is a $3PC(y_1, m)$. Finally, $a$ is adjacent to $b$, else there is a $3PC(a, b)$. Now the graph induced by $V(\Pi) \cup V(Q) \cup V(R)$ is an $R_{10}$ configuration. It follows that $G$ is not weakly balanced, a contradiction. \[\square\]

### 6.6 Stabilized Parachutes

In the remainder of this subsection, we consider wheel-free weakly balanced graphs $G$ that contain a parachute with long sides and short middle. In this subsection, we make the further assumption that $G$ contains a stabilized parachute, as defined below.
Figure 17: Connections $Q$ and $R$

**Definition 6.7** A stabilized parachute $(\Pi, R)$, see Figure 18, consists of a parachute $\Pi = \text{Par}(P_1, P_2, M, T)$ with long sides $P_1 = v_1, a, \ldots, z$ and $P_2 = v_2, \ldots, z$, a short middle $M = v, m, z$ and of a chordless path $R = r_1, \ldots, r_k$ (possibly $k = 1$), where $r_i \in V \setminus V(\Pi)$ for $i = 1, \ldots, k$, such that node $r_1$ is adjacent to node $a$ and node $r_k$ is adjacent to $v$. Nodes $r_1$ and $r_k$ do not have any other adjacencies in $\Pi$ than those just mentioned and nodes $r_i$ for $i = 2, \ldots, k - 1$, are not adjacent to any node of $\Pi$. Furthermore,

(i) any strongly adjacent node of Type $f[6.1]$ relative to $\Pi$ that is adjacent to $v_2$ must also be adjacent to $v_1$, and

(ii) any node in $V \setminus (V(\Pi) \cup V(R))$ that has two neighbors in $T$ and is adjacent to $r_k$ must also be adjacent to $m$.

We now prove that if $G$ contains a stabilized parachute, then $G$ has an extended star cutset.

**Lemma 6.8** Let $G$ be a wheel-free weakly balanced graph. If $\Pi$ is a stabilized parachute then the only possible direct connections from bottom to top avoiding $S(\Pi)$ are of Type $b, c, d, e[6.4]$ or Type $b_1, b_2, c_1[6.5]$.

**Proof:** A direct connection of Type o[6.1] or Type g or j[6.4] cannot occur since a stabilized parachute has middle path of length 2. Similarly for Types f1, g1 and j1[6.5]. Now we show that a direct connection $Q = x_1, \ldots, x_n$ of Type $a, f, h, i$ or $k[6.4]$ and Type $a1[6.5]$ cannot occur.

**Case 1** Path $Q$ is of Type $a[6.4]$.

It follows from Condition (i) of Definition 6.7 that $x_1$ is adjacent to $v_1, m$ and a node in $P_2$. Nodes $x_2, \ldots, x_n$ are not adjacent or equal to any of the nodes $r_1, \ldots, r_{k-1}$, else
Figure 18: A stabilized parachute

there is a direct connection from bottom to top that contradicts Theorem 6.5. Clearly, $x_1$ is not in $R$ and $r_k$ is not in $Q$.

If $r_k$ is not adjacent to any node in $\tilde{Q}$, there is a wheel with center $v_1$ whether or not $x_1$ is adjacent at least one node in the set $\{r_1, \ldots, r_k\}$.

If $r_k$ is adjacent to at least one node in $\tilde{Q}$, let $x_j$ be the node of $\tilde{Q}$ that is adjacent to $m$ and let $q$ be a neighbor of $r_k$ in $Q_{x_j}v_n$ such that $Q_{x_j}q$ contains no other neighbor of $r_k$. Note that $x_j, q, x_1 \in V^c$, so $H = x_j, Q_{x_j}q, q, r_k, R, r_1, a, v_1, x_1, m, x_j$ is a hole. Now $(H, v)$ is a wheel.

**Case 2** Path $Q$ is of Type $a1[6.5]$. 

By Theorem 6.5, $x_n$ is not adjacent or equal to $r_1, \ldots, r_{k-1}$ and obviously, $x_n \neq r_k$.

By Condition (ii) of Definition 6.7, node $x_n$ is not adjacent to $r_k$. Therefore, after parachute modification, we are back in Case 1.

**Case 3** Path $Q$ is of Type $f, h, i$ or $k[6.4]$.

If some node of $V(Q) \setminus \{x_1\}$ is adjacent or equal to at least one node of $V(R) \setminus \{r_k\}$, then there is a direct connection from the bottom to the top of $\Pi$ violating Theorem 6.5(ii). So no such adjacency exists. Clearly, $x_1$ is not in $R$ and $r_k$ is not in $Q$. If $Q$ is of Type $h, i$ or $k[6.4]$ and $x_1$ is adjacent to a node in $R$, there is a wheel with center $x_1$. If $Q$ is of Type $f[6.4]$, then $x_1$ is adjacent to at most one node of $R$, else there is a wheel with center $x_1$. If $x_1$ is adjacent to exactly one node of $R$, there is a $3PC(x_1, a)$, where $a$ is the neighbor of $v_1$ in $P_1$. So, in all cases, $x_1$ is not adjacent to a node in $R$. If $r_k$ is not adjacent to a node in $Q$, then there is a wheel with center $v_1$. So $r_k$ is adjacent to a node in $Q$. Now, if $Q$ is of Type $f[6.4]$, there is a wheel with center $v$, and if $Q$ is of Type $h, i$ or $k[6.4]$, there is a $3PC(x_1, r_k)$.

$\square$
Lemma 6.9 Let $G$ be a wheel-free weakly balanced graph. If $G$ contains a parachute $\Pi$ with long sides having a direct connection of Type $b$, $c$, $d[6.4]$ or $b1$, $b2$, $c1[6.5]$, then $G$ has a stabilized parachute with top shorter than the top of $\Pi$.

Proof: If the direct connection is of Types $c$, $d[6.4]$ or $b1$, $b2$, $c1[6.5]$ there exists a parachute with long sides with a direct connection of Type $b[6.4]$. So we assume that the direct connection $Q = x_1, \ldots, x_n$ is of Type $b[6.4]$. Assume w.l.o.g. that $x_1$ is adjacent to the neighbor $b$ of $v_2$ in $P_2$. Construct the parachute $\Pi'$ as follows. The middle path of $\Pi'$ is $M' = x_n, m, z$. The top path $T'$ of $\Pi'$ is the subpath of $T$ connecting the two neighbors of $x_n$ in $T$, say $t \in \bar{T}$ and $v_2$. The side path $P'_1$ is identical to $P_2$ and the side path $P'_1$ connects $t$ to $z$, using nodes of $V(T) \cup V(P_1)$. We will show that $\Pi'$, with extra path induced by $Q_{x_1x_{n-1}}$, defines a stabilized parachute with shorter top than $\Pi$. In order to prove that $\Pi'$ defines a stabilized parachute, we need to check Conditions (i) and (ii) of Definition 6.7. Condition (i) holds since $t \in V(\bar{T})$ and a node $w$ of Type $f[6.1]$ relative to $\Pi'$ that is adjacent to $t$ must also be adjacent to $v_2$, else $w$ violates Theorem 6.1 relative to $\Pi$. To see that Condition (ii) holds, consider a node $y$ adjacent to $x_{n-1}$ and to two nodes of $T'$. There is a direct connection that violates Theorem 6.5(i) with respect to $\Pi$, unless node $y$ is adjacent to $m$. This completes the proof that $\Pi'$ is a stabilized parachute.

Theorem 6.10 Let $G$ be a wheel-free weakly balanced graph. If $G$ contains a stabilized parachute, then $G$ has an extended star cutset.

Proof: Among all parachutes that give rise to a stabilized parachute, let $\Pi$ be one with shortest top. If $\Pi$ has no extended star cutset, every direct connection from bottom to top avoiding $S(\Pi)$ is of Type $e[6.4]$ by Lemmas 6.8 and 6.9.

Consider $Q = x_1, \ldots, x_n$ of Type $e[6.4]$ and assume w.l.o.g. that $x_1$ and $x_n$ are adjacent to $v_1$. Then the first node $v_1$ of the extra path $R = r_1, \ldots, r_k$ is adjacent to the neighbor of $v_1$ in $P_1$, by Condition (i) of Definition 6.7. Note that the nodes $x_2, \ldots, x_n$ are not adjacent or equal to $r_1, \ldots, r_{k-1}$, because otherwise $\Pi$ would have a direct connection of Type $b[6.5]$, which contradicts, by Lemma 6.9, the fact that $\Pi$ is a stabilized parachute with shortest top.

If $r_k$ is not adjacent to any node in $Q_{x_2x_n}$, there is a wheel with center $v_1$ whether or not $x_1$ has neighbors in $R$.

If $r_k$ is adjacent to at least one node in $Q_{x_2x_n}$, then there is a parachute with shorter top obtained by replacing the center node $v$ by the node $x_n$ and replacing the extra path $R$ by a chordless path from $x_n$ to $r_1$ only involving nodes of $(V(Q) \setminus \{x_1\}) \cup V(R)$. The new parachute satisfies Condition 6.7(i) as any node violating it would violate Theorem 6.1 with respect to $\Pi$. To see that Condition (ii) holds, consider a node $w$ adjacent to $x_{n-1}$ and to two nodes of the new top. There is a direct connection that violates Theorem 6.5(i) with respect to $\Pi$, unless node $w$ is adjacent to $m$.

6.7 Parachutes with Long Top and Long Sides

In this subsection, we show that, if $G$ is a wheel-free weakly balanced graph that contains a parachute with long top and long sides, then $G$ has an extended star cutset.
**Lemma 6.11** Let $G$ be a wheel-free weakly balanced graph that contains no stabilized parachute. Let $\Pi$ be a parachute with long sides having a direct connection of Type a[6.4]. Then $\Pi$ has short top and a direct connection of Type f[6.4].

**Proof:** Let $Q = x_1, \ldots, x_n$ be a direct connection of Type a[6.4] where $x_1$ is adjacent to $v_1$ and let $x_j$ be the node of $Q$ adjacent to $m$. Consider the parachute $\Pi'$ in $P_2$ with center $m$, side nodes $x_1$ and $x_j$ and bottom node $v_2$. Then $\Pi'$ has long sides and short middle. Let $R'$ be the path in $P_2$ from $m$ to the neighbor of $x_1$ in $P_2$. By assumption, $\Pi' \cup R'$ is not a stabilized parachute. If $w$ would be a node violating Condition 6.7(i) with respect to $\Pi'$, then it would be adjacent to $x_j$, $v$ and some node in $P_2$. So, there would be a wheel $(H, m)$ with $H = w, x_j, Q_{x,j}, v_1, v, w$. Hence $\Pi'$ satisfies Condition 6.7(i). So Condition 6.7(ii) is not satisfied for $\Pi'$ and $R'$. Hence, there exists a node $w$ adjacent to the neighbor of $m$ distinct from $v$ in $P_2$, to two neighbors $x_f, x_i$ on $Q$ for $1 \leq f < i \leq j$, and not adjacent to $v$. So $w, x_i, Q_{x_i,x_f}, x_n$ is a direct connection from bottom to top of $\Pi$. By Theorem 6.5, it has to be of Type f[6.4], since it contains $x_j$. \hfill \Box

**Lemma 6.12** Let $G$ be a wheel-free weakly balanced graph that contains no stabilized parachute. Then a parachute with long sides cannot have a direct connection of Type e[6.4] or Type a1[6.5].

**Proof:** If $Q = x_1, \ldots, x_n$ is a direct connection of Type e[6.4], where $x_1$ is adjacent to $v_2$, $m$ and $b \in P_1$ and $x_n$ is adjacent to $v_2$ and $t \in T$. We show that a stabilized parachute occurs by taking $v_2$ as the center node, $x_1, x_n$ as the side nodes and $v_1$ as the bottom node. There exists no pair $d_1, d_2$ of strongly adjacent nodes of Type f[6.1] relative to this parachute such that $d_1$ is adjacent to $x_1$ but not to $x_n$ and $d_2$ is adjacent to $x_n$ but not to $x_1$ (else there is a wheel with center $v$). Since there are two possibilities for the path $R$, namely the subpath of $T$ from $v_2$ to $t$ and the path from $v_2$ to $b$, Condition (i) of Definition 6.7 is satisfied by one of the choices for $R$. Next, we consider Condition (ii). First, consider the case when $R$ is the subpath of $T$ connecting $v_2$ to $t$. If there is a node $w$ adjacent to two nodes in $Q$ and to the neighbor of $v_2$ in $T$ and $w$ is not adjacent to $v$, then there is a direct connection from bottom to top of $\Pi$ that contradicts Theorem 6.5, for $w \in V^*$ cannot be adjacent to $m$. Therefore Condition (ii) holds.

Now, consider the case when $R$ connects $v_2$ to $b$. If there is a node $w$ adjacent to two nodes in $Q$ and to the neighbor of $v_2$ in $P_2$, then one of three possibilities occurs.

**If $w$ is also adjacent to $v$, Condition (ii) holds.**

**If $w$ is not adjacent to $v$ but is adjacent to at least one node of $V(\Pi) \setminus \{v, q\}$, then there is a direct connection from bottom to top of $\Pi$ that contradicts Theorem 6.5.**

**If $w$ is not adjacent to any node of $V(\Pi) \setminus \{q\}$, then there is a direct connection of Type b[6.4] from bottom to top of $\Pi$, and the result follows from Lemma 6.9.**

If the direct connection is of Type a1[6.5], we get, after a parachute modification at the top, a parachute with long sides, long top and a direct connection of Type a[6.4], contradicting Lemma 6.11. \hfill \Box

**Theorem 6.13** Let $G$ be a wheel-free weakly balanced graph. If $G$ contains a parachute with long top and long sides, then $G$ has an extended star cutset.
Proof: By Theorem 6.5(i), II must have a direct connection \( Q \) of Type a, b, c, d, e\[6.4\] or Type a1, b1, b2, c1\[6.5\]. Now, we have a stabilized parachute. Indeed, if \( Q \) is of Type b, c, d\[6.4\] or Type b1, b2, c1\[6.5\], this is guaranteed by Lemma 6.9. If \( Q \) is of Type a\[6.4\], this is guaranteed by Lemma 6.11 since II has long top. Finally, if \( Q \) is of Type e\[6.4\] or Type a1\[6.5\], this follows from Lemma 6.12. Now, by Theorem 6.10, \( G \) contains an extended star cutset. □

6.8 Parachutes with Short Middle Path

In this subsection, we assume again that \( G \) is a wheel-free weakly balanced graph. We show that, if \( G \) contains a parachute with long sides and short middle but \( G \) contains no connected squares, then \( G \) has an extended star cutset.

A direct connection \( Q = x_1, \ldots, x_n \) from bottom to top of a parachute II is of Type \( f_{\text{short}} \) if \( Q \) is of Type f\[6.4\] and \( x_2 \) is adjacent to \( m \).

Lemma 6.14 Let \( G \) be a wheel-free weakly balanced graph that contains no connected squares. Suppose \( G \) contains no extended star cutset and let II be a parachute with long sides and short middle. Then II has short top and each direct connection from bottom to top is of Type a\[6.4\], \( f_{\text{short}} \) or \( h, i\[6.4\] \). Moreover, II has at least one direct connection of Type \( f_{\text{short}} \) or \( h, i\[6.4\] \).

Proof: By Theorem 6.13, II has short top. Since \( S(II) \) is not an extended star cutset of \( G \) and II has a short middle path, by Theorem 6.5(ii), its direct connections are of Type a, f, h, i, k\[6.4\]. As there are no connected squares, II has no direct connection of Type k\[6.4\]. By Theorem 6.10, \( G \) contains no stabilized parachute. By Lemma 6.11, II has a direct connection of Type f\[6.4\] if it has one of Type a\[6.4\]. Now let \( Q \) be a direct connection of Type f\[6.4\]. Let II' be the unique parachute in II \( \cup \) \( Q \) with center node \( m \), middle path \( m, v, v_2 \) and side nodes \( z \) and the neighbor \( x_j \) of \( m \) on \( Q \). The top of II' is the subpath of \( Q \) connecting \( x_j \) with \( z \) and this top must be short by Corollary 6.13. So \( Q \) is of Type \( f_{\text{short}} \) relative to II. □

Definition 6.15 For \( k \geq 2 \), a \( k \)-parachute II\( ^k \) is defined as follows, see Figure 19. For \( k \) even, say \( k = 2p \), II\( ^k \) consists of nodes \( v, m, v_1, \ldots, v_{p+1}, x_1, \ldots, x_{p+1}, y_1, \ldots, y_p, z_1, \ldots, z_p \) and chordless paths \( P_j \), for \( j = 1, \ldots, 2p \) where:

- node \( v \) is adjacent to nodes \( m \) and \( v_1, \ldots, v_{p+1} \),
- node \( m \) is adjacent to nodes \( v \) and \( z_1, \ldots, z_p \),
- for \( t = 1, \ldots, p+1 \), node \( x_t \) is adjacent to nodes \( v_1, \ldots, v_t \),
- for \( t = 1, \ldots, p \), node \( y_t \) is adjacent to nodes \( z_1, \ldots, z_t \),
- for \( t = 1, \ldots, p \), path \( P_{2t-1} \) connects \( x_t \) to \( z_t \) and path \( P_{2t} \) connects \( y_t \) to \( v_{t+1} \),
- for \( i \neq j \), \( V(P_i) \cap V(P_j) = \emptyset \),
- there are no adjacencies between the nodes of II\( ^k \) other than those indicated above.
(2p + 1)-parachute

Figure 19:
For $k$ odd, say $k = 2p + 1$, the $k$-parachute $\Pi^k$ is obtained from $\Pi^{k-1}$ by adding a node $z_{p+1}$ adjacent to $m$, a node $y_{p+1}$ adjacent to nodes $z_1, \ldots, z_{p+1}$ and a chordless path $P_{2p+1}$ connecting $x_{p+1}$ to $z_{p+1}$ whose inner nodes are distinct from $V(\Pi^{k-1}) \cup \{z_{p+1}, y_{p+1}\}$ and are not adjacent to $\{y_{p+1}\} \cup V(\Pi^{k-1}) \setminus \{x_{p+1}\}$.

This definition implies that a 2-parachute is a parachute with long side paths $P_1, P_2$, short middle path $v$, $m$, $z_1$ and short top path $v_1, y_{p+1}, v_2$.

**Theorem 6.16** Assume that $G$ is a wheel-free weakly balanced graph that contains no connected squares. If $G$ contains a parachute with long sides and short middle, then $G$ has an extended star cutset.

**Proof:** Assume that $G$ has no extended star cutset. By Theorem 6.10, $G$ contains no stabilized parachute. Let $\Pi^k$ be a $k$-parachute with $k$ maximal.

**Claim 1** $\Pi^k$ exists and $k \geq 3$.

**Proof of Claim 1:** By Lemma 6.14, each parachute with long sides and short middle has short top, so is a 2-parachute. Hence, $\Pi^k$ exists.

Suppose $k = 2$. A parachute with long sides and short middle and with a direct connection of Type $f_{\text{short}}$ induces a 3-parachute. So, by Lemmas 6.11 and 6.14, we get

(*) Direct connections of parachutes with long sides and short middle are of Type $h$, $i[6,4]$.

Let $\Pi$ be a parachute with long sides and short middle. Then the top of $\Pi$ is short. Let $t$ denote its single node. Let $Q = x_1, \ldots, x_n$ be a direct connection from the bottom to the top of $\Pi$. By (*), $Q$ is of Type $h$, $i[6,4]$. Hence for both $\ell = 1$ and $\ell = 2$, $x_1$ has a neighbor $y_{\ell}$ on $P_1$. Moreover, $m$ has two neighbors $x_i$ and $x_j$ on $Q$ (with $j < i$, say). Note that $j = 1$ if $Q$ is of Type $h[6,4]$. W.l.o.g. assume that $\Pi$ and $Q$ are chosen such that, if possible, $Q$ is of Type $h[6,4]$.

For $\ell = 1$ and $\ell = 2$, we define the parachute $\Pi_\ell$ with long sides and short middle as follows: the middle path is $m, v, v_0, v_1, \ldots, v_n$, the top path is $Q_{x_1, x_n}$, and the side paths are $Q_{x_1 x_n} \cup \{t, v_1\}$ and $Q_{x_1 x_n} \cup \{P_1 v_0\}$. Let $R = r_1, \ldots, r_n'$ be the shortest path that is a direct connection from the bottom to the top of $\Pi_1$ or $\Pi_2$. Then, by (*), $R$ is of Type $h$ or $i[6,4]$. By construction, $R$ is node disjoint from $Q \cup \Pi$. So $R$ is a direct connection from the bottom to the top of both $\Pi_1$ and $\Pi_2$ and it has the same type relative to both $\Pi_1$ and $\Pi_2$. If $R$ is of Type $i[6,4]$ with respect to both $\Pi_1$ and $\Pi_2$, then $r_1$ is adjacent to $t$, a node of $P_1$ and a node of $P_2$. But this contradicts Theorem 6.1. So $R$ is of Type $h[6,4]$ with respect to both $\Pi_1$ and $\Pi_2$ and, therefore, $r_1$ is adjacent to $v$. By Theorem 6.1, $r_1$ cannot have neighbors in both $P_1$ and $P_2$. So $r_1$ has a neighbor in $Q_{x_1, x_n} \setminus \{x_j\}$. This implies that $Q$ is of Type $i[6,4]$ with respect to $\Pi$. But this contradicts the choice of $\Pi$, $Q$ since $R$ is of Type $h[6,4]$ with respect to $\Pi_1$. This proves Claim 1.

From now on, we will assume that $k$ is odd ($k = 2p + 1$ say), as the proof of the even case is essentially the same. Let $\Pi^*$ be the parachute with top path $z_{p+1}, y_{p+1}, z_p$, middle path $m, v, v_{p+1}$ and side paths $P_{2p+1} \cup \{v_{p+1}\}$ and $P_2 \cup \{z_p\}$. Let $P_{2p+2}$ be a direct connection of $\Pi^*$ of Type $f_{\text{short}}, h$ or $i[6,5]$. Denote by $v_{p+2}$ the node of $P_{2p+2}$ adjacent to $v$ and closest to $y_{p+1}$. Moreover, let $x_{p+2}$ be the first node of $P_{2p+2}$ (so adjacent to the bottom of $\Pi^*$).
For $i = 1, \ldots, p$, we denote by $\Pi_i$ the parachute with middle path $m, v, v_i$, side paths $P_{2i-1} \cup \{z_{i-1}\}$ and $P_{2p+1} \cup \{v_i\}$ and top path $z_{p+1}, y_{p+1}, z_i$, and we denote by $\Pi_i'$ the parachute with middle path $m, v, v_i$, side paths $P_{2i-1} \cup \{v_i\}$ and $P_{2p+1} \cup \{v_i\}$ and top path $z_{p+1}, y_{p+1}, z_i$.

Claim 2: Let $i = 1, \ldots, p$ and $\Pi \in \{\Pi_i, \Pi'_i\}$. Then $P_{2p+2}$ is a direct connection for $\Pi$.

Moreover, $P_{2p+2}$ is of Type $f_{\text{short}}$ for $\Pi$ if and only if it is of Type $f_{\text{short}}$ for $\Pi'$.

Proof of Claim 2: If $P_{2p+2}$ would contain no direct connection for $\Pi$, then $P_{2p+2}$ would be of Type $f_{\text{short}}$ for $\Pi'$ and $P_{2p+2} \cup \{v_{p+1}\}$ would be a direct connection for $\Pi$ that is not of Type $a[6.4], f_{\text{short}}$ or $h, i[6.4]$. So $P_{2p+2}$ contains a direct connection $Q$ for $\Pi$.

Suppose that $Q$ is a proper subpath of $P_{2p+2}$. Then $Q$ is not a direct connection for $\Pi'$, hence it has to be of Type $f_{\text{short}}$ for $\Pi$. However this implies that $Q \cup \{v_i\}$ is a direct connection for $\Pi'$ that is not of Type $a[6.4], f_{\text{short}}$ or $h, i[6.4]$, a contradiction. So $Q = P_{2p+2}$, and $P_{2p+2}$ is a direct connection for $\Pi$. The remainder of the claim now follows because, for both $\Pi'$ and $\Pi$, we have that $P_{2p+2}$ is of Type $f_{\text{short}}$ if and only if $P_{2p+2}$ has no neighbor on $P_{2p+1}$. This proves Claim 2.

Claim 3: $P_{2p+2}$ is of Type $f_{\text{short}}$ for all the parachutes $\Pi_i$ and $\Pi'_i$ with $i = 1, \ldots, p$.

Proof of Claim 3: Suppose not. Then, by Claim 2, $P_{2p+2}$ is not of Type $f_{\text{short}}$ for $\Pi'$ and $\Pi'_i$. Hence $x_{p+2}$, the first node on $P_{2p+2}$, has a neighbor on $P_{2p}$ and one on $P_{2p-1}$ and either it is adjacent to $v$ (if $P_{2p+2}$ is of Type $h[6.4]$ for $\Pi'$) or to $x_{p+1}$ (if $P_{2p+2}$ is of Type $i[6.4]$ for $\Pi'$).

In any case $x_{p+2} \in V'$. But then, $x_{p+2}$ is a strongly adjacent node violating Theorem 6.1 with respect to the parachute with top path $v_{p+1}, x_{p+1}, v_p$, middle path $v, m, z_p$ and side paths $P_{2p} \cup \{z_{p}\}$ and $P_{2p-1} \cup \{v_p\}$. This proves Claim 3.

From the last claim, it follows immediately that $\Pi^k \cup P_{2p+2}$ is a $k + 1$-parachute, contradicting our choice of $k$.

6.9 The Parachute Theorem

Theorem 6.17 Let $G$ be a weakly balanced graph that is not strongly balanceable. Assume $G$ contains no extended star cutset. Then $G$ is wheel-free and contains a parachute with long sides. Furthermore, if $G$ contains no connected squares, any parachute $\Pi$ with long sides has a short top, a long middle and a direct connection of Type $j[6.4]$ or $j[6.5]$. In addition, any direct connection from bottom to top avoiding $S(\Pi)$ is of one of these two types.

Proof: Assume $G$ contains no extended star cutset. By Theorems 4.6, 5.1 and 6.6, $G$ is wheel-free and contains a parachute with long sides. Let $\Pi$ be such a parachute and let $Q$ be a direct connection from bottom to top avoiding $S(\Pi)$. There exists a parachute with long sides and short middle in $\Pi \cup Q$ when $Q$ is of Type $a[6.1]$, Types $a, b, c, d, e, f, g, h$ or $i[6.4]$ or $a1, b1, b2, c1, f1, g1[6.5]$. So by Theorem 6.16, none of these direct connections can occur. Now, by Theorem 6.5, $Q$ must be of Type $j, k[6.4]$ or $j[6.5]$. This proves the theorem, since Type $k[6.4]$ yield connected squares.
Connected Squares

A Classification of Nodes and Paths

In this section, we prove the following result:

Theorem 7.1 Let $G$ be a wheel-free weakly balanced graph that contains connected squares. Then $G$ has a biclique cutset or a 2-join.

Throughout this section, $G$ denotes a wheel-free weakly balanced graph that contains connected squares, and $CS(P_1^r, P_2^c; P_1^c, P_2^r)$ denotes connected squares in $G$, see Figure 20. We assume $s_1, s_2, t_1, t_2 \in V^c$ and $s_1', s_2', t_1', t_2' \in V^r$. Recall that $P_i$ denotes the subpath obtained from $P_i$ by removing its endnodes $s_i, t_i$. The subpaths $P_2^r, P_1^r, P_2^c$ are analogously defined.

The following lemma characterizes the strongly adjacent nodes to connected squares.

Lemma 7.2 Let $\Sigma = CS(P_1^r, P_2^c; P_1^c, P_2^r)$ be connected squares and $v \in V(G) \setminus V(\Sigma)$ be a strongly adjacent node to $\Sigma$. Then one of the following holds:

- Node $v$ has exactly two neighbors in $\Sigma$, both contained in $P_1^r$ or in $P_2^c$ or in $P_1^c$ or in $P_2^r$.

- Node $v$ is of one of the following types, see Figure 21:

  Type a Node $v$ has three neighbors in $\Sigma$, two of them being $s_1, s_2$ or $t_1, t_2$ or $s_1', s_2'$ or $t_1', t_2'$. If $v \in V^c$, the third neighbor is in $P_1^c$ or in $P_2^c$ and if $v \in V^r$, the third neighbor is in $P_1^r$ or in $P_2^r$.

  By extension, we define $s_1, s_2, s_1', s_2', t_1, t_2, t_1', t_2'$ to be Type a nodes.
**Type b** Node \( v \) has exactly two neighbors in \( \Sigma \) that are \( s'_1, s'_2 \) or \( t'_1, t'_2 \) or \( s'_1, s'_2 \) or \( t'_1, t'_2 \).

**Type c** Node \( v \) has exactly two neighbors in \( \Sigma \) and if \( v \in V^c \), then \( v \) has one neighbor in \( P^c_1 \) and one in \( P^c_2 \). If \( v \in V^c \), then \( v \) has one neighbor in \( P^c_1 \) and one in \( P^c_2 \).

**Figure 21:** Strongly adjacent nodes

**Proof:** If all the neighbors of \( v \) in \( \Sigma \) belong to one of the paths \( P^c_1, P^c_2, P^c_3, P^c_4 \), then \( v \) has two neighbors in this path, else \( v \) is the center of a wheel. If \( v \) has neighbors in more than one path, then \( v \) has a unique neighbor in each of these paths, else \( v \) is again the center of a wheel.

In the rest of the proof, assume w.l.o.g. that \( v \in V^c \). If \( v \) has (unique) neighbors in both \( P^c_1 \) and \( P^c_2 \), say \( v_1 \) and \( v_2 \), then \( v \) has no neighbors in \( P^c_1 \) and \( P^c_2 \). For, if \( v \) is adjacent to, say \( v_3 \) in \( P^c_1 \), then assume w.l.o.g. that \( v_3 \) is distinct from \( t'_1 \). Then \( v_1, v_2 \) and \( v_3 \) are intermediate nodes in the three paths of a \( 3PC(v, t'_1) \). So \( v \) is of Type c in this case.

Consider now the case in which \( v \) has no neighbors in \( P^c_1 \) or \( P^c_2 \), say \( P^c_3 \). If \( v \) has (unique) neighbors in both \( P^c_1 \) and \( P^c_2 \), say \( v_3 \) and \( v_4 \), then either \( v_3 = s'_1 \) and \( v_4 = s'_2 \) or \( v_3 = t'_1 \) and \( v_4 = t'_2 \). For, if not, assume \( v_3 \neq s'_1 \). Then \( v_3 = t'_2 \), else nodes \( t'_2, s'_1 \) and \( s'_2 \) are intermediate nodes in a \( 3PC(v_3, s'_2) \). The same argument, applied to \( v_4 \neq s'_2 \), shows \( v_4 = t'_1 \). Thus node \( v \) is of Type a if it has a neighbor in \( P^c_1 \), and \( v \) is of Type b if it has no neighbor in \( P^c_1 \). Finally, if \( v \) has no neighbors in \( P^c_1 \) or \( P^c_2 \), assume w.l.o.g. that \( v \) has no neighbor in \( P^c_3 \), is adjacent to \( v_1 \) in \( P^c_1 \), to \( v_3 \) in \( P^c_1 \) and \( v_3 \neq t'_1 \). Then \( v_1 \) is adjacent to \( s'_1 \), else we have a \( 3PC(v_1, s'_1) \). Now \( s'_1 \) is the center of an odd wheel.

Let \( S^c \) comprise all Type a and Type b[7.2] nodes adjacent to \( s'_1 \) and \( s'_2 \). Node sets \( S^c \), \( T^c \) and \( T^c \) are defined analogously. Let \( S = S^c \cup S^e \) and \( T = T^c \cup T^e \).

**Lemma 7.3** The sets \( S \) and \( T \) are disjoint and no node of \( S \) is adjacent to a node of \( T \).
Proof: The first property follows immediately from Lemma 7.2. If the second property does not hold, there is a 3-path configuration connecting a node in \( \{s_1, s_2, s'_1, s'_2\} \) and a node in \( \{t'_1, t'_2, t'_3, t'_4\} \).

Let \( v \) be a Type b[7.2] node in \( S \) and let \( \mathcal{P}_{v}(\Sigma) \) be the family of direct connections between \( v \) and \( T \), avoiding the set \( S \setminus \{v\} \). When no confusion arises, we write \( \mathcal{P}_{v} \) instead of \( \mathcal{P}_{v}(\Sigma) \). Consider the following classification of the paths in \( \mathcal{P}_{v} \).

**Classification 7.4** Let \( P = x_1, x_2, \ldots, x_n \) be a direct connection in \( \mathcal{P}_{v} \) where \( x_1 \) is adjacent to a Type b[7.2] node \( v \) in \( S \) and \( x_n \) is adjacent to a node in \( T \).

- **P is attached** if \( x_n \) is adjacent to a Type a[7.2] node in \( T \).
- **P is detached** if \( x_n \) is not adjacent to any Type a[7.2] node in \( T \). Hence \( x_n \) is adjacent only to Type b[7.2] nodes in \( T \).

The above classification induces a classification of the strongly adjacent nodes of Type b[7.2]:

**Classification 7.5** Let \( v \) be a Type b[7.2] node in \( S \).

- **Node \( v \) is attached** if \( v \) has at least one attached direct connection in \( \mathcal{P}_{v} \).
- **Node \( v \) is detached** if \( \mathcal{P}_{v} \) is nonempty and all the direct connections in \( \mathcal{P}_{v} \) are detached.
- **Node \( v \) is separable** if \( \mathcal{P}_{v} \) is empty.

Similarly, each Type b[7.2] node \( w \in T \) is classified as attached, detached, or separable, based on the direct connections in \( \mathcal{P}_{v} \) between \( w \) and \( S \), avoiding \( T \setminus \{w\} \).

An attached direct connection \( P \in \mathcal{P}_{v} \) is **minimal** if there exists no direct connection \( P' \in \mathcal{P}_{v} \) with nodes in \( V(P) \cup V(\Sigma) \) such that \( V(P') \setminus V(\Sigma) \subset V(P) \setminus V(\Sigma) \).

**Lemma 7.6** Let \( v \in S \) be an attached Type b[7.2] node, and let \( P = x_1, x_2, \ldots, x_n \) be a minimal attached direct connection in \( \mathcal{P}_{v} \), where \( x_n \) is adjacent to a Type a[7.2] node \( t \) in \( T^c \). Say \( t \) has a neighbor in \( P^c \). Let \( x_h \) be the node of highest index in \( V(P) \setminus V(\Sigma) \). Then the following holds:

(i) \( N(x_h) \cap V(\Sigma) \subset V(P^c) \).

(ii) \( v \in S^c \).

(iii) At most one node of \( \{x_1, \ldots, x_h\} \) is adjacent to \( s_1 \) and none is adjacent to \( s_2, s'_1 \) or \( s'_2 \).

(iv) **Node \( x_n \) is not adjacent to any Type a[7.2] node \( t' \) with a neighbor in \( P^c \).**

**Proof:** Since \( P \in \mathcal{P}_{v} \) is minimal, no node in \( \{x_1, \ldots, x_{h-1}\} \) is adjacent to a node in \( V(\Sigma) \setminus \{s_1, s_2, s'_1, s'_2\} \).

**Claim 1** If \( x_h \) is a strongly adjacent node of Type c[7.2], then no node in \( \{x_1, \ldots, x_{h-1}\} \) is adjacent to a node in \( \{s_1, s_2, s'_1, s'_2\} \).
Proof of Claim 1: By Lemma 7.2, \( h < n \). As \( P \in \mathcal{P}_v \) and \( t \in T^c \), node \( x_h \) has one neighbor in \( \bar{P}_1 \), say \( z_1 \), and one in \( \bar{P}_2 \), say \( z_2 \). Let \( x_i, i < h \), be the node of highest index adjacent to a node \( x^* \in \{ s_1, s_2, s_3, s_4 \} \). By symmetry, we may assume \( x^* = s_1 \), then nodes \( z_1, z_2 \) and \( x_i \) are intermediate nodes in the three paths of a \( 3PC(x_h, s_1) \). So \( x^* = s_1 \). Now \( z_1 \) is adjacent to \( s_1 \), else there is a \( 3PC(z_1, s_1) \). Let \( Q \) be the shortest path from \( x_i \) to \( x_h \), contained in \( V(P_{x_i,x_h}) \cup \{ v, s_2 \} \). Then the hole \( H = x_h, P_{x_h,x_i}, x_i; Q, s_1, P_1, t_1, t_2, (P_1)_h^c, z_1, x_h \) induces a wheel with center \( z_1 \). This proves Claim 1.

Claim 2 \( N(x_h) \cap V(\Sigma) \subseteq V(P_1^c) \).

Proof of Claim 2: Assume not. Then, by Lemma 7.2, \( x_h \) is a strongly adjacent node of Type [7.2] with neighbors in \( P_1^c \) and \( P_2^c \) and, by Claim 1, \( P_{x,x_{n-1}} \) has no neighbors in \( \Sigma \). If \( v \) would be in \( S^c \), there would be a \( 3PC(x_h, s_1) \). So \( v \in S^c \). If \( v \) is adjacent to \( x_h \), there is a wheel with center \( x_h \). If \( v \) is not adjacent to \( x_h \), there is a \( 3PC(x_h, v) \). This proves Claim 2.

Claim 3 No node in \( \{ x_1, \ldots, x_{n-1} \} \) is adjacent to \( s_2, s_3 \) or \( s_4 \).

Proof of Claim 3: Let \( x_i, i < h \), be the node of highest index adjacent to a node \( x^* \in \{ s_2, s_3, s_4 \} \). By symmetry we may assume \( x^* = s_1 \). However, if \( x^* = s_1 \), there is a \( 3PC(t_1, s_1) \) and if \( x^* = s_2 \), there is a \( 3PC(t_1, s_1) \). This proves Claim 3.

By Claim 2, (i) holds. If \( v \) would be in \( S^c \) then, by Claim 3, there would be a \( 3PC(t_1, s_1) \) in \((V(\Sigma) \cup V(P)) \setminus \{ s_1 \}\). So (ii) follows as well. Claims 2 and 3 establish the second part of (iii). The first part of (iii) holds since, when \( s_1 \) has more than one neighbor in \( \{ x_1, \ldots, x_h \} \), there is a wheel with center \( s_1 \). Finally, suppose (iv) is false. Then by (iii) and symmetry between \( t \) and \( t' \), no node among \( x_1, \ldots, x_{n-1} \) has a neighbor in \( \Sigma \). So there is a \( 3PC(t, s_1) \).

Lemma 7.6 shows that, up to symmetry, Figure 3 depicts the possible attached direct connections in \( \mathcal{P}_v \) where, in Figure 22(a), node \( s_1 \) may not be adjacent to a node \( x_i \) of \( P \) and node \( x_h \) may have two neighbors in \( \bar{P}_1^c \).

We now characterize the direct connections in \( \mathcal{P}_v \), when \( v \) is a detached Type b[7.2] node.

Lemma 7.7 Let \( P = x_1, x_2, \ldots, x_n \) be a direct connection in \( \mathcal{P}_v \), where \( x_1 \) is adjacent to a detached Type b[7.2] node \( v \in S \) and \( x_n \) is adjacent to a Type b[7.2] node \( t \in T^c \). Then \( P \) satisfies the following properties:

- No node \( x_i, 1 \leq i \leq n \), is adjacent to a node in \( \Sigma \).
- Node \( v \) belongs to \( S^c \).

Proof: Since \( v \) is a detached node, no node \( x_i, 1 \leq i \leq n \), is adjacent to a node in \( V(\Sigma) \setminus \{ s_1, s_2, s_3, s_4 \} \). Let \( x_i \) be the node with highest index adjacent to a (unique) node \( x^* \in \{ s_1, s_2, s_3, s_4 \} \). By symmetry, we may assume \( x^* = s_1 \). If \( x^* = s_1 \), there is a \( 3PC(s_1, t_1) \) and if \( x^* = s_1 \), there is a \( 3PC(s_1, t) \). Hence the first part of the lemma follows. The second part now follows immediately for, if \( v \in S^c \), there is a \( 3PC(t_1, s_1) \).
Figure 22: Attached direct connections

7.2 Extreme Connected Squares

**Definition 7.8** A subgraph $\Sigma$ of $G$ is called extreme connected squares if it has the following properties and no other subgraph $\Sigma'$ of $G$ with $V(\Sigma) \subset V(\Sigma')$ does.

Two disjoint node sets $S, T \subset V(\Sigma)$ induce bicliques $K_S$ and $K_T$ respectively, and $E(K_S) \cup E(K_T)$ is a 2-join of $\Sigma$. Each connected component $G_j$ of $\Sigma \setminus (E(K_S) \cup E(K_T))$ has a nonempty intersection $S_j$ with $S$ and a nonempty intersection $T_j$ with $T$, and either $S_j \cup T_j \subset V^r$ or $S_j \cup T_j \subset V^r$. Furthermore, at least two connected components $G_j$ satisfy $S_j \cup T_j \subset V^r$ and at least two satisfy $S_j \cup T_j \subset V^r$. Finally, every node of $\Sigma$ belongs to a chordless path of $\Sigma \setminus (E(K_S) \cup E(K_T))$ with endnodes in $S$ and $T$.

Using the notation introduced in Definition 7.8, we let $S^r = S \cap V^r$ and we define $S^c$, $T^c$ and $T^r$ analogously. $\Sigma^r$ denotes the subgraph of $\Sigma$ comprising all connected components $G_j$ that have nonempty intersection with $S^r \cup T^r$. We define $\Sigma^c$ analogously. A chordless path of $\Sigma \setminus (E(K_S) \cup E(K_T))$ with endnodes in $S$ and $T$ is called an $ST$-path. It follows from the definition of extreme connected squares that, if $P_j^r$, $P_j^c$ are ST-paths in distinct components of $\Sigma^c$ and $P_k^r$, $P_k^c$ are ST-paths in distinct components of $\Sigma^r$, then $V(P_j^r) \cup V(P_j^c) \cup V(P_k^r) \cup V(P_k^c)$ induces connected squares. Connected squares $CS(P_j^r, P_j^c; P_k^r, P_k^c)$ constructed in this fashion are said to be contained in $\Sigma$. Let $S^c$ be the set of nodes in $G$ adjacent to all the nodes in $S^r$. The sets $S^r$, $T^c$ and $T^r$ are analogously defined. Let $S = S^c \cup S^r$ and $T = T^c \cup T^r$. Note that $S \subset S$ and $T \subset T$.

**Lemma 7.9** Let $\Sigma$ be extreme connected squares and $\Sigma_{i,j,k,l} = CS(P_i^r, P_j^c; P_k^r, P_l^c)$ be connected squares contained in $\Sigma$. Let $Q_i = s, x_1, \ldots, x_n, t$ be a path in $G$ where

- node $s$ is adjacent to all nodes of $S^r$, node $t$ is adjacent to all the nodes of $T^r$ and these are the only adjacencies between nodes of $Q_i$ and $\Sigma^r$. 

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• at least one node of $Q_i$ is coincident with or adjacent to a node in the component $G_i$ that contains $P_i^c$, and

• no node of $Q_i$ is adjacent to a node of $V(\Sigma^c) \setminus V(G_i)$.

Then $V(Q_i) \subseteq V(G_i)$.

Proof: Let $\Sigma'$ be the graph obtained from $\Sigma$ by adding $V(Q_i)$ to $V(G_i)$. Define $S' = S \cup \{s\}$, $T' = T \cup \{t\}$. The first condition in Lemma 7.9 shows that $E(K_{S'}) \cup E(K_{T'})$ is a 2-join of $\Sigma'$. The second and third conditions in Lemma 7.9 guarantee that the number of connected components of $\Sigma' \setminus (E(K_{S'}) \cup E(K_{T'}))$ is the same as for $\Sigma$. Finally, every node of $\Sigma'$ belongs to a chordless path of $\Sigma' \setminus (E(K_{S'}) \cup E(K_{T'}))$ with endnodes in $S'$ and $T'$. So $\Sigma'$ satisfies all the properties of Definition 7.8. Thus $\Sigma' = \Sigma$, since $\Sigma$ is extreme. □

Lemma 7.10 Let $\Sigma$ be extreme connected squares and let $v$ be a Type $a[7.2]$ node with respect to some connected squares contained in $\Sigma$. Then $v \in S \cup T$.

Proof: Let $v$ be a Type $a[7.2]$ node with respect to $\Sigma_{ijkl} = CS(P^c_i; P^c_j; P^c_k; P^c_l)$ contained in $\Sigma$. Assume w.l.o.g. that $v$ is adjacent to a node in $P^c_i$ and to the endnodes $s^c_i, s^c_l$ of $P^c_l$ and $P^c_k$ in $S^c$. There exists a unique chordless path $Q_i$ contained in $(V(P^c_i) \setminus \{s^c_i\}) \cup \{v\}$, where $s^c_i$ is the endnode of $P^c_i$ in $S^c$. By Lemma 7.9 applied to $\Sigma_{ijkl}$ and $Q_i$, it suffices to show that

(i) $v$ is adjacent to all the nodes in $S^c$ and no other node of $\Sigma^c$, and

(ii) $v$ has no neighbor in $V(\Sigma^c) \setminus V(G_i)$.

If (i) does not hold, then either $v$ is not adjacent to some node $s^c_m \in S^c$ (by Definition 7.8, $s^c_m$ belongs to an $ST$-path $P^c_m$), or $v$ has a neighbor in the interior of an $ST$-path $P^c_m$ in $\Sigma^c$. Assume w.l.o.g. that the component $G_m$ that contains $P^c_m$ is distinct from the component $G_k$ that contains $P^c_k$. Now, $v$ is a strongly adjacent node violating Lemma 7.2 in connected squares $CS(P^c_i; P^c_j; P^c_k; P^c_l)$.

If (ii) does not hold, $v$ has a neighbor in an $ST$-path $P^c_m$ in $\Sigma^c \setminus G_i$. Then $v$ is a strongly adjacent node violating Lemma 7.2 in connected squares $CS(P^c_i; P^c_m; P^c_k; P^c_l)$.

Lemma 7.11 Let $\Sigma$ be extreme connected squares and let $v$ be an attached Type $b[7.2]$ node with respect to some connected squares contained in $\Sigma$. Then $v \in S \cup T$.

Proof: By Lemma 7.10, we can assume that $v$ is not a Type $a[7.2]$ node with respect to any connected squares contained in $\Sigma$. Choose $\Sigma_{ijkl} = CS(P^c_i; P^c_j; P^c_k; P^c_l)$ such that $v$ is attached in $\Sigma_{ijkl}$ with an attached direct connection $P = x_1, \ldots, x_n$ and $V(P) \setminus V(\Sigma_{ijkl})$ has smallest cardinality among all possible choices of $\Sigma_{ijkl}$ and $P$. Let $s^c_i$ and $t^c_i$ denote the endnodes of $P^c_i$ in $S^c$ and $T^c$ respectively. $s^c_j$, $t^c_j$, $s^c_k$, $t^c_k$, $s^c_l$, $t^c_l$ are defined analogously. Let $x_h$ be the node of highest index in $V(P) \setminus V(\Sigma_{ijkl})$ (possibly $h = n$). By Lemma 7.10, we can also assume that $x_h$ is adjacent to $t^c_l$, $t^c_j$, $t^c_i$ or $t^c_k$, say $t^c_l$.

By Lemma 7.9 applied to $\Sigma_{ijkl}$ and the chordless path contained in $(V(P^c_i) \setminus \{s^c_i\}) \cup V(P) \cup \{v\}$, it suffices to show the following claims.

Claim 1 No node of $P$ is adjacent to a node in $\Sigma^c$.

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Proof of Claim 1: Assume that a node of \( P \) is adjacent to a node \( z \in \Sigma^r \) and let \( P_m^r \) be an \( ST \)-path containing \( z \). Let \( s_m^r \) and \( t_m^r \) be the endnodes of \( P_m^r \). Assume w.l.o.g. that \( k \neq m \). Let \( \Sigma_{ijkm} = CS(P_i^r; P_j^r; P_k^r, P_m^r) \) and choose the shortest subpath \( P' \) of \( P \) with endnodes \( x, y \) satisfying the following property. \( \emptyset \neq N(x) \cap V(\Sigma_{ijkm}) \subseteq V(P_i^r) \), \( \emptyset \neq N(y) \cap V(\Sigma_{ijkm}) \subseteq V(P_m^r) \) and \((N(x) \cup N(y)) \cap V(\Sigma_{ijkm}) \neq \{s_i^r, s_m^r\}, \{t_i^r, t_m^r\} \). Note that the existence of such a path is possible when \( h < n \), or when \( h = n \) and \( x_n \) is strongly adjacent to \( \Sigma_{ijkm} \). If \( h = n \) and \( x_n \) has \( t_i^r \) as unique neighbor in \( \Sigma_{ijkm} \), then \( s_i^r \) must have a neighbor in \( P \), otherwise there is a \( 3PC(s_i^r, t_i^r) \). So again a path \( P' \) with the desired property exists. Since \( x \) has neighbors only in \( P_i^r \), and \( y \) has neighbors only in \( P_m^r \), by possibly shortening \( P' \) and modifying \( P_i^r \) and \( P_m^r \) accordingly, we can assume that \( x \) has a unique neighbor \( x^* \) in \( P_i^r \) and \( y \) has a unique neighbor \( y^* \) in \( P_m^r \). Lemma 7.2 shows that even after shortening, \( P' \) has length at least 1.

To complete the proof of Claim 1, we show that, for any connected squares \( CS(P_i^r; P_j^r; P_k^r, P_m^r) \), the existence of a path \( P' \) with the above properties leads to a contradiction.

Assume first that no intermediate node of \( P' \) is adjacent to \( s_m^r, t_m^r \) or \( t_m^r \). Then \( x^*, y^* \) belong to the same side of the bipartition, else \( V(\Sigma_{ijkm}) \cap V(P') \) contains a \( 3PC(x^*, y^*) \). So we can assume w.l.o.g. that \( x^*, y^* \in V_r \) and \( y^* \neq s_m^r \). Let \( H = x^*, x, P, y, y^*(P_m^r)_{y^*}, t_m^r, t_j^r, t_k^r, P_k^r, s_m^r, s_i^r, (P_i^r)_{x^*}, x^* \). Now, \( t_i^r \) has two neighbors in \( H \), namely \( t_k^r, t_m^r \). So, if \( t_i^r \) is adjacent to \( x^* \), we have an odd wheel, and if \( t_i^r \) is not adjacent to \( x^* \), we have a \( 3PC(t_i^r, x^*) \).

So \( s_i^r, s_m^r \) or \( t_m^r \) is adjacent to some intermediate node of \( P' \). We assume w.l.o.g. that the last case occurs and \( x^* = t_i^r \) by the minimality of \( P' \). Then \( y^* \in V(P_m^r) \setminus \{t_m^r\} \). Now \((H', t_m^r) \) is a wheel where \( H' = x^*, x, P, y, y^*(P_m^r)_{y^*}, s_m^r, s_i^r, (P_i^r)_{x^*}, x^* \). This proves Claim 1.

Claim 2 Node \( v \) is adjacent to all the nodes in \( S^r \) and to no other node in \( \Sigma^r \).

Proof of Claim 2: Let \( P_m^r \) be any \( ST \)-path in \( \Sigma^r \). By Lemma 7.2, if \( v \) has a neighbor in \( P_m^r \), then this neighbor is unique, namely \( s_m^r \). So assume \( v \) has no neighbor in \( P_m^r \). It follows from Claim 1 that \( v, s_m^r \) and \( t_m^r \) are intermediate nodes in the three paths of a \( 3PC(s_i^r, t_i^r) \). This proves Claim 2.

Claim 3 No node of \( V(\Sigma^r) \cup \{v\} \) is adjacent to a node in \( V(\Sigma^r) \setminus V(G_i) \).

Proof of Claim 3: Node \( v \) is not adjacent to any node in \( V(\Sigma^r) \) since we have assumed that it is not of Type a[7.2] with respect to any connected squares contained in \( \Sigma \). Assume next that a node \( x_q \) of \( P \) is adjacent to a node \( v_m \in V(\Sigma^r) \setminus V(G_i) \). Let \( P_m^r \) be an \( ST \)-path containing \( v_m \) and let \( \Sigma_{imkl} = CS(P_i^r; P_m^r; P_k^r, P_l^r) \). By the choice of \( P \) and \( \Sigma_{ijkl} \), \( x_q = x_h \) and no node \( x_1, \ldots, x_{k-1} \) is adjacent to a node of \( V(P_i^r) \setminus V(P_m^r) \setminus V^r \). Now \( v \) is an attached Type b[7.2] node for \( \Sigma_{imkl} \) and \( P \) is a minimal attached direct connection in \( \Sigma_{imkl} \) violating Lemma 7.6(i). Finally, assume that a node \( x_q \) of \( P \) is adjacent to a node \( v_m \in V(G_i) \). Let \( P_m^r \) be an \( ST \)-path containing \( v_m \) and let \( \Sigma_{imkl} = CS(P_i^r; P_m^r; P_k^r, P_l^r) \). Then again \( v \) is an attached Type b[7.2] node for \( \Sigma_{imkl} \) and \( P \) is a minimal attached direct connection in \( \Sigma_{imkl} \) violating Lemma 7.6(iii).

\[\square\]

7.3 Biclique Cutsets and 2-Joins

In this subsection, we prove the following theorem:
Theorem 7.12 Let \( \Sigma \) be extreme connected squares in a wheel-free weakly balanced graph \( G \). Then either \( E(K_S) \cup E(K_T) \) is a 2-join of \( G \) separating \( \Sigma^c \) from \( \Sigma^r \), or \( K_S \) or \( K_T \) is a biclique cutset of \( G \).

Proof: Note first that by Definition 7.8, the only edges connecting a node of \( \Sigma^c \) and a node in \( \Sigma^r \) are the ones in \( K_S \) and \( K_T \). So if \( E(K_S) \cup E(K_T) \) is not a 2-join of \( G \) separating \( \Sigma^c \) from \( \Sigma^r \), then \( \Sigma^c \) and \( \Sigma^r \) are joined by a direct connection in \( G \setminus (E(K_S) \cup E(K_T)) \).

Claim 1 No node of \( G \setminus \Sigma \) is adjacent to a node of \( \Sigma^c \) and a node of \( \Sigma^r \).

Proof of Claim 1: Let \( v \) be a node contradicting the claim. Let \( \Sigma_{ijkl} = CS(P^c_i, P^r_j; P^r_k, P^c_l) \) be connected squares contained in \( \Sigma \) such that \( v \) has neighbors in \( P^c_i \) and in \( P^r_k \). Then, by Lemma 7.2, node \( v \) is of Type [7.2] with respect to \( \Sigma_{ijkl} \). Now by Lemma 7.10, node \( v \) belongs to \( \Sigma \), a contradiction. This proves Claim 1.

Claim 2 If \( E(K_S) \cup E(K_T) \) is not a 2-join of \( G \) separating \( \Sigma^c \) from \( \Sigma^r \), and neither \( K_S \) nor \( K_T \) is a biclique cutset of \( G \), then there exists a path \( P = x_1, x_2, \ldots, x_n, n > 1 \), with at least one of the following properties:

- The path \( P \) is a direct connection between \( \Sigma^c \setminus S^c \) and \( \Sigma^r \setminus T^c \), avoiding \( S^c \cup T^c \), such that no node \( x_h, 1 < h < n \), is adjacent to a node in \( T^c \).
- The path \( P \) is a direct connection between \( \Sigma^c \setminus S^c \) and \( \Sigma^r \setminus T^c \), avoiding \( S^c \cup T^c \), such that no node \( x_h, 1 < h < n \), is adjacent to a node in \( S^c \).
- The path \( P \) is a direct connection between \( \Sigma^c \setminus T^c \) and \( \Sigma^r \setminus S^c \), avoiding \( T^c \cup S^c \), such that no node \( x_h, 1 < h < n \), is adjacent to a node in \( T^c \).
- The path \( P \) is a direct connection between \( \Sigma^c \setminus T^c \) and \( \Sigma^r \setminus S^c \), avoiding \( T^c \cup S^c \), such that no node \( x_h, 1 < h < n \), is adjacent to a node in \( S^c \).

Proof of Claim 2: If \( E(K_S) \cup E(K_T) \) is not a 2-join of \( G \), then \( G \setminus (E(K_S) \cup E(K_T)) \) contains a direct connection \( P = x_1, x_2, \ldots, x_n \) between \( \Sigma^c \) and \( \Sigma^r \), where \( x_1 \) is adjacent to a node in \( \Sigma^c \) and \( x_n \) is adjacent to a node in \( \Sigma^r \). (Note that by Claim 1, \( n > 1 \)).

If \( (N(x_1) \cup N(x_n)) \cap V(\Sigma) \not\subseteq S \) and \( (N(x_1) \cup N(x_n)) \cap V(\Sigma) \not\subseteq T \), then \( P \) belongs to at least one of the above four families of direct connections and we are done. So assume w.l.o.g. that \( (N(x_1) \cup N(x_n)) \cap V(\Sigma) \subseteq S \), that is, the set \( N(x_1) \cap V(\Sigma) \) is contained in \( S^c \) and the set \( N(x_n) \cap V(\Sigma) \) is contained in \( S^r \).

Since \( K_S \) is not a biclique cutset, separating \( P \) from \( \Sigma \setminus S \), \( G \) contains a direct connection \( Q = y_1, y_2, \ldots, y_m \) between \( V(P) \) and \( V(\Sigma) \setminus S \) avoiding \( S \), where \( y_i \) is adjacent to a node of \( P \) and \( y_m \) is adjacent to a node of \( \Sigma \setminus S \). Note that for all \( 1 < i < m \), we have that \( N(y_i) \cap V(\Sigma) \subseteq S \) and by Claim 1, we can assume w.l.o.g. that \( N(y_m) \cap V(\Sigma) \subseteq V(\Sigma^r) \).

If some intermediate node of \( Q \) is adjacent to a node in \( S^c \), let \( y_i \neq y_m \) be such a node with highest index. Then \( Q_{y_iy_m} \) is a direct connection between \( \Sigma^c \setminus T^r \) and \( \Sigma^r \setminus S^c \), avoiding \( T^c \cup S^c \). Note that by construction, an intermediate node of such subpath cannot be adjacent to a node in \( T^c \).

If no intermediate node in \( Q \) is adjacent to a node in \( S^c \), let \( x_j \) be the node of lowest index in \( P \), adjacent to \( y_1 \). Then the path \( R = x_1, P_{x_jx_j}, x_j, y_1, Q, y_m \) is a direct connection between \( \Sigma^c \setminus T^c \) and \( \Sigma^r \setminus S^r \), avoiding \( T^c \cup S^r \), such that no intermediate node in \( R \) is adjacent to a node in \( T^c \). This completes the proof of Claim 2.
Let $P = x_1, x_2, \ldots, x_n$ be a direct connection between $\Sigma^c \setminus T^c$ and $\Sigma^r \setminus S^c$, avoiding $T^c \cup S^c$ such that no node $x_h$, $1 < h < n$, is adjacent to a node in $S^c$. So the intermediate nodes of $P$ have no neighbors in $V(\Sigma) \setminus T^c$. By Claim 1, $n > 1$, so $x_1$ has no neighbor in $\Sigma^r$ and $x_n$ has no neighbor in $\Sigma^c$. There exist $i$ and $k$ such that $x_1$ has a neighbor in $G_i \setminus T^c$ and $x_n$ has a neighbor in $G_k \setminus S^c$. Let $P_i^c$ be an $ST$-path of $G_i$ such that $x_1$ has a neighbor in $P_i^c$ distinct from its endnode $t_i^c$ in $T_i$ and let $P_k^r$ be an $ST$-path of $G_k$ such that $x_n$ has a neighbor in $P_k^r$ distinct from its endnode $s_k^r$ in $S_k$. For any $ST$-path of $\Sigma^c$ in $G_j$, $j \neq i$, say $P_j^c$, and any $ST$-path of $\Sigma^r$ in $G_l$, $l \neq k$, say $P_l^r$, the connected squares $\Sigma_{ijkl} = CS(P_i^c, P_j^c; P_k^r, P_l^r)$ have the following properties: $x_1$ has a neighbor in $P_i^c$ distinct from $t_i^c$ and possibly a neighbor in $P_j^c$, $x_n$ has a neighbor in $P_k^r$ distinct from $s_k^r$ and possibly a neighbor in $P_l^r$, and intermediate nodes of $P$ may be adjacent to $t_i^c$ and to the endnode $t_j^c$ of $P_j^c$ in $T_j$ but to no other node of $\Sigma_{ijkl}$.

Claim 3 Neither $x_1$ nor $x_n$ are strongly adjacent nodes of Type $c[7,2]$ with respect to $\Sigma_{ijkl}$.

Proof of Claim 3: Assume that $x_1$ is Type $c[7,2]$ with respect to $\Sigma_{ijkl}$. Let $s_i^c$, $s_j^c$ and $s_k^r$ denote the endnodes of $P_i^c$, $P_j^c$ and $P_k^r$ in $S_i$, $S_j$ and $S_k$ respectively. Then $s_i^c$, $s_j^c$ and $x_2$ are intermediate nodes in a $3PC(x_1, s_i^c)$. The proof for $x_n$ is identical.

We now distinguish two cases:

Case 1 Either $N(x_1) \cap (V(\Sigma) \setminus (S^c \cup T^r)) \neq \emptyset$ or $N(x_1) \cap (V(\Sigma) \setminus T^c) \subseteq S_i$.

Claim 4 $N(x_1) \cap V(\Sigma) \subseteq V(G_i)$ and $x_1$ has a unique neighbor, say $x_0$, in $P_i^c$.

Proof of Claim 4: Assume first $N(x_1) \cap (V(\Sigma) \setminus T^c) \subseteq S_i$. If $N(x_1) \cap V(\Sigma) \subseteq V(G_i)$, node $x_1$ is adjacent to $t_i^c \in T_i$, for some $q \neq i$. Choose an $ST$-path $P_i^c$ in $G_q$ containing $t_i^c$ as endnode. Now $x_1$ is a strongly adjacent node, violating Lemma 7.2 in $CS(P_i^c, P_j^c; P_k^r, P_l^r)$. So $N(x_1) \cap V(\Sigma) \subseteq V(G_i)$. If $x_1$ has more than one neighbor in connected squares $\Sigma_{ijkl}$, (i.e. $x_1$ is adjacent to $t_i^c$ and $s_i^c$), then since $N(x_1) \cap V(\Sigma) \subseteq V(G_i)$ and $\Sigma$ is extreme, $x_1 \in V(\Sigma)$, a contradiction.

Assume now that $N(x_1) \cap (V(\Sigma) \setminus (S^c \cup T^r)) \neq \emptyset$. We assume w.l.o.g. that $N(x_1) \cap (V(G_i) \setminus (S^c \cup T^r)) \neq \emptyset$ and we choose an $ST$-path $Q_i^c$ in $G_i$ so that $x_1$ has a neighbor $x_0$ that is an intermediate node of $Q_i^c$. Then $N(x_1) \cap V(\Sigma) \subseteq V(G_i)$, for otherwise we can choose an $ST$-path $P_i^c$ in $G_q$ containing a neighbor of $x_1$, for some $q \neq i$. By Claim 1, $x_1$ cannot be of Type $a[7,2]$ in $CS(Q_i^c, P_i^c; P_j^c, P_l^c)$ and by Claim 3, $x_1$ cannot be of Type $c[7,2]$. So $x_1$ violates Lemma 7.2 in $CS(Q_i^c, P_i^c; P_j^c, P_l^c)$. Now $x_1$ can have only one neighbor in $P_i^c$, for otherwise $N(x_1) \cap V(\Sigma) \subseteq V(G_i)$ and the fact that $\Sigma$ is extreme would imply that $x_1 \in V(\Sigma)$, a contradiction. This completes the proof of Claim 4.

Claim 5 No intermediate node of $P$ is adjacent to a node in $T^c$.

Proof of Claim 5: Let $x_m$ be a node of $P$ with lowest index $m > 1$, adjacent to a node in $T^c$. If $x_m$ is adjacent to $t_q^c \in T_q$, for some $q \neq i$, let $CS(P_i^c, P_j^c; P_k^r, P_l^r)$, be connected squares such that $t_i^c \in P_j^c$. If $x_m$ is adjacent to both $t_i^c$, $t_i^c$, then $x_m$ is a Type $b[7,2]$ node with respect to $CS(P_i^c, P_j^c; P_k^r, P_l^r)$ and the direct connection $P_{x_1,x_{m-1}}$ violates Lemma 7.6. (Note that, since by Claim 4, $x_0$ is the unique neighbor of $x_1$ in $CS(P_i^c, P_j^c; P_k^r, P_l^r)$, then $x_m$ is attached in $CS(P_i^c, P_j^c; P_k^r, P_l^r)$). Node $x_m$ cannot be adjacent to $t_i^c$ only for, if $x_0 \in V^c$, we have a $3PC(x_0, t_i^c)$ and if $x_0 \in V^c$, we have a $3PC(x_0, t_i^c)$. So $x_m$ has no neighbor in $T^c \setminus T_i$. 

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Let \( t'_i \in T_i \) be a neighbor of \( x_m \), let \( P' \) be the \( x_0 s_i / \)-subpath of \( P_i \) and \( Q \) be the chordless path made up by the concatenation of \( P' \) and \( P_{x_0s_i} \). Then \( x_m \) is the unique neighbor of \( t'_i \) in \( Q \), for otherwise \( t'_i \) has a neighbor in \( P' \) and \( (C, t'_i) \) is a wheel, where \( C \) is defined as follows:

- If \( P \) contains an intermediate node adjacent to \( t'_i \), for \( q \neq i \), let \( x_h, m < h \), be such a node with lowest index. Let \( C = s_i, Q, x_m, P_{x_0x_m}, x_h, t'_q, t'_q, P_k, s'_h, s'_i \).

- If \( P \) contains no intermediate node adjacent to \( t'_i \), for \( q \neq i \), let \( x_{n+1} \) be the neighbor of \( x_n \), closest to \( t'_k \) in \( P_{r_i} \). Since \( x_{n+1} \neq s'_i \), then by Lemma 7.2, \( x_n \) is not adjacent to \( s'_i \). Let \( C = s'_i, Q, x_m, P_{x_0x_m}, x_n, (P_k), x_{n+1}, t'_k, t'_k, P', s'_i, s'_i \).

Since \( N(x_1) \cap V(\Sigma) \subseteq V(G_i) \) by Claim 4, \( N(x_m) \cap V(\Sigma) \subseteq T_i \) and \( x_m \) is the unique neighbor of \( t'_i \) in \( Q \), then \( V(Q) \subset V(\Sigma) \) and, since \( \Sigma \) is extreme, this contradicts \( x_1 \notin V(\Sigma) \). This completes the proof of Claim 5.

We show that \( N(x_n) \cap V(\Sigma) \subseteq V(G_k) \). For, if not, we can choose \( P_i \) and \( P'_i \) so that \( x_n \) has neighbors in both. Now by Claim 1, \( x_n \) cannot be of Type a[7,2] in \( \Sigma_{ijkl} \) and by Claim 3, \( x_n \) cannot be of Type c[7,2]. Finally \( x_n \) cannot be of Type b[7,2], for by Claims 4 and 5, \( P_{x_0x_1} \) shows that \( x_n \) is an attached Type b[7,2] node, a contradiction to Lemma 7.11.

Finally, \( x_n \) has a unique neighbor, say \( x_{n+1} \), in \( P_i \) for, if not, since \( N(x_n) \cap V(\Sigma) \subseteq V(G_k) \), then \( x_n \in V(\Sigma) \). Since \( x_0 \neq t'_i \) and \( x_{n+1} \neq s'_i \), there is a 3PC\((t'_i, s'_i)\). So Case 1 cannot happen.

**Case 2** \( N(x_1) \cap (\Sigma - T^c) \subseteq S^c \), \( N(x_1) \cap S^c_x \neq \emptyset \) and \( N(x_1) \cap S^c \neq \emptyset \).

Choose \( P_i \) and \( P'_i \) so that \( x_1 \) is a Type b[7,2] node in \( \Sigma_{ijkl} \), adjacent to \( s'_i \) and \( s'_i \). Now, since \( x_1 \notin V(\Sigma) \), Lemma 7.11 shows that \( x_1 \) is not attached in \( \Sigma_{ijkl} \). So no subpath of \( P \) is an attached direct connection for \( x_1 \) in \( \Sigma_{ijkl} \). Since \( P_x = (\Sigma_{ijkl}) \neq \emptyset \), \( P \) contains a node \( x_m \) such that \( P_{x_0x_m} \) is a detached direct connection for \( x_1 \) in \( \Sigma_{ijkl} \). So, by Lemma 7.7, \( x_m \) is adjacent to both \( t'_i \) and \( t'_j \) and no intermediate node of \( P_{x_0x_m} \) is adjacent to \( t'_i \) or \( t'_j \).

**Claim 6** \( N(x_1) \cap V(\Sigma) = S^c \), \( N(x_m) \cap V(\Sigma) = T^c \) and no intermediate node of \( P_{x_0x_m} \) is adjacent to a node of \( \Sigma \).

**Proof of Claim 6:** We first show \( N(x_1) \cap V(\Sigma) = S^c \). If not, choose \( P_x \) not containing a neighbor of \( x_1 \) and by symmetry we can assume \( q \neq i \). Then in \( \Sigma_{ijkl} = CS(P^i, P^q, P^r, P^t) \), \( s'_i \) is the unique neighbor of \( x_1 \). Now \( x_1 \) satisfies Case 1 in the subgraph \( G' \) of \( G \) induced by \( V(\Sigma_{ijkl}) \cup V(P) \) and \( \Sigma_{ijkl} \) is extreme in \( G' \). This is not possible since Case 1 cannot happen.

Assume now that a node \( x_h, 1 < h < m \), is adjacent to \( t'_q \) (we assume that \( h \) is the lowest such index) or \( x_m \) is not adjacent to \( t'_i \). We assume w.l.o.g. that \( q \neq i \) and we choose \( P_x \) containing \( t'_q \). Then since \( N(x_1) \cap V(\Sigma) = S^c \), node \( x_1 \) is a Type b[7,2] adjacent to \( s'_i \) and \( s'_i \) in \( \Sigma_{ijkl} = CS(P^i, P^q, P^r, P^t) \) and \( P_{x_0x_m} \) in the first case, \( P_{x_0x_m} \) in the second case, shows that \( x_1 \) is attached in \( \Sigma_{ijkl} \). By Lemma 7.11, \( x_1 \in V(\Sigma) \) and this completes Claim 6.

Let \( \Sigma' \) be the graph induced by \( V(\Sigma) \cup V(P_{x_0x_m}) \). Define \( S' = S \cup \{x_m\} \) and \( T' = T \cup \{x_1\} \). By Claim 6, \( S' \) and \( T' \) induce bicliques and \( E(K_{S'}) \cup E(K_{T'}) \) is a 2-join of \( \Sigma' \). The connected components of \( \Sigma' \setminus (E(K_{S'}) \cup E(K_{T'})) \) satisfy the conditions of Definition 7.8 and every node of \( \Sigma' \) belongs to an \( S'T' \)-path of \( \Sigma' \). Now \( V(\Sigma) \subset V(\Sigma') \) contradicts the choice of \( \Sigma \) as extreme connected squares. 

\( \square \)
8 Goggles

8.1 Introduction

Goggles are formed by a parachute with short top, long sides and long middle, together with a direct connection from bottom to top of Type $][6.4]$. In this section, we assume that $G$ is a weakly balanced graph that contains goggles $\Gamma = Go(P, Q, R, S, T)$, see Figure 23. We use the following notation: $P = x, \ldots, i, h$, $Q = a, \ldots, v$, $R = b, \ldots, v$, $S = u, \ldots, j, h$ and $T = h, k, \ldots, v$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{goggles.png}
\caption{Goggles}
\end{figure}

The paths $P, Q, R, S$ have length greater than 1, but the path $T$ may be of length 1 in which case $h \in N(v)$ and $k = v$. It will be convenient to denote the length of $T$ by $|T|$, since we will often distinguish the cases $|T| = 1$ and $|T| > 1$. Assume w.l.o.g. that $a \in V^r$. Then $x, a, v \in V^c$ and $b, h \in V^r$. Further, we assume that $G$ contains

- no connected squares,
- no extended star cutset.

Since $G$ does not contain an extended star cutset, it follows from Sections 3 and 4 that $G$ contains

- no wheel,
- no parachute with long sides and long top,
- no parachute with long sides, short top and short middle.

The main result of this section is that $G$ has a 2-join.
This is achieved by considering goggles $\Gamma$ with shortest possible top path $T$ (among all goggles in $G$) and, subject to this, it is assumed that $\Gamma$ has the fewest number of nodes. Throughout this section, this assumption is made about $\Gamma$.

We show how certain 2-joins of $\Gamma$ extend to the full graph $G$. This requires an understanding of the possible paths connecting nodes of $\Gamma$. In order to identify these paths, we first list the strongly adjacent nodes to $\Gamma$. This is done in the next subsection. In Subsection 8.3, we describe paths connecting nodes of $\Gamma$, starting from a strongly adjacent node, and in Subsection 8.4, those starting from node $h$. In Subsection 8.5, we identify candidate bicliques for a 2-join of $G$. Finally, in Subsections 8.6 and 8.7, we prove the 2-join theorem.

Our assumptions on $G$ imply the following results about parachutes:

**Corollary 8.1** If $\Pi$ is a parachute with long sides, long middle, short top $v_1, t, v_2$ and center node $v$ then, in $G$, every direct connection between the bottom of $\Pi$ and the top node $t$ avoiding $N(v) \cup (N(v_1) \cap N(v_2)) \setminus \{t\}$ is of Type $j[6.4]$ or $j1[6.5]$. Furthermore, $\Pi$ has at least one such direct connection.

**Proof:** This is a consequence of Theorem 6.17 and the assumption that $G$ contains no extended star cutset and no connected squares. \hfill \qed

**Lemma 8.2** Let $\Pi$ be a parachute with long sides, long middle, short top $v_1, t, v_2$ and center node $v$. There exists no chordless path $x_1, \ldots, x_m$ such that

(i) $m \geq 2$ and $x_p \in V(G) \setminus V(\Pi)$, $1 \leq p \leq m$,

(ii) nodes $x_1$ and $x_m$ each have a unique neighbor in $\Pi$, and these two neighbors are distinct nodes in the set $\{t, v, v_1, v_2\}$,

(iii) for $2 \leq p \leq m - 1$, node $x_p$ has no neighbor in $\Pi$.

**Proof:** Assume such a path exists.

**Case 1:** Node $x_1$ is adjacent to $v_1, v_2$ or $v$, and $x_m$ is adjacent to $t$.

Since $G$ has no extended star cutset, there exists a direct connection $Y = y_1, \ldots, y_n$ between the bottom of $\Pi$ and $\{x_1, \ldots, x_m\}$ avoiding $N(v) \cup (N(v_1) \cap N(v_2))$. This implies a direct connection $W = w_1, \ldots, w_l$ from the bottom of $\Pi$ to $\{t\}$ avoiding $N(v) \cup (N(v_1) \cap N(v_2)) \setminus \{t\}$. By Corollary 8.1, the direct connection $W$ is of Type $j[6.4]$ or $j1[6.5]$.

If $W$ is of Type $j[6.4]$, let $s \in M$ be the neighbor of $y_1$. Since $s$ is not adjacent to $v$, there is a $3PC(v, s)$ irrespective of whether $Y$ contains neighbors of $v_1, v_2$ or $t$.

If $W$ is of Type $j1[6.5]$, there is a $3PC(v, y_1)$.

**Case 2:** Node $x_1$ is adjacent to either $v_1$ or $v_2$, say $v_1$, and $x_m$ is adjacent to $v$ or $v_2$.

If $x_m$ is adjacent to $v_2$, there is a parachute with long sides and long top $x_1, \ldots, x_m$, a contradiction. Consider now the case where $x_m$ is adjacent to $v$. By Corollary 8.1, there exists a direct connection $Y = y_1, \ldots, y_n$ of Type $j[6.4]$ or $j1[6.5]$ between the bottom of $\Pi$ and $t$. Let $s \in M$ be the neighbor of $y_1$ closest to the bottom node $z$ of $\Pi$. No node of $Y$ is coincident with or adjacent to one of the nodes $x_1, \ldots, x_m$, for otherwise there is a $3PC(v, s)$ or $3PC(v, y_1)$. Consider the parachute with top path $v_1, v, v_2$, same side paths as $\Pi$ and middle path $t, y_n, \ldots, y_1, s, M_{xz}, z$. Now the result follows from Case 1. \hfill \qed
8.2 Strongly Adjacent Nodes

Lemma 8.3 Let $w \in V(G) \setminus V(\Gamma)$ be a strongly adjacent node to $\Gamma$. Then, one of the following holds:

(i) Node $w$ is a twin of a node of $\Gamma$, relative to $\Gamma$.

(ii) Node $w$ is of one of the following types, see Figure 24.

Type a Node $w$ has exactly two neighbors in $\Gamma$ and $w$ is adjacent to $x$ and $u$ (a and $b$ resp.).

Type b Node $w$ has exactly two neighbors in $\Gamma$ and is adjacent to the two neighbors of $h$ ($v$ resp.) in $P, S$ ($Q, R$ resp.).

Type c Node $w$ has exactly two neighbors in $\Gamma$, one of them is the neighbor of $h$ ($v$ resp.) in $T$ and the other is the neighbor of $h$ ($v$ resp.) in either $P$ or $S$ ($Q$ or $R$ resp.).

Type d $w \in V^c$ ($w \in V^r$ resp.) has exactly two neighbors in $\Gamma$, one of them in $P \setminus \{h\}$ and the other in $S \setminus \{h\}$ ($Q \setminus \{v\}$ and $R \setminus \{v\}$ resp.).

Proof: We consider first the case where $w$ has two neighbors in $\Gamma$ and then the case where $w$ has three or more neighbors.

Case 1 Node $w$ has two neighbors in $\Gamma$, say $\alpha$ and $\beta$.

If $\alpha$ and $\beta$ belong to $T$, then $w$ must be a twin, otherwise there are goggles with shorter top. If $\alpha$ and $\beta$ belong to the same path $P, Q, R, S$, then $w$ must be a twin, otherwise there are goggles with fewer nodes and same top. Now assume no path $P, Q, R, S, T$ contains both $\alpha$ and $\beta$. Because of the symmetry between paths $P, Q, R$ and $S$, we can assume w.l.o.g. that $\alpha \in V(P)$.

Assume first that $\beta \in V(S)$. If $w \in V^c$, then $w$ is of Type d. Suppose now $w \in V^r$.

If $\alpha = x$, then $\beta = u$ for otherwise we have a 3PC($x, h$). Thus node $w$ is of Type a.

Suppose now $\alpha \neq x$. By symmetry, $\beta \neq u$. Now, if $\alpha$ ($\beta$ resp.) is not adjacent to $h$, there is a 3PC($h, \alpha$) (3PC($h, \beta$ resp.)). Hence both $\alpha$ and $\beta$ are neighbors of $h$. Thus node $w$ is of Type d.

If $\beta \in V(T)$, then $w \in V^r$, otherwise there are goggles with shorter top path $T_{\emptyset}$. If $\alpha$ ($\beta$ resp.) is not adjacent to $h$, there is 3PC($h, \alpha$) (3PC($h, \beta$ resp.)). Hence, both $\alpha$ and $\beta$ are neighbors of $h$. Thus node $w$ is of Type c.

Finally, if $\beta \in V(Q) \cup V(R)$, because of symmetry, we can assume that $\beta \in V(Q)$ and $w \in V^c$. If $\beta$ is not adjacent to $v$, there is a 3PC($v, \beta$). Hence $\beta \neq a$. Now if $\alpha \neq h$ or if $|T| > 1$, there is a 3PC($x, h$). Hence it follows that $\alpha = h$ and $\beta, h \in N(v)$. Then $w$ is of Type c.

Case 2 Node $w$ has three or more neighbors in $\Gamma$.

Clearly $w$ has at most one neighbor in each of the paths $P, Q, R, S, T$, for otherwise there is a wheel. We now consider two cases depending upon whether $|N(w) \cap T| = 0$ or 1.
Figure 24: Strongly adjacent nodes
Case 2.1 $|N(w) \cap V(T)| = 1$.

Now $w$ has neighbors in at least two different paths $P, Q, R, S$. Because of symmetry, we assume that $w$ has exactly one neighbor in $V(P) \setminus \{h\}$. It follows that $w$ is not adjacent to $h$ nor to $(V(Q) \cup V(R)) \setminus \{v\}$ for otherwise there is a wheel. This implies that $w$ has exactly one neighbor in $V(S) \setminus \{h\}$ and three neighbors in $T$. If $w \in V^c$, there is a $3PC(w, h)$. Hence $w \in V^c$. Let $\alpha, \beta, \gamma$ be the neighbors of $w$ in $P, S$ and $T$ respectively. We now consider the following two subcases.

Assume first $\alpha$ or $\beta$ is a neighbor of $h$, say $\alpha \in N(h)$. If $\beta = u$, there is a parachute with short middle path $w, u, a$, where $w$ is the center node and $a$ the bottom node. Suppose now $\beta \neq u$. If $\gamma \notin N(h)$, there are goggles with a shorter top path, and if $\gamma \in N(h)$ and $\beta \notin N(h)$, there are goggles with fewer nodes but a top path of the same length as $T$. So $w$ must be a twin of $h$.

Assume now that neither $\alpha$ nor $\beta$ is a neighbor of $h$. If $\alpha \neq x$ or $\beta \neq u$, say $\alpha \neq x$, we have a parachute with long top $\gamma, T_{\gamma, h}$, $hP_{ho}$, $a$ and long sides, with center node $w$ and bottom node $a$. If $\alpha = x$ and $\beta = u$, consider the parachute with side paths $P$ and $S$, center node $w$, middle path $w, \gamma, T_{\gamma, h}$, and top path $x, b, u$. If $\gamma = k$, this parachute has long sides and short middle path. So $|T| \geq 2$ and $\gamma \neq k$. Now, by Corollary 8.1, there exists a direct connection $Y$ of Type $[j][6.4]$ or $[j][6.5]$ between $b$ and $T_{\gamma, h} \setminus \{\gamma, h\}$. This parachute and the path $Y$ induce goggles with a shorter top path than $T$.

Case 2.2 $|N(w) \cap V(T)| = 0$.

Clearly, $h, v \notin N(w)$. Suppose $w$ has four neighbors in $T$, one in each of the paths $P, Q, R, S$. Because of symmetry, we can assume w.l.o.g. that $w \in V^c$. This implies a $3PC(w, h)$. Consequently we can assume w.l.o.g. that $w$ has no neighbor in $Q$ and has exactly one neighbor in $P, R$ and $S$. If $w \in V^c$, there is a $3PC(w, h)$. Hence $w \in V^c$. Let $\alpha, \beta$ and $\gamma$ be the neighbors of $w$ in $P, S$ and $R$ respectively. If $\alpha = x$ and $\beta = u$, then either $w$ is a twin of $a$ or $b$ or there are goggles with fewer nodes than $T$ but same top. Suppose now $\alpha \neq x$ or $\beta \neq u$, say $\alpha \neq x$. If $\beta \notin N(h)$, there is a $3PC(a, \alpha)$. If $\beta \in N(h)$ and $\alpha \notin N(h)$, there is a $3PC(a, \beta)$. Hence $\alpha, \beta \in N(h)$. Now there are connected squares, which is a contradiction.

\[\square\]

Remark 8.4 There cannot exist nodes $w$ and $z$ of Type $b[8.3]$ or Type $c[8.3]$ having exactly one common neighbor in $T$.

Proof: Otherwise there is a wheel with center $h$ or $v$. \[\square\]

Remark 8.5 When $|T| = 1$, if $w$ and $z$ are Type $c[8.3]$ nodes adjacent to $h$ and $v$ respectively, then $w$ and $z$ are adjacent.

Proof: If $w$ and $z$ are not adjacent, there is a violation of Lemma 8.2, as follows. W.l.o.g. assume $w$ is adjacent to the neighbor of $v$ in $Q$, say $t$, and assume that $z$ is adjacent to $i$. The parachute has top $h, w, t$, side paths $S$ and $t, Q_{i\alpha}, a, u$, center node $v$ and middle path $v, R, b, u$. The extra path is $h, i, z, v$. \[\square\]
8.3 Direct Connections from Strongly Adjacent Nodes of Type a, b and c

Lemma 8.6 Let \( w \) be a Type a\([8.3]\) node adjacent to \( a \) and \( b \). Let \( W \) be the node set consisting of the twins of \( a \) and \( b \) and Type a\([8.3]\) nodes, but not nodes \( a \), \( b \) and \( w \). In \( G \setminus \{wa, wb\} \), every direct connection \( X = x_1, \ldots, x_n \) between \( w \) and \( V(\Gamma) \) avoiding \( W \) is one of the following types, see Figure 25.

**Type 1** \( n = 1 \) and node \( x_1 \) is adjacent to \( u \) or \( x \) but not strongly adjacent to \( \Gamma \), or \( n = 2 \) and \( x_2 \) is a twin of \( u \) or \( x \).

**Type 2** Node \( x_n \) has a unique neighbor \( t \in V^r \) in \( \Gamma \) and \( t \in V(P) \cup V(S) \), or node \( x_n \in V^r \) is a twin of a node in \( V(P) \cup V(S) \).

Furthermore, in \( G \setminus \{wa, wb\} \), there exists a direct connection of Type 2 between \( w \) and \( V(\Gamma) \).

![Figure 25: Direct connections from a Type a node](image)

**Proof:** Let \( \Pi_x \) be the parachute with top path \( a, w, b, \) side paths \( Q \) and \( R \), and middle path \( x, P, h, T, v \). Similarly, let \( \Pi_u \) be the parachute with top path \( a, w, b, \) side paths \( Q \) and \( R \), and middle path induced by \( u, S, h, T, v \).

Assume first that \( x_n \) is adjacent to \( u \) or \( x \), say \( x \). Since \( X \) avoids \( W \), \( x_n \) is not a twin of \( a \) or \( b \) nor a Type a\([8.3]\) node. So either \( x_n \) is a twin of the neighbor of \( x \) in \( P \), and then \( X \) is of Type 2, or \( x \) is the unique neighbor of \( x_n \) in \( \Gamma \), and then \( X \) is of Type 1, since \( n > 1 \) would violate Lemma 8.2 in \( \Pi_x \).

Assume next that \( x_n \) is a twin of \( u \) or \( x \), say \( x \). Then \( X \) is of Type 1, since \( n > 2 \) would imply a violation of Lemma 8.2 in the parachute obtained from \( \Pi_x \) by substituting \( x_n \) for \( x \).

Assume now that \( x_n \) is not adjacent to \( u \) or \( x \), and is not a twin of \( u \) or \( x \). If the neighbors of \( x_n \) in \( \Gamma \) all lie in \( S \) or all lie in \( R \), then the path \( X \) is of Type 2. If \( x_n \) has \( a \) or \( b \) as unique neighbor in \( \Gamma \), there is a violation of Lemma 8.2 in \( \Pi_x \). So \( x_n \) has a neighbor in \( \Gamma \) distinct.
from $a$, $b$, $x$ and the neighbor of $x$ in $P$. This implies that $X$ is a direct connection from the bottom of $\Pi_{x}$ to the top avoiding $N(x) \cup ((N(a) \cap N(b)) \setminus \{w\})$. Therefore, by Corollary 8.1, $X$ is of Type j[6,4] or j1[6,5] in the parachute $\Pi_{x}$. Similarly, $X$ is of Type j[6,4] or j1[6,5] in the parachute $\Pi_{a}$. So $x_{n}$ has its neighbors in $V(P) \cup V(S) \cup V(T)$. Furthermore, if $x_{n}$ has two neighbors in $V(P) \cup V(T)$ or in $V(S) \cup V(T)$, then $x_{n} \in V^{c}$. And, if $x_{n}$ has only one neighbor in $V(P) \cup V(T)$ and one in $V(S) \cup V(T)$, then $x_{n} \in V^{c}$. It follows that, if the only neighbors of $x_{n}$ are in $V(T) \setminus \{h\}$, there are goggles with shorter top. It also follows that $x_{n}$ cannot be of Type b or of Type c[8,3]. If $x_{n}$ is of Type d[8,3], there is a 3PC($x_{n}, h$). Thus path $X$ is of Type 2.

To complete the proof of the theorem, note that, by applying Corollary 8.1 to $\Pi_{x}$, a path $X$ of Type 2 must exist.

\begin{lem}
Let $w$ be a Type b[8,3] node adjacent to $i$ and $j$. Let $W$ be the node set consisting of the twins of $i$ and $j$ and Type b[8,3] nodes adjacent to $i$ and $j$, but not nodes $i$, $j$ and $w$. Then the top path $T$ has length greater than 1 and, in $G \setminus \{wi, wj\}$, every direct connection $X$ between $w$ and $V(1)$, avoiding $W$ is one of the following types, see Figure 26.

Type 1 $n = 1$ and node $x_{1}$ is adjacent to $h$ but not strongly adjacent to $\Gamma$, or $n = 2$ and $x_{2}$ is a twin of $h$.

Type 2 Either $x_{n}$ has a unique neighbor $t \in V^{c}$ in $\Gamma$ and $t$ is in $T$ or $x_{n} \in V^{c}$ is a twin of a node in $T$.

Furthermore, in $G \setminus \{wi, wj\}$, there exists a direct connection of Type 2 such that $x_{n}$ is not adjacent to $k$ and $x_{n}$ is not a twin of $k$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure26.png}
\caption{Direct connections from a Type b node}
\end{figure}

\textbf{Proof}: Let $\Pi_{a}$ be the parachute with top path $i$, $w$, $j$, side paths $i$, $P_{x}$, $x$, $a$ and $j$, $S_{z}$, $u$, $a$, center node $h$ and middle path $h$, $T$, $v$, $Q$, $a$. $\Pi_{b}$ is defined similarly, with bottom node $b$ and middle path $h$, $T$, $v$, $R$, $b$. 86
Assume first that $x_n$ is adjacent to $h$. Since $X$ avoids $W$, $x_n$ is not a twin of $i$ or $j$. So there are three possibilities. When $h$ is the unique neighbor of $x_n$ in $\Gamma$, the path $X$ is of Type 1, since $n > 1$ would violate Lemma 8.2 in $\Pi_s$. When $x_n$ is a twin of $k$, the path $X$ is of Type 2. Finally, when $|T| = 1$ and $x_n$ is of Type $c[8,3]$, there is an odd wheel with center $h$.

Assume next that $x_n$ is a twin of $h$. Then $X$ is of Type 1, since $n > 2$ would imply a violation of Lemma 8.2 in the parachute obtained from $\Pi_a$ by substituting $x_n$ for $h$.

Assume now that $x_n$ is not adjacent to $h$ and is not a twin of $h$. If $x_n$ has $k$ as unique neighbor in $\Gamma$, path $X$ is of Type 2. If $x_n$ has $i$ or $j$ as unique neighbor in $\Gamma$, there is a violation of Lemma 8.2 in $\Pi_a$. By Remark 8.4, $x_n$ is not of Type $c[8,3]$ with neighbors $i$ and $j$. So, $x_n$ has a neighbor in $\Gamma$ distinct from $h$, $i$, $j$, $k$. This implies that $X$ is a direct connection from the bottom of $\Pi_a$ (and $\Pi_b$) to the top, avoiding $N(h) \cup (((N(i) \cap N(j)) \setminus \{w\})$.

Therefore, by Corollary 8.1, $X$ is of Type $j[6,4]$ or $j[6,5]$ in $\Pi_a$ and in $\Pi_b$. $x_n$ cannot be of Type $a[8,3]$ by Lemma 8.6, it cannot of Type $d[8,3]$ since there would be a $3PC(x_n, v)$, and it cannot be of Type $b$ or $c[8,3]$ since $x_n \in V^e$ would violate Corollary 8.1 in $\Pi_a$ or $\Pi_b$. If $x_n$ is adjacent to $(V(Q) \cup V(R)) \setminus \{v\}$, there are connected squares. So $x_n$ must have all its neighbors in $T$. Thus path $X$ is of Type 2.

Finally, by Corollary 8.1 applied to $\Pi_a$, a path $X$ of Type 2 must exist such that $x_n$ is not adjacent to $k$ and $x_n$ is not a twin of $k$. This implies that $T$ is of length greater than 1.

\begin{lemma}
Let $w$ be a Type $c[8,3]$ node adjacent to $i$ and $k$. Let $W$ be the node set consisting of the twins of $i$ and $k$ and Type $c[8,3]$ nodes adjacent to $i$ and $k$, but not nodes $i$, $k$ and $w$. In $G \setminus \{wi, wk\}$, every direct connection $X$ between $w$ and $V(\Gamma)$, avoiding $W$ is one of the following types, see Figure 27.

**Type 1** Node $x_1$ is adjacent to $h$ but not strongly adjacent to $\Gamma$, or $n = 2$ and $x_2$ is a twin of $h$.

**Type 2** Either $x_n$ has a unique neighbor $t \in V^e$ in $\Gamma$ and $t$ is in $S$ or $x_n \in V^e$ is a twin of a node in $S$.

**Type 3** Node $x_1$ is a Type $c[8,3]$ node. The top path $T$ of $\Gamma$ is of length 1. Node $x_1$ is adjacent to $h$ and to the node in $Q \cap N(v)$ or to the node in $R \cap N(v)$.

Furthermore, in $G \setminus \{wi, wj\}$, there exists a direct connection of Type 2 such that $x_n$ is not adjacent to $j$ and $x_n$ is not a twin of $j$.

\end{lemma}

**Proof:** Let $\Pi_a$ be the parachute with top path $i, w, k$, side paths $i, P_{ix, x, a}$ and $k, T_{kv, v, Q, a}$, center node $h$ and middle path $h, S, u, a, 1$. $\Pi_b$ is defined similarly, with bottom node $b$.

Assume first that $x_n$ is adjacent to $h$. Since $X$ avoids $W$, $x_n$ is not a twin of $i$ or $k$. So, there are three possibilities. When $h$ is the unique neighbor of $x_n$ in $\Gamma$, the path $X$ is of Type 1, since $n > 1$ would violate Lemma 8.2 in $\Pi_a$. When $x_n$ is a twin of $j$, the path $X$ is of Type 2. Finally, when $|T| = 1$ and $x_n$ is of Type $c[8,3]$, $n = 1$ by Remark 8.5 and the path $X$ is of Type 3.
Figure 27: Direct connections from a Type c node
Assume next that \( x_n \) is a twin of \( h \). Then \( X \) is of Type 1, since \( n > 2 \) would imply a violation of Lemma 8.2 in the parachute obtained from \( \Pi_a \) by substituting \( x_n \) for \( h \).

Assume now that \( x_n \) is not adjacent to \( h \) and is not a twin of \( h \). If \( x_n \) has \( j \) as unique neighbor in \( \Gamma \), we have a path \( X \) of Type 2. If \( x_n \) has \( i \) or \( k \) as unique neighbor in \( \Gamma \), there is a violation of Lemma 8.3 in \( \Pi_a \). So \( x_n \) has a neighbor in \( \Gamma \) distinct from \( h, i, j, k \). This implies that \( X \) is a direct connection from the bottom of \( \Pi_a \) (and \( \Pi_b \)) to the top, avoiding \( N(h) \setminus (N(i) \cap N(j)) \setminus \{w\} \). Therefore, by Corollary 8.1, \( X \) is of Type \([6.4] \) or \([6.5] \) in \( \Pi_a \) and in \( \Pi_a \) \( x_n \) cannot be of Type a[8.3] by Lemma 8.6. Thus path \( X \) is of Type 2.

Finally, by Corollary 8.1 applied to \( \Pi_a \), a path \( X \) of Type 2 must exist such that \( x_n \) is not adjacent to \( j \) and \( x_n \) is not a twin of \( j \).

\[ \square \]

### 8.4 Partition of the Neighbors of \( h \)

Let \( Z(h) \) comprise the nodes of \( N(h) \) together with the nodes with at least two neighbors in \( \{i, j, k\} \). By Remark 8.4, \( Z(h) \) is an extended star with center \( h \).

Let \( H(h) \) be the set of nodes of \( G \setminus V(\Gamma) \) that have \( h \) as unique neighbor in \( \Gamma \).

**Lemma 8.9** Suppose \( |\Gamma| > 1 \) and let \( y \in H(h) \). Every direct connection \( Y = y_1, \ldots, y_n \) from \( y \) to \( V(\Gamma) \setminus \{h, i, j, k\} \) avoiding \( Z(h) \setminus \{y\} \) is one of the following types, see Figure 28.

- **Type 1** Node \( y_n \) either has a unique neighbor in \( V(P) \cup V(S) \setminus \{h, i, j\} \) or is a twin of a node in \( V(P) \cup V(S) \setminus \{h, i, j\} \). Furthermore, if \( y_n \) has a neighbor in \( \Gamma \), then no node of \( Y \) is adjacent to \( j \) or \( k \), and if \( y_n \) has a neighbor in \( \Gamma \), then no node of \( Y \) is adjacent to \( i \) or \( k \).

- **Type 2** Node \( y_n \) is of Type a[8.3], adjacent to \( a \) and \( b \) and no node of \( Y \) is adjacent to \( i, j \) or \( k \).

- **Type 3** Node \( y_n \) is adjacent to \( p \in V(\Gamma) \setminus \{h, k\} \) but is not of Type c[8.3], or \( y_n \) is a Type b[8.3] node adjacent to the neighbors of \( v \) in \( Q \) and \( R \). If \( y_n \) is adjacent to \( p \in V(\Gamma) \setminus \{h, k\} \), then no node of \( Y \) is adjacent to \( i \) or \( j \). If \( y_n \) is a Type b[8.3] node, then no node of \( Y \) is adjacent to \( i, j, k \).

**Proof:** There are two cases to consider.

**Case 1** Node \( y_n \) has only one neighbor \( p \) in \( V(\Gamma) \setminus \{h, i, j, k\} \).

Suppose \( p \in (P \cup S) \setminus \{h, i, j\} \). W.l.o.g. assume \( p \in P \setminus \{h, i\} \). If any of the nodes in \( Y \) is adjacent to \( j \) or \( k \), there is a \( 3PC(a, j) \) or \( 3PC(a, k) \). Thus path \( Y \) is of Type 1.

Suppose \( p \in T \setminus \{h, k\} \). If \( Y \) has a node adjacent to \( i \) or \( j \), there is a \( 3PC(a, i) \) or a \( 3PC(a, j) \). Hence path \( Y \) is of Type 3.

Suppose \( p \in (Q \cup R) \setminus \{v\} \). W.l.o.g. assume \( p \in Q \setminus \{v\} \). If \( Y \) has a node adjacent to \( i \) or \( j \), there is a \( 3PC(a, i) \) or a \( 3PC(a, j) \). If none of the nodes in \( Y \) is adjacent to \( k \), there is a \( 3PC(h, v) \) if \( p \neq a \) and a \( 3PC(a, v) \) if \( p = a \). So, let \( t \) be the largest index such that \( y_t \) is adjacent to \( k \). If \( p \in V^r \), there is a \( 3PC(k, p) \). So \( p \) is not adjacent to \( v \) and there are edges with a shorter top path \( h, k \) obtained from \( \Gamma \) by replacing \( Q \) by \( a, Q_{ap}, p, y_n, Y_{y_n, y_t}, y_t, k \) and \( R \) by \( b, R, v, T_{vb}, k \).

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Figure 28: Direct connections from $y \in H(h)$ when $|T| > 1$
Case 2 Node $y_n$ has at least two neighbors in $V(\Gamma) \setminus \{h, i, j, k\}$.

Assume first that $y_n$ is a twin of a node $d$ of $\Gamma$. Let $\Gamma'$ be the goggle obtained from $\Gamma$ by substituting $y_n$ for $d$. If $n = 1$, node $y_1$ must be adjacent to $i, j$ or $k$, for otherwise $y$ is a strongly adjacent node violating Lemma 8.3 in $\Gamma'$. If $y_1$ is adjacent to $k$, we get a path of Type 3, otherwise we get a path of Type 1. Now assume $n \geq 2$. If $y_{n-1}$ has a unique neighbor in $\Gamma'$, applying Case 1 to $Y' = y_1, \ldots, y_{n-1}$ and $\Gamma'$, we get that $Y$ is of Type 1 or 3 in $\Gamma$. If $y_{n-1}$ is strongly adjacent to $Y'$, then $y_{n-1}$ is adjacent to $i, j$ or $k$ and, by Lemma 8.3 applied to $\Gamma'$, $y_{n-1}$ is a twin of a node $d'$ in $\Gamma'$. Now, either the path $y_1, \ldots, y_{n-1}$ is reduced to a single node, in which case $Y$ is of Type 1 or 3 in $\Gamma$, or $y_{n-2}$ has a unique neighbor in the goggle $\Gamma''$ obtained from $\Gamma'$ by replacing $d'$ by $y_{n-1}$ and again, by applying Case 1 to $Y'' = y_1, \ldots, y_{n-2}$ and $\Gamma''$, we obtain that $Y$ is of Type 1 or 3 in $\Gamma$.

Assume now node $y_n$ is a Type $a[8,3]$ node. Suppose $y_n$ is adjacent to $x$ and $u$. There is a contradiction to Lemma 8.6, irrespective of whether any of the nodes in $Y$ is adjacent to $i, j$ or $k$. Suppose $y_n$ is adjacent to $a$ and $b$. If any of the nodes in $Y$ is adjacent to $i, j$ or $k$, we have a violation of Lemma 8.6. Otherwise we have a Type 2 path.

Assume node $y_n$ is a Type $b[8,3]$ node adjacent to the neighbors of $v$ in $Q$ and $R$, say $q$ and $r$. Then no node in $Y$ is adjacent to $i$ or $j$, otherwise there is a $3PC(a, i)$ or $3PC(a, j)$. Now no node in $Y$ is adjacent to $k$, otherwise there is a $3PC(q, k)$. Thus path $Y$ is of Type 3.

Assume node $y_n$ is a Type $c[8,3]$ node adjacent to the neighbors of $v$ in $Q$ and $T$. This contradicts Lemma 8.8, irrespective of whether or not a node of $Y$ is adjacent to $i, j$ or $k$.

Finally, assume node $y_n$ is a Type $d[8,3]$ node. Irrespective of whether or not a node of $Y$ is adjacent to $i, j$ or $k$, there is a $3PC(y_n, x)$ if the neighbors of $y_n$ are in $V(Q) \cup V(R)$, and there is a $3PC(a, y_n)$ if they are in $V(P) \cup V(S)$.

\[ \square \]

**Lemma 8.10** Suppose $|T| = 1$ and let $y \in H(h)$. Every direct connection $Y = y_1, \ldots, y_n$ from $y$ to $V(\Gamma) \setminus \{h, i, j, v\}$ avoiding $Z(h) \setminus \{y\}$ is one of the following types, see Figure 29.

**Type 1** Node $y_n$ either has a unique neighbor in $(V(P) \cup V(S)) \setminus \{h, i, j\}$ or is a twin of a node in $(V(P) \cup V(S)) \setminus \{h, i, j\}$. Furthermore, if $y_n$ has a neighbor in $P$, then no node of $Y$ is adjacent to $j$ or $v$, and if $y_n$ has a neighbor in $S$, then no node of $Y$ is adjacent to $i$ or $v$.

**Type 2** Node $y_n$ is of Type $a[8,3]$, adjacent to $a$ and $b$, and no node of $Y$ is adjacent to $i, j$ or $v$.

**Type 3** Node $y_n$ is of Type $a[8,3]$, adjacent to $x$ and $u$, and no node of $Y$ is adjacent to $i, j$.

Node $y_n$ is the unique node of $Y$ adjacent to $v$.

**Proof:** There are two cases to consider.
Figure 29: Direct connections from $y \in H(h)$ when $|T| = 1$
Case 1 Node $y_h$ has only one neighbor $p$ in $V(\Gamma) \setminus \{h, i, j, v\}$.
Suppose $p \in (P \cup S) \setminus \{h, i, j\}$. W.l.o.g. assume $p \in P \setminus \{h\}$. If any of the nodes in $Y$ is adjacent to $j$ or $v$, there is a $3PC(a, j)$ or $3PC(a, v)$. Thus path $Y$ is of Type 1.
Suppose $p \in (Q \cup R) \setminus \{v\}$. W.l.o.g. assume $p \in Q \setminus \{v\}$. If $Y$ has a node adjacent to $i$ or $j$, there is a $3PC(a, i)$ or a $3PC(a, j)$. At least one node of $Y$ is adjacent to $v$, otherwise there is a parachute with long top if $p$ is adjacent to $v$ and a $3PC(h, x)$ if not. Now, exactly one node of $Y$ is adjacent to $v$, say $y_l$, and $p$ is not adjacent to $v$ since, otherwise, there would be a wheel with center $v$. Consider the parachute $\Pi_1$, with top $h, y, y_1, y, x$, side paths $P$ and $y, y, y, y_n, p, Q, a, x$, center node $v$ and middle path $v, R, b, x$. If $t \neq 1$, $\Pi_1$ has long top. If $t = 1$, applying Corollary 8.1, we have that $G$ contains a path $X = x_1, \ldots, x_m$ of Type $j[6.4]$ or $j[6.5]$ relative to $\Pi_1$ where $x_m$ is adjacent to a node of $R$ distinct from $v$ and its neighbor. No node of $X$ can be adjacent to a node of $S$ for otherwise there is a violation of Corollary 8.1 applied to the parachute $\Pi_2$ obtained from $\Pi_1$ by replacing the side path $P$ by $S$ and the bottom node $x$ by $u$. Now there is a $3PC(h, x)$.

Case 2 Node $y_h$ has at least two neighbors in $V(\Gamma) \setminus \{h, i, j, v\}$.
Assume first that node $y_h$ is a twin of a node $d$ of $\Gamma$. Let $\Gamma'$ be the goggles obtained from $\Gamma$ by substituting $y_h$ for $d$. If $n = 1$, node $y_l$ must be adjacent to $i, j$ or $v$, for otherwise $y$ is a strongly adjacent node violating Lemma 8.3 in $\Gamma'$. If $y_l$ is adjacent to $i$ or $j$, we have a Type 1 path. If $y_l$ is adjacent to $v$, assume w.l.o.g that $d \in Q$. Node $y$ is of Type c[8.3] in $\Gamma'$ and, by Lemma 8.8 applied to $\Gamma'$, there exists a path $X = x_1, \ldots, x_m$ of Type 2[8.8] from $y$ to $R$ such that $x_m$ has a neighbor $s$ in $R$ that is distinct from the neighbor of $v$ in $R$. Now if $d$ has no neighbor in $X$ let $H = v, Q, a, u, b, R, s, x_n, X, x_m, x_1, y, h, v$ and $(H, y_l)$ is a odd wheel. If $d$ has at least one neighbor in $X$, there is a $3PC(d, u)$. If $n \geq 2$, we can apply Case 1 or 2 to $\Gamma'$, so $Y$ is of Type 1.
Assume now node $y_h$ is a Type a[8.3] node. Suppose $y_h$ is adjacent to $x$ and $u$. No node of $Y$ is adjacent to $i$ or $j$ for otherwise there is $3PC(y_h, i)$ or a $3PC(y_h, j)$. Now a node of $Y$ must be adjacent to $v$ otherwise there is a $3PC(h, x)$. Furthermore, since there is no wheel, node $v$ has a unique neighbor, say $y_l$ in $Y$. If $t \neq 1$, there is a parachute with long top $h, y, y_1, y, y, y, y, y, y_n, p, Q, a, x$, center node $v$ and middle path $v, R, b, x$. Hence $t = 1$. Thus path $Y$ is of Type 3. Suppose $y_h$ is adjacent to $a$ and $b$. If any of the nodes in $Y$ is adjacent to $i, j$ or $v$, we have a violation of Lemma 8.6. Otherwise we have a Type 2 path.
By Lemma 8.7, node $y_h$ cannot be of Type b[8.3] since $|T| = 1$.
Node $y_h$ cannot be of Type c[8.3] since such nodes belong to $Z(h)$.
Assume node $y_h$ is a Type d[8.3] node. Irrespectively of whether or not a node of $Y$ is adjacent to $i, j$ or $v$, there is a $3PC(y_h, x)$ when the neighbors of $y_h$ are in $Q \cup R$ and there is a $3PC(a, y_h)$ when they are in $P \cup S$.

\[ \square \]

**Definition 8.11** If $|T| > 1$, let
\[ H_{PS}(h) = \{ y \in H(h) : \text{there is a Type 1 or Type 2[8.9] path from } y \text{ to } V(\Gamma) \}, \]

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\[ H_{QR}(h) = \{ y \in H(h) : \text{there is a Type 3[8.9] path from } y \text{ to } V(\Gamma) \}. \]

If \(|T| = 1\), let

\[ H_{PS}(h) = \{ y \in H(h) : \text{there is a Type 1[8.10] path from } y \text{ to } V(\Gamma) \} \cup \{ y \in H : \text{there is a Type 2[8.10] but no Type 3[8.10] path from } y \text{ to } V(\Gamma) \}, \]

\[ H_{QR}(h) = \{ y \in H(h) : \text{there is a Type 3[8.10] path from } y \text{ to } V(\Gamma) \}. \]

**Lemma 8.12** \( H_{PS}(h) \cap H_{QR}(h) = \emptyset \).

**Proof:** We consider two cases depending on the length of \( T \).

**Case 1** \(|T| > 1\).

Suppose the lemma is false, i.e. there exists \( y \) in \( H_{PS}(h) \cap H_{QR}(h) \). Let \( X = x_1, \ldots, x_n \) be a Type 1 or 2[8.9] path and \( Y = y_1, \ldots, y_m \) a Type 3[8.9] path.

In fact, \( X \) is a Type 1[8.9] path for, otherwise, there is a violation of Lemma 8.6 if a node of \( X \) is coincident with or adjacent to a node of \( Y \), and there is a 3PC\((a, y)\) if not.

W.l.o.g. assume that \( x_n \) is adjacent to \( p \in V(S) \setminus \{ h, j \} \). Note that by Lemma 8.9, no node in \( V(X) \cup V(Y) \) is adjacent to \( i \).

A node of \( X \) is coincident with or adjacent to a node of \( Y \), otherwise there is a 3PC\((a, y)\).

If \( y_m \) is a Type b[8.3] node adjacent to the neighbors of \( v \) in \( Q \) and \( R \), there are connected squares since by Lemma 8.9, no node of \( V(X) \cup V(Y) \) is adjacent to \( k \). So \( y_m \) is adjacent to \( V(T) \setminus \{ h, k \} \).

If \( y_m \) is a twin of node \( d \), then \( y_m \) has no neighbor in \( X \) and \( m > 1 \), else there is a wheel with center \( y_m \). Therefore \( m > 2 \) and we can replace \( d \in V(T) \) by \( y_m \) and \( Y \) by the shorter path \( y_1, \ldots, y_{m-1} \). By repeating this shortening argument if necessary, we can assume w.l.o.g. that \( y_m \) has a unique neighbor \( t \) in \( \Gamma \) and \( t \in V(T) \setminus \{ h, k \} \).

Now, if node \( k \) is not adjacent to any node in \( Y \), then there is a 3PC\((h, t)\) if \( t \in V^c \) and there are goggles with shorter top than \( \Gamma \) if \( t \in V^r \). So \( k \) has a neighbor in \( Y \). If some node of \( Y \setminus \{ y_m \} \) is coincident with or adjacent to a node of \( X \), there is a 3PC\((k, a)\). Otherwise, \( y_m \) is adjacent to a node of \( X \) but no other node of \( Y \) is, therefore there is a 3PC\((k, y_m)\) or a 3PC\((a, y_m)\).

**Case 2** \(|T| = 1\).

Suppose the lemma is false. By Definition 8.11, both a Type 1[8.10] direct connection \( X \) and a Type 3[8.10] direct connection \( Y \) exist, from \( y \). Let \( X = x_1, \ldots, x_n \) where \( x_n \) is adjacent to \( P \) and let \( Y = y_1, \ldots, y_m \) where \( y_m \) is adjacent to \( x \) and \( u \). If no node of \( X \) is adjacent to or coincident with a node of \( Y \), there is a 3PC\((y_m, y)\). If \( y_1 \) is the only node of \( Y \) adjacent to a node of \( X \), then there is a violation of Lemma 8.10, since \( y_1 \in N(x) \). So, some node of \( X \) is coincident with or adjacent to a node of \( V(Y) \setminus \{ y_1 \} \). By Lemma 8.6, \( x_n \) is a twin of \( x \) and it is the only node of \( X \) with a neighbor in \( V(Y) \setminus \{ y_1 \} \). Furthermore, \( x_n \) is not adjacent to \( y_m \) (else there is a violation of Lemma 8.6 in the goggles obtained from \( \Gamma \) by substituting \( x_n \) for \( x \)). Now there is a 3PC\((x_n, y_m)\). \( \square \)
8.5 Bicliques

Lemma 8.13 (i) Nodes $u, x, a, b$, their twins and the Type $a[8.3]$ nodes adjacent to $u$ and $x$ form a biclique.

(ii) Nodes $u, x, a, b$, their twins and the Type $a[8.3]$ nodes adjacent to $a$ and $b$ form a biclique.

(iii) When $|T| > 1$, nodes $u, x, a, b$, their twins and all Type $a[8.3]$ nodes form a biclique.

Proof: (i) If a twin of $x$ is not adjacent to a twin of $a$ or $b$, there is a $3PC(h, u)$. Let $w$ be a Type $a[8.3]$ node adjacent to $u$ and $x$ and let $Y$ be a path of Type $2[8.6]$. Suppose that $w$ is not adjacent to a twin $x^*$ of $x$. If $x^*$ has no neighbor in $Y$, there is a $3PC(h, u)$ and otherwise there is a $3PC(h, x^*)$.

(ii) follows by symmetry.

(iii) Let $w$ be a Type $a[8.3]$ node adjacent to $a$ and $b$, and let $y$ be a Type $a[8.3]$ node adjacent to $a$ and $x$. Suppose $w$ and $y$ are not adjacent. By Lemma 8.6, there exist a direct connection $X = x_1, \ldots, x_n$ of Type $2[8.6]$ from $w$ to $P$ or $S$, say $P$, and a direct connection $Y = y_1, \ldots, y_m$ of Type $2[8.6]$ from $y$ to $Q$ or $R$, say $Q$. No node of $Y$ is adjacent to a node of $X$, otherwise there is a $3PC(h, v)$. If $w$ is adjacent to a node in $Y$, there is a $3PC(h, w)$. If $y$ is adjacent to $X$, there is a $3PC(y, v)$. So no node of $Y \cup \{y\}$ is adjacent to a node of $X \cup \{w\}$. But now there is a $3PC(u, h)$. □

Lemma 8.14 (i) Nodes $h, i, j, k$ and their twins form a biclique.

(ii) Nodes $h, i, j$, their twins and the Type $b[8.3]$ nodes adjacent to $i$ and $j$ form a biclique.

(iii) Nodes $h, i, k$, their twins and the Type $c[8.3]$ nodes adjacent to $i$ and $k$ form a biclique.

(iv) When $|T| = 1$, the nodes $h, v$, their twins and all Type $c[8.3]$ nodes form a biclique.

Proof: (i) Suppose a twin $h^*$ of $h$ is not adjacent to a twin $i^*$ of $i$. Node $h^*$ is of Type $c[8.3]$ in the goggle $\Gamma$ obtained from $\Gamma$ by substituting $i^*$ for $i$. So, by Lemma 8.8, there exists a path $Y = y_1, \ldots, y_n$ of Type $2[8.8]$ from $h^*$ to $P$ such that $y_n$ is not adjacent to $i^*$ and is not a twin of $i^*$. Now there is a wheel with center $i$. By symmetry, $h^*$ is also adjacent to the twins of $j$. Now suppose $h^*$ is not adjacent to a twin $k^*$ of $k$. Node $h^*$ is of Type $b[8.3]$ in the goggle $\Gamma^*$ obtained from $\Gamma$ by substituting $k^*$ for $k$. So, by Lemma 8.7, there exists a path $Y = y_1, \ldots, y_n$ of Type $2[8.7]$ from $h^*$ to $T$ such that $y_n$ is not adjacent to $k^*$ and is not a twin of $k^*$. Now there is a wheel with center $k$.

To prove (ii), we show that any Type $b[8.3]$ node $w$ adjacent to $i$ and $j$ is also adjacent to all the twins of $i$ and $j$. Let $Y = y_1, \ldots, y_n$ be a path of Type $2[8.7]$ from $w$ to $T$ such that $y_n$ is not adjacent to $k$ and is not a twin of $k$. Suppose that $w$ is not adjacent to a twin $i^*$ of $i$. If $i^*$ has no neighbor in $Y$, there is a wheel with center $i$, and otherwise there is a $3PC(i^*, a)$. By symmetry, $w$ is also adjacent to the twins of $j$.

To prove (iii), consider a Type $c[8.3]$ node $w$ adjacent to $i$ and $k$, and let $Y = y_1, \ldots, y_n$ be a path of Type $2[8.8]$ from $w$ to $S$ such that $y_n$ is not adjacent to $j$ and is not a twin of $j$. If $w$ is not adjacent to a twin $i^*$ of $i$, there is a wheel with center $i$ if $i^*$ has no neighbor in $Y$, and a $3PC(i^*, a)$ otherwise. Similarly, if $w$ is not adjacent to a twin $k^*$ of $k$, there is a wheel with center $k$ if $k^*$ has no neighbor on $Y$, and a $3PC(k^*, a)$ otherwise.

Finally, (iv) follows from (iii) and Remark 8.5. □
The next result shows that, if \(|T| > 1\), then node \(h\) and its twins form a biclique with \(i, j\), their twins and the nodes in \(H_{PS}(h)\), or with \(k\), its twins and the nodes in \(H_{QR}(h)\).

For a twin \(h'\) of \(h\), the node sets \(H_{PS}(h')\) and \(H_{QR}(h')\) are subsets of \(H(h')\) defined as in Definition 8.11, but relative to the goggles \(\Gamma'\) obtained from \(\Gamma\) by substituting \(h'\) for \(h\).

**Lemma 8.15** If \(|T| > 1\), then either \(H_{PS}(h') = H_{PS}(h)\) for every twin \(h'\) of \(h\), or \(H_{QR}(h') = H_{QR}(h)\) for every twin \(h'\) of \(h\), or both.

**Proof:** Assume that \(h\) has a twin \(h^*\) such that \(H_{PS}(h^*) \neq H_{PS}(h)\). To prove the lemma, we will show that \(H_{QR}(h') = H_{QR}(h)\) for every twin \(h'\) of \(h\).

**Claim 1:** There exist a node \(y \in H_{PS}(h) \setminus H_{PS}(h^*)\) and a direct connection \(Y\) of Type 1 or 2[8,9] from \(y\) to \(\Gamma\) such that \(h^*\) has no neighbor in \(Y\) and there exist a node \(z \in H_{PS}(h^*) \setminus H_{PS}(h)\) and a direct connection \(Z\) of Type 1 or 2[8,9] from \(z\) to \(\Gamma\) such that \(h^*\) has no neighbor in \(Z\).

**Proof:** Among all possible choices of \(y \in H_{PS}(h) \setminus H_{PS}(h^*)\) and of direct connection \(Y\) of Type 1 or 2[8,9] from \(y\) to \(\Gamma\), choose \(y\) and \(Y = y_1, \ldots, y_m\) such that \(Y\) is shortest. We assume w.l.o.g. that \(y \in H_{PS}(h) \setminus H_{PS}(h^*)\). Next we show that \(h^*\) has no neighbor in \(V(Y) \cup \{y\}\). Assume otherwise. Then \(h^*\) has a unique neighbor \(y^*\) in \(Y \cup \{y\}\), else there is a wheel with center \(h^*\). Since \(y \notin H_{PS}(h^*)\), we have that \(y^* \neq y\). If \(y^* = y_m\), then by applying Lemma 8.3 to the goggles \(\Gamma^*\) obtained from \(\Gamma\) by substituting \(h^*\) for \(h\), we get that node \(y^*\) must be a twin of \(i\) or \(j\) in \(\Gamma^*\). Now, by Lemma 8.14, \(y^*\) is adjacent to \(h\), a contradiction. So \(y^* = y_j\), \(j < m\), and \(Y_{w_j y_m}\) is shorter than \(Y\), a contradiction to our choice of \(y\). So, \(h^*\) has no neighbor in \(V(Y) \cup \{y\}\).

When \(Y\) is of Type 1[8,9], assume w.l.o.g. that \(y_m\) has its neighbors in \(P\). Let \(X = h, y, \ldots, a\) be a shortest path from \(h\) to \(a\), whose intermediate nodes are the ones in \(Y\) and possibly part of \(P\). Consider the parachute \(\Pi\) whose top is \(j, h^*, k\), middle path \(X\) and side paths \(k, T_{k v} v, Q, a\) and \(j_s, w, a\). Applying Corollary 8.1 to \(\Pi\), we obtain a path \(Z^* = z, z_1, \ldots, z_n\) from \(h^*\) to \(X\). Let \(\Pi'\) be the parachute obtained from \(\Pi\) by substituting \(a\) with \(h\) and \(Q\) with \(R\). By applying Corollary 8.1 to \(\Pi'\), we obtain that \(Z^*\) has no neighbor in \(R\). So the only neighbors of \(V(Z^*) \setminus \{z_n\}\) in \(\Gamma\) may be in \(V(P) \setminus \{V(X) \cup \{h\}\}\) and the only neighbors of \(z_n\) in \(V(Y) \cup V(\Gamma)\) are in \(V(Y) \cup V(P) \setminus \{h\}\). So \(z \in H_{PS}(h^*) \setminus H_{PS}(h)\) and a subpath \(Z\) of \(V(Z^*)\) is the required direct connection. This proves Claim 1.

**Claim 2:** \(H_{QR}(h^*) = H_{QR}(h)\).

**Proof:** Among all possible choices of \(w \in H_{QR}(h) \setminus H_{QR}(h^*)\) and of direct connection \(W\) of Type 3[8,9] from \(w\) to \(\Gamma\), choose \(w\) and \(W = w_1, \ldots, w_p\) such that \(W\) is shortest. We assume w.l.o.g. \(w \in H_{QR}(h) \setminus H_{QR}(h^*)\). Next we show that \(h^*\) has no neighbor in \(W \cup \{w\}\). Assume otherwise. Then \(h^*\) has a unique neighbor \(w^*\) in \(W \cup \{w\}\), else there is a wheel with center \(h^*\). Since \(w \notin H_{QR}(h^*)\), we have that \(w^* \neq w\). If \(w^* = w_p\), then by applying Lemma 8.3 to the goggles \(\Gamma^*\) obtained from \(\Gamma\) by substituting \(h^*\) for \(h\), we get that node \(w^*\) must be a twin of \(k\) in \(\Gamma^*\). Now, by Lemma 8.14, \(w^*\) is adjacent to \(h\), a contradiction. So \(w^* = w_j\), \(j < p\), and \(W_{w_j w_p}\) is shorter than \(W\), a contradiction to our choice of \(w\). So, \(h^*\) has no neighbor in \(V(W) \cup \{w\}\).
By Claim 1, there exist \( z \in H_{PS}(h^*) \setminus H_{PS}(h) \) and a direct connection \( Z = z_1, \ldots, z_n \) of Type 1 or 2[8.9] from \( z \) to \( \Gamma \) such that \( h \) has no neighbor in \( Z \). No node of \( V(Z) \cup \{ z \} \) is adjacent to a node of \( V(W) \cup \{ w \} \) for the following reason.

- If some node of \((V(Z) \setminus \{ z_n \}) \cup \{ z \}\) has a neighbor in \( W \), then \( z \in H_{PS}(h^*) \cap H_{QR}(h^*) \), a contradiction to Lemma 8.12.
- If some node of \((V(W) \setminus \{ w_p \}) \cup \{ w \}\) has a neighbor in \( Z \), then \( w \in H_{PS}(h) \cap H_{QR}(h) \), a contradiction to Lemma 8.12.
- If \( z_n \) is adjacent to \( w_p \), node \( z_n \) cannot be of Type a[8.3], for otherwise there would be a violation of Lemma 8.6. So \( Z \) is of Type 1[8.9] and at least one neighbor of \( z_n \) is in \( V(P) \setminus \{ h, i \} \). If \( w_p \) is a Type b[8.3] node, then there is a \( 3PC(w_p, a) \). So all the neighbors of \( w_p \) are in \( T \). Node \( w_p \) has a unique neighbor \( \pi \) in \( T \), else there is a wheel with center \( w_p \). Node \( \pi \) is distinct from \( v \), else there is a \( 3PC(v, a) \). Now, there are goggles with shorter top \( T_{x_0} \) in \( V(\Gamma) \cup \{ w_p, z_n \} \), contradicting our choice of \( \Gamma \).

Now there is a \( 3PC(j, a) \) with the following paths.

\[
j, S_{j_0}, a, j, h, w, W, \ldots, Q, a \qquad j, h^*, z, Z, \ldots, a.
\]

This completes the proof of Claim 2.

Suppose now that \( H_{QR}(h') \neq H_{QR}(h) \) for some twin \( h' \) of \( h \). This implies \( H_{PS}(h') = H_{PS}(h) \) by Claim 2. In addition \( H_{QR}(h') \neq H_{QR}(h^*) \) implies \( H_{PS}(h') = H_{PS}(h^*) \). But this contradicts \( H_{PS}(h^*) \neq H_{PS}(h) \).

**Lemma 8.16** (i) If there exists a Type b[8.3] node adjacent to \( i \) and \( j \), denote by \( D \) the node set comprising \( i, j \), their twins and \( H_{PS}(h) \) and denote by \( F \) the node set comprising \( h \), its twins and the Type b[8.3] nodes adjacent to \( i \) and \( j \). Then \( D \cup F \) induces a biclique.

(ii) If there exists a Type c[8.3] node adjacent to \( i \) and \( k \) or if \( |T| = 1 \), denote by \( D \) the node set comprising \( h \), its twins and the Type c[8.3] nodes adjacent to \( i \) and \( k \) and denote by \( F \) the node set comprising \( k \), its twins, \( H_{QR}(h) \) and, when \( T = 1 \), the Type c[8.3] nodes adjacent to \( h \). Then \( D \cup F \) induces a biclique.

**Proof:** (i) By Lemma 8.14(ii), it suffices to show that \( y \in H_{PS}(h) \) is adjacent to any twin \( h^* \) of \( h \) and to any Type b[8.3] node \( w \) adjacent to \( i \) and \( j \).

By Lemma 8.7, \( |T| > 1 \) and there exists a Type 2[8.7] path \( X = x_1, \ldots, x_m \) from \( w \) to \( V(\Gamma) \setminus \{ h, i, j, k \} \). By Definition 8.11, there is a Type 1 or Type 2[8.9] path \( Y = y_1, \ldots, y_n \) from \( y \) to \( V(\Gamma) \setminus \{ h, i, j, k \} \). If \( Y \) is of Type 1[8.9], \( y_n \) is adjacent to \( a \) and \( b \), nodes \( y_a \) and \( x_m \) are not adjacent, otherwise there is a violation of Lemma 8.6. If \( Y \) is of Type 1[8.9], assume w.l.o.g. that \( y_h \) is adjacent to \( p \in V(S) \setminus \{ h, j \} \). Again, nodes \( y_h \) and \( x_m \) are not adjacent, otherwise there is a wheel with center \( x_m \) if \( x_m \) is a strongly adjacent node, or a \( 3PC(t, a) \) if \( x_m \) has a unique neighbor \( t \in V^c \) in \( \Gamma \). In both cases, no node of \( Y \cup \{ y \} \) is adjacent to a node of \( X \) and no node of \( Y \) is adjacent to \( w \), otherwise there is a violation of Lemma 8.12 or Lemma 8.7. Now there is a \( 3PC(a, i) \) unless \( w \) and \( y \) are adjacent.

Suppose now that \( y \) is not adjacent to a twin \( h^* \) of \( h \). No node of \( X \) and any node of any node of \( Z \) is adjacent to \( h^* \), otherwise there is a wheel with center \( h^* \). Now there is a \( 3PC(i, a) \)
whether or not \( h^* \) has a neighbor in \( Y \). Indeed, when \( h^* \) has no neighbor in \( Y \), the three paths are
\[
i, P_{ia} x, a \quad i, h^*, k, T_{ikv} v, Q, a \quad i, w, y, y, \ldots, u, a
\]
and when \( h^* \) has a neighbor \( y_j \) in \( Y \), the three paths are
\[
i, P_{ia} x, a \quad i, w, X, \ldots, v, Q, a \quad i, h^*, y, y_j, y, \ldots, u, a.
\]

(ii) Case 1 \( |T| > 1 \) and there is a Type c[8.3] node \( w \) adjacent to \( i \) and \( k \).

By Lemma 8.14(iii), it suffices to show that \( y \in H_{Q}(h) \) is adjacent to any twin \( h^* \) of \( h \) and to any Type c[8.3] node \( w \) adjacent to \( i \) and \( k \).

By Lemma 8.8, there exists a Type 2[8.8] path \( X = x_1, \ldots, x_m \) from \( w \) to \( V(\Gamma) \setminus \{h, i, j, k\} \).
By Definition 8.11, there is a Type 3[8.9] path \( Y = y_1, \ldots, y_n \) from \( y \) to \( V(\Gamma) \setminus \{h, i, j, k\} \).
If \( y_n \) has a unique neighbor \( p \in V(T) \setminus \{h, k\} \), nodes \( y_n \) and \( x_m \) are not adjacent for, otherwise, there are goggles with shorter top if \( p \in V^c \) and there is a 3PC \((a, p)\) if \( p \in V^c \).
If \( y_n \) is a Type b[8.3] node, \( y_n \) and \( x_m \) are not adjacent, otherwise there is a violation of Lemma 8.7.
In both cases, no node of \( V(Y) \cup \{y\} \) is adjacent to a node of \( X \) and no node of \( Y \) is adjacent to \( w \), otherwise there is a violation of Lemma 8.12 or Lemma 8.8.
Now there is a 3PC \((a, i)\) unless \( w \) and \( y \) are adjacent.

Suppose now that \( y \) is not adjacent to \( h^* \). No node of \( X \) is adjacent to \( h^* \), otherwise there is a wheel with center \( h^* \). Now there is a 3PC \((a, i)\) whether or not \( h^* \) is adjacent to a node of \( Y \).

Case 2 \( |T| = 1 \).

By Lemma 8.14(iv), it suffices to show that \( y \in H_{Q}(h) \) is adjacent to any twin \( h^* \) of \( h \) and to any Type c[8.3] node \( w \) adjacent to \( i \) and \( v \), if such a node exists.

By Definition 8.11, there is a Type 3[8.10] path \( Y = y_1, \ldots, y_n \) from \( y \) to \( V(\Gamma) \setminus \{h, i, j, v\} \).
Consider the parachute with top path \( h, y_1, v \), side paths \( P \) and \( y_1, Y, y_n, x \), center node \( v \) and middle path \( v, Q, a, x \). By Corollary 8.1, there must be a Type j[6.4] or Type j[6.5] direct connection \( X = x_1, \ldots, x_m \) from node \( y \) to a node of \( V(Q) \setminus \{v\} \).
No node of \( X \) is adjacent to a node of \( Y \) or of \( V(P) \setminus \{x\} \). Moreover, no node of \( X \) is adjacent to a node of \( V(S) \setminus \{u\} \), for otherwise a direct connection contradicting Corollary 8.1 would exist.
Finally, using the middle path \( v, R, b, x \) instead of \( v, Q, a, x \) above, and by Lemma 8.3, \( x_m \) must be a Type a[8.3] node adjacent to \( a \) and \( b \).

No node of \( Y \) is adjacent to \( h^* \), for otherwise \( y_n \) would violate Lemma 8.6.
Node \( x_m \) is not adjacent to \( h^* \), else there is a violation of Lemma 8.3 in the goggles \( \Gamma^* \) obtained from \( \Gamma \) by replacing \( h \) with \( h^* \).
Now, suppose \( y \) is not adjacent to \( h^* \). Then no node of \( X \) is adjacent to \( h^* \). For, otherwise, let \( x^* \) be the neighbor of \( h^* \) in \( X \), closest to \( x_m \).
Consider the hole \( H = y, y_1, Y, y_n, x, a, x_m, X_{x_{max}}, x^*, h^*, i, h, y \). Node \( v \) has three neighbors in \( H \), namely \( y_1 \), \( h \) and \( h^* \) and \( (H, v) \) is an odd wheel.
But then the nodes in \( X \cup \{y, y_1\} \) induce a direct connection from a Type a[8.3] node that violates Lemma 8.6 in \( \Gamma^* \).
So \( y \) is adjacent to \( h^* \).

Suppose \( y \) is not adjacent to a Type c[8.3] node \( w \). By Lemma 8.8, \( w \) has no neighbor on \( Y \). Consider the parachute with the top path \( h, y_1, v \), side paths \( h, S, a \) and \( y_1, Y, y_n, u \) and middle path \( v, Q, a, u \). Then the path \( v, w, i, h \) contradicts Lemma 8.2 with respect to that parachute. \( \Box \)
8.6 One More Lemma

**Lemma 8.17** Let \( Z \) be the node set comprising nodes \( i, j \), their twins and the nodes in \( H_{PS}(h) \). Let \( W \) be the node set comprising node \( k \), its twins, the nodes in \( H_{QR}(h) \) and, when \( |T| = 1 \), the Type c[8.3] nodes adjacent to \( h \). Finally, let \( S = (V(\Gamma) \cup N(\Gamma)) \setminus (Z \cup W) \). Then there exists no direct connection from \( Z \) to \( W \) avoiding \( S \).

**Proof:** Suppose the contrary. Then there exists a direct connection \( Y = y_1, \ldots, y_m \), from \( z \in Z \) to \( w \in W \) avoiding \( S \).

**Case 1** \( z \in H_{PS}(h) \) and \( w \in H_{QR}(h) \).

Since no node of \( Y \) belongs to the extended star \( Z(h) \) (the nodes of \( N(h) \), together with the nodes with at least two neighbors in \( \{i, j, k\} \)), there must be a direct connection \( X \) from \( Y \) to \( V(\Gamma) \setminus \{h, i, j, k\} \) avoiding \( Z(h) \). Now \( Y \cup X \) contains either a direct connection establishing \( z \in H_{QR}(h) \) or a direct connection establishing \( w \in H_{PS}(h) \), both contradicting Lemma 8.12.

**Case 2** \( z = i \) or \( j \) or a twin of \( i \) or \( j \), and \( w = k \) or a twin of \( k \) or, when \( |T| = 1 \), a Type c[8.3] node adjacent to \( h \).

If \( y_m \) is adjacent to a Type c[8.3] node adjacent to \( h \), then path \( Y \) contradicts Lemma 8.8. Now we claim that \( m \geq 2 \). When \( z = i \) or \( j \), and \( w = k \), this follows from the definition of \( S \). When \( z \) is a twin of \( i \) or \( j \), and \( w = k \), the claim follows from Lemma 8.14(iii) applied to the goggles obtained from \( \Gamma \) by replacing \( i \) by \( z \). Similarly, the claim follows when \( w \) is a twin of \( k \). Now, \( m \geq 2 \) implies that there is a parachute with long top \( Y \) and long sides, a contradiction.

**Case 3** \( z \in H_{PS}(h) \) and \( w = k \) or a twin of \( k \) or, when \( |T| = 1 \), a Type c[8.3] node adjacent to \( h \).

It follows from Case 2 that \( Y \) contains no node adjacent to \( i \) or \( j \). Since no node of \( Y \) belongs to the extended star \( Z(h) \), there must be a direct connection \( X \) from \( Y \) to \( V(\Gamma) \setminus \{h, i, j, k\} \) avoiding \( Z(h) \). By Lemma 8.9 or 8.10 and by Lemma 8.12, \( Y \cup X \) contains a direct connection of Type 1 or Type 2[8.9 or 8.10] for \( z \). Now there is a \( 3PC(w, a) \).

**Case 4** \( z = i \) or \( j \) or a twin of \( i \) or \( j \), and \( w \in H_{QR}(h) \).

It follows from Case 2 that \( Y \) contains no node adjacent to \( k \). Since no node of \( Y \) belongs to the extended star \( Z(h) \), there must be a direct connection \( X \) from \( Y \) to \( V(\Gamma) \setminus \{h, i, j, k\} \) avoiding \( Z(h) \). By Lemma 8.9 or 8.10 and by Lemma 8.12, \( Y \cup X \) contains a direct connection of Type 3[8.9 or 8.10] for \( w \). Now, there is a \( 3PC(z, a) \).

\( \square \)

8.7 2-Join Theorem

Now we prove the main result of this section.
Theorem 8.18 Suppose $G$ is a weakly balanced graph that contains goggles but no wheel, no connected squares and no extended star cutset. Then $G$ contains a 2-join.

Proof: Among the goggles of $G$, let $T$ be one with shortest top path $T$ and, subject to this condition, with the fewest number of nodes. By Lemmas 8.15 and 8.16(ii), $H_{PS}(h') = H_{PS}(h)$ or $H_{QR}(h') = H_{QR}(h)$ for every twin $h'$ of $h$. The proof will distinguish between these two cases. Note that, if there is a Type b[8.3] node adjacent to $i$ and $j$, then the first case always occurs. Indeed, by Lemma 8.16(i), each node in $H_{PS}(h)$ is adjacent to each twin $h'$ of $h$. Hence $H_{PS}(h) \subseteq H_{PS}(h')$ for each twin $h'$ of $h$. Replacing $h$ by $h'$ in the goggles yields the reverse inclusion. Similar arguments can be used to show that the second case always occurs when $|T| = 1$ or when there is a Type c[8.3] node adjacent to $k$. In both cases we define six disjoint sets $A, B, D, F, M$ and $N$ such that the nodes in $A \cup B$ induce a biclique $K_{AB}$ and the nodes in $D \cup F$ induce a biclique $K_{DF}$. We then prove that $E(K_{AB}) \cup E(K_{DF})$ is a 2-join of the graph, separating the nodes in $A \cup D \cup M$ from the nodes in $B \cup F \cup N$.

Let $U_{ab} = \{w \mid w$ is a Type a[8.3] node adjacent to $a$ and $b\}$. Similarly, let $U_{ux} = \{w \mid w$ is a Type a[8.3] node adjacent to $u$ and $x\}$. Let $U_1$ be the nodes in $U_{ab}$ that are adjacent to all nodes in $U_{ux}$ and let $U_2 = U_{ab} \setminus U_1$.

- The set $A$ comprises $x, u$, their twins and $U_1$.
- The set $B$ comprises $a, b$, their twins and $U_{ux}$.

By Lemma 8.13(i, ii), the nodes in $A \cup B$ induce a biclique $K_{AB}$. In addition, by Lemma 8.13(iii), $U_{ab} = U_1$ when $|T| > 1$.

Case 1 $H_{PS}(h') = H_{PS}(h)$ for every twin $h'$ of $h$ (if $h'$ exists). Furthermore, $|T| > 1$ and there exists no Type c[8.3] node adjacent to $k$.

- The set $D$ comprises $i, j$, their twins and $H_{PS}(h)$.
- The set $F$ comprises $h$, its twins and the Type b[8.3] nodes adjacent to $i$ and $j$.
- The set $M$ comprises the nodes in $P \cup S \setminus \{h, i, j, u, x\}$.
- The set $N$ comprises the nodes in $Q \cup R \cup T \setminus \{h, a, b\}$.

$D \cup F$ induces a biclique $K_{DF}$. Indeed, if there is a Type b[8.3] node, this follows from Lemma 8.16(i). Else, it follows from Lemma 8.14(i) and the fact that $H_{PS}(h') = H_{PS}(h)$ for every twin $h'$ of $h$.

Case 2 $H_{QR}(h') = H_{QR}(h)$ for every twin $h'$ of $h$ (if $h'$ exists). Furthermore, there exists no Type b[8.3] node adjacent to $i, j$.

By Remark 8.4, we may assume that no Type c[8.3] node is adjacent to $j$.

- The set $D$ comprises $h$, its twins and the Type c[8.3] nodes adjacent to $i$ and $k$.
- The set $F$ comprises $k$, its twins, $H_{QR}(h)$ and, when $|T| = 1$, the Type c[8.3] node adjacent to $h$.
- The set $M$ comprises the nodes in $P \cup S \setminus \{h, u, x\}$.
• The set $N$ comprises the nodes in $Q \cup R \cup T \setminus \{h, k, a, b\}$.

$D \cup F$ induces a biclique $K_{DF}$. Indeed, if there is a Type c[8,3] node adjacent to $k$ or if $|T| = 1$, this follows from Lemma 8.16(ii). Else, it follows from Lemma 8.14(i) and the fact that $H_{QR}(h') = H_{QR}(h)$ for every twin $h'$ of $h$.

Lemmas 8.15 and 8.16 show that the above two cases exhaust all possibilities.

If the theorem is false, there must be a direct connection $Y = y_1, \ldots, y_m$ between $A \cup D \cup M$ and $B \cup F \cup N$ in the partial graph $G \setminus (E(K_{AB}) \cup E(K_{DF}))$. Let $y_0$ be a neighbor of $y_1$ in $A \cup D \cup M$. W.l.o.g. choose $y_0$ in $M$ if possible, and if $y_1$ has no neighbor in $M$, choose $y_0$ in $V(T) \cap (A \cup D)$ if possible. Similarly, let $y_{m+1}$ be a neighbor of $y_m$ in $B \cup F \cup N$ and choose $y_{m+1} \in N$ if possible or, when $y_m$ has no neighbor in $N$, choose $y_{m+1}$ in $V(T) \cap (B \cup F)$ if possible. We show that such a direct connection $Y$ violates one of the lemmas proved in the earlier subsections.

**Claim 1:** $m \geq 2$

*Proof:* Assume $m = 1$. If $y_0 \in M$ and $y_2 \in N$, then $y_1$ violates Lemma 8.3. If $y_0 \in A$ or $y_2 \in B$, then $y_1$ violates Lemma 8.3 or $Y$ violates of Lemma 8.6. So, $y_0 \in D$ or $y_2 \in F$. Now again, $y_1$ violates Lemma 8.3 unless $y_0$ is of Type c[8,3] (in Case 2) or in $H_{PS}(h)$ (in Case 1) or $y_2$ is of Type b[8,3] (in Case 1) or in $H_{QR}(h)$ (in Case 2) or of Type c[8,3] adjacent to $h$ (in Case 2 and $|T| = 1$).

If $y_0$ or $y_2$ is of Type c[8,3], $Y$ contradicts Lemma 8.8. In particular, $Y$ is not of Type 3[8,8] since $|T| = 1$ implies that Case 2 occurs and, in this case, the Type c[8,3] nodes adjacent to $h$ belong to $F$, the Type c[8,3] nodes adjacent to $v$ belong to $D$ and all the edges between them are removed in the 2-join.

If $y_2$ is of Type b[8,3], $Y$ contradicts Lemma 8.7.

Finally, if $y_0 \in H_{PS}(h)$ or $y_2 \in H_{QR}(h)$, $Y$ contradicts Lemma 8.12. This proves Claim 1.

**Claim 2:** $y_1$ has at most one neighbor in $\Gamma$ or is a twin of a node in $M$, and $y_m$ has at most one neighbor in $\Gamma$ or is a twin of a node in $N$.

*Proof:* Suppose not. Then by Lemma 8.3, $y_1$ or $y_m$ is a node of Type a, b, c or d[8,3].

For $y_1$, the only possibility is Type d[8,3], by the definition of sets $A, B, D, F$. In this case, if $y_{m+1}$ is a twin of $h$ or a Type b[8,3] node adjacent to $i$ and $j$, there is a 3PC($y_1, y_{m+1}$). Otherwise, there is a 3PC($y_1, h$).

There are four possibilities for $y_m$.

If $y_m$ is of Type a[8,3], by Lemma 8.13(iii), $|T| = 1$, $y_m \in U_2$ and Case 2 applies. Furthermore, by Lemma 8.6, either $V(Y) \setminus \{y_m\}$ induces a Type 1 or 2[8,6] direct connection from $y_m$ to $V(\Gamma)$, or $y_0$ is a Type a[8,3] node in $A$. Let $z$ be a Type a[8,3] node adjacent to $u$ and $z$ but not to $y_m$. By Lemma 8.6, there exists a direct connection $Z = z_1, \ldots, z_p$ of Type 2[8,6] from $z$ to $V(Q) \cup V(R)$. Consider first the case where $V(Y) \setminus \{y_m\}$ induces a path of Type 2[8,6] from $y_m$ to $V(P) \cup V(S)$. Let $y = y_0$ or $y_1$, whichever belongs to $V$. If no node of $Z$ is adjacent to a node of $Z$, then there is a 3PC($x, y$). On the other hand, by Lemma 8.6 applied to $z$ and to $y_m$, the only adjacency between $Y$ and $Z$ is between $y_1$ and $z_p$. So $y_1$ and $z_p$ are adjacent. If $z_p \in V^c$, let $z^* = z_p$, and if $z_p \in V^c$, let $z^*$ be the unique
neighboring neighbor of \( z_p \) in \( V(Q) \cup V(R) \). Now \( y_0 = h \) or a twin of \( h \), and \( z^* = v \), otherwise there is a 3PC \((y^*, z^*)\). But this implies \( y_1 \in H_{QR}(h) \), a contradiction. Consider now the case where either \( y_0 \) is a Type a[8.3] node in \( A \) or \( V(Y) \setminus \{y_m\} \) induces a direct connection of Type 1[8.6] from \( y_m \) to \( u \) or \( x \) or a twin of \( u \) or \( x \), say \( y_0 = x \) or a twin of \( x \). No node of \( Z \) is adjacent to a node of \( Y \), otherwise there is a 3PC \((y_0, h)\) or there is a direct connection from \( y_m \) to \( V(Y) \) violating Lemma 8.6. Assume w.l.o.g. that \( z_p \) has a neighbor in \( R \). Hence we have a violation of Lemma 8.2 for the parachute with top path \( a, y_0, z \), middle node \( u \), bottom node \( v \) and the extra path is \( y_0, y_1, y, y_m, a \) or a subpath of it, in case \( Y \) contains a neighbor of \( z \). So \( y_m \) is not of Type a[8.3].

If \( y_m \) is of Type b[8.3], then its neighbors in \( \Gamma \) are the neighbors of \( v \) in \( Q \) and \( R \). By Lemma 8.7, \(|\Gamma| \geq 1 \) and \( y_1 \) is adjacent to a node \( y_0 \) in \( D \). So \( y_0 \in H_{PS}(h) \) (in Case 1) or \( y_0 = h \) or a twin of \( h \) (in Case 2). Now, in Case 1, the path \( Y \) shows that \( y_0 \in H_{QR}(h) \), a contradiction to Lemma 8.12. In Case 2, the path \( Y \) shows that \( y_1 \in H_{QR}(h) \) and therefore \( y_1 \in F \), a contradiction.

If \( y_m \) is of Type c[8.3], path \( Y \) contradicts Lemma 8.8.

If \( y_m \) is of Type d[8.3], there is a 3PC \((y_m, v)\). This completes the proof of Claim 2.

Claim 3: \( y_0 \in A \) and \( y_m+1 \in B \) cannot occur.

Proof: Suppose \( y_0 \in A \) and \( y_m+1 \in B \). If \( y_0 = u \) or \( x \), say \( x \), then by Claim 2, \( y_1 \) is not adjacent to \( u \). Similarly, if \( y_0 \) is a twin of \( u \) or \( x \), say a twin of \( x \), then \( y_1 \) cannot be adjacent to \( u \) or \( x \) by our choice of \( y_0 \). Now, the path \( Y \) contradicts Lemma 8.2 in a parachute with side paths \( P \), \( S \), bottom node \( h \) and top node \( y_m+1 \). So \( y_0 \) cannot be \( x \), \( u \) or one of their twins. Such a contradiction also occurs when \( y_m+1 \) is \( a, b \) or one of their twins. So \( y_0 \in U_{ab} \) and \( y_m+1 \in U_{ac} \). By Lemma 8.6, there exists a path \( Z = z_1, \ldots, z_p \) of Type 2[8.6] from \( y_m+1 \) to \( Q \) or \( R \), say \( Q \). Now no node of \( V(Y) \setminus \{y_0\} \) is adjacent to a node of \( Z \), otherwise there is a direct connection from \( y_0 \) to \( \Gamma \) violating Lemma 8.6. Now we have a violation of Lemma 8.2 for the following parachute. The top path is \( b, y_0, y_m+1 \), the side paths are \( R \) and \( y_m+1, Z, z_1, \ldots, v \), the center node is \( u \) and the middle path is \( a, S, h, T, v \). The extra path is \( Y \). This completes the proof of Claim 3.

Claim 4: In Case 1, \( y_0 \in D \) and \( y_m+1 \in F \) cannot occur.

Proof: Assume first \( y_m+1 \) is node \( h \) or a twin of \( h \). Since \( y_m \notin H_{PS}(h) \) and \( H_{PS}(h) = H_{PS}(y_m+1) \), it follows that \( y_m \in H_{QR}(y_m+1) \). Now, Lemma 8.17 applied to \( y_0 \) and \( y_m \) in the goggles \( \Gamma^* \) obtained from \( \Gamma \) by replacing \( h \) by \( y_m+1 \) (when \( y_m+1 \) is a twin of \( h \)) contradicts the existence of \( Y \). Indeed, by Lemma 8.14, \( y_0 \) belongs to the set \( Z \) relative to the goggles \( \Gamma^* \).

Now assume \( y_m+1 \) is of Type b[8.3]. By Lemma 8.7, \( y_0 \) is node \( i, j \) or one of their twins. W.l.o.g. \( y_0 = i \) or one of its twins. Then there is a violation of Lemma 8.2 in parachute with center node \( h \), top path \( i, y_m+1, y_0 \), where \( Y \) is the extra path. This completes the proof of Claim 4.

Claim 5: In Case 2, \( y_0 \in D \) and \( y_m+1 \in F \) cannot occur.

Proof: Assume first \( y_0 \) is node \( h \) or a twin of \( h \). Since \( y_1 \notin H_{QR}(h) \) and \( H_{QR}(h) = H_{QR}(y_0) \), it follows that \( y_1 \in H_{PS}(y_0) \). Now, \( Y \) contradicts Lemma 8.17 applied to \( y_1 \) and
in the goggles $\Gamma^*$ obtained from $\Gamma$ by replacing $h$ by $y_0$ (when $y_0$ is a twin of $h$). Indeed, by Lemma 8.14, $y_{m+1}$ belongs to the set $W$ relative to the goggles $\Gamma^*$.

Now assume $y_0$ is of Type $[8.3]$. By Lemma 8.8, $y_{m+1}$ is node $k$ or one of its twins, or a Type $[8.3]$ node adjacent to $h$. Then there is a violation of Lemma 8.2 in parachute with center node $h$, top path $i$, $y_0$, $y_{m+1}$, where $Y$ is the extra path. This completes the proof of Claim 5.

**Claim 6:** $y_0 \not\in A$.

**Proof:** Assume $y_0 \in A$. By Claim 3, $y_{m+1} \in F \cup N$. If $y_0 = u$, $x$ or one of their twins, there is a $3PC(y_0, h)$ if $y_{m+1}$ is not a twin of $h$ and a $3PC(y_0, y_{m+1})$ if $y_{m+1}$ is a twin of $h$. So consider $y_0 \in U_1$. In Case 1, by Lemma 8.6, the only possibility is that $y_{m+1} = h$ or a twin of $h$. So $y_{m} \in H_{PS}(y_{m+1})$, since $Y$ is a path of Type 2[8.9]. But then $y_{m} \in D$, a contradiction. In Case 2, by Lemma 8.6, the only possibility is that $y_{m+1} \in H_{QR}(h)$. Now, $Y$ is a Type 2[8.9] or 8.10 path from $y_{m+1}$. So, when $[T] > 1$, $y_{m+1} \in H_{PS}(h)$ and this contradicts Lemma 8.12. Now consider $[T] = 1$. Since $y_{m+1} \in H_{QR}(h)$, there exists a Type 3[8.10] path $X = x_1, \ldots, x_n$ from $y_{m+1}$ where $x_n$ is adjacent to $x$ and $u$. There is no adjacency between $V(Y) \cup \{y_0\}$ and $V(X) \setminus \{x_n\}$, else we get a violation of Lemma 8.6 for $y_0$. Now, nodes $x_n$ and $y_0$ are not adjacent, else there is a $3PC(x_n, y_{m+1})$. This contradicts the assumption $y_0 \in U_1$. This completes the proof of Claim 6.

**Claim 7:** $y_0 \not\in D$.

**Proof:** Assume $y_0 \in D$. By Claims 4 and 5, $y_{m+1} \in B \cup N$. In Case 1, $y_0 \in H_{PS}(h)$ or $y_0$ coincides with $i$ or $j$ or with a twin of $i$ or $j$. If $y_{m}$ is a twin of $k$ or $y_{m}$ has $k$ as its unique neighbor in $\Gamma$, the path $Y$ contradicts Lemma 8.17. Otherwise, if $y_0 = i$ or $j$ or a twin of $i$ or $j$, there is a $3PC(y_0, a)$, and if $y_0 \in H_{PS}(h)$, there is a violation of Lemma 8.12.

In Case 2, if $y_0$ is a Type $c[8.3]$ node adjacent to $i$ and $k$, there is a violation of Lemma 8.8. Now, suppose $y_0$ is $h$ or one of its twins. Then $y_0 \not\in F$ implies that $y_0 \in H_{PS}(h)$. $V(Y) \setminus \{y_0\}$ induces a direct connection from $y$ to $V(Y) \setminus \{h, i, j, k\}$ satisfying Lemma 8.9 or 8.10. By Lemma 8.12, this path cannot be of Type $3[8.9]$ or $[8.10]$. Therefore, it must be of Type 2[8.9] or 8.10 since $y_{m+1} \in B \cup N$. In fact, the only possibility is $y_{m} \in U_2$, but this was excluded in Claim 2. This completes the proof of Claim 7.

**Claim 8:** $y_0 \not\in M$.

**Proof:** Suppose $y_1$ is a twin of $p \in M$ or has a unique neighbor $y_0 = p \in M$. W.l.o.g. $y_0 \in P \setminus \{x, h\}$.

If $y_{m+1}$ is $a$, $b$ or one of their twins, there is a $3PC(y_{m+1}, v)$.

If $y_{m+1} \in U_{ax}$, there is a violation of Lemma 8.6.

In Case 1, if $y_{m+1}$ is $h$ or a twin of $h$, then $y_{m} \in H_{QR}(y_{m+1})$ since $y_{m} \not\in D$. But $y_{m} \in H_{PS}(y_{m+1})$ since $V(Y) \setminus \{y_{m}\}$ is of Type 1[8.9] for $y_{m}$. This contradicts Lemma 8.12. If $y_{m+1}$ is of Type $b[8.3]$, there is a contradiction of Lemma 8.7.

In Case 2, if $y_{m+1}$ is $k$, a twin of $k$, $y_{m+1} \in H_{QR}(h)$ or a Type $c[8.3]$ node adjacent to $h$, then Lemma 8.17 is violated when $p = i$, or there is a $3PC(y_{m+1}, a)$ otherwise.

Now, consider the case when $y_{m}$ is a twin of a node in $N$ (in this case, let $q = y_{m}$) or has a unique neighbor $y_{m+1} \in N$ (in this case, let $q = y_{m+1}$). If $q \in T$, there is a $3PC(q, a)$.
if \( q \in V^c \), there are goggles with a shorter top if \( q \in V^c \) and \( p \neq i \) and \( j \), and there is a \( 3PC(p, a) \) or \( 3PC(y_1, a) \) if \( q \in V^r \) and \( p = i \) or \( j \). If \( q \in Q \cup R \setminus \{v\} \), there is a \( 3PC(u, b) \) if \( q \) is not adjacent to \( v \) or a \( 3PC(x, q) \) otherwise.

This completes the proof of the theorem. \( \square \)

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References


