Representation of faithful normal expectations in von Neumann algebras

Andre De Korvin
Carnegie Mellon University
REPRESENTATION OF FAITHFUL NORMAL EXPECTATIONS IN von NEUMANN ALGEBRAS

A. de Korvin

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Introduction.

Let $G$ and $IB$ be two $C^*$ algebras with identity. Suppose $8 \in G$. Let $\varphi$ be a positive linear map of $G$ on $IB$ such that $\varphi$ preserves the identity and such that $\varphi(BX) = B\varphi(X)$ for all $B$ in $IB$ and all $X$ in $G$. $\varphi$ is then defined to be an expectation of $G$ on $IB$. The extension of the notion of an expectation in the probability theory sense, to expectations on finite von Neumann algebra is largely due to J. Dixmier and H. Umegaki [1]. In [4] Tomiyama considers an expectation on von Neumann algebras to be a projection of norm one. If $\varphi$ is an expectation in the sense $\varphi(BX) = B\varphi(X)$, $\varphi$ positive and $\varphi$ preserves identities, then $\varphi(XB) = \varphi(X)B$ for all $X$ in $G$, $B$ in $B$. $IB$ is the set of fixed points of $\varphi$. By writing $\varphi[(X - (\varphi(X))^*(X - (\varphi(X)))] \geq 0$ we have $\varphi(X^*X) \geq \varphi(X)^*\varphi(X)$. In particular $\varphi$ is a bounded map.

Let $h$ and $k$ be two Hilbert spaces, $h \odot k$ will denote the tensor product of $h$ and $k$. Let $G$ be a von Neumann algebra acting on $h$, by an ampliation of $G$ in $h \odot k$ one means a map $ij$ of $G$ in $L(h \odot k)$ such that $S(A) = A \odot I$, where $I$ denotes the identity operator on $k$. The image of $G$ by an ampliation is then a von Neumann subalgebra of $L(h \odot k)$. In what follows $CT$ will designated the image of $G$ by an ampliation 0 and $\tilde{A}$ will stand for $i)(A)$.

In this paper expectations of a particular type are considered. If $IB$ is a subalgebra of $G$ and if $IB$ is the range of a faithful, normal expectation $\varphi$ defined on $G$, then it will be shown that
there exists an ampliation of $G$ in $h (D \mathcal{L})$, independent of $B$
and of $\varphi$, such that $\varphi 0 I_k$ is a spatial isomorphism of $S$
This result extends a result by Nakamera, Takesaki, and Umegaki [2], which consider the case when $G$ is a finite von Neumann algebra.

**Definitions.**

Let $M$ and $N$ be C* algebras and $\varphi$ a positive linear map
of $M$ on $N$. Let $M$ be the set of all $n \times n$ matrices whose
entries are elements of $M$, call those entries $A_{ij}$. Define for
each $n$, $\varphi^*(A_{ij}) = (\varphi(A_{ij}))^*$ and $\varphi^n$ is then a map of $M$ on $N$.

$\varphi$ is called completely positive if each $\varphi$ is.

Let $G$ and $\mathcal{B}$ be two von Neumann algebras, with $G \subset \mathcal{B}$. Let
$\varphi$ be an expectation of $R^\infty$ on $G$. $\varphi$ is called faithful if for
any $T$ in $G^\gamma$, $\varphi(T^\gamma) = 0$ implies $T = 0$. Let $A^\alpha$ be a net of
uniformly bounded self adjoint operators in $G$. $\varphi$ is called normal
if $\sup_{\alpha} \varphi(A^\alpha) = \varphi(\sup A^\alpha)$.

The ultra-weak topology on $G$ will be the weakest which will
make all $E_w (A) = L(Ax_i, y_i)$ continuous where $S||x_i||^2 < \infty$
and $S||y_i|| < \infty$. In what follows if $N$ is arbitrary von Neumann
algebra, $N^\mathcal{T}$ will denote the commutant of $N$. If $h$ is any Hilbert
space, $\dim h$ will denote the cardinality of the dimension of $h$.

**Proposition 1.**

Let $M$ and $N$ be two von Neumann algebras acting on $h^M$
and $h^\mathcal{N}$. Let $\varphi$ be a * isomorphism of $M$ on $B$. Let $k$ be a
Hilbert space such that $\dim k >_\mathcal{N} \max (\dim h^M, \dim h^\mathcal{N})$, then
$\varphi \otimes I_k$ is a spatial isomorphism.

This theorem says that there exists a isometry $V$ of $M \otimes k$ on
$h \otimes N$ such that $\varphi \otimes I_k (A \otimes k) = (\varphi(A) \otimes k) I_k = V (A \otimes I_k) V^\gamma (= V^\gamma V^\star)$. Tomiyama has shown this result in [5].
Proposition 2.

Let $M$ and $N$ be two C*-algebras with identities. Let $\psi$ be an expectation of $M$ on $N$, then $\psi$ is completely positive. This result was shown by Nakamura, Takesaki, and Umegaki in [2].

One of the tools for the proof of the theorem will be the Stinespring construction which is given in [3] and which will be sketched here for completeness sake.

Let $M$ be any von Neumann algebra acting on $h$. Let $M \otimes h$ denote the tensor product of $M$ and $h$ as linear spaces. Let $N$ be von Neumann algebra of $M$ which is the range of a faithful, normal expectation $\psi$. On $M \otimes h$ define an inner product by:

$$<T^i \otimes x^i, I^j \otimes y^j> = I^{\psi(x^i y^j)}$$

where $a_i \otimes x_i$ are in $M$, $b_j \otimes y_j$ are in $h$ and where $(\cdot)$ denotes the inner product in $h$. Now:

$$\sum_{i, j} (a_i^* x_i, y_j) = (I^{a_i^* x_i} I^{a_i x_i})^{\psi}.$$ 

Let $A$ be in $M_n$ with $\cdot A^* j = a_j^* a^*$ then if $x = (x_1, x_2, \ldots, x_n)$

$$(A x, x) = \sum_i (a_j^* a_i x_i, x_i) \geq 0.$$ 

By proposition 2,

$$\sum (a_j^* a_i x_i, x_j) \geq 0.$$ 

Hence the product defined on $M \otimes h$ is bilinear and positive.

However it is possible to have $<\xi, \xi> = 0$ with $\xi \neq 0$. Divide
out the space $M_0 h$ by all vectors of norm zero. Then taking
the completion of that space, one obtains a Hilbert space which
will be denoted $M_Q h$.

Lemma 3. $h$ is imbedded as a Hilbert space in $M \otimes h$.

Proof: In fact we shall show that $h$ is isomorphic to $N \otimes h$.

Let $a_i, i = 1, 2, \ldots, n$ be operators in $B$, consider the map

$$S : \bigotimes_{i=1}^{n} a_i \otimes x \mapsto \bigotimes_{i=1}^{n} a_i x.$$ 

then

$$< \bigotimes_{i=1}^{n} a_i \otimes x, \bigotimes_{i=1}^{n} a_i \otimes y >$$

$$= \bigotimes_{i,j} (\phi(a_i^* a_j) x_i, x_j)$$

$$= \bigotimes_{i,j} (a_i^* a_j x_i, x_j)$$

$$= S(a^\wedge a \otimes x, x)$$

$$= \left( \sum_{i=1}^{n} a_i x_i \right)$$

Hence $S$ is an isometry of $N \otimes h$ on $h$. In particular then,
once can view $h$ as a subspace of $M \otimes h$.

Lemma 4.

$\phi$ defines a self adjoint projection $E$ of $M \otimes h$ on $N \otimes h$.

Proof: Let $a_i, i = 1, 2, \ldots, n$ be operators of $M$. Define

$$E(\sum_{i=1}^{n} a_i \otimes x_i) = \sum_{i=1}^{n} \phi(a_i) \otimes x_i$$

the proof in [2] shows that $E$ is a well defined self adjoint
projection of $M \otimes h$ on $N \otimes h$. Recall for example how self
adjointness is checked out.

\[ \langle E(\sum_{i} a_i \otimes x_i), \sum_{j} b_j \otimes y_j \rangle = \langle \sum_{i} \varphi(b_i^*) a_i x_i, y_j \rangle = 1 \]

\[ = \sum_{i,j} (\varphi(\varphi(b_j^*) a_i) x_i, y_j) \]

\[ = \langle \sum_{i} a_i \otimes x_i, \sum_{j} \varphi(b_j) \otimes y_j \rangle = \langle \sum_{i} a_i \otimes x_i, E(\sum_{j} b_j \otimes y_j) \rangle \]

**Lemma 5.**

There exists an ultra-weakly continuous representation \( I \) of \( M \) in \( L(M^{\otimes i}) \) such that \( t(b) E = El(b) \) for all \( b \) in \( N \).

Moreover if \( h \) and \( N(2) h \) are identified by the isometry \( S \) of lemma 3, then \( \varphi(a) = El(a) E \) for all \( a \) in \( M \).

**Proof:** For each \( a \) in \( M \) define

\[ I(a) (\sum_{i} a_i \otimes x_i) = \sum_{i} a a_i \otimes x_i. \]

\( I \) is then a representation of \( M \) in \( L(M^{\otimes h}) \). Let \( b_i, i=1,2,\ldots,n \) be operators in \( N \) then:

\[ Et(a)(\sum_{j} b_j \otimes x_j) = E(\sum_{j} ab_j \otimes x_j) \]

\[ = \sum_{j} \varphi(a) b_j \otimes x_j = \varphi(a) (\sum_{j} b_j \otimes x_j) \]

identifying \( \sum_{j} b_j \otimes x_j \) with \( \sum_{j} b_j x_j \). this shows that \( Ef(a)E = \varphi(a). \)

Let \( b \) be in \( N \) then

\[ t(b) E(\sum_{i} a_i \otimes x_i) = t(b)(\sum_{i} \varphi(a_i) \otimes x_i) \]

\[ = b \varphi(a_i) 0x_i = Et(b)(\sum_{i} a_i \otimes x_i). \]
So \( l(b)E = Ep(b) \), for all \( b \) in \( N \). To show now that \( I \) is u. W. continuous. Let

\[
\xi_k = \sum_{i=1}^{n_k} a_i^{(k)} \otimes x_i^{(k)} \quad \text{and} \quad \eta_h = \sum_{j=1}^{n_h} b_j^{(h)} \otimes y_j^{(h)}
\]

with \( \sum C_k \| \| < \infty \) and \( \| \| \| Y \| \| h \| \| < \infty \). Let \( \alpha \) be a net converging u.w. to \( a \) in \( M \). Then it is sufficient to show that \( A \) tends to zero where

\[
A = \sum_{k,h} \langle \lambda(a - a'n) \xi_k , \eta_h \rangle.
\]

We have

\[
A = \sum_{k,h} \left( \sum_{i,j} \left( \sum_{j} a_i^{(k)} (a - a'n) a_j^{(k)} \right) \right) \left( \sum_{j} b_j^{(h)} \otimes y_j^{(h)} \right) \left( \sum_{i} \sum_{j} \left( \sum_{i} a_i^{(k)} (a - a'n) a_j^{(k)} \right) \otimes x_i^{(k)} \otimes y_j^{(h)} \right)
\]

Now \( \left( \sum_{i} a_i^{(k)} (a - a'n) a_j^{(k)} \right) \) tends to zero u.w. As \( \langle \rho \rangle \) is normal, \( A \) tends to zero. Let \( N \subset M \) be two von Neumann algebras acting on \( h \). Let \( \rho \) be a faithful, normal expectation of \( M \) on \( N \).

**Proposition 6.**

There exists a Hilbert space \( k \) such that:

1. \( h \) can be imbedded in \( k \)
2. There exists an u.w. continuous representation \( I \) of \( M \) in \( L(k) \) such that \( \rho(A) = p_t(A)p \), where \( p_t \) is the projection of \( k \) on \( h \).
3. \( t \) is \( a^* \) isomorphism.
4. \( p \) commutes with all \( I(b) \) with \( b \) in \( N \).

Proof: Let \( k = M \otimes h \), if \( I(a) = 0 \) then \( t(a^*a) = 0 \) so \( \rho(a^*a) = 0 \).
By faithfulness of $\langle p \rangle$ this implies $a = 0$. Hence $I$ is a * isomorphism of $M$ in $L(k)$. The rest of proposition 6 is a restatement of lemma 5.

Theorem 7.

There exists an ampliation of $M$ in $h \otimes k$ such that if $^* \subseteq Y$ von Neumann subalgebra of $M$ which is the range of $JL$ faithfully normal expectation $\langle p \rangle$ then there exists an isometry $V$ in $(N \otimes I_k)^f$ such that $\langle p \otimes I_k, \widetilde{\alpha} \rangle = V \tilde{V}^*$, $VV^* = I$. Then putting $V^*V = P$, then $J^*s$ in $(N \otimes I^f)^i$, $\langle p \otimes I, \alpha \rangle P = P \tilde{P}$. For all $^*$ positive, $\tilde{P} = 0$ implies $\tilde{\alpha} = 0$.

Proof: Let $s$ be a Hilbert space with cardinality greater or equal to the maximum of $x_1$ and cardinality of a Hammel basis of $M(x)^h$. Define $T(\tilde{\alpha}) = i(\alpha)0I_s \otimes s = \langle p \otimes I, \tilde{\alpha} \rangle$. Then:

$\tilde{\varphi}(\tilde{\alpha}) = (P, (x)1_s)T(\tilde{\alpha}) (P \otimes I_s)$. By proposition 1, $X$ is spatial, there exists an isometry $U$ of $h \otimes s$ onto $k (x) s$ such that $\tilde{\varphi}(\tilde{\alpha}) = U(\tilde{\alpha})U^*$. Hence

$$\tilde{\varphi}(\tilde{\alpha}) = P_{h \otimes s}U(A \otimes I_s)U^*P_{h \otimes s}$$

where $P_{h \otimes s}$ denotes the projection of $k \otimes s$ on $h \otimes s$. Moreover $P_{h \otimes s}$ commutes with all $\tilde{\alpha}B$ as $B$ ranges over $N$ (proposition 6). So $U^*P_{h \otimes s}U$ commutes with all $\tilde{B}$ for $B$ in $N$.

Let $V = P_{h \otimes s}^e$ then $W^* = P_{h \otimes s}$ (= $I_{h \otimes s}$) * Define $V^*V = P = ^*P_{h \otimes s}U$. Then $P$ is in $(N \otimes I_s)^i$. So $\tilde{\varphi}(\tilde{\alpha}) = V \tilde{V}^*$ for all $A$ in $M$; Claim: $V$ is in $(N \otimes I_s)^i$. Let $B$ be in $N$ $\tilde{B} = \tilde{\varphi}(\tilde{B}) = V \tilde{V}^$ so $V^*B = P\tilde{B}^* = \tilde{B}P^V = \tilde{B}V \otimes s$ so $V$ is in $N$. Now
\[ \text{PAP} = V^*V\tilde{\phi}(\tilde{A})V \]
\[ = V^*(\tilde{\phi}(\tilde{A}))V \]
\[ = V^*V\tilde{\phi}(\tilde{A}) = P\tilde{\phi}(\tilde{A}) \text{ Also,} \]

\[ \tilde{\phi}(\tilde{A}) = \tilde{\phi}(\tilde{A})V^*V \]

Let \( \tilde{P} \) be now the central carrier of \( P \), \( I - P = (I - \tilde{P}) = (I - \tilde{P})P \) then \( 0 \). So \( \tilde{P}^* = I \). Hence if \( A(\tilde{B}) = P\tilde{B}\tilde{P}^* \) then \( \tilde{P}^* = I \). By faithfulness if \( A \) is positive \( A = 0 \).
References


Carnegie-Mellon University-
Pittsburgh, Pennsylvania