A Certain Series Associated with Catalan’s Constant

Victor S. Adamchik
Carnegie Mellon University

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Victor Adamchik
adamchik@cs.cmu.edu
Carnegie Mellon University

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Abstract

A parametric class of series generated by integration of complete elliptic integrals

\[ \sum_{k=0}^{\infty} \frac{\left( \frac{2k}{k} \right)^2}{(k + r) 16^k} \]

is evaluated in closed form. Alternative proofs to results of Ramanujan and others are given. A particular case of the Saalschutzian hypergeometric series \( {}_4F_3(1) \) is derived.

Keywords: summation, elliptic functions, hypergeometric functions, Catalan’s constant

AMS subject classification: 33C, 33E, 11Y
0. Preamble

The subject of our interest is the hypergeometric series generated by elliptic integrals

\[ S(r) = \sum_{k=0}^{\infty} \frac{\left( \frac{2k}{k+r} \right)^2}{(k+r)16^k} = \frac{1}{r} 3F_2 \left( \frac{1}{2}, \frac{1}{2}, r; 1, r+1; 1 \right) \]  

This series has a long and interesting story. About a century ago Ramanujan [1, p. 351 and 2, p. 39] in his first letter to Hardy stated without proof a particular case of (1), when the parameter \( r = n \) is a positive integer, namely

\[ S(n) = \frac{16^n}{\pi n^2} \left( \frac{2n}{n} \right)^2 \sum_{k=0}^{n-1} \frac{\left( \frac{2k}{k} \right)^2}{16^k} \]  

In 1927, when Ramanujan's collected papers were published and result (2) became publicly known, it attracted a great deal of attention. Different proofs were given by Watson [3] and Darling [4], later Bailey [5] and Hodgkinson [6] generalized (2) to

\[ 3F_2(a, b, c+n-1; c, a+b+n; 1) = \frac{\Gamma(n) \Gamma(a+b+n)}{\Gamma(a+n) \Gamma(b+n)} \sum_{k=0}^{n-1} \frac{(a)_k (b)_k}{k! (c)_k} \]  

which gives Ramanujan's result when \( a = b = \frac{1}{2} \) and \( c = 1 \). Ramanujan [7, pp. 237-239 and 2, p. 45] also stated a complementary formula to (2), when the parameter \( r = n + \frac{1}{2} \) is a half integer, namely

\[ S(n + \frac{1}{2}) = \frac{4}{\pi} \frac{\left( \frac{2n}{n} \right)^2}{16^n} \left( 2G + \sum_{k=0}^{n-1} \frac{16^k}{\left( \frac{2k}{k} \right)^2 (2k+1)^2} \right) \]  

Here \( G \) is Catalan's constant defined by

\[ G = \frac{1}{2} \int_{0}^{1} K(k) \, dk = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \]  

and \( K \) is the complete elliptic integral of the first kind, given by

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\[ K(k) = \int_{0}^{\frac{\pi}{2}} \frac{d \theta}{\sqrt{1 - k^2 \sin^2 \theta}} \]

As mentioned in [2, p. 47], Ramanujan’s proofs of formulas (2) and (4) most likely were based on the recurrence equation

\[
\left(r + \frac{1}{2}\right)^2 S(r + 1) - r^2 S(r) = \frac{1}{\pi} \quad (5)
\]

subject to initial conditions. This equation is derived from the fact that \( S(r) \) is generated by integration of complete elliptic integrals as

\[
S(r) = \frac{2}{\pi} \int_{0}^{1} z^{r-1} K(z) \, dz, \quad \text{Re}(r) > 0. \quad (6)
\]

In 1981, unawared of Ramanujan’s equation (5), Dutka [8] employed by (6) rediscovered formulas (2) and (4). In section 3 we outline the derivation of equation (5), as well as its solution. In view of (5), it’s pretty straightforward to see that for any rational \( r = n + p \), where \( n \) is a positive integer and \( 0 < p \leq 1 \), series (1) has a closed form representation

\[
S(n + p) = \frac{(p)_n^2}{(p + \frac{1}{2})_n^2} \left( S(p) + \frac{1}{\pi p^2} \sum_{k=0}^{n-1} \frac{(p + \frac{1}{2})_k^2}{(p + 1)_k^2} \right) \quad (7)
\]

Here \((p)_n = p(p + 1) ... (p + n - 1)\) is the Pochhammer symbol. There are only three known cases when the function \( S(p) \) is expressible in terms other than hypergeometric functions, namely \( p = 1, \frac{1}{2}, \frac{1}{4} \):

\[
S(1) = _3F_2\left(\frac{1}{2}, \frac{1}{2}, 1; 1, 2; 1\right) = _2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; 1\right) = \frac{4}{\pi}
\]

\[
S\left(\frac{1}{2}\right) = _3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; 1\right) = \frac{8G}{\pi}
\]

\[
S\left(\frac{1}{4}\right) = _3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}; 1, \frac{5}{4}; 1\right) = \frac{\Gamma\left(\frac{1}{4}\right)^4}{4\pi^2}
\]

where \( \Gamma(z) \) is the Euler gamma function. All these cases are due to Ramanujan. L.Glasser [9] made a conjection that it is possible to express \( S\left(\frac{1}{2^r}\right) \) for \( k \geq 3 \) in finite terms, however that is remained to be seen.

It does not appear to have been previously studied the case when the parameter \( r \) in (1) is a negative integer (assuming that the term \( r = -k \) is dropped from summation):
\[ S(r) = \sum_{k=0}^{\infty} \frac{\left( \frac{2k}{k} \right)^2}{(k + r) 16^k}. \] (8)

A few particular cases of (8) appeared in the handbooks by E.P. Adams and R.L. Hippisley [10] and by E.R. Hansen [11]:

\[ S(-1) = -\frac{2G + 1}{\pi} + \log(2) - \frac{1}{2} \]
\[ S(-2) = -\frac{18G + 13}{16\pi} + \frac{9}{16} \log(2) - \frac{21}{64} \]

In the present paper, using contour integration technique, we will show that for negative integer \( r \), sum (8) is solvable in closed form by

\[ S(r) = -S\left(\frac{1}{2} - r\right) + \frac{4}{16-r} \left( \frac{-2r}{r} \right)^2 (H_{-r} - H_{-2r} + \log 2), \quad r = 0, -1, -2, \ldots \] (9)

where \( H_n \) are the harmonic numbers \( H_n = \sum_{k=1}^{n} \frac{1}{k} \).

As a consequence of this result, in section 2, we derive the new representation for Saalschutzian \( _4F_3(1) \) series with special set of the parameters

\[ \left( n - \frac{1}{2} \right) _4F_3\left( 1, 1, n + \frac{1}{2}, n + \frac{1}{2}; 2, n + 1, n + 1; 1 \right) = \]
\[ \frac{4n^2}{2n-1} (H_{n-1} + \log 4) - \frac{16^n}{\left( \frac{2n}{n} \right)^2} _2F_2\left( \frac{1}{2}, \frac{1}{2}, n - \frac{1}{2}; 1, n + \frac{1}{2}; 1 \right) \] (10)

**1. Evaluation**

We consider two cases, when \( r \) is positive and negative. We denote \( S^+(r) = S(r) \) for \( \Re(r) > 0 \) and \( S^-(r) = S(r) \) for \( \Re(r) \leq 0 \).

Let \( r \) be a positive integer. We transform series (1) to a definite integral involving complete elliptic integrals. Multiplying the summand by \( x^{k+r} \) and differentiating it with respect to \( x \), we get

\[ g(r, x) = x^{r-1} \sum_{k=0}^{\infty} \left( \frac{2k}{k} \right)^2 \frac{x^k}{16^k} = \frac{2}{\pi} x^{r-1} K(x), \quad |x| < 1. \] (11)
where \(K(x)\) is the elliptic integral. Integrating both sides of (11), we arrive at

\[
S^+(r) = \int_0^1 g(r, x) \, dx = \frac{2}{\pi} \int_0^1 x^{r-1} K(x) \, dx, \quad Re(r) > 0.
\] (12)

In the next subsections we evaluate \(S^+(r)\), by first developing a recurrent equation for \(S^+(r)\), and then solving it by iteration. The result depends on the disparity of \(r\).

Now let us consider the second case when \(r\) is a negative integer. We split the series \(S^+(r)\) into two sums as

\[
S^-(r) = \sum_{k=0}^{\infty} \frac{(2k)^2}{(k + r) 16^k} = \left(\sum_{k=0}^{r-1} + \sum_{k=-r+1}^{\infty}\right) \frac{(2k)^2}{(k + r) 16^k}.
\]

Leaving the first sum unchanged, and converting the second sum into an elliptic integral (by applying the same reasoning as above), we obtain

\[
S^-(r) = \sum_{k=0}^{r-1} \frac{(2k)^2}{(k + r) 16^k} + \int_0^1 x^{r-1} \left(\frac{2}{\pi} K(x) - \sum_{k=0}^{r-1} \frac{(2k)^2}{16^k} x^k\right) \, dx,
\] (13)

\[Re(r) \leq 0\]

In subsection 1.3, using the contour integration technique, we establish a functional relation between \(S^-(r)\) into \(S^+(r)\).

1.1 \(S^+(r)\) for \(r\) a non-negative integer

Consider the system of indefinite integrals

\[
\begin{align*}
k_p(x) &= \int x^p K(x) \, dx \\
e_p(x) &= \int x^p E(x) \, dx
\end{align*}
\] (14)

where the parameter \(p\) is a positive integer or zero, and \(E(x)\) and \(K(x)\) are complete elliptic integrals. Using integration by parts, the above integral system can be reduced to the system of coupled recurrent equations

\[
\begin{align*}
k_p(x) &= x^p k_0(x) - 2 \, p \, (k_p(x) - k_{p-1}(x) + e_{p-1}(x)) \\
e_p(x) &= x^p e_0(x) - \frac{2}{3} \, p \, (e_{p-1}(x) + e_p(x) + k_p(x) - k_{p-1}(x))
\end{align*}
\]
with initial conditions

\[
\begin{align*}
2k_0(x) &= E(x) + (x - 1)K(x) \\
\frac{3}{2}e_0(x) &= (x + 1)E(x) + (x - 1)K(x)
\end{align*}
\]

Eliminating \(e_{p-1}(x)\) from the first equation, and \(k_{p-1}(x)\) and \(k_p(x)\) from the second, the system is simplified to

\[
\begin{align*}
k_p(x) &= \frac{4p^2}{(2p+1)^2}k_{p-1}(x) + \frac{2x^pE(x) + 2(2p+1)(x-1)x^pK(x)}{(2p+1)^2}, \quad p \geq 1 \\
e_p(x) &= \frac{4p^2}{(2p+1)(2p+3)}e_{p-1}(x) + \frac{2(1-2p+2p+1)x^pE(x) + 2(x-1)x^pK(x)}{(2p+1)(2p+3)}
\end{align*}
\]

Now we compute the values of \(k_p(x)\) and \(e_p(x)\) at the limiting points \(x = 0\) and \(x = 1\). We get two recurrent equations

\[
k_p(1) = \frac{4p^2}{(2p+1)^2}k_{p-1}(1) + \frac{2}{(2p+1)^2}, \quad p \geq 1
\]

\[
k_p(0) = 0, \quad p \geq 0
\]

\[
k_0(1) = 2.
\]

and

\[
e_p(1) = \frac{4p^2}{(2p+1)(2p+3)}e_{p-1}(1) + \frac{4}{(2p+1)(2p+3)}, \quad p \geq 1
\]

\[
e_p(0) = 0, \quad p \geq 0
\]

In view of formulas (12) and (15), we conclude that

\[
S^+(r) = \frac{2}{\pi}(k_{r-1}(1) - k_{r-1}(0)) = \frac{2}{\pi}k_{r-1}(1)
\]

where \(S^+(r)\) satisfies the recurrence relation

\[
\left( r + \frac{1}{2} \right)^2S^+(r + 1) - r^2S^+(r) = \frac{1}{\pi}, \quad r \geq 1
\]

\[
S^+(1) = \frac{4}{\pi}.
\]

Equation (18) can be solved by iteration (see section 3 for details). We have proven
Proposition 1.1 Let n be a positive even. Then S(n) defined by (1) evaluates to

\[ S(n) = \frac{16^n}{\pi n^2 \left(\frac{2n}{n}\right)^2} \sum_{k=0}^{n-1} \binom{2k}{k}^2 \frac{1}{16^k}. \]  \hfill (19)

1.2 \quad S^+(r) for r a positive half-integer

Consider slightly different (than (14)) system of indefinite integrals

\[ \begin{aligned}
\hat{k}_r(x) &= \int x^{p - \frac{1}{2}} K(x) \, dx \\
\hat{e}_r(x) &= \int x^{p - \frac{1}{2}} E(x) \, dx
\end{aligned} \]  \hfill (20)

where the parameter p is a positive integer or zero, and E(x) and K(x) are complete elliptic integrals. Using integration by parts, we transform (20) to the system of recurrent equations

\[ \begin{aligned}
p^2 \hat{k}_r(x) &= (p - \frac{1}{2})^2 \hat{k}_{p-1}(x) + \frac{1}{2} x^{p - \frac{3}{2}} \left( E(x) + 2 p (x - 1) K(x) \right) \\
p (p + 1) \hat{e}_r(x) &= (p - \frac{1}{2})^2 \hat{e}_{p-1}(x) + x^{p - \frac{3}{2}} \left( (p - 1) E(x) + \frac{x - 1}{2} K(x) \right) \end{aligned} \]  \hfill (21)

where

\[ \begin{aligned}
\hat{k}_0(x) &= \pi \sqrt{x} \, _3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; x\right) \\
\hat{e}_0(x) &= \pi \sqrt{x} \, _3F_2\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; x\right)
\end{aligned} \]

and \(_3F_2(x)\) is the hypergeometric function. By computing the limits at \( x = 0 \) and \( x = 1 \), system (21) yields (assuming \( p > 0 \))

\[ \begin{aligned}
\hat{k}_p(1) &= \left( p - \frac{1}{2} \right)^2 \hat{k}_{p-1}(1) + \frac{1}{2 p^2}, \quad p \geq 1 \\
\hat{k}_p(0) &= 0, \quad p \geq 0 \\
\hat{k}_0(1) &= 4 G
\end{aligned} \]  \hfill (22)

where G is Catalan's constant. Therefore,

\[ S^+(p + 1) = \frac{2}{\pi} \hat{k}_p(1), \quad p \geq 0 \]  \hfill (23)
The sequence $S^+(r)$, where $r$ is a positive half integer, satisfies the same recurrence equation (18), but with a different initial condition:

$$
\left(r + \frac{1}{2}\right)^2 S^+(r + 1) - r^2 S^+(r) = \frac{1}{\pi}
$$

$$
S^+\left(\frac{1}{2}\right) = \frac{8 G}{\pi}
$$

Solving this recurrence by iteration (see section 3 for details), we have proven

**Proposition 1.2** Let $n$ be a positive integer. Then $S(n + \frac{1}{2})$ defined by (1) evaluates to

$$
S(n + \frac{1}{2}) = \frac{4}{\pi} \left(\frac{2n}{n}\right)^2 \left(2 G + \sum_{k=0}^{n-1} \frac{16^k}{\left(\begin{array}{c} 2k \\ k \end{array}\right)^2 (2k + 1)^2}\right)
$$

### 1.3 $S^-(r)$, for $r$ a negative integer

Recall formula (13). Observing that the finite sum inside of the integrand

$$
\sum_{k=0}^{-r} \left(\begin{array}{c} 2k \\ k \end{array}\right)^2 \frac{x^k}{16^k}
$$

is the Taylor expansion of $\frac{2}{\pi} K(x)$ at $x = 0$, we pull that sum out of integration, by understanding integration in the Hadamard sense (finite part). Computing limits at the end points and obliterating logarithmic and polynomial order singularities, we get

$$
S^-(r) = f \cdot p \cdot \int_0^1 x^{-r-1} K(x) \, dx.
$$

Comparing this integral with formula (12) immediately implies that

$$
S^-(r) = S^+(r) + F(r)
$$

where $F(r)$ is an unknown function. The necessity of $F$ becomes obvious once we recall that in the original series we skip the term $k = -r$, when $r$ is a negative integer. In order to find $F$, we derive a contour integral representation for the sum $S(r)$ as

$$
S(r) = \frac{1}{2 \pi i} \int_{\gamma - i \infty}^{\gamma + i \infty} \frac{\Gamma(s) \, \Gamma\left(\frac{1}{2} - s\right)}{\Gamma(1 - s) \, \Gamma\left(\frac{1}{2} + s\right) \, (r - s)} \, ds.
$$
The contour $(\gamma - i \infty, \gamma + i \infty)$ is a straight line lying in the strip $0 < \gamma = \text{Re}(s) < \frac{1}{2}$. Integral (26) evaluates to (1) by summing residues at single poles $s = 0, -1, -2, \ldots$, lying to the left of the contour. However, if $r$ is a negative integer, the integrand in (26) has a double pole at $s = r$. According to the definition of $S(r)$ we must skip this pole. Thus, we have

$$S^-(r) = \frac{1}{2 \pi i} \int_{\gamma - i \infty}^{\gamma + i \infty} \frac{\Gamma(s) \Gamma(\frac{1}{2} - s)}{\Gamma(1 - s) \Gamma(\frac{1}{2} + s)} \frac{ds}{(r-s)} \quad (27)$$

As a matter of fact, the contour integral herein can also be computed via residues at the poles $s = \frac{1}{2}, \frac{3}{2}, \ldots$, lying to the right of the contour. Evaluating the integral via those poles allows us to avoid the double pole at $s = r$. This yields

$$\frac{1}{2 \pi i} \int_{\gamma - i \infty}^{\gamma + i \infty} \frac{\Gamma(s) \Gamma(\frac{1}{2} - s)}{\Gamma(1 - s) \Gamma(\frac{1}{2} + s)} \frac{ds}{(r-s)} =$$

$$- \sum_{k=0}^{\infty} \frac{(2k)!^2}{k!^4 (k - r + \frac{1}{2}) 16^k} = -S^+(\frac{1}{2} - r). \quad (28)$$

Finally, computing the residue

$$\text{res}_{s=r} \left( \frac{\Gamma(s) \Gamma(\frac{1}{2} - s)}{\Gamma(1 - s) \Gamma(\frac{1}{2} + s)} \frac{1}{(r-s)} \right) = \frac{4}{16^r} \left( -\frac{2}{r} \right)^2 \left( H_{-2r} - H_{-r} - \log 2 \right)$$

we establish

**Proposition 1.3** Let $r$ be a negative integer or zero. Then

$$S^-(r) = -S^+ \left( \frac{1}{2} - r \right) - \frac{4}{16^r} \left( -\frac{2}{r} \right)^2 \left( H_{-r} - H_{-2r} + \log 2 \right) \quad (29)$$

where $S^+ \left( \frac{1}{2} - r \right)$ is defined in Proposition 1.2.

### 1.4 $S^-(r)$ for $r$ a negative half integer

This case immediately follows from the previous subsection, taking into consideration that the integrand in (26) has only a single pole at $s = r$. 
Proposition 1.4 Let $n$ be a positive integer. Then

$$S^\ast(-n + \frac{1}{2}) = -S^\ast(n).$$

(30)

2. Special cases of hypergeometric functions

In this section we derive a particular case of the Saalschutzian hypergeometric series $\genfrac{[}{]}{0pt}{}{4}{3}H_1$. We begin by recalling that the hypergeometric series $\genfrac{[}{]}{0pt}{}{p+1}{p}F_p(a_0, a_1, ..., a_p; b_1, ..., b_p; 1)$ is called Saalschutzian if parameters $a_i$ and $b_i$ satisfy the relation

$$1 + a_1 + a_2 + ... + a_{p+1} = b_1 + ... + b_p$$

Proposition 2.1 Let $n$ be a positive integer. Then

$$\frac{(2n-1)^2}{8n^2} \genfrac{[}{]}{0pt}{}{4}{3}F_3\left(1, 1, n + \frac{1}{2}, n + \frac{1}{2}; 2, n + 1, n + 1; 1\right) =$$

$$-\frac{4G}{\pi} + H_{n-1} + \log 4 - \frac{2}{\pi} \sum_{k=0}^{n-2} \frac{16^k}{(2k+1)^2 \binom{2k}{k}^2}$$

(31)

where $G$ is Catalan’s constant, and $H_n$ are harmonic numbers.

Proof.

In view of formula (29) with $r = -n$, $n = 0, 1, 2, ...$, we have

$$S^\ast(-n) = -S^\ast(n + \frac{1}{2}) - \frac{4}{16\pi} \left(\frac{2n}{n}\right)^2 \left(H_n - H_{2n} + \log 2\right)$$

(32)

where $S^\ast(n + \frac{1}{2})$ is defined in (25). On the other hand, if we evaluate the original sum (8) by means of the hypergeometric function, we obtain

$$S^\ast(-n) = \sum_{k=0}^{n-1} \frac{\left(\frac{2k}{k}\right)^2}{(k-n) 16^k} +$$

$$\frac{(2n+2)^2}{n+1} \frac{1}{16^{n+1}} \genfrac{[}{]}{0pt}{}{4}{3}F_3\left(1, 1, n + \frac{3}{2}, n + \frac{3}{2}; 2, n + 2, n + 2; 1\right).$$

(33)
The finite sum in the right-hand side of (33) can be evaluated in terms of harmonic numbers (see Proposition 3.2) as

\[
16^n \sum_{k=0}^{n-1} \frac{\left( \frac{2}{k} \right)^2}{16^k (n-k)} = 4 \left( \frac{2}{n} \right)^2 \sum_{k=0}^{n-1} \frac{1}{2k+1} = 2 \left( \frac{2}{n} \right)^2 \left( 2H_{2n-1} - H_{n-1} \right) \tag{34}
\]

Combining formulas (32) and (33), and replacing \( n \) by \( n - 1 \), we arrive at (31).

**Remark.** By using different ideas, formula (31) was first proved in [13].

3. **Addendum**

In this section we provide a solution to equations (18) and (24)

**Proposition 3.1** The solutions to the recurrence relation

\[
(2n+1)^2 x_{n+1} - (2n)^2 x_n = a, \quad x_1 = b
\]

is

\[
x_n = \frac{16^n}{4n^2 \left( \frac{2}{n} \right)^2} \left( b + a \sum_{k=1}^{n-1} \frac{\left( \frac{2}{k} \right)^2}{16^k} \right) \tag{36}
\]

**Proof.**

We solve recurrence (35) by iteration. Iterating it \( n - 1 \) times, we get

\[
x_{n+1} = b \prod_{j=0}^{n-1} \frac{(2n-2j)^2}{(2n+1-2j)^2} + a \sum_{k=0}^{n-1} \frac{\prod_{j=0}^{k-1} (2n-2j)^2}{\prod_{j=0}^{k} (2n+1-2j)^2}. \tag{37}
\]

In pretty straightforward manner the finite products herein can be converted to the binomial coefficients by using Euler’s product representation for the Gamma function. We obtain

\[
\prod_{j=0}^{n-1} \frac{(2n-2j)}{(2n-2j+1)} = \frac{4^{n+1}}{2(n+1) \left( \frac{2n+2}{n+1} \right)}
\]
and
\[
\sum_{k=0}^{n-1} \prod_{j=0}^{k-1} \frac{(2n-2j)^2}{(2n-2j+1)^2} = \frac{16^{n+1}}{4(n+1)^2} \frac{16}{45} \sum_{k=1}^{n} \frac{(2k)^2}{16^k}.
\]

Substituting them into (37) yields the desired result.

**Proposition 3.2** Let \( n \) be a positive integer. Then
\[
\frac{16^n}{4 \left( \frac{2n}{n} \right)^2} \sum_{k=0}^{n-1} \frac{(2k)^2}{16^k (n-k)} = \sum_{k=0}^{n-1} \frac{1}{2k+1}.
\]

**Proof.**

We rearrange the terms in the sum in the left-hand side of (38), by summing them in the opposite order from \( n - 1 \) to 0. We get
\[
\sum_{k=0}^{n-1} \frac{(2k)^2}{(n-k) 16^k} = \sum_{k=1}^{n} \frac{(2n-2k)^2}{k 16^{n-k}}.
\]

Since the summand evaluates to zero for \( k > n \), we extend the range of summation to infinity. Using the definition of the hypergeometric series, we rewrite that sum in terms of \( 4F3 \) as
\[
\frac{16^n}{4 \left( \frac{2n}{n} \right)^2} \sum_{k=1}^{\infty} \frac{(2n-2k)^2}{k 16^{n-k}} \frac{1}{4F3} \left( 1, 1-n, 1-n; 2, \frac{3}{2}-n, \frac{3}{2}-n; 1 \right)
\]

The latter further simplifies to polygamma functions by formula 7.5.3.43 from [12] as
\[
\frac{2n^2}{(2n-1)^2} 4F3 \left( 1, 1-n, 1-n; 2, \frac{3}{2}-n, \frac{3}{2}-n; 1 \right) = \psi \left( n + \frac{1}{2} \right) - \psi \left( \frac{1}{2} \right) = \sum_{k=0}^{n-1} \frac{2}{2k+1}
\]

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