Graphs without Odd Holes, Parachutes or Proper Wheels:  
A Generalization of Meyniel Graphs  
and of Line Graphs of Bipartite Graphs

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Abstract

We prove that the strong perfect graph conjecture holds for graphs that do not contain parachutes or proper wheels. This is done by showing the following theorem:

If a graph $G$ contains no odd hole, no parachute and no proper wheel, then $G$ is bipartite or the line graph of a bipartite graph or $G$ contains a star cutset or an extended strong 2-join or $G$ is disconnected.

To prove this theorem, we prove two decomposition theorems which are interesting in their own rights. The first is a generalization of the Burlet-Fonlupt decomposition of Meyniel graphs by clique cutsets and amalgams. The second is a precursor of the recent decomposition theorem of Chudnovsky, Robertson, Seymour and Thomas for Berge graphs that contain a line graph of a bipartite subdivision of a 3-connected graph.

Key words: perfect graph, odd hole, strong perfect graph conjecture, decomposition, star cutset, 2-join, Meyniel graph, line graph of bipartite graph

Running head: WP-FREE GRAPHS

1 Introduction

A graph is perfect if, in all its induced subgraphs, the size of a largest clique is equal to the chromatic number. A hole is a chordless cycle of length at least four. A hole is odd (even) if it contains an odd (even) number of nodes. A long standing conjecture of Berge [1] states that a graph $G$ is perfect if and only if neither $G$ nor its complement contains an odd hole. (The complement $\bar{G}$ of $G$ has node set $V(G)$ and two nodes are adjacent in $\bar{G}$ if and only if they are not adjacent in $G$). Berge's conjecture is known as the Strong Perfect Graph Conjecture. It was proved recently by Chudnovsky, Robertson, Seymour and Thomas [3]. This conjecture was already known to hold for several special classes of perfect graphs.

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For example, Meyniel [13] showed that if every odd cycle of \( G \) is a triangle or contains at least two chords, then \( G \) is perfect. These graphs are known as Meyniel graphs.

Another well-known example is the following. A graph \( G \) is the line graph of a graph \( H \) if \( V(G) = E(H) \) and \( v_i, v_j \in V(G) \) are adjacent if \( e_i, e_j \in E(H) \) have a common endnode. If \( G \) is the line graph of a bipartite graph \( H \), then \( G \) is perfect. (Indeed, the maximum degree of a node in \( H \) is equal to the chromatic index of \( H \) and this implies that the chromatic number of \( G \) equals the size of its largest clique).

In this paper we introduce WP-free graphs (W stands for proper Wheel and P stands for Parachute: They will be defined later) and characterize the WP-free graphs that are perfect. Meyniel graphs and line graphs of bipartite graphs are perfect WP-free graphs.

WP-free graphs do not contain the complement of a hole \( H, |H| \geq 7 \). We show that if a WP-free graph contains no odd hole, then it is perfect. This is achieved by proving a structural theorem for even-signable WP-free graphs, a class of graphs that is larger than the class of WP-free graphs containing no odd hole. The proof of this theorem follows from two independent decomposition theorems, each interesting in its own right. The first is a generalization of the Burlet-Fonlupt decomposition of Meyniel graphs by clique cutsets and amalgams [2]. The second is a precursor of the recent decomposition theorem of Chudnovsky, Robertson, Seymour and Thomas [3] for Berge graphs that contain a line graph of a bipartite subdivision of a 3-connected graph.

### 1.1 Wheels, Parachutes and WP-Free Graphs.

A wheel \((H, v)\) consists of a hole \( H \) together with a node \( v \), called the center, that has at least three neighbors in \( H \). If \( v \) has exactly \( k \) neighbors in \( H \), the wheel is called a \( k \)-wheel.

**Definition 1.1** A T-wheel (or twin wheel) is a 3-wheel \((H, v)\) such that the three neighbors of \( v \) in \( H \) are consecutive.

A wheel \((H, v)\) is a \( \Delta \)-free wheel (or triangle-free wheel) if the neighbors of \( v \) in \( H \) induce a stable set. That is, the graph induced by \((H, v)\) is a triangle-free graph.

A wheel \((H, v)\) is a universal wheel if \( v \) is adjacent to every node of \( H \).

A wheel \((H, v)\) is an L-wheel (or line wheel) if \((H, v)\) is the line graph of a cycle \( C \) with a unique chord and \( V(C) \) induces a triangle-free graph, i.e. the unique chord of \( C \) is not a triangular chord. So \( v \) has neighbors \( a_1, a_2, b_1 \) and \( b_2 \) in \( H, H = a_1, P_1, b_1, b_2, P_2, a_2, a_1 \) and \( P_1, P_2 \) are paths of length greater than 1.

A wheel that is in none of the above four classes is called a proper wheel.

**Definition 1.2** An L-parachute \( LP(a_3b_1, a_2b_2, a_3, z) \) is a graph induced by an L-wheel \((H, a_3)\) where \( H = a_1, b_1, \ldots, z, \ldots, b_2, a_2, \ldots, a_1 \), where \( a_1, a_2, b_1, b_2 \) are the neighbors of \( a_3 \) in \( H \), together with a chordless path \( P = a_3, \ldots, z \) of length greater than 1. No node of \( H \setminus \{z, b_1\} \) may be adjacent to an intermediate node of \( P \).

A T-parachute \( TP(a_3, a_2b_1, b_2, z) \) is a graph induced by a T-wheel \((H, a_2)\) where \( H = b_1, a_1, b_2, \ldots, z, \ldots, b_1 \), where \( b_1, a_1, b_2 \) are the neighbors of \( a_2 \) in \( H \), together with a chordless path \( P = a_2, \ldots, z \) of length greater than 1. No node of \( H \setminus \{z, b_1\} \) may be adjacent to an intermediate node of \( P \).

A parachute is either an L-parachute or a T-parachute.
For an L-parachute or a T-parachute, let $P_1$, $P_2$ be respectively the $b_1$-path and the $b_2$-path in $H \setminus a_1$ and $C_1$, $C_2$ be the cycles induced by $P \cup P_1$ and $P \cup P_2$. Note that in a T-parachute or an L-parachute, the paths $P_1$ and $P_2$ may have length one.

In the definition below and throughout the rest of the paper, $G$ contains $G'$ if $G'$ is an induced subgraph of $G$ and $G$ is $G'$-free if $G$ does not contain $G'$.

**Definition 1.3** A graph is WP-free if it contains neither a proper wheel nor a parachute.

**Lemma 1.4** Let $G$ be an L-parachute LP$(a_1, b_1, a_2, b_2, a_3, z)$ with the property that no proper subgraph of $G$ is a parachute or a proper wheel. Then $G$ is of one of the following types, see Figure 1.

**Proof:** If no intermediate node of $P$ is adjacent to $b_1$ or $b_2$, $G$ is of type a). Suppose an intermediate node of $P$ is adjacent to $b_1$, and $b_1$ is not adjacent to $z$. If the neighbor of $a_3$ in $P$ is the only intermediate node of $P$ that is adjacent to $b_1$, there is a smaller proper wheel with center $a_2$. Otherwise there is a smaller L-parachute. So $b_1$ must be adjacent to $z$ and therefore $(C_2, b_1)$ is a wheel, which is not proper by assumption and is not universal since $b_1$ and $b_2$ are nonadjacent. So $(C_2, b_1)$ is either a $\Delta$-free wheel or a T-wheel or an L-wheel and we have types b) or c) or d) in these three cases. \hfill $\Box$

**Lemma 1.5** Let $G$ be a T-parachute TP$(a_1, a_2, b_1, b_2, z)$ that is not an L-parachute and such that no proper subgraph of $G$ is a parachute or a proper wheel. Then $G$ is one of the following graphs, see Figure 2.

**Proof:** If no intermediate node of $P$ is adjacent to $b_1$ or $b_2$ we have type a). Assume an intermediate node of $P$ is adjacent to $b_1$. Since no proper induced subgraph of $G$ is a parachute or a proper wheel, then $b_1$ is adjacent to $z$ and therefore $(C_2, b_1)$ is a wheel, which is not proper by assumption and is not universal since $b_1$ and $b_2$ are nonadjacent. If $(C_2, b_1)$ is an L-wheel then $G$ is also an L-parachute of type c. So $(C_2, b_1)$ is either a $\Delta$-free wheel or a T-wheel, and we have types b) or c). \hfill $\Box$
Figure 1: L-parachutes

Figure 2: T-parachutes
A cap is a cycle $C$ of length at least 5 with a unique chord that is a triangular chord of $C$. A cap is odd if $C$ is odd.

**Remark 1.6** A graph $G$ is Meyniel if and only if $G$ contains no odd hole and no odd cap.

**Proof:** $G$ is not a Meyniel graph if and only if $G$ contains an odd cycle that is not a triangle and has at most one chord. Let $C$ be a smallest such cycle. $C$ is either an odd hole or an odd cap. \qed

If $G$ contains a cap, $G$ contains an odd hole or an odd cap. So the class of cap-free graphs contains the class of Meyniel graphs. The structure of cap-free graphs is very similar to the structure of Meyniel graphs and was studied in [7]. Since every proper wheel and parachute contains a cap, the class of WP-free graphs contains the class of cap-free graphs.

A **diamond** is a cycle of length 4 with a unique chord. A **claw** is a graph on 4 nodes, one of them with degree 3 and the others with degree 1. The following characterization of the line graphs of bipartite graphs is due to Harary and Holtzmann [11]. It can be proven following the arguments of the proof of Remark 3.2.

**Remark 1.7** $G$ is the line graph of a bipartite graph if and only if $G$ contains no odd hole, no claw and no diamond.

It is straightforward to check that if $G$ is a proper wheel or a parachute, then $G$ contains a claw or a diamond. This implies the following remark:

**Remark 1.8** The class of WP-free graphs containing no odd hole includes the class of Meyniel graphs and the class of line graphs of bipartite graphs.

### 1.2 Even-Signable Graphs

We study even-signable WP-free graphs, a class of graphs that includes WP-free graphs containing no odd hole.

A graph $G$ is **signed** if its edges are given odd or even labels. A subset of $E(G)$ is odd (resp. even) if it contains an odd (resp. even) number of edges labeled odd. A graph $G$ is even-signable if there exists a signing of its edges such that every triangle is odd and every hole is even. These graphs were introduced in [6]. More results can be found in [7]. Note that, if $G$ contains no odd hole, then $G$ is even-signable since all its edges can be labeled odd. Also, if $G$ is triangle-free, then $G$ is even-signable since all its edges can be labeled even. It is shown in [7] that, if one can efficiently test whether $G$ is even-signable, then one can also efficiently test whether $G$ contains an odd hole.

The graphs in Figure 3 are relevant in this paper. Solid lines represent edges and dotted lines represent paths of length at least one. The first three graphs are referred to as 3-path configurations ($3PC$’s). The first graph is called a $3PC(x, y)$ (or $3PC(\cdot, \cdot)$), where node $x$ and node $y$ are connected by three paths $P_1$, $P_2$ and $P_3$. The second is called a $3PC(xyz, u)$ (or $3PC(\Delta, \cdot)$), where $xyz$ is a triangle and $P_1$, $P_2$ and $P_3$ are three paths with endnodes $x$, $y$ and $z$ respectively and a common endnode $u$. The third is called a $3PC(xyz, uvw)$ (or $3PC(\Delta, \Delta)$), consists of two node disjoint triangles $xyz$ and $uvw$ and paths $P_1$, $P_2$ and $P_3$.
Figure 3: 3-path configurations and wheel

with endnodes $x$ and $u$, $y$ and $v$ and $z$ and $w$ respectively. In all three cases, the nodes of $P_i \cup P_j$ induce a hole for $i \neq j$. This implies that all paths of a $3PC(\Delta, \cdot)$ have length greater than one, and at most one path of a $3PC(\Delta, \cdot)$ has length one.

A wheel $(H, v)$ is an odd wheel if it contains an odd number of triangles: Since $H$ is a hole, every triangle of $(H, v)$ contains $v$ and two adjacent nodes of $H$. So a wheel $(H, v)$ is odd if the subgraph of $H$, induced by the neighbors of $v$, contains an odd number of edges.

A consequence of a theorem of Truemper [14] is the following co-NP characterization of even-signable graphs.

**Theorem 1.9** A graph is even-signable if and only if it contains no $3PC(\Delta, \cdot)$ and no odd wheel.

A derivation of this result and a discussion of Truemper’s theorem can be found in [7] and [8]. We find it convenient to work with even-signable graphs because the graphs of Theorem 1.9 are easy to spot when proving results.

1.3 The Main Theorem

In a graph $G$, a node set $S$ is a cutset if the graph $G \setminus S$ is disconnected. A node set $S$ is a star if it consists of a node $x$ and neighbors of $x$. Chvátal [4] showed that a minimally imperfect graph cannot contain a star cutset.

A graph $G$ has an extended 2-join if $V(G)$ can be partitioned into subsets $V_A$, $V_B$ and $U$ ($U$ possibly empty), such that $A_1, A_2 \in V_A$, $B_1, B_2 \in V_B$ are nonempty disjoint sets with the following properties: (i) every node of $A_1$ is adjacent to every node of $B_1$, every node of $A_2$ is adjacent to every node of $B_2$ and these are the only adjacencies between $V_A$ and $V_B$, (ii) every node of $U$ is adjacent to $A_1 \cup A_2 \cup B_1 \cup B_2$ and possibly to other nodes in $V(G)$, (iii) the connected components of $G(V_A)$ meet both $A_1$ and $A_2$ and, if $|A_1| = |A_2| = 1$ then $V_A$ does not induce a chordless path and, (iv) the connected components of $G(V_B)$ meet both $B_1$ and $B_2$ and, if $|B_1| = |B_2| = 1$ then $V_B$ does not induce a chordless path.

An extended 2-join is called extended strong 2-join when, in addition, both $A_1 \cup B_1$, $A_2 \cup B_2$ induce cliques. When $U = \emptyset$, the extended 2-join reduces to the 2-join introduced by Cornuéjols and Cunningham [9].
In this paper, we prove the following result.

**Theorem 1.10** Let $G$ be an even-signable WP-free graph that is not a triangle-free graph nor the line graph of a triangle-free graph. Then $G$ contains a star cutset or an extended strong 2-join or $\tilde{G}$ is disconnected.

**Corollary 1.11** Let $G$ be a WP-free graph that contains no odd hole. Then $G$ is a bipartite graph or the line graph of a bipartite graph or $G$ contains a star cutset or an extended strong 2-join or $\tilde{G}$ is disconnected.

This result, together with the next two theorems, implies that the Strong Perfect Graph Conjecture holds for WP-free graphs.

**Theorem 1.12** [4] A minimally imperfect graph cannot contain a star cutset.

The following theorem follows from a result of Conforti, Cornu{é}jols, Gasparyan and Vuškovi{ć} [5] on universal 2-amalgams.

**Theorem 1.13** [5] A minimally imperfect graph cannot contain an extended strong 2-join.

**Theorem 1.14** A WP-free graph is perfect if and only if it contains no odd hole.

*Proof:* The “if” part is obvious. We prove the “only if” statement. Let $G$ be a minimally imperfect WP-free graph that contains no odd hole. Then $G$ is even-signable. By Theorem 1.12, $G$ does not contain a star cutset and by Theorem 1.13, $G$ does not contain an extended strong 2-join. Furthermore, $\tilde{G}$ is connected. Hence, by Corollary 1.11, $G$ is a bipartite graph or the line graph of a bipartite graph. In both cases $G$ is perfect, a contradiction. \qed

### 1.4 Proof Outline of the Main Theorem

A graph $G$ has an *amalgam* if $V(G)$ can be partitioned into subsets $V_A$, $V_B$ and $U$ ($U$ possibly empty), such that $A_1 \subset V_A$, $B_1 \subset V_B$ are nonempty sets with the following properties: (i) every node of $A_1$ is adjacent to every node of $B_1$ and these are the only adjacencies between $V_A$ and $V_B$, (ii) $U$ is a clique and every node of $U$ is adjacent to $A_1 \cup B_1$ and possibly to other nodes in $V(G)$, (iii) $|V_A| \geq 2$ and $|V_B| \geq 2$.

The notion of amalgam was introduced by Burlet and Fonlupt [2]. The *join* introduced by Cunningham and Edmonds [10] is an amalgam with $U = \emptyset$.

A node $u$ is universal for a graph $H$ if $u$ is adjacent to all the nodes in $H$.

Theorem 1.10 is in fact the consequence of the following stronger results.

**Theorem 1.15** Let $G$ be an even-signable WP-free graph that does not contain an L-wheel nor a $3PC(\Delta, \Delta)$. Then either $G$ is a triangle-free graph plus at most one universal node or $G$ contains a clique cutset or an amalgam.

This theorem is proved in Section 2.

**Theorem 1.16** Let $G$ be an even-signable WP-free graph that contains an L-wheel or a $3PC(\Delta, \Delta)$. Then either $G$ is the line graph of a triangle-free graph or $G$ contains a star cutset or an extended strong 2-join or $\tilde{G}$ is disconnected.

This theorem is proved in Section 3.
2 GM-graphs

**Definition 2.1** A graph $G$ is a GM-graph (Generalized Meyniel graph) if $G$ is an even-signable WP-free graph and $G$ does not contain an $L$-wheel or a $3PC(\Delta, \Delta)$.

In this section we prove Theorem 1.15 which states that every GM-graph $G$ is a triangle-free graph plus at most one universal node or $G$ contains a clique cutset or an amalgam. This theorem is interesting in its own right. Indeed, when specialized to Meyniel graphs, this result is a famous theorem of Burlet and Fonlupt [2]: every Meyniel graph $G$ is a bipartite graph plus at most one universal node or $G$ contains a clique cutset or an amalgam. In addition, Theorem 1.15 has algorithmic consequences that we do not develop in this paper.

We first introduce some definitions.

For $S \subseteq V(G)$, we let $G(S)$ be the subgraph of $G$ induced by the nodes in $S$. We let $N(S)$ denote the set of nodes with at least one neighbor in $S$. Two nodes $u$, $v$ are twins with respect to $S$ if $u$ and $v$ are adjacent and $N(u) \cap (S \setminus \{u, v\}) = N(v) \cap (S \setminus \{u, v\})$. If $u$ and $v$ are twins with respect to $V(G)$, we simply say that $u$ and $v$ are twins.

We denote a cap by $(H, x)$ where $H$ is a hole and $x$ is a node adjacent to consecutive node $a$, $b$ in $H$. The nodes $a$, $b$ are called the attachments of the cap.

Given three disjoint node sets $A$, $B$ and $C$ such that no node of $A$ is adjacent to a node of $B$, a direct connection between $A$ and $B$ is a minimal path $P$ (in terms of its node set) between a node in $A$ and a node in $B$. The direct connection $P$ avoids the set $C$ if no node of $P$ is in $C$.

We will need the following technical lemma about caps in GM-graphs.

**Lemma 2.2** Let $G$ be a GM-graph that contains no clique cutset but contains a cap $(H, x)$ with attachments $a$, $b$. Then $G$ has the following properties:

(i) In every direct connection $P = x_1, \ldots, x_n$ from $x$ to $V(H) \setminus \{a, b\}$ in $G \setminus (V(H) \cup \{x\})$, node $x_n$ is a universal node for $H$ or is a twin of $a$ or $b$ with respect to $H$.

(ii) Let $U$ be the set of universal nodes for $H$ that are endnodes of some such direct connection and let $T$ be the set of twins of $a$ or $b$ that are endnodes of some direct connection. Then $T$ is a clique, every node of $U$ is adjacent to every node of $T$ and $U$ contains two nonadjacent nodes $u$ and $u'$.

(iii) There exists a node $x'$ adjacent to $u$ and $u'$ such that $(H, x')$ is a cap with attachments $a$ and $b$.

**Proof:** Suppose that (i) does not hold. Among all caps $(Q, y)$ with attachments $\{a, b\}$ and direct connection $P = x_1, \ldots, x_n$ from $y$ to $V(Q) \setminus \{a, b\}$ in $G \setminus (V(Q) \cup \{y\})$, such that $x_n$ is neither a universal node for $Q$ nor a twin of $a$ or $b$ with respect to $Q$, choose $(Q, y)$ and $P$ such that $P$ is shortest possible. It follows from this choice of $(Q, y)$ and $P$ that no node $x_j$ with $j \leq n - 1$ is adjacent to both $a$ and $b$. Also, at least one of the nodes $a$, $b$ is not adjacent to any of the nodes $x_j$ for $2 \leq j \leq n - 1$ (otherwise $Q$ can be modified, $P$ shortened and (i) still does not hold). Assume w.l.o.g. that $b$ is not adjacent to any of the nodes $x_j$ for $2 \leq j \leq n - 1$. By construction, $x_n$ has at least one neighbor $z$ in $V(Q) \setminus \{a, b\}$. 

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Assume first that \(x_n\) has one or two neighbors in \(Q\). We only sketch the proof since checking the various cases is routine. If \(n = 1\), there is a \(3PC(\Delta, \cdot)\) or an odd wheel or a T-parachute or a \(3PC(\Delta, \Delta)\). So \(n \geq 2\). Since \(G\) does not contain an L-wheel or a \(3PC(\Delta, \Delta)\) or a \(3PC(\Delta, \cdot)\), it follows that \(a\) is adjacent to some node \(x_j\) for \(j \leq n - 1\) or \(b\) is adjacent to \(x_1\). Let \(S\) be the hole containing \(V(P) \cup \{b\}\) and possibly nodes of \((V(Q) \setminus \{a\}) \cup \{y\}\). Since \((S, a)\) is neither a proper wheel nor an L-wheel, either \(a\) or \(b\) is adjacent to \(x_1\). But now, there is a T-parachute or a \(3PC(\Delta, \cdot)\) or a proper wheel or a \(3PC(\Delta, \Delta)\), a contradiction.

So \(x_n\) has at least three neighbors in \(Q\). Assume that \(x_n\) is adjacent to at most one of the nodes \(a, b, \) and let \(S\) denote the hole with nodes in \(V(Q) \cup \{x_n\}\) that contains \(a, b\) and \(x_n\). If \(n \geq 2\), we have a contradiction to the choice of \((Q, y)\) and \(P\). If \(n = 1\), we have a T-parachute if \(x_1\) is adjacent to \(a\) or \(b\) and a proper wheel otherwise. So \(x_n\) is adjacent to both \(a\) and \(b\) and at least one other node of \(Q\). Since \((Q, x_n)\) is not a proper wheel nor a line wheel, \(x_n\) must be universal for \(Q\) or a twin of \(a\) or \(b\) with respect to \(Q\). This completes the proof of (i).

Suppose that (ii) does not hold. Let \(x_n\) and \(x_m'\) be the last nodes of direct connections \(P\) and \(P'\) where \(x_n \in T\) and \(x_m' \in T \cup U\) are not adjacent. Assume w.l.o.g. that \(x_n\) is a twin of \(b\) with respect to \(H\). If \(x_m'\) is a twin of \(a\), then \(V(H) \cup \{x_n, x_m'\}\) induces a T parachute, a contradiction. So we can assume w.l.o.g. that both \(x_n\) and \(x_m'\) are adjacent to \(a, b\) and the neighbor \(b'\) of \(b\) in \(V(H) \setminus \{a\}\).

If \(P\) and \(P'\) have no common node nor adjacent nodes, let \(C\) denote the hole induced by \(V(P) \cup V(P') \cup \{b', x\}\). Now \((C, b)\) is a proper wheel unless \(C\) is of length four, i.e. \(x\) is adjacent to \(x_n\) and \(x_m'\). But then there is a T-parachute induced by \((V(H) \setminus \{b\}) \cup \{x, x_n, x_m'\}\).

So \(P\) and \(P'\) have a common node or adjacent nodes. Let \(Q\) be a shortest path from \(x_n\) to \(x_m'\) in \(P \cup P'\). There is a T-parachute with top node \(b'\), side nodes \(x_n\) and \(x_m'\) and side paths contained in \(Q\).

So \(x_n\) and \(x_m'\) are adjacent. This shows that \(T\) is a clique and every node of \(T\) is adjacent to every node of \(U\). Since \(U \cup T\) is not a clique cutset separating \(x\) from \(V(H) \setminus \{a, b\}\), there must exist two nodes in \(U\) that are nonadjacent, say \(u\) and \(u'\). This completes the proof of (ii).

Now we prove (iii). Let \(P\) and \(P'\) be direct connections from \(x\) to \(V(H) \setminus \{a, b\}\) in \(G \setminus (V(H) \cup \{x\})\) that end in \(u\) and \(u'\) respectively.

If \(P\) and \(P'\) have no common node nor adjacent nodes, let \(C\) denote the hole induced by \(V(P) \cup V(P') \cup \{b', x\}\), where \(b'\) is the neighbor of \(b\) in \(V(H) \setminus \{a\}\). Since \((C, b)\) is not a proper wheel, \(b\) must be adjacent to every node of \(P\) and \(P'\). By symmetry, \(a\) is adjacent to every node of \(P\) and \(P'\). Since \((C, a)\) is not a proper wheel, it follows that \(C\) has length four. So (iii) holds in this case.

Now assume that \(P\) and \(P'\) have a common node or adjacent nodes. Let \(Q\) be a shortest path from \(u\) to \(u'\) in \(P \cup P'\), let \(C\) be the hole induced by \(V(Q) \cup \{u, u', b'\}\) and \(C'\) the hole induced by \(V(Q) \cup \{u, u', a'\}\) where \(a'\) is the neighbor of \(a\) in \(V(H) \setminus \{b\}\). If \(Q\) contains an intermediate node adjacent to \(b\), then \(Q\) has length two, otherwise \((C, b)\) or \((C', b)\) is a proper wheel. By symmetry, the same holds for \(a\). Furthermore, when \(Q\) has length two, the claim holds if its intermediate node is adjacent to both \(a\) and \(b\). So, whether \(Q\) has length two or not, we can assume w.l.o.g. that \(b\) is not adjacent to any intermediate node of \(Q\). Let \(M\) be a shortest path from \(b\) to \(Q\) in \(V(P) \cup V(P') \cup \{x\}\). Let \(m\) be the node of \(M\) adjacent to \(Q\). By
the choice of \( Q \), \( m \) has at most three neighbors in \( Q \). If \( m \) has two adjacent neighbors \( q_1, q_2 \) in \( Q \), there is a \( 3PC(mq_1q_2, b) \). So we can assume w.l.o.g. that \( m \) has only one neighbor \( z \) in \( Q \) since, otherwise we can modify \( Q \) to get the desired property. Now there is a parachute with side nodes \( u \) and \( u' \), side paths \( Q_{u,z} \) and \( Q_{u',z} \), top node \( b' \), center node \( b \) and middle path \( M \). This completes the proof of (iii). \( \square \)

2.1 D-structures

**Definition 2.3** A D-structure \((C_1, C_2, K)\) of \( G \) consists of disjoint sets of nodes \( C_1 \), \( C_2 \) and \( K \), where \( |C_1| \geq 2 \), \( |C_2| \geq 2 \) and the nodes of \( K \) induce a clique of \( G \) (possibly \( K \) is empty). Furthermore, the subgraph \( G(C_1) \) is connected and every node in \( C_1 \) is universal for \( C_2 \cup K \), every node in \( C_2 \) is universal for \( C_1 \cup K \) and there exists no node in \( V(G) \setminus (C_1 \cup C_2 \cup K) \) adjacent to a node in \( C_1 \) and a node in \( C_2 \).

This notion was introduced in [7], where it was shown that, if a cap-free graph \( G \) contains a D-structure, then \( G \) contains an amalgam. Here, we show the following result.

**Theorem 2.4** If \( G \) is a GM graph that contains a D structure, then \( G \) contains a clique cutset or an amalgam.

**Proof:** Let \( U \) be the set of nodes in \( V(G) \setminus (C_1 \cup C_2 \cup K) \) that are adjacent to \( C_1 \) and are connected to a node in \( C_2 \) by a path with nodes in \( V(G) \setminus (C_1 \cup K) \).

**Claim 1:** If \( G \) contains no clique cutset, every node in \( U \) is universal for \( C_1 \).

**Proof:** Assume not and choose \( u \in U \) contradicting the claim and \( c_2 \in C_2 \) connected by a shortest possible path with nodes in \( V(G) \setminus (C_1 \cup K) \) and among all these paths, let \( P = x_0 = u, x_1, \ldots, x_n, x_{n+1} = c_2 \) be one with the largest number of nodes adjacent to \( C_1 \). Since \( C_1 \) and \( C_2 \) belong to a D-structure, then \( n \geq 1 \). By our choice, intermediate nodes of \( P \) are either nonadjacent to \( C_1 \) or universal for \( C_1 \). Since \( u \) is adjacent but not universal to \( C_1 \) and \( G(C_1) \) is connected, \( C_1 \) contains adjacent nodes \( a, b \) such that \( u \) is adjacent to \( a \) but not to \( b \).

We now show that \( G(V(P) \cup \{a,b\}) \) contains a cap \((H,x)\) where \( H = a, x_i, P_{x_i,x_j}, x_j, a \). Assume that \( P \) contains consecutive nodes that are both adjacent to \( a \) and let \( x_i, x_{i+1} \) be such nodes with highest index. Then \( i < n \) by the definition of D-structure, so \( P_{x_i,x_{i+1}} \) contains a node adjacent to \( a \). Let \( x_j \) be such a node, of lowest index and let \( H = a, x_{i+1}, P_{x_{i+1},x_j}, x_j, a \). Now \((H,x_j)\) is a cap. If \( P \) does not contain consecutive nodes that are both adjacent to \( a \), let \( x_i \) be the node of lowest index \( i \geq 1 \) adjacent to \( a \) (and \( b \)) and let \( H = a, x_0, P_{x_0,x_i}, x_i, a \). Now \((H,b)\) is a cap.

Let \((H,x)\) be a cap where \( H = a, x_i, P_{x_i,x_j}, x_j, a \) and \( j \geq i + 2 \). Since \( G \) contains no clique cutset, by Lemma 2.2, \( G \) contains nonadjacent nodes \( z, z' \), universal for \( H \) (possibly adjacent to \( x \)) and, since \( K \) is a clique, at least one of these nodes, say \( z \), is not in \( K \). Now \( z \not\in C_1 \), since otherwise \( x_{j-1} \) is adjacent to \( z \in C_1 \) but not \( a \in C_1 \) and so, if \( j = n + 1 \), the definition of D-structure is contradicted, and if \( j \leq n \), the choice of \( u \) is contradicted. Furthermore \( x_i \) is adjacent to \( a \in C_1 \) and \( z \in C_2 \), a contradiction to the definition of D-structure.
So \( z \in V(G) \setminus (C_1 \cup C_2 \cup K) \). Now \( j = i + 2 \) and \( z \) is universal for \( C_1 \), else the minimality of \( P \) is contradicted. Let \( P' \) be obtained from \( P \) by removing \( x_{i+1} \) and adding \( z \). Now \( P \) and \( P' \) have the same length and \( P' \) contradicts our assumption that \( P \) has the largest number of neighbors in \( C_1 \). So this completes the proof of Claim 1.

Let \( K' \) contain the nodes in \( K \) that are not universal for \( U \) and \( K'' = K \setminus K' \). Define \( A = C_1, B = C_2 \cup K' \cup U \). We show that, if \( G \) contains no clique cutset, \( (A, B, K'') \) is an amalgam of \( G \). Claim 1 shows that every node in \( B \) is universal for \( A \) and by definition of \( K'' \), every node in \( K'' \) is universal for \( U \). Since \( (C_1, C_2, K) \) is a D-structure, every node in \( K'' \) is universal for \( C_1 \cup C_2 \cup K' \).

**Claim 2:** Let \( G' \) be the graph obtained from \( G \) by removing all edges with one endnode in \( A \) and the other in \( K' \). If \( G \) contains no clique cutset, in \( G'(V(G) \setminus (C_2 \cup K'' \cup U)) \) no path connects a node of \( K' \) and a node of \( C_1 \).

**Proof:** Let \( P = x, v_1, \ldots, v_p, k \) be a shortest path connecting \( x \in C_1 \) and \( k \in K' \) and contradicting the claim. No intermediate node of \( P \) is adjacent to a node in \( C_2 \) else, by the definition of \( U \), \( v_1 \) belongs to \( U \). If \( p \geq 2 \), let \( c_2 \) be any node in \( C_2 \) and \( H = k, x, v_1, \ldots, v_p, k \). Then \((H,c_2)\) is a cap and since \( G \) contains no clique cutset, by Lemma 2.2, \( G \) contains two nonadjacent nodes universal for \( H \) and one of them, say \( z \), is not in \( K \). Since \( v_1 \) is adjacent to \( x \in C_1 \) and \( z \) is not in \( C_2 \), \( z \notin C_1 \), else \( v_p \) is adjacent to \( z \in C_1 \) and \( k \) and \( P' = z, v_p, k \) contradicts the minimality of \( P \). Now since \( v_1 \in U \) and \( v_1 \) is adjacent to \( z \), \( z \) is also not in \( U \). So \( z \in V(G) \setminus (C_1 \cup C_2 \cup K \cup U) \) and \( P' = x, z, k \) again contradicts the minimality of \( P \).

So \( P = x, v_1, k \). Since \( k \) is not universal for \( U \), \( U \) contains a node not adjacent to \( k \). Let \( u \) be such a node, connected in \( G \setminus (C_1 \cup K) \) to a node of \( C_2 \), say \( c_2 \), by a shortest possible path and among these paths, let \( Q = x_1 = u, \ldots, x_m = c_2 \) have the largest number of neighbors of \( C_1 \). Note that \( Q \) contains several nodes that are universal for \( C_1 \), so let \( u_1, \ldots, u_n \) be such nodes of \( Q \), with \( u_t \) closer to \( u \) than \( u_{t+1} \) \((u_1 = x_1 = u \text{ and } u_n = x_m = c_2)\). Note that all nodes \( u_1, \ldots, u_{n-1} \) belong to \( U \).

We now show that no two consecutive nodes of \( Q \) are universal for \( C_1 \). For, let \( u_{i-1}, u_i \), be consecutive nodes of highest index. Note that \( i < n - 1 \) by the definition of \( D \)-structure. So let \( H = x, u_i, Q_{u_i, u_{i+1}}, u_{i+1}, x \), and \((H, u_{i-1})\) is a cap and again since \( G \) contains no clique cutset, by Lemma 2.2, there exists a node \( z \) not in \( K \) universal for \( H \). Since \( u_t \) is adjacent to \( x \in C_1 \) and \( z \), then \( z \notin C_2 \). Let \( x_j \) be the neighbor of \( u_{i+1} \) in \( Q_{u_i, u_{i+1}} \). Now \( z \notin C_1 \), else since \( x_j \) is adjacent to \( z \), then \( x_j \in U \) and, since \( x_j \) is not adjacent to \( x \), Claim 1 is contradicted. So since \( z \) is adjacent to \( x \) and to \( x_j \), then \( z \) is in \( U \). Now \( Q_{u_t u_{i+1}} \) has length 2 and \( z \) has no neighbor in \( V(Q) \setminus V(Q_{u_t u_{i+1}}) \) else the minimality of \( P \) is contradicted. Let \( P' \) be obtained from \( P \) by removing \( x_j \) and adding \( z \). Now \( P \) and \( P' \) have the same length and \( P' \) contradicts the fact that \( P \) has the largest number of neighbors in \( C_1 \). So no two consecutive nodes of \( Q \) are universal for \( C_1 \).

Let \( x_{i} \) be the node of smallest index adjacent to \( k \). Since by our choice, \( k \) is not adjacent to \( u_1 \) but is adjacent to all the nodes \( u_2, \ldots, u_n = x_m \), such a node exists and it belongs to \( Q_{x_{i+1} u_{i+1}} \) (possibly \( n = 2 \)). If \( x_i = u_2 \), let \( H = x, u_1, Q_{u_1 u_2}, u_2, x \), and \((H, k)\) is a cap. So by the same argument as above, there exists a node \( z \) not in \( K \) universal for \( H \). Again, the above argument rules out the existence of such a node \( z \) and so \( x_i \) is an intermediate node of \( Q_{u_1 u_2} \). Let \( H = x, u_1, Q_{u_1 x_i}, x_i, k, x \). Since \( v_1 \notin U \), \( v_1 \) is not adjacent to any node in
\( Q_u, x \). Now \((H, v)\) is a cap and so there exists a node \( z \) not in \( K \) universal for \( H \). Since \( x_1 = u \) is adjacent to \( x \in C_1 \) and \( z \notin C_2 \). Since \( x_2 \) is adjacent to \( z \) but not to \( x \in C_1 \), by Claim 1, \( z \notin C_1 \). So the same argument as above rules out the existence of such a node \( z \) when \( z \) is adjacent to \( x_1 \), \( x_2 \) and \( x_3 \). So \( i = 2 \) and \( z \) is adjacent to \( x_1 \), \( x_2 \) but not \( x_3 \). By Lemma 2.2(i), \( G \setminus \{ x, k \} \) contains a chordless path \( R = v_1 = r_1, \ldots, r_q = z \). Note that intermediate nodes of \( R \) may be adjacent to \( x \) or \( k \) but not to \( x_1 \) or \( x_2 \). At least one node of \( R \) belongs to \( C_1 \cup K \), otherwise there exists a path from \( v_1 \) to \( C_2 \) whose intermediate nodes are in \( V(R) \cup V(Q) \) and this path contains no node of \( C_1 \cup K \), thus proving that \( v_1 \in U \), a contradiction. So let \( r_i \) be the node of \( R \) with lowest index in \( C_1 \cup K \). Then \( r_i \) is adjacent to \( c_2 \). So let \( S = s_1 = v_1, \ldots, s_{n-1} = x_3, s_n = x_2 \) be a shortest \( v_1, x_2 \)-path whose nodes are in \( R_{r_1, r_i} \cup Q_{x_2, x_m} \). Since \( s_{n-1} \) is a direct connection from \( v_1 \) to \( H \), avoiding \( x \) and \( k \), by Lemma 2.2(i), \( x_3 \) must be a twin of node \( k \) with respect to \( H \) (indeed, \( x_3 \) is not adjacent to \( x_1 \), so it can be neither universal for \( H \) nor a twin of \( x \)). Now, by Lemma 2.2(ii), \( z \) is adjacent to \( x_3 \), a contradiction. This completes the proof of Claim 2.

The following claim shows that \((A, B, K''')\) is an amalgam of \( G \).

**Claim 3:** Let \( G'' \) be obtained from \( G \) by removing all edges with one endnode in \( A \) and the other in \( B \). Then in \( G''(V(G) \setminus K''') \), no path connects a node in \( A \) and a node in \( B \).

**Proof:** Let \( P = x_1, \ldots, x_n \) be a chordless path between \( x_1 \) in \( A \) and \( x_n \) in \( B \) and contradicting the claim. Claim 1 shows that if \( x_n \in C_2 \), then \( x_2 \in U \), a contradiction. Claim 2 shows \( x_n \notin K' \). So \( x_n \in U \) and let \( P_{x_n} \) be a path with nodes in \( V(G) \setminus (C_1 \cup K) \) connecting \( x_1 \) and a node in \( C_2 \). Now there is a path with nodes in \( V(G) \setminus (C_1 \cup K) \) between \( x_2 \) and a node in \( C_2 \) only using nodes of \( V(P_{x_n}) \cup V(P) \). So \( x_2 \) must belong to \( U \), a contradiction. \( \square \)

### 2.2 M-structures

M-structures were first introduced by Burlet and Foilupt [2] in their study of Meyniel graphs.

An induced subgraph \( G(V_i) \) of \( G \) is called an M-structure (multipartite structure) if \( \tilde{G}(V_i) \) contains at least two connected components each with at least two nodes. Let \( W_1, \ldots, W_k \) be the node sets of these connected components. The proper subclasses of \( G(V_i) \) are the sets \( W_i \) of cardinality greater than or equal to 2. The partition of an M-structure is denoted by \( (W_1, \ldots, W_r, K) \) where \( K \) is the union of all non-proper subclasses. Note that \( K \) induces a clique in \( G \).

**Lemma 2.5** An M-structure \( G(V_i) \) of \( G \) is maximal with respect to node inclusion, if and only if there exists no node \( v \in V(G) \setminus V_i \) such that \( v \) is universal for a proper subclass of \( G(V_i) \).

**Proof:** Let \( G(V_i \cup \{ u \}) \) be an M-structure. Assume node \( u \) is not universal for any proper subclass of \( G(V_i) \). In \( G(V_i \cup \{ u \}) \) node \( u \) is adjacent to at least one node in each of the proper subclasses. Thus there exists only one proper subclass in \( G(V_i \cup \{ u \}) \), contradicting the assumption.

Conversely let node \( u \) be universal for some proper subclass \( W_i \) of \( G(V_i) \). Then \( \tilde{G}(V_i \cup \{ u \}) \) has at least two components with more than one node, the graph induced by \( W_i \) and at least one component with more than one node in \( (V_i \cup \{ u \}) \setminus W_i \). \( \square \)
The above proof yields the following:

**Corollary 2.6** Let \( G(V_1) \) and \( G(V_2) \) be \( M \)-structures with \( V_1 \subseteq V_2 \). Let \( W_i \) and \( Z_j \) be connected components of \( G(V_1) \) and \( G(V_2) \) respectively having nonempty intersection. Then \( W_i \subseteq Z_j \).

**Lemma 2.7** Let \( G(V_1) \) be a maximal \( M \)-structure of a GM-graph \( G \) that has no clique cutset. Then no node in \( V(G) \setminus V_1 \) can be adjacent to two distinct proper subclasses of \( G(V_1) \).

**Proof:** Assume node \( x' \in V(G) \setminus V_1 \) is adjacent to two proper subclasses \( W_1 \) and \( W_2 \) of \( G(V_1) \). Since \( G(V_1) \) is maximal, by Lemma 2.5 node \( x' \) is not universal for either of the classes. Also since \( G(W_1) \) is connected, \( W_1 \) contains a pair of nonadjacent nodes \( x_1, y_1 \), such that \( x' \) is adjacent to \( x_1 \) but not to \( y_1 \). Similarly \( W_2 \) contains a pair of nonadjacent nodes \( x_2, y_2 \) such that \( x' \) is adjacent to \( x_2 \) but not to \( y_2 \). Let \( H = x_1, x_2, y_1, y_2, x_1 \). Then \( (H, x') \) is a cap. Since \( G \) has no clique cutset, by Lemma 2.2(iii), \( G \) contains a node \( x \) (possibly \( x' = x \)) adjacent to \( x_1, x_2 \) but not to \( y_1, y_2 \) and two nonadjacent nodes \( u, u' \) that are universal for the cap \( (H, x) \).

**Claim 1:** Nodes \( u \) and \( u' \) are universal for \( W_1 \) and \( W_2 \) and neither \( u \) nor \( u' \) is in \( W_1 \cup W_2 \).

**Proof:** Note first that the edges of \( G(V_1) \) that have their endnodes in \( \{x_1, y_1, x_2, y_2, u, u'\} \) are \( x_1y_1, x_2y_2, xy_1, xy_2 \) and \( uu' \). If the claim does not hold, then \( W_1 \) or \( W_2 \), say \( W_1 \), has the property that, in \( G \), \( u \) or \( u' \) has a neighbor in \( W_1 \), or \( u \) or \( u' \) is in \( W_1 \). In both cases, \( G(W_1 \cup \{u, u'\}) \) is connected. Consider a shortest path in this graph between \( x \) and \( u \) or \( u' \). W.l.o.g. let \( P = u, z_1, \ldots, z_n, x \) be such a path. Now if \( n = 1 \) and \( u' \) is adjacent to \( z_1 \) in \( G \), then \( G \) contains a triangle \( z_1, u, u' \) together with a chordless path \( z_1, x, y_2, x_2 \) and no other edge connects the triangle and the path. This is the complement of a \( T \)-parachute on six nodes. Otherwise, if \( n > 1 \) or \( u' \) is not adjacent to \( z_1 \) in \( G \), then \( u, z_1, \ldots, z_n, x, y_2, x_2 \) contains a chordless path of length five. Again, this is the complement of a \( T \)-parachute on six nodes and the proof of Claim 1 is complete.

So by Lemma 2.5, since \( u, u' \) are nonadjacent, they must belong to the same proper subclass of \( G(V_1) \), say \( W_3 \), which is distinct from \( W_1, W_2 \).

**Claim 2:** Node \( x \) is universal for \( W_3 \).

**Proof:** Assume not. Then \( G(W_3 \cup \{x\}) \) is connected. Let \( P = u, z_1, \ldots, z_n, x \) be a shortest path in this graph between \( u, u' \), say \( u \), and \( x \). Now the same proof as in Claim 1 shows the existence of a parachute.

So, by Lemma 2.5, \( x \) belongs to \( G(V_1) \). However, in \( G(V_1) \), \( x \) is adjacent to \( y_1 \in W_1 \) and \( y_2 \in W_2 \), a contradiction to the fact that \( W_1, W_2 \) are distinct proper subclasses of \( G(V_1) \).

**Theorem 2.8** If \( G \) is a GM-graph containing an \( M \)-structure either with at least three proper subclasses, or with at least one proper subclass which is not a stable set, then \( G \) contains a clique cutset or an amalgam.

**Proof:** If \( G \) contains a D-structure \((C_1, C_2, K)\) then, by Lemma 2.4, \( G \) contains an amalgam. So the theorem follows from the proof of the following statement:

If \( G \) is a GM-graph containing an \( M \)-structure either with at least three proper subclasses, or with at least one proper subclass which is not a stable set, then \( G \) contains a D-structure \((C_1, C_2, K)\).
Let $G(V_1)$ be an $M$-structure of $G$ satisfying the above property and $G(V_2)$ a maximal $M$-structure with $V_1 \subseteq V_2$.

**Claim 1:** The $M$-structure $G(V_2)$ either contains at least three proper subclasses or contains exactly two proper subclasses not both of which are stable sets.

**Proof:** If $G(V_1)$ has a proper subclass, say $W_i$, which is not a stable set, by Corollary 2.6, there exists a proper subclass, say $Z_j$ of $G(V_2)$ such that $W_i \subseteq Z_j$. Then $Z_j$ is not a stable set. If all proper subclasses of $G(V_1)$ are stable sets, then $G(V_1)$ has at least three proper subclasses say $W_1, W_2, \ldots, W_k$. If $G(V_2)$ has only two proper subclasses, say $Z_1, Z_2$, then by Corollary 2.6, we may assume w.l.o.g. that $W_1 \cup W_2 \subseteq Z_1$. Then $Z_1$ is not a stable set, since every node in $W_1$ is adjacent to a node in $W_2$. This completes the proof of Claim 1.

**Claim 2:** Suppose that $G(V_2)$ is a maximal $M$-structure of $G$ with partition $(W_1, W_2, K)$, where $W_1$ is not a stable set. Then $G$ contains a $D$-structure $(C_1, C_2, K)$.

**Proof:** Let $C_1$ be a connected component of $G(W_1)$ with more than one node. Let $C_2 = W_2$. Then $(C_1, C_2, K)$ is a $D$-structure, since by Lemma 2.7 no node of $V(G) \setminus V_2$ is adjacent to a node in $C_1$ and a node in $C_2$, and $|C_2| \geq 2$, since $W_2$ is a proper subclass of $G(V_2)$. This completes the proof of Claim 2.

**Claim 3:** Suppose that $G(V_2)$ is a maximal $M$-structure of $G$ with at least three proper subclasses. Then $G$ contains a $D$-structure $(C_1, C_2, K)$.

**Proof:** Let $W_1, W_2, \ldots, W_l$, $l \geq 3$ be the proper subclasses of $G(V_2)$ and let $K$ be the collection of all non-proper subclasses. Let $C_1$ be the nodes in two proper subclasses of $G(V_2)$ (note that $G(C_1)$ is a connected graph), $C_2$ be the nodes in all the other proper subclasses of $G(V_2)$. Then $(C_1, C_2, K)$ is a $D$-structure since $|C_1| \geq 2$, $|C_2| \geq 2$ and Lemma 2.7 shows that the only nodes having neighbors in both $C_1$ and $C_2$ belong to $K$. So the proof of Claim 3 is complete.

**Corollary 2.9** Let $G$ be a GM-graph that contains a cap. Then $G$ contains a clique cutset or an amalgam.

**Proof:** Assume $G$ contains a cap but no clique cutset. By Lemma 2.2, $G$ contains a cap $(H, x)$ and nonadjacent nodes $u, u'$ universal for $(H, x)$. Since $G(V(H) \cup x)$ is connected, $G$ contains an $M$-structure with proper subclasses $W_1 = \{u, u'\}$ and $W_2 = \{V(H) \cup x\}$ and $W_2$ is not a stable set. By Theorem 2.8, $G$ contains an amalgam. □

In [7], it was shown that, if $G$ is a cap-free graph, then $G$ contains an amalgam or $G$ is a triangle-free graph plus at most one universal node. Theorem 1.15 follows from this result and Corollary 2.9. Here, for the sake of completeness, we give a direct proof (without using [7]).

### 2.3 Expanded Holes

An expanded hole consists of nonempty sets of nodes $S_1, \ldots, S_n$, $n \geq 4$, not all singletons, such that, for all $1 \leq i \leq n$, the graphs $G(S_i)$ are connected. Furthermore, every $s_i \in S_i$ is adjacent to $s_j \in S_j$, $i \neq j$, if and only if $j = i + 1$ or $j = i - 1$ (modulo $n$).
Lemma 2.10 Let $G$ be a cap-free graph and let $H$ be a hole of $G$. If $s$ is a node having two adjacent neighbors in $H$, then either $s$ is universal for $H$ or $s$ together with $H$ induces an expanded hole.

Proof: Let $s$ be a node with two adjacent neighbors in $H$. If $s$ has no other neighbors on $H$, then $s$ induces a cap with $H$. Let $H = x_1, \ldots, x_n, x_1$ with node $s$ adjacent to $x_1$ and $x_n$. If $s$ is not universal for $H$, and does not induce an expanded hole together with $H$, then let $k$ be the smallest index for which $s$ is not adjacent to $x_k$. Let $l$ be the smallest index such that $l > k$ and $s$ is adjacent to $x_l$. Now node $x_{k-2}$ ($x_n$ if $k = 2$) together with the hole $s, x_{k-1}, \ldots, x_l, s$ forms a cap.

Lemma 2.11 Let $G$ be a cap-free graph and let $S = \bigcup_{i=1}^n S_i$, $n > 4$, be a maximal expanded hole in $G$ with respect to node inclusion. Either $G$ contains an $M$-structure with a proper subclass that is not a stable set of $G$, or all nodes that are adjacent to a node in $S_i$ and a node in $S_{i+1}$ ($S_{n+1} = S_1$) for some $i$, are universal for $S$ and induce a clique of $G$.

Proof: Let $u$ be a node adjacent to $s_1 \in S_1$ and $s_2 \in S_2$. By applying Lemma 2.10 to any hole that contains $s_1$ and $s_2$ and a node each from the sets $S_j$, $j > 2$, we have that $u$ is adjacent to all nodes in $S \setminus (S_1 \cup S_2)$, else the maximality of $S$ is contradicted. Now since node $u$ is adjacent to $s_1, s_2$ and is universal for all sets $S_j$, $j > 2$, Lemma 2.10 shows that $u$ is universal for $S_1$ and $S_2$, hence for $S$.

Let $u$ and $v$ be two nonadjacent nodes that are universal for $S$. Then $u, v$ together with $s_1 \in S_1$, $s_2 \in S_2$ and $s_4 \in S_4$ induces an $M$-structure with proper sets $W_1 = \{u, v\}$ and $W_2 = \{s_1, s_2, s_4\}$. Furthermore $W_2$ is not a stable set of $G$.

Theorem 2.12 A cap free graph that contains an expanded hole contains a clique cutset or an amalgam.

Proof: Let $S = \bigcup_{i=1}^n S_i$ be a maximal expanded hole in $G$. First assume that $n = 4$. Then the node set $S$ induces an $M$-structure with proper subclasses $S_1 \cup S_3$ and $S_2 \cup S_4$. $S_2 \cup S_4$ is not a stable set because, say, $|S_2| \geq 2$ and $G(S_2)$ is connected. Hence by Theorem 2.8 we are done. Now assume that $n > 4$. By Theorem 2.8 we may assume that $G$ does not contain an $M$-structure with a proper subclass that is not a stable set of $G$. By Theorem 2.4, it is sufficient to show that $G$ contains a D-structure $(C_1, C_2, K)$. Assume w.l.o.g. that $|S_2| \geq 2$ and let $K$ be the set of nodes that are universal for $S$. Lemma 2.11 shows that $K$ is a clique of $G$. Let $C_1 = S_2$ and $C_2 = S_1 \cup S_3$. Lemma 2.11 shows that every node that is adjacent to a node of $C_1$ and a node of $C_2$ is universal for $S$ and hence belongs to $K$. Therefore $(C_1, C_2, K)$ is a D-structure.

2.4 A Proof of Theorem 1.15

Now we are ready to prove Theorem 1.15.

Proof: If $G$ contains a cap, by Corollary 2.9, $G$ contains a clique cutset or an amalgam.

Assume that $G$ is a connected cap-free graph. If $G$ is a triangulated graph, $G$ is either a clique or it contains a clique cutset. If $G$ is a clique and contains at least four nodes, $G$
contains a join and if $G$ contains less than four nodes, then $G$ is a triangle-free graph plus at most one universal node.

Assume now that $G$ is a connected cap-free graph that contains a hole. Let $F$ be a maximal node set inducing a biconnected triangle-free subgraph of $G$. Assume that $G$ does not have a clique cutset or an amalgam.

**Claim 1:** Every node in $V(G) \setminus F$ that has at least two neighbors in $F$ is universal for $F$.

**Proof:** Let $u$ be a node in $V(G) \setminus F$ having at least two neighbors in $F$. The graph induced by $F \cup \{u\}$ contains a triangle $u, x, y$ else the maximality of $F$ is contradicted. Let $H$ be a hole in $G(F)$ containing $x$ and $y$. ($H$ exists since, by biconnectedness, $x$ and $y$ belong to a cycle and since $G(F)$ contains no triangle, a smallest cycle containing $x$ and $y$ is a hole). Lemma 2.10 shows that either $u$ is universal for $H$ or forms an expanded hole with $H$. Theorem 2.12 rules out the latter possibility. Let $F' \subseteq F$ be a maximal set of nodes such that $G(F')$ contains $H$, is biconnected and such that node $u$ is universal for $F'$. If $F \neq F'$, then since $G(F)$ and $G(F')$ are biconnected, some $z \in F \setminus F'$ belongs to a hole that contains an edge of $G(F')$. Let $H'$ be such a hole. By Lemma 2.10 and Theorem 2.12, node $u$ is adjacent to all the nodes of $H'$. Let $F'' = F' \cup V(H')$. $G(F'')$ is biconnected, $u$ is universal for $F''$. Hence $F''$ contradicts the maximality of $F'$. Hence $u$ is universal for $F$ and the proof of Claim 1 is complete.

**Claim 2:** Let $U$ be the set of universal nodes for $F$. Then the nodes in $U$ induce a clique of $G$.

**Proof:** Let $w, z \in U$ be two nonadjacent nodes of $U$ and let $v_1, \ldots, v_n, v_1$ be a hole of $G(F)$. Then nodes $w, z$ together with $v_1, v_2, v_3$ and $v_4$ induce an M-structure, either with two proper subclasses not both of which are stable if $v_1$ and $v_4$ are not adjacent, or with three proper subclasses. By Theorem 2.8, $G$ contains an amalgam. This completes the proof of Claim 2.

**Claim 3:** $V(G) = F \cup U$.

**Proof:** Let $S = V(G) \setminus (F \cup U)$. By Claim 1, every node in $S$ has at most one neighbor in $F$. Let $C$ be a connected component of $G(S)$. By maximality of $F$, there is at most one node in $F$, say $y$, that has a neighbor in $C$. If such a node $y$ exists, let $C_1, \ldots, C_l$ be the connected components of $G(S)$ adjacent to $y$. Let $V_1 = C_1 \cup \ldots C_l \cup \{y\}$, $A = \{y\}$, $K = U$, $V_2 = V(G) \setminus (V_1 \cup K)$ and $B$ be the set of neighbors of $y$ in $F$. Then $(A, B, K)$ is an amalgam of $G$, separating $V_1$ from $V_2$.

If no component of $G(S)$ is adjacent to a node of $F$, let $V_1 = U \cup S$, $A = U$, $V_2 = B = F$. Then $(A, B, \emptyset)$ is an amalgam of $G$. This completes the proof of Claim 3.

If $U$ contains at least two nodes, then let $V_1 = A = U$, $V_2 = B = F$ and $(A, B, \emptyset)$ is an amalgam of $G$. If $U$ contains at most one node, then $G$ is a triangle-free graph plus at most one universal node.

\[\square\]

3 Line graphs of triangle-free graphs and extensions

In this section, we prove Theorem 1.16.
3.1 L-graphs

If $G$ is the line graph of a graph $H$, the nodes in a maximal clique of $G$ correspond either to the edges in a triangle of $H$ or to the edges incident with a node of $H$.

A graph $G$ is $L\Delta$-free if $G$ is the line graph of a triangle-free simple graph. In this case, there obviously is a one to one correspondence between maximal stars of $H$ and maximal cliques of $G$.

Harary and Holtzmann [11] characterize the line graphs of bipartite simple graphs. In Remark 3.1 below we characterize $L\Delta$-free graphs in a similar way. The proof can be easily deduced from the proof of Remark 3.2.

Remark 3.1 The following three conditions are equivalent.
1) $G$ is $L\Delta$-free.
2) $G$ contains no claw and no diamond.
3) Every node $v \in V(G)$ belongs to at most two maximal cliques $C_1$ and $C_2$, and no node of $C_1 \setminus \{v\}$ is adjacent to a node in $C_2 \setminus \{v\}$.

Maffray and Reed [12] characterize the line graphs of bipartite multigraphs. The following remark has a similar proof and characterizes the line graphs of triangle-free multigraphs. A gem is a graph induced by a 5-cycle $a, b, c, d, e, a$ with chords $ac$ and $ad$.

Remark 3.2 The following three conditions are equivalent.
1) $G$ is the line graph of a triangle-free multigraph $H$.
2) $G$ contains no claw, no gem and no universal 4-wheel.
3) Every node $v \in V(G)$ belongs to at most two maximal cliques $C_1$ and $C_2$, and $C_1 \cap C_2$ consists of $v$ and all its twins. No node of $C_1 \setminus C_2$ is adjacent to a node in $C_2 \setminus C_1$.

Proof: Assume $G$ is the line graph of a triangle-free multigraph $H$. Since an edge of $H$ has at most two endnodes, $G$ is claw-free. Assume $G$ contains a gem $G'$ with $V(G') = \{a, b, c, d, e\}$ and $E(G') = \{ab, bc, cd, de, ea, ac, ad\}$. Since $\{b, c, a, d\}$ induce a diamond of $G$, the edges $e_a, e_c$ of $H$ corresponding to the nodes $a$ and $c$ of $G$ are parallel edges with endnodes $s, t$, while $c_0$ has $s$ but not $t$ as endnode and $c_d$ has $t$ but not $s$ as endnode. By the same argument applied to the diamond induced by $\{a, c, d, e\}$, $e_a, e_d$ are parallel, a contradiction. So $G$ cannot contain a gem. The same argument shows that $G$ cannot contain a universal 4-wheel and 1) $\rightarrow$ 2).

Assume that $G$ satisfies 2) and suppose first that $v$ belongs to three maximal cliques, $C_1, C_2, C_3$. Since every pair of cliques contains nonadjacent nodes, $C_1$ contains (possibly coincident) nodes $a_2, a_3$. $C_2$ contains (possibly coincident) nodes $b_1, b_3$ and $C_3$ contains (possibly coincident) nodes $c_1, c_2$ where $b_1$ and $c_1, a_2$ and $c_2, a_3$ and $b_3$ are nonadjacent. Together with $v$, these nodes induce a graph that contains a claw, a gem or an universal 4-wheel. For, choose $a_2, a_3, b_3, b_1, c_1, c_2$ so that they form the maximum number of coincident pairs. If all three pairs are coincident, there is a claw. If two of the pairs are coincident, there is a gem. Otherwise there is a universal 4-wheel. So every node of $G$ is in at most two maximal cliques. $C_1 \cap C_2$ obviously contains all twins of $v$. If a node in $C_1 \cap C_2$ is not a twin of $v$ then it belongs to three maximal cliques, a contradiction. Finally, if a node of $C_1 \setminus C_2$
is adjacent to a node in $C_2 \setminus C_1$, $v$ is in at least three maximal cliques, again a contradiction and $2) \rightarrow 3$).

Assume that $G$ satisfies $3)$ and construct $H$ as follows: $V(H)$ corresponds to the set of maximal cliques of $G$. For every node belonging to a unique maximal clique $C_1$, add to $H$ a pendant edge attached to the node $v_{C_1}$. For every node belonging to maximal cliques $C_1, C_2$, add to $H$ an edge with endnodes $v_{C_1}$ and $v_{C_2}$, so that the nodes in $C_1 \cap C_2$ are associated to parallel edges. Since no node of $C_1 \setminus C_2$ is adjacent to a node of $C_2 \setminus C_1$, $G$ is the line graph of $H$ and $H$ is a triangle-free multigraph. So $3) \rightarrow 1)$. 

Let $G$ be an $L\Delta$-free graph. Let $G'$ be an induced subgraph of $G$ and $K'$ be a clique of $G'$ with at least two nodes. Since $G$ contains no diamond by Remark 3.1, there exists a unique clique $K$ of $G$ containing $K'$. We say that $K$ is the extension of $K'$.

We say that a clique $K$ of $G$ is big if $K$ has more than two nodes and $K$ is flat if $K$ contains exactly two nodes. Unless otherwise specified, all the cliques will be maximal.

A connected graph $G$ has a 2-node cutset $\{u, v\}$ if $G \setminus \{u, v\}$ is a disconnected graph.

**Definition 3.3** A graph $G$ is an L-graph if it is an $L\Delta$-free graph and it satisfies the following properties.

- a) $G$ is connected, contains a big clique and every node of $G$ is in two cliques. (Equivalently, $H$ is connected, contains a node of degree at least 3 and every node has degree at least 2).

- b) $G$ contains no join. (Equivalently, $H$ contains no cutnode).

- c) For every 2-node cutset of $G$, one of the components is an induced path. (Equivalently, if $H$ contains two edges whose removal disconnects $H$, then one of the two components is a path).

It follows from this definition that if $G$ is an L-graph, then $G$ contains at least two big cliques. In fact, every hole of $G$ has at least two edges belonging to big cliques.

A segment $S$ of an L-graph $G$ is a maximal induced connected subgraph of $G$ such that no pair of nodes of $S$ belongs to the same big clique of $G$. Note that a segment is a chordless path of $G$ and may have length one or zero. Every node $x$ of $G$ is in exactly one segment, that we call $S_x$, so the segments of $G$ partition $V(G)$. A segment $S$ is long if $|V(S)| \geq 3$, short if $|V(S)| = 2$ and atomic if $|V(S)| = 1$. Furthermore if a segment $S$ is short and $K_x, K_y$ are the big cliques containing the endnodes of $S$, no atomic segment is in $K_x \cap K_y$ (i.e. $K_x \cap K_y$ is empty) since $G$ contains no diamond.

Every L-graph has at least three segments. If $G$ is a $3PC(\Delta, \Delta)$ or an L-wheel, then $G$ is an L-graph and $G$ is minimal with this property. These two graphs are called elementary L-graphs.

**Lemma 3.4** Let $S_1, S_2, S_3$ be three segments in an L-graph $G$. Then $G$ contains an elementary L-graph $B$, such that $S_1, S_2$ and $S_3$ are all in $B$ and $S_1, S_2$ are contained in distinct segments of $B$. 

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Proof: By Definition 3.3 b), $H$ is 2-connected and therefore $H$ contains a cycle going through any two given edges. This implies the existence of a hole in $G$ going through nodes $x_1 \in S_1$ and $x_2 \in S_2$. Since all cliques of $S_1$, $S_2$ are atomic or flat in $G$ and $S_1$, $S_2$ are maximal with this property, it follows that, in every hole $C = P_1, S_1, P_2, S_2$ of $G$ going through $x_1$ and $x_2$, at least one edge of $P_1$ and at least one edge of $P_2$ are extendable to big cliques of $G$.

Assume first that $G$ contains a hole $C$ going through $x_1$ and $x_2$ such that $S_3 \subseteq C$. Let $C = x_1, Q_1, x_2, Q_2, x_1$, where $P_1 \subseteq Q_1$ and $P_2 \subseteq Q_2$. By Definition 3.3 c), $Q_1 \setminus \{x_1, x_2\}$ and $Q_2 \setminus \{x_1, x_2\}$ are in the same connected component of $G \setminus \{x_1, x_2\}$, so they are connected by a shortest path $P$ in $G \setminus \{x_1, x_2\}$. Since every node of $C$ is in two cliques and the cliques of $S_1$, $S_2$ are atomic or flat in $G$, then $P = y_1, \ldots, y_m$ (possibly $m = 1$), where $y_1$ belongs to an extension of a clique in $P_1$ and $y_m$ belongs to an extension of a clique in $P_2$. If $m = 1$, $C \cup P$ induce an L-wheel and if $m > 1$, $C \cup P$ induce a $3PC(\Delta, \Delta)$.

Assume now that no hole $C$ going through $x_1$ and $x_2$ contains $S_3$. By Definition 3.3 b), c) $S_3$ belongs to a path $P - y_1, \ldots, y_m$ (possibly $m = 1$), where $y_1$ belongs to an extension of a clique of $P_1$ and $y_m$ belongs to an extension of a clique $P_2$. This shows that $C \cup P$ induce an elementary L-graph of $G$.

Lemma 3.5 Let $C$ be a hole of an L-graph $G$. For every segment $S_3$ of $G \setminus C$, there is a path $P$ in $G$ containing $S_3$ such that $C \cup P$ is an elementary L-graph $G_1$ in $G$. The segments of $G_1$ are $P$ and two subpaths of $C$.

Proof: The proof is identical to the previous one.

3.2 Tripods

A triad is a graph consisting of three internally node-disjoint paths $t, \ldots, x; t, \ldots, y$ and $t, \ldots, z$ of length greater than one, where $t$, $x$, $y$, $z$ are distinct nodes. Furthermore, the graph induced by the nodes of the triad contains no other edge than those of the three paths. Node $t$ is the meet of the triad.

A fan is a graph consisting of a path $P = x, \ldots, y$ of length greater than one, together with a node $z$ not in $P$ adjacent to at least one intermediate node in $P$ and not adjacent to $x$ and $y$. Node $z$ is the center of the fan and the edges connecting $z$ to $P$ are the spokes. Furthermore, the graph induced by the nodes of the fan contains no other edge than those of $P$ and the spokes.

A stool consists of a triangle $z' y' z'$ together with three node disjoint paths $x', \ldots, x; y', \ldots, y$ and $z', \ldots, z$ of length at least one. Furthermore, the graph induced by the nodes of the stool contains no other edges than those of the triangle and of the three paths.

A tripod is a triad or a stool or a fan. Nodes $x$, $y$ and $z$ are called the attachments of the tripod.

Lemma 3.6 Let $G$ be a node-minimal graph with the following properties.

(i) $G$ contains nodes $x$, $y$, $z$ such that no edge has both endnodes in $\{x, y, z\}$.
(ii) $V(G) \setminus \{x, y, z\}$ is nonempty.
(iii) $G$ and $G \setminus \{x, y, z\}$ are both connected.

Then $G$ is a tripod with attachments $x$, $y$ and $z$.
Proof: Let $G$ be a graph with the above properties and let $P_{xy} = x = y_1, \ldots, y_m = y$ be a shortest $xy$-path in $G \setminus \{z\}$, $P_x$ and $P_y$ similarly defined. Assume w.l.o.g. that $P_{xy}$ is not shorter than any of the other two. If $P_{xy}$ contains an intermediate node that is a neighbor of $z$, then by the minimality of $G$, $V(G) = V(P_{xy}) \cup \{z\}$ and $G$ is a fan.

Otherwise let $P = z_1, \ldots, z_n$ be a direct connection between $z$ and $V(P_{xy}) \setminus \{x, y\}$ ($P$ exists since $G$ and $G \setminus \{x, y, z\}$ are both connected), and let $P_z = z, z_1, \ldots, z_n$. By the minimality of $G$, $V(G) = V(P_{xy}) \cup V(P_z)$ and $z_n$ either has a unique neighbor in $P_{xy}$ or $z_n$ has two neighbors in $P_{xy}$, and these neighbors are adjacent.

By the minimality of $G$, at most one among $x$ and $y$ has a neighbor in $P_z$. Assume $x$ has a neighbor in $P_z$. Then by the minimality of $G$, $z_n$ is adjacent to the neighbor of $x$ in $P_{xy}$, possibly to $x$ and to no other node of $P_{xy}$. Now $P_z$ is longer than $P_{xy}$, contradicting our choice. So by symmetry, neither $x$ nor $y$ have a neighbor in $P_z$ and therefore if $z_n$ has two neighbors in $P_{xy}$, neither of these nodes is $x$ or $y$ and we have a stool in this case. Finally, if $z_n$ has a unique neighbor in $P_{xy}$, say $t$, then $t$ is not adjacent to $x$ or $y$ else our choice of $P_{xy}$ is again contradicted and in this case we have a triad.

\[\square\]

3.3 Links

Let $G$ be a graph that contains an L-wheel or a 3PC($\Delta, \Delta$).

Let $G'$ be an L-graph that is an induced subgraph of $G$. A link of $G'$ is a chordless path $P = x_1, \ldots, x_n$ in $G \setminus G'$ (possibly $n = 1$) such that $x_1$ has a neighbor $x_0$ in $G'$, $x_n$ has a neighbor $x_{n+1}$ in $G'$, and $x_0, x_{n+1}$ are nonadjacent nodes in distinct segments of $G'$. Furthermore $P$ is minimal with the above property.

Lemma 3.7 Let $G'$ be an L-graph that is an induced subgraph of a graph $G$. Let $U$ be a connected component of $G \setminus G'$ such that $N(U) \cap G'$ is not contained in a clique of $G'$ and is not contained in a segment of $G'$. Then

a) either $U$ contains a link, or

b) $N(U) \cap V(G') = \{x, y, z\}$ where $x$ and $y$ are the distinct endnodes of a long segment and $z$ is an atomic segment such that $z = K_x \cap K_y$, where $K_x$ and $K_y$ are the big cliques containing $x$ and $y$.

If $G$ is a WP-free graph, only a) can occur.

Proof: If $N(U) \cap G'$ contains two nodes that are nonadjacent and in distinct segments, then a) holds. So we may assume that this is not the case.

Since $N(U) \cap G'$ contains two nodes, say $x$ and $z$, that are in distinct segments, then $x$ and $z$ belong to some big clique $K_x$ of $G'$. Since $N(U) \cap G'$ contains a node $y$ not in $K_x$, by Remark 3.1 3) we can assume that $y$ is not adjacent to $x$. Since a) does not hold, the segments $S_x, S_y$ coincide, so $S_z, S_y$ are distinct. This implies that $z$ and $y$ belong to some big clique $K_y$ and $z = K_x \cap K_y$ is an atomic segment of $G'$. Now all the other nodes of $G'$ are readily seen to be nonadjacent to $U$. So b) holds.

Now, we show that if $G$ is a WP-free graph, only a) can occur. Assume now that b) holds and let $S_x$ be the long segment of $G'$ containing $x$ and $y$ and $S_z$ the atomic segment containing $z$. Since $G'$ is an L-graph, by Lemma 3.4, $G'$ contains an elementary L-graph $G''$.
containing $S_x$ and $S_z$ in distinct segments and $G''$ must be an L-wheel with atomic segment $S_z$.

Let $U'$ be the subgraph of $G$ induced by $V(U) \cup \{x, y, z\}$ with edges $xz, yz$ removed. By Lemma 3.6 $U'$ contains an induced subgraph $T$ which is a tripod with attachments $x$, $y$, $z$. We show that the graph $G'' \cup T$ contains a proper wheel or a parachute or a $3PC(\Delta, \Delta)$. Therefore if $G$ is a WP-free graph, b) cannot occur and a) is the only possibility.

If $T$ is a triad, then $G'' \cup T$ contains an L-parachute $LP(x'x, y' y, z, t)$, where $t$ is the meet of the triad.

If $T$ is a stool with triangle $abc$, then $G'' \cup T$ contains a $3PC(abc, z)$.

If $T$ is a fan with center $z$, then $G'' \cup T$ contains a proper wheel with center $z$.

If $T$ is a fan with center $x$ or $y$ but not $z$, say $x$, let $P_{yz}$ be a shortest $yz$-path in $T$ and $C_{yz}$ be the chordless cycle closed by edge $yz$ with $P_{yz}$. If $x$ has two neighbors in $C_{yz}$, then these neighbors are nonadjacent (since $T$ is not a fan with center $z$). So, in this case, we have an L-parachute of type a) $LP(x'x, y' y, z, t)$, where $t$ is the neighbor of $x$ distinct from $z$. Now assume that $x$ has at least three neighbors on $P_{yz}$. Since $(C_{yz}, x)$ is a wheel which is not proper, $(C_{yz}, x)$ is either a $\Delta$-free wheel or a T-wheel or an L-wheel. We then have an L-parachute $LP(x'x, y' y, z, t)$ where $t$ is the neighbor of $x$ closest to $y$ in $P_{yz}$. This L-parachute is of types b), c) or d). (Remark that in this proof we have not used the fact that $G$ contains no T-parachute).

So it is important to study the links of an L-graph $G'$ of $G$. The following lemma gives a list of all possible links, when $G'$ is an elementary L-graph.

**Lemma 3.8** Let $G$ be an even-signable WP-free graph, $G'$ an elementary L-graph in $G$ and $P = x_1, \ldots, x_n$ be a link of $G'$. Then

a) either $G' \cup P$ is an L-graph, or

b) $n = 1$ and $x_1$ is either universal for $G'$ or the twin of an endnode of a segment of $G'$.

**Proof:** We use the following notation: The two big cliques of $G'$ are $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. The segments of $G'$ are $P_1 = a_1, \ldots, b_1, P_2 = a_2, \ldots, b_2$ and $P_3 = a_3, \ldots, b_3$.

If $G'$ is an L-wheel, then $a_3 = b_3$ and the segment $P_3$ is atomic, while $P_1$ and $P_2$ are long segments. Otherwise $G'$ is a $3PC(\Delta, \Delta)$ and its segments are either long or short. The nodes in distinct segments $P_i$ and $P_j$ induce a hole of $G'$, that we denote with $H_{ij}$.

**Case 1** $n = 1$.

Let $x_0, x_2$ be neighbors of $x_1$ that are nonadjacent and in distinct segments, say $P_i$ and $P_j$, of $G'$. Then either $x_0, x_2$ are the unique neighbors of $x_1$ in $H_{ij}$ or $(H_{ij}, x_1)$ is a wheel.

**Case 1.1** $x_0, x_2$ are the unique neighbors of $x_1$ in $H_{ij}$, or $(H_{ij}, x_1)$ is a $\Delta$-free wheel.

**Case 1.1.1** $G'$ is a $3PC(\Delta, \Delta)$.

Assume w.l.o.g. that $i = 1$ and $j = 3$. Now $x_1$ has more that two neighbors in $G'$, else we have a $3PC(A, x_0)$ or a $3PC(B, x_0)$. If $x_1$ has at most one neighbor in $A$ and at most one neighbor in $B$, we have a $3PC(A, x_1)$ or a $3PC(B, x_1)$. Since no two neighbors of $x_1$ in $H_{13}$ are adjacent, then $x_1$ cannot be adjacent to both $a_1$ and $a_3$ or both $b_1$ and $b_3$, so by
symmetry we may assume that \( x_1 \) is adjacent to \( a_2, a_3 \) but not to \( a_1 \). Let \( x_0 \) be the neighbor of \( x_1 \), closest to \( a_1 \) in \( P_1 \). Then we have a T-parachute \( TP(a_2, a_3, x_1, a_1, x_0) \) of type a or b.

**Case 1.1.2** \( G' \) is an L-wheel.

Assume first that \( i = 1 \) and \( j = 2 \). If \( x_1 \) has at most one neighbor in \( A \) and at most one neighbor in \( B \), we have a 3PC(\( A, x_1 \)) when \( x_1 \) has at least three neighbors in \( G' \) and when \( x_1 \) has two neighbors in \( G' \) we have a 3PC(\( A, x_0 \)) or a 3PC(\( B, x_0 \)). So by symmetry, we may assume that \( x_1 \) is adjacent to \( b_3 = a_3, b_2 \) but not \( b_1 \). Let \( x_0 \) be the neighbor of \( x_1 \), closest to \( b_1 \) in \( P_1 \). Now \( x_0 \) is distinct from \( a_1 \), else let \( C = a_1, P_1, b_1, b_2, x_1, a_1 \) and \( (C, b_3) \) is an odd wheel. Now we have a T-parachute \( TP(b_2, b_3, x_1, b_1, x_0) \) of type a or b.

Assume now that \( i = 1 \) and \( j = 3 \). If \((H_{13}, x_1)\) is a \( \Delta \)-free wheel and \( x_1 \) has no neighbor in \( P_2 \), then there is a proper wheel with center \( a_3 \) and if \( x_1 \) is adjacent to \( a_3, x_0 \) and to no node in \( P_2 \), then we have an L-parachute \( LP(a_2 a_1, b_2 b_1, a_3, x_0) \). So \( x_1 \) has at least one neighbor in \( P_2 \). Now \( x_1 \) is adjacent to \( a_2 \) or \( b_2 \), say \( b_2 \), else we have a 3PC(\( A, x_1 \)). Then we have a T-parachute \( TP(b_2, b_3, x_1, b_1, x_0) \), where \( x_0 \) is the neighbor of \( x_1 \), closest to \( b_1 \) in \( P_1 \).

**Case 1.2** \((H_{ij}, x_1)\) is a universal wheel.

We may assume that \( x_1 \) is not universal for \( G' \), else b) holds.

**Case 1.2.1** \( G' \) is a 3PC(\( \Delta, \Delta \)).

Assume w.l.o.g. that \((H_{13}, x_1)\) is a universal wheel. Then both \( P_1 \) and \( P_3 \) have length less than three, otherwise \((H_{12}, x_1)\) or \((H_{23}, x_1)\) is a proper wheel. Furthermore, since \((H_{13}, x_1)\) is an even wheel, either both \( P_1 \) and \( P_3 \) have length one or both \( P_1 \) and \( P_3 \) have length two. Assume \( P_1 = a_1, t_1, b_1 \) and \( P_3 = a_3, t_3, b_3 \). Then \( x_1 \) has no neighbor in \( P_2 \), else \((H_{23}, x_1)\) is a proper wheel. Now the graph induced by \( V(G') \setminus \{a_3, t_3\} \cup \{x_1\} \) is a T-parachute \( TP(b_3, b_1, x_1, b_2, a_1) \) of type c.

Assume \( P_1 = a_1, b_1 \) and \( P_3 = a_3, b_3 \). Then \( x_1 \) has neighbors in \( P_2 \), else \( V(G') \setminus \{a_1\} \cup \{x_1\} \) induces an odd wheel with center \( b_3 \). So \((H_{12}, x_1)\) is an L-wheel or a T-wheel. If \((H_{12}, x_1)\) is a L-wheel, let \( x_2, x_3 \) be the neighbors of \( x_1 \), where \( x_2 \) is closest to \( b_2 \) in \( P_2 \). Then \( V(G') \setminus \{a_1\} \cup \{x_1\} \) induces an L-parachute of type c \( LP(x_3, x_2, a_3, b_3, x_1, b_2) \). If \((H_{12}, x_1)\) is a T-wheel, with \( x_1 \) adjacent to, say \( b_2 \), we have a T-parachute \( TP(b_3, x_1, a_3, b_2, a_2) \) of type c.

**Case 1.2.2** \( G' \) is an L-wheel.

If \((H_{12}, x_1)\) is a universal wheel and \( x_1 \) is not adjacent to \( a_3 = b_3 \), then \( P_1 \) and \( P_2 \) have both length 2, else \((H_{13}, x_1)\) or \((H_{23}, x_1)\) is a proper wheel. But then, we have a T-parachute \( TP(a_2, u_1, x_1, a_3, b_1) \) of type c.

If \((H_{13}, x_1)\) is a universal wheel, then \( x_1 \) has at least one neighbor in \( P_2 \) since otherwise \((H, a_3)\) is a proper wheel where \( H = a_2, P_2, b_2, b_1 \). But now, since \( x_1 \) is not universal for \( G' \), \((H_{19}, x_1)\) is a proper wheel.

**Case 1.3** \((H_{ij}, x_1)\) is an L-wheel or a T-wheel.

If \( x_1 \) has at most one neighbor in \( A \) and at most one neighbor in \( B \), then \((H_{ij}, x_1)\) is an L-wheel and \( x_1 \) has no neighbor in \( P_k, k \neq i, j \), for otherwise we have a 3PC(\( A, x_1 \)), and in this case a) holds. So by symmetry, we assume that \( x_1 \) has at least two neighbors in \( B \). Furthermore \( x_1 \) is adjacent to both \( b_i \) and \( b_j \), since otherwise, if \( x_1 \) is adjacent to \( b_i \) but not \( b_j \), then \((H_{ij}, x_1), (H_{ik}, x_1)\) or \((H_{jk}, x_1)\), \( k \neq i, j \), is a proper wheel or \( H_{jk} \) together with \( b_i \) and \( x_1 \) induces a T-parachute of type c.

**Case 1.3.1** \( G' \) is an L-wheel.
Assume \( i = 1 \) and \( j = 3 \). Then \( x_1 \) is adjacent to \( b_1 \) and \( b_3 \).

If \((H_{13}, x_1)\) is an L-wheel, then \( b_2 \) is the only neighbor of \( x_1 \) in \( P_2 \), else \((H_{12}, x_1)\) is a proper wheel, and \( a) \) holds.

If \((H_{13}, x_1)\) is a T-wheel, then either \( x_1 \) is adjacent to the neighbor \( b'_1 \) of \( b_1 \) in \( P_1 \), or \( x_1 \) is adjacent to \( a_1 \).

If \( x_1 \) is adjacent to \( b'_1 \), then \( x_1 \) has a neighbor in \( P_2 \), else we have an L-parachute \( LP(a_2a_1, b_2b_1, a_3, b'_1) \) of type \( c \). Since \( x_1 \) is not adjacent to \( a_1 \), \((H_{12}, x_1)\) is an L-wheel or a T-wheel. If \((H_{12}, x_1)\) is a T-wheel, then \( b_2 \) is the only neighbor of \( x_1 \) in \( P_2 \), so \( x_1 \) is a twin of \( b_1 \) and \( b) \) holds. If \((H_{12}, x_1)\) is an L-wheel, then the neighbors of \( x_1 \) in \( P_2 \) are \( a_2 \) and its neighbor in \( P_2 \), else \((H_{23}, x_1)\) is a proper wheel. But now we have a T-parachute \( TP(b_1, x_1, b_3, b'_1, a_1) \) of type \( c \).

If \( x_1 \) is adjacent to \( a_1 \), then \( x_1 \) has a neighbor in \( P_2 \), else we have a proper wheel with center \( b_3 \), so \((H_{12}, x_1)\) must be an L-wheel, \( x_1 \) is a twin of \( a_3 \) and \( b) \) holds.

Assume \( i = 1 \) and \( j = 2 \). Then \( x_1 \) is adjacent to \( b_1 \) and \( b_2 \).

If \((H_{12}, x_1)\) is an L-wheel then \( x_1 \) is adjacent to \( a_1, a_2, b_1, b_2 \) and no other node of \( H_{12} \), else there is a proper wheel with center \( a_3 \). If \( x_1 \) is adjacent to \( b_2 \), it is a twin of \( b_3 \) and \( b) \) holds, and if \( x_1 \) is not adjacent to \( b_3 \), we have a T-parachute \( TP(b_2, b_1, a_3, x_1, a_1) \) of type \( a \).

If \((H_{12}, x_1)\) is a T-wheel then \( x_1 \) is w.l.o.g. adjacent to \( b_1, b_2, b'_1 \) and no other node of \( H_{12} \).

If \( x_1 \) is adjacent to \( b_3 \), it is a twin of \( b_1 \) and \( b) \) holds, and if \( x_1 \) is not adjacent to \( b_3 \), we have an odd wheel with center \( b_3 \).

**Case 1.3.2** \( G' \) is a 3PC(\( \Delta, \Delta \)).

Assume w.l.o.g. that \( i = 1 \) and \( j = 3 \). Then \( x_1 \) is adjacent to \( b_1 \) and \( b_3 \). If \((H_{13}, x_1)\) is an L-wheel and \( x_1 \) has two neighbors in \( P_1 \setminus \{b_1\} \) or \( P_3 \setminus \{b_3\} \), say \( P_1 \setminus \{b_1\} \), then \( b_2 \) is the only neighbor of \( x_1 \) in \( P_2 \), else \((H_{12}, x_1)\) is a proper wheel, and \( a) \) holds. If \( x_1 \) is adjacent to \( a_1 \) and \( a_3 \), then \( x_1 \) has a neighbor in \( P_2 \), else we have a T-parachute \( TP(a_1, a_3, a_2, x_1, b_3) \). If \((H_{23}, x_1)\) is a \( \Delta \)-free wheel, there is a T-parachute. So \((H_{23}, x_1)\) must be an L-wheel, \( x_1 \) is adjacent to both \( a_2 \) and \( b_2 \) and \( a) \) holds.

If \((H_{13}, x_1)\) is a T-wheel, then assume w.l.o.g. that \( x_1 \) is adjacent to \( b'_1 \). Node \( x_1 \) has a neighbor in \( P_2 \) since, otherwise, there is an odd wheel with center \( b_1 \). Now suppose that \((H_{12}, x_1)\) is a universal wheel. Then, there is a T-parachute \( TP(a_1, a_2, a_3, x_1, b_2) \). So \((H_{12}, x_1)\) is an L-wheel or a 1-wheel. \((H_{12}, x_1)\) is a T-wheel, else \((H_{23}, x_1)\) is a proper wheel. If \( P_1 \) has length one and \( x_1 \) is adjacent to \( a_1 \) and \( a_2 \), there is a T-parachute \( TP(a_1, a_2, a_3, x_1, b_2) \). So \( b_2 \) is the only neighbor of \( x_1 \) in \( P_2 \) and \( b) \) holds.

**Case 2** \( n > 1 \).

By Lemma 3.7, since \( P \) is a link, the neighbors of \( x_1 \) in \( G' \) are either contained in a big clique \( A \) or \( B \) or in a segment of \( G' \) and the same holds for \( x_n \).

**Claim 1** No intermediate node of \( P' \) has a neighbor in \( G' \).

Assume node \( y \) of \( G' \) is adjacent to an intermediate node \( x_j \) of \( P \). Since \( P \) is a link, by Lemma 3.7, \((N(x_1) \cap G') \cup \{y\}\) is contained in a big clique or a segment of \( G' \) and the same holds for \((N(x_n) \cap G') \cup \{y\}\). So either \( y = a_3 = b_2 \), and the neighbors of one endnode of \( P \), say \( x_1 \), are contained in \( A \) while the neighbors of \( x_n \) are contained in \( B \) or \( y \) is the endnode of a nonatomic segment say \( P_1 \), the neighbors of one endnode of \( P \), say \( x_1 \), are contained in \( P_1 \) and the neighbors of \( x_n \) are contained in a big clique, say \( A \). This shows that such a node \( y \) is unique.
Assume first that \( y = a_3 = b_3 \), so \( G' \) is an \( L \)-wheel. We also assume w.l.o.g. that \( x_1 \) is adjacent to \( a_1 \), while \( x_n \) is adjacent to \( b_2 \). Now \( x_1 \) is adjacent to \( a_2 \) but not to \( a_3 \), since otherwise \((H_1, y)\) is a proper wheel where \( H = a_2, a_1, x_1, P, x_n, b_2, P, a_2 \). By symmetry, \( x_n \) is adjacent to \( b_1 \) and not to \( a_1 \). Let \( x_j \) be the node of lowest index adjacent to \( a_3 \). Now we have a \( T \)-parachute \( TP(a_2, a_1, x_3, x_j) \).

Assume now that \( y = a_1 \), the neighbors of \( x_1 \) are contained in \( P_1 \) and the neighbors of \( x_n \) are contained in \( A \), so \( x_n \) is adjacent to \( a_2 \) or \( a_3 \).

If \( x_n \) is adjacent to \( a_2 \), let \( Q \) be a shortest path between \( x_1 \) and \( b_1 \), whose intermediate nodes (if any) are in \( P_1 \). Let \( H_1 = x_n, a_2, P_2, b_2, b_1, Q, x_1, P, x_n \). Now either \((H_1, a_1)\) is a wheel or \( a_1 \) has two neighbors in \( H_1 \) and they are nonadjacent. Now \( x_n \) is also adjacent to \( a_3 \), else let \( H_2 = x_n, a_2, a_3, P_3, b_3, b_1, Q, x_1, P, x_n \), then \((H_2, a_1)\) is a proper wheel.

Finally \( x_n \) is also adjacent to \( a_1 \), else we have a \( T \)-parachute \( TP(a_2, a_3, a_1, x_n, x_j) \), where \( x_j \) is the node of highest index adjacent to \( a_1 \). Let \( H_2' = x_n, a_3, P_3, b_3, b_1, Q, x_1, P, x_n \). Now \((H_1, a_1)\) and \((H_2', a_1)\) are either both \( L \)-wheels or both \( T \)-wheels. If \( x_1 \) has a unique neighbor \( x_0 \) in \( P_1 \), we have either an \( L \)-parachute \( LP(x_j x_{j-1}, x_n, a_3, a_1, x_0) \) or a \( T \)-parachute \( TP(x_n, a_1, x_{n-1}, a_3, x_n) \). If \( x_1 \) has several neighbors in \( P_1 \), we have a \( 3PC(\Delta, a_1) \) or an \( L \)-parachute \( LP(x_j x_{j-1}, x_n, a_3, a_1, x_1) \) or a \( T \)-parachute \( TP(x_n, a_1, x_{n-1}, a_3, x_1) \).

If \( x_n \) is adjacent to \( a_3 \) and \( a_3 \neq b_3 \), the proof is identical. Finally, consider the case where \( x_n \) is adjacent to \( a_3 \) and \( a_3 = b_3 \). If \( x_1 \) has exactly two neighbors on \( P_1 \) and they are adjacent, there is a \( 3PC(\Delta, a_1) \) or a proper wheel with center \( a_1 \). Otherwise, there is an \( L \)-parachute \( LP(a_2 a_1, b_2 b_1, a_3, x_0) \) or \( LP(a_2 a_1, b_2 b_1, a_3, x_i) \) where \( x_i \) is the node of lowest index adjacent to \( a_1 \). This completes the proof of Claim 1.

**Case 2.1** All the neighbors of \( x_1 \) in \( G' \) are contained in a big clique, say \( A \), and all the neighbors of \( x_n \) are contained in \( B \).

We assume w.l.o.g. that \( x_1 \) is adjacent to \( a_1 \) and \( x_n \) is adjacent to \( b_2 \).

**Case 2.1.1** \( G' \) is an \( L \)-wheel.

Assume \( x_1 \) is adjacent to \( a_1 \) only. Then \( x_n \) is adjacent to \( a_3 = b_3 \), else we have an odd wheel with center \( a_3 \). Let \( H = a_1, x_1, P, x_n, b_2, b_1, P_1, a_1 \) if \( x_n \) is not adjacent to \( b_1 \), and \( H = a_1, x_1, P, x_n, b_1, P, a_1 \) if \( x_n \) is adjacent to \( b_1 \). Then \((H, a_3)\) is a proper wheel.

Assume \( x_1 \) is adjacent to \( a_2 \) but not to \( a_3 \). If \( x_n \) is adjacent to \( a_3 \), we have an odd wheel with center \( a_3 \) and if \( x_n \) is not adjacent to \( a_3 \) we have a \( T \)-parachute \( TP(a_1, a_2, x_1, a_3, b_2) \).

Assume \( x_1 \) is adjacent to \( a_1, a_3 \) but not to \( a_2 \). Let \( H = a_1, x_1, P, x_n, b_2, P, a_2, a_1 \). Then \((H, a_3)\) is a proper wheel.

So \( x_1 \) is adjacent to \( a_1, a_2, a_3 \) and by symmetry, \( x_n \) is adjacent to \( b_1, b_2, b_3 \), and \( a \) holds in this case.

**Case 2.1.2** \( G' \) is a \( 3PC(\Delta, \Delta) \).

Assume \( x_1 \) is adjacent to \( a_1 \) only. If \( x_n \) is adjacent to \( b_2 \) only, we have a \( 3PC(\Delta, a_1) \). If \( x_n \) is adjacent to \( b_1 \), we have a \( 3PC(\Delta, b_2, a_3, a_1) \). If \( x_n \) is adjacent to \( b_3 \) and not \( b_1 \), we have a \( T \)-parachute \( TP(b_3, b_2, x_n, b_1, a_1) \).

Assume \( x_1 \) is adjacent to \( a_1, a_2 \) but not to \( a_3 \). If \( x_n \) is not adjacent to \( b_1 \), we have a \( 3PC(a_1, a_2, x_1, b_2) \). If \( x_n \) is adjacent to \( b_1 \), we have a \( T \)-parachute \( TP(a_2, a_1, x_1, a_3, b_1) \) if \( x_n \) is not adjacent to \( b_3 \) and a \( 3PC(b_1, b_3, x_n, a_1) \) if \( x_n \) is adjacent to \( b_3 \).

Assume \( x_1 \) is adjacent to \( a_1, a_3 \) but not to \( a_2 \). By symmetry, we may assume that \( x_n \) is not adjacent to \( b_3 \) and we have a \( T \)-parachute \( TP(a_1, a_3, x_1, a_2, b_2) \).
So $x_1$, is adjacent to $a_1$, $a_2$ and $a_3$. Again by symmetry, $x_n$ is adjacent to $b_1$, $b_2$, and $b_3$ and a) holds in this case.

**Case 2.2** All the neighbors of $x_1$ in $G'$ are contained in a big clique, say $A$, and all the neighbors of $x_n$ are contained in a segment, say $P_1$.

Then $x_1$ is adjacent to $a_2$ or $a_3$.

**Case 2.2.1** $G'$ is an L-wheel.

We first show that $x_n$ has two neighbors in $P_1$ and these neighbors are adjacent. If not, either $P_1$ contains two neighbors of $x_n$ and these neighbors are nonadjacent or $P_1$ has a unique neighbor of $x_n$. Assume the first possibility holds: If $x_1$ is adjacent to $a_2$ only, we have a $3PC(A,x_n)$. If $x_1$ is adjacent to $a_3$ only, we have an L-parachute $LP(a_2,a_1,b_1,b_1,a_3,x_n)$. If $x_1$ is adjacent to $a_1$, $a_3$ and possibly $a_2$ we have a $3PC(x_1,a_1,a_3,x_n)$ if $x_n$ is not adjacent to both $a_1$ and $x_1$, and a $T$-parachute $TP(x_1,a_1,a_2,x_n,t)$, where $t$ is the neighbor of $x_n$ in $P_1$ that is closest to $b_1$, if $x_n$ is adjacent to both $a_1$ and $x_1$. If $x_1$ is adjacent to $a_1$, $a_2$ but not $a_3$, there is a proper wheel with center $a_3$. Finally if $x_1$ is adjacent to $a_2$, $a_3$ but not $a_1$, we have a T-parachute $TP(a_2,a_3,a_1,x_1,x_n)$. So $P_1$ cannot contain two nonadjacent neighbors of $x_n$.

The same proof rules out the case where $P_1$ has a unique neighbor of $x_n$, so $x_n$ has two adjacent neighbors, say $y$ and $z$, in $P_1$ and $y$ is closer than $z$ to $a_1$ in $P_1$.

Assume $x_1$ is adjacent to $a_3$. Then $x_1$ is adjacent to $a_1$, else we have a $3PC(x_nyz,a_3)$. Now $x_2$ is also adjacent to $a_2$, else we have a $3PC(x_nyz,a_3)$ when $a_1 \neq y$ and an L-parachute $LP(x_nz,x_1a_3,a_1,b_1)$ of type $d$ when $y = a_1$ and $n > 2$. When $y = a_1$ and $n = 2$, we have an odd wheel with center $a_1$. So a) holds in this case.

Assume finally $x_1$ is adjacent to $a_2$ but not $a_3$. Then $x_1$ adjacent to $a_1$, else we have a $3PC(x_nyz,a_2)$. Now we have a $3PC(x_nyz,a_1)$ when $a_1 \neq y$ and when $y = a_1$, we have a proper wheel with center $a_1$.

**Case 2.2.2** $G'$ is a $3PC(\Delta, \Delta)$.

We assume w.l.o.g. that $x_1$ is adjacent to $a_2$. $x_1$ is adjacent to $a_3$ since, otherwise, there is a $3PC(B,a_3)$.

Assume that $x_1$ is not adjacent to $a_1$. If $x_n$ has two nonadjacent neighbors in $P_1$, there is a $T$-parachute $TP(a_2,a_3,a_1,x_1,x_n)$. If $x_n$ has a unique neighbor $x_{n+1}$ in $P_1$, there is a $T$-parachute $TP(a_2,a_3,a_1,x_1,x_{n+1})$. If $x_n$ has exactly two neighbors in $P_1$, say $y$, $z$, and they are adjacent, there is a $3PC(x_nyz,a_2)$.

So $x_1$ is adjacent to $a_1$, $a_2$ and $a_3$. If $x_n$ has a unique neighbor $x_{n+1}$ in $P_1$, there is a $3PC(x_1a_3a_2,x_{n+1})$. If $x_n$ has two nonadjacent neighbors in $P_1$, there is a $3PC(x_1a_3a_2,x_n)$ if $x_n$ is not adjacent to both $x_1$ and $a_1$, and a $T$-parachute $TP(x_1,a_1,a_2,x_n,t)$ otherwise, where $t$ is the neighbor of $x_n$ closest to $b_1$. So, $x_n$ has exactly two neighbors in $P_1$, say $y$, $z$, and they are adjacent. Let $y$ be the one that is closest to $a_1$ in $P_1$. If $y = a_1$, there is a proper wheel with center $a_1$. So $y \neq a_1$ and a) holds in this case.

**Case 2.3** All the neighbors of $x_1$ are contained in a segment, say $P_1$ and all the neighbors of $x_n$ are contained in a segment, say $P_2$.

Note that the choice of $P_1$ and $P_2$ is done w.l.o.g. by assuming that we are not in Case 2.2. We show that $x_1$ has exactly two neighbors and these two neighbors are adjacent. Assume $x_1$ has exactly one neighbor $y$ in $P_1$. Since $x_n$ has a neighbor in $P_2 \setminus \{b_2\}$, there is a $3PC(A,y)$.
If $x_1$ has two nonadjacent neighbors in $P_1$, replace $P_1$ by the chordless $a_1b_1$-path containing $x_1$ and nodes of $P_1$ and let $P' = x_2, \ldots, x_n$. The proof of Case 1 shows that $P'$ has length bigger than 2 and $x_2$ now has $x_1$ as unique neighbor in $P'_1$ and by the above argument, this is impossible. So $x_1$ has exactly two neighbors in $P_1$ and they are adjacent. By symmetry, $x_n$ has exactly two neighbors in $P_2$ and they are adjacent. So a) holds in this case. □

**Lemma 3.9** Let $G$ be an even-signable WP-free graph, $G'$ be an L-graph in $G$ and $P = x_1, \ldots, x_n$ be a link of $G'$. Then

a) either $G' \cup P$ is an L-graph, or

b) $n = 1$ and $x_1$ is either universal for $G'$ or the twin of an endnode of a segment of $G'$.

**Proof:** Since $P$ is a link of $G'$, $x_1, x_n$ have neighbors $x_0, x_{n+1}$ that are nonadjacent and in distinct segments $S_{x_0}$ and $S_{x_{n+1}}$ of $G'$. Let $S_3$ be any other segment of $G'$. By Lemma 3.4, $G'$ contains an elementary L-graph $G_1$, with $S_{x_0}$ and $S_{x_{n+1}}$ in distinct segments of $G_1$ and containing $S_3$. So $P$ is a link of $G_1$, and by Lemma 3.8, the statement holds when $G' = G_1$.

**Case 1** $n = 1$.

Assume $x_1$ is a universal node for $G_1$ and $x_1$ is not adjacent to node $y$ of $G'$. Let $C$ be any hole of $G_1$. By Lemma 3.5, $G'$ contains an elementary L-graph $G_2$, containing $C$ and segment $S_y$. Since at least two nodes of $C$ are nonadjacent and in distinct segments of $G_2$, $x_1$ is a link of $G_2$. Since $(C, x_1)$ is a universal wheel but $x_1$ is not universal for $G_2$, Lemma 3.8 is contradicted.

Assume $x_1$ is a link of $G_1$ and is adjacent to all nodes in distinct cliques $K'_1, K'_2$ of $G_1$, not in the same segment of $G_1$ and to no other node of $G_1$. (This happens both when $x_1$ is a twin of an endnode of a segment of $G_1$ and when $G_1 \cup \{x_1\}$ is an L-graph.) Let $K_1, K_2$ be the cliques of $G'$ that extend $K'_1, K'_2$.

Assume $x_1$ is adjacent to node $y$ in $G' \setminus (K_1 \cup K_2)$ and let $C$ be a hole of $G_1$ containing two nodes of $K'_1$ and two nodes of $K'_2$. By Lemma 3.5, $G'$ contains an elementary L-graph $G_2$, containing $C$ and segment $S_y$. Now $x_1$ is a link of $G_2$, for $x_1$ is adjacent to $y$, and no node of $C$ is in the same segment as $y$ and at least one neighbor of $x_1$ in $C$ is nonadjacent to $y$. Since $(C, x_1)$ is an L-wheel or a T-wheel and $x_1$ is adjacent to $y$, Lemma 3.8 is contradicted in $G_2$.

Assume $x_1$ is not adjacent to node $z$ in $K_1$. Let $C$ be a hole containing two nodes of $K'_1$ and two nodes of $K'_2$. By Lemma 3.5, $G'$ contains an elementary L-graph $G_2$, containing $C$ and segment $S_z$. Since $G_2$ contains at least three nodes of $K_1$, its restriction $K_1^*$ to $G_2$ is a big clique of $G_2$ and since $x_1$ is adjacent to two nodes in $K_1^*$, two other nodes of $C$ and no other node of $G_2$, $x_1$ is a link of $G_2$, violating Lemma 3.8.

So $x_1$ is adjacent to all nodes in $K_1 \cup K_2$ and is adjacent to no other node of $G'$. If $K_1, K_2$ have a common node in $G'$, then $x_1$ is a twin of such a node and b) holds. Otherwise $G' \cup \{x_1\}$ is an L-graph and a) holds.

**Case 2** $n > 1$.

Assume that node $y$ of $G'$ is adjacent to an intermediate node $x_j$ of $P$. By Lemma 3.4, $G'$ contains an elementary graph $G_2$ containing $S_y$, where $S_{x_0}$ and $S_{x_{n+1}}$ are in distinct segments of $G_2$, So $P$ is a link of $G_2$ (the minimality of $P$ follows from the fact that $P$ is a link of
Since an intermediate node of \( P \) is adjacent to \( y \), Lemma 3.8 is violated in \( G_2 \). So no intermediate node of \( P \) has a neighbor in \( G' \).

Since \( P \) is a link of \( G_1 \), by Lemma 3.8, \( x_1 \) and \( x_n \) are adjacent to all the nodes in cliques \( K'_1, K'_2 \), not in the same segment of \( G_1 \). Let \( K_1, K_2 \) be the cliques of \( G' \) that extend \( K'_1, K'_2 \).

Assume \( x_1 \) is adjacent to node \( y \) in \( G' \setminus K_1 \). By Lemma 3.4, \( G' \) contains an elementary L-graph \( G_2 \), containing \( S_y \), where \( S_{x_0} \) and \( S_{x_{n+1}} \) are in distinct segments of \( G_2 \). So \( P \) is a link of \( G_2 \), contradicting Lemma 3.8. So all the neighbors of \( x_1 \) in \( G' \) are contained in \( K_1 \) and by symmetry, all the neighbors of \( x_n \) in \( G' \) are contained in \( K_2 \).

Assume \( x_1 \) is not adjacent to node \( z \) in \( K_1 \). By Lemma 3.4, \( G' \) contains an elementary L-graph \( G_2 \), containing \( S_z \), where \( S_{x_0} \) and \( S_{x_{n+1}} \) are in distinct segments of \( G_2 \). So \( P \) is a link of \( G_2 \), contradicting Lemma 3.8. So \( x_1 \) belongs to the extension of \( K_1 \) and by symmetry, \( x_n \) belongs to the extension of \( K_2 \) and a) holds in this case.

\[ \square \]

3.4 A Proof of Theorem 1.16

A twin class of a graph \( G \) is a maximal subset of \( V(G) \) with the property that every pair of nodes in it are twins. The twin classes of \( G \) partition \( V(G) \) into cliques. A restriction of \( G \) is an induced subgraph \( H \) of \( G \) obtained by keeping exactly one node in each twin class. All the restrictions of \( G \) are obviously isomorphic graphs.

Let \( G \) be a graph. An induced subgraph \( G' \) of \( G \) is an extended L-graph if any restriction \( H \) of \( G' \) is an L-graph and every twin class of \( G' \) containing an intermediate node in a segment of \( H \) contains no other node. A segment of \( G' \) is a segment of one of its restrictions \( H \) together with all the nodes in the twin classes of its endnodes. A path \( P \) in \( G' \setminus G' \) is a link of \( G' \) if \( P \) is a link of some restriction \( H \) of \( G' \). By Lemma 3.9, it follows that if \( P \) is a link of \( G' \), then \( P \) is a link of all the restrictions of \( G' \).

**Theorem 3.10** Let \( G \) be an even-signable WP-free graph, \( G' \) a node-maximal extended L-graph in \( G \) and \( P = x_1, \ldots, x_n \) a link of \( G' \). Then \( n = 1 \) and \( x_1 \) is a universal node for \( G' \).

**Proof:** Follows by applying Lemma 3.9 to all the possible restrictions of \( G' \) and the maximality of \( G' \).

\[ \square \]

We can now prove Theorem 1.16.

**Proof:** Let \( G \) be an even-signable WP-free graph that contains an L-wheel or a 3PC(\( \Delta, \Delta \)) as induced subgraph. Then \( G \) contains a node-maximal extended L-graph \( G' \) and let \( U \) be the set of nodes that are universal for \( G' \).

Assume first that \( V(G) \cap V(G') \cup U \). If \( U \neq \emptyset \), then \( G \) is disconnected. If \( U = \emptyset \) and at least one twin class of \( G \) contains at least two nodes, then \( G \) contains a star cutset. Finally, if \( U = \emptyset \) and every twin class of \( G \) contains a single node, \( G \) is the line graph of a triangle-free graph.

Assume now \( G' \setminus (V(G') \cup U) \) is nonempty and let \( C_1, \ldots, C_n \) be the connected component of \( G' \setminus (V(G') \cup U) \). By Theorem 3.10 and Lemma 3.7, for every connected component \( C_i \), \( N(C_i) \cap G' \) is either contained in a clique of \( G' \) or in a segment of \( G' \) which is not atomic.
Assume first that a component, say $C_1$, has its neighbors in $G'$ contained in a clique $K$. Then the removal of the nodes in $K \cup U$ separates $C_1$ from $G' \setminus K$ and we have a star cutset.

Assume now that no component has its neighbors in $G'$ contained in a clique of $G'$ and let $K_1$, $K_2$ be the two big cliques of $G'$ that contain the endnodes of the nonatomic segment $S$ containing the neighbors of $C_1$. Let $A_1$, $A_2$ be the subsets of $K_1$, $K_2$ that are the twin classes of the endnodes of $S$ and let $B_1 = K_1 \setminus A_1$, $B_2 = K_2 \setminus A_2$. Since $K_1$, $K_2$ are cliques of an extended L-graph, $A_1$, $A_2$ are nonempty and disjoint, while $B_1 \setminus B_2$, $B_2 \setminus B_1$ are both nonempty. If $B_1 \cap B_2$ is empty, we have an extended strong 2-join separating $S \cup C_1$ from $G' \setminus S$, and if $B_1 \cap B_2$ is nonempty, we have a star cutset separating the same sets. \qed

References


