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Frame-Free Continuum Thermomechanics

Brian Seguin
Carnegie Mellon University

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Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213

Thesis Committee:
Walter Noll, Chair
Juan Schäffer
David Owen
Eliot Fried

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Abstract

The principle of material frame-indifference is a fundamental concept in classical physics. In 1972 Walter Noll showed that one can formulate constitutive laws without using any external frames of reference. Constitutive laws formulated in this way vacuously satisfy the principle of material frame-indifference. This thesis describes the basic concepts involved in formulating frame-free continuum thermomechanics and presents a framework for formulating constitutive laws for simple materials that take into account thermal effects. The restrictions coming from the second law of thermodynamics on the constitutive laws for materials with fading memory are found using the Coleman–Noll procedure. Also, it is shown that materials with fading memory, when subjected to slow processes, can be approximated by what I call thermoelasto-viscous materials, which are also studied in detail.

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Contents

0 Scope of Monograph

Thermomechanics is not an uncommon word in the physics community[1]. This is not surprising since it contains elements of two well established branches of physics — mechanics and thermodynamics. Mechanics[2] is the branch of physics concerned with the behavior of physical bodies when subjected to forces or displacements, and the subsequent effect of the bodies on their environment[3]. Thermodynamics, on the other hand, concerns itself with the interplay between heat and energy. Although most mechanicians know a thing or two about energy, they usually do not concern themselves with the other aspects of thermodynamics.

Thermomechanics can best be thought of as an expansion of mechanics, rather then a merger of mechanics and thermodynamics. There are several reasons to believe that this is the case. Mechanics is by far the older of the two disciplines, going all the way back to Aristotle, and thus its foundations are much better understood then those of thermodynamics. Mechanics is also a much broader subject in the sense that many more problems that arise in mechanics can be solved then those that arise in thermodynamics. Even the word thermomechanics seems to indicate that it is the thermodynamics that is being added to mechanics rather then an equal merging of the two fields.

This thesis is by no means an attempt to discuss the entire field of thermomechanics. We only consider the thermomechanics of continuous bodies. This presentation emphasizes the basic concepts of continuum thermomechanics with a focus on the formulation of constitutive laws. What makes this treatment of thermomechanics different from any other is the role that frames of reference play. A frame of reference is the background on which we view physical phenomena[4]. In what follows no fixed frame of reference, or *physical space*, is used. Also, when at all possible, concepts are defined without using any frame of reference. It is in this sense that this is a "frame-free" treatment of continuum thermomechanics.

This work is divided into four main chapters. Chapter I describes in detail the basic concepts that are used throughout this work. Namely, the concepts of force, motion and body are precisely defined in the context of continuous bodies. After the foundations of mechanics are laid down energy, entropy and heat are introduced. This part ends with a brief discussion of constitutive laws and the role of the second law of thermodynamics. Here we take the point of view put forward by Coleman and Noll in [CNP], that the second law is a restriction on constitutive laws and not on the possible processes the body can undergo.

Chapter II defines simple thermomechanical elements. The material presented there extends Noll's new theory of simple materials [NTSM], which only dealt with mechanical

[1]A Google search yields about 68,000 hits, however, a definition of thermomechanics is hard to find in a dictionary.

[2]Here, and in the rest of this thesis, the word mechanics refers to classical mechanics. No relativistic or quantum effects will be considered.

[3]This definition is taken from Wikipedia on 5/2/10. Although using Wikipedia for academic purposes is questionable, I find this definition of mechanics to be quite good.

[4]For a detailed discussion about frames of reference see Part I of [FC], *The Illusion of Physical Space*.

phenomena, to include thermal phenomena. Here the constitutive laws are formulated using a state space and without a fixed frame of reference or a reference placement. Material symmetry is also discussed without employing a fixed frame of reference. This frame-free approach to material symmetry removes any possibility of confusing which restrictions on constitutive laws come from the principle of material frame-indifference and which come from material symmetry.

Chapter III deals with memory in materials, with an emphasis on materials with fading memory as defined by Coleman and Noll in [ATCM]. Here we model memory by assuming the response of the material is continuous with respect to a uniformity on the set of condition processes. This assumption can be used to generate a topology on the state space which differs from the basic topology introduced in Chapter II. This chapter then goes on to find the restrictions imposed on the constitutive laws defining materials with fading memory that come from the second law of thermodynamics. Finally, an approximation theorem is proved that says that when materials with fading memory are subjected to slow processes of long duration they can be approximated by what are called *thermoelasto-viscous materials*.

Chapter IV deals with thermoelasto-viscous materials in detail[5]. A thermoelasto-viscous material is defined by a set of constitutive laws in which the stress, entropy, heat flux and free energy are functions of the present configuration, temperature, temperature gradient and on the rate of change of all three of these. The Coleman–Noll procedure is carried out, and the symmetry group of the material is also discussed, both without using a frame of reference. It is then shown what form the constitutive laws of a fluid thermoelasto-viscous material take when a frame of reference is considered. Finally, the governing equations for these materials are explicitly obtained and discussed.

The last Chapter discusses possible future projects that could come out of this work.

[5]Many of the results presented in this chapter are published in [TEVM].

Chapter I

Basic Concepts

1 Introduction

This chapter is a revision of the paper *Basic Concepts of Thermomechanics* [BCT] by Noll and myself and could serve as a blueprint for the first few chapters of future textbooks on continuum mechanics and continuum thermomechanics. Here we try to give a clear mathematical formulation of the basic concepts that are needed in continuum mechanics and continuum thermomechanics. The material presented here may be considered an update of the paper *Lectures on the Foundations of Continuum Mechanics and Thermodynamics* [LFCM] by Noll, published in 1973, and an elaboration of topics treated in Part 3, entitled *Updating the Non-Linear Field Theories of Mechanics*, of the booklet [FC] also by Noll. However, there is material presented here that has not appeared elsewhere, for example, the frame-free definition of a continuous body given in Section 4.

The present presentation differs from most existing textbooks on the subject in several important respects:

1. It uses the mathematical infrastructure based on sets, mappings, and families, rather than the infrastructure based on variables, constants, and parameters. (For a detailed explanation, see *The Conceptual Infrastructure of Mathematics* by Noll [CIM].) See the Appendix of [BCT] for a brief introduction to this infrastructure. For a full treatment of this infrastructure see [FDS].
2. It is completely coordinate-free and $\mathbb{R}^n$-free when dealing with basic concepts.
3. It does not use a fixed *physical space*. Rather, it employs an infinite variety of *frames of reference*, each of which is a Euclidean space. Motivation for avoiding physical space can be found in Part 1, entitled *On the Illusion of Physical Space*, of the booklet [FC]. Here, the basic laws are formulated without the use of a physical space or any external frame of reference.
4. It considers inertia as only one of many external forces and does not confine itself to using only inertial frames of reference. Hence kinetic energy, which is a potential for inertial forces, does not appear separately in the energy balance equation.

Several important issues are not dealt with here. For example:

1. Internal interactions at a distance, both forces and heat transfers, are absent. They should be included in a more encompassing analysis because they are important, for example, in applications of continuum thermomechanics to astrophysics.
2. The basic properties of concepts such as force, stress, energy, heat transfer, temperature, and entropy are taken for granted and we do not deal with the large and important literature that tries to derive them from more primitive assumptions[1].
3. Phase transitions are absent.
4. Diffusion, i.e., the intermingling of different substances, is not dealt with here.
5. The connection between chemical reactions and continuum thermomechanics is not discussed.
6. Electromagnetic effects are absent.

I hope that, in the future, the topics just described will be treated in the same spirit as is done here, in particular by using the mathematical infrastructure based on sets, mappings, and families and without using a fixed physical space.

2 Physical Systems

The concept of a materially ordered set was first introduced by Noll in the context of a mathematical model for physical systems (see [LFCM]). The present description is taken from [MBAM].

Here Ω is considered to consist of the whole system and all of its parts. Given $a, b \in \Omega$, $a \prec b$ is read "a is a part of b". The maximum ma is the "material all", i.e. the whole system, and the minimum mn is the "material nothing". The $\inf\{a, b\}$ is the overlap of a and b, and a^{rem} is the part of the whole system ma that remains after a has been removed. With this in mind, the two conditions (MO3) and (MO4) below are very natural.

Definition 2.1 *An ordered set Ω with order $\prec$ is said to be* **materially ordered** *if the following axioms are satisfied:*

(MO1) *Ω has a maximum* ma *and a minimum* mn.

(MO2) *Every doubleton has an infimum.*

(MO3) *For every $p \in \Omega$ there is exactly one member of Ω, denoted by p^{rem}, such that* $\inf\{p, p^{\text{rem}}\} = \text{mn}$ *and* $\sup\{p, p^{\text{rem}}\} = \text{ma}$.

(MO4) $(\inf\{p, q^{\text{rem}}\} = \text{mn}) \implies p \prec q \quad \text{for all} \quad p, q \in M$.

The mapping $\text{rem} := (p \mapsto p^{\text{rem}}) : \Omega \longrightarrow \Omega$ *is called the* **remainder mapping** *in Ω.*

Theorem 2.2 *Let Ω be a materially ordered set. Then every doubleton has a supuremum and Ω has the structure of a Boolean algebra with*

$$p \wedge q := \inf\{p, q\} \quad \text{and} \quad p \vee q := \sup\{p, q\} \quad \text{for all} \quad p, q \in \Omega, \tag{2.1}$$

[1]See, for example, the work of Gurtin and Williams in [CDI] or Coleman and Owen in [MFT].

which means that the following relations:

$$\mathrm{mn} = \mathrm{ma}^{\mathrm{rem}} \tag{2.2}$$

$$p \wedge \mathrm{ma} = p \tag{2.3}$$

$$p \wedge p^{\mathrm{rem}} = \mathrm{mn} \tag{2.4}$$

$$(p^{\mathrm{rem}})^{\mathrm{rem}} = p \tag{2.5}$$

$$p \wedge q = q \wedge p \tag{2.6}$$

$$(p \wedge q) \wedge r = p \wedge (q \wedge r) \tag{2.7}$$

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) \tag{2.8}$$

$$(p \vee q)^{\mathrm{rem}} = p^{\mathrm{rem}} \wedge q^{\mathrm{rem}} \tag{2.9}$$

are valid for all p, q and r in M.

The symbol $p \wedge q$ is read as p **meet** q, and the symbol $p \vee q$ is read as p **join** q.

The formulas (2.2)–(2.7) are intuitively very plausible. The formulas (2.8) and (2.9) are less plausible. The proofs are highly non-trivial. The best version of these can be found in [MBAM].

Theorem 2.3 *Let Ω be a materially ordered set and $p \in \Omega$ be given. Then $\Omega_p := \{q \in \Omega \mid q \prec p\}$ is a materially ordered set with order $\prec$ and the remainder mapping in Ω_p is given by*

$$\mathrm{rem}_p := (a \mapsto a^{\mathrm{rem}} \wedge p). \tag{2.10}$$

The proof is easy.

3 Additive Mappings and Interactions

Every linear space will be over $\mathbb{R}$ and will be finite dimensional unless stated otherwise. Every Euclidean space will also be finite dimensional unless stated otherwise.

Let Ω be a materially ordered set and let $\mathcal{W}$ be a linear space. We say that the parts p and q are **separate** if $p \wedge q = \mathrm{mn}$. We use the notation

$$(\Omega^2)_{\mathrm{sep}} := \{(p, q) \in \Omega^2 \mid p \wedge q = \mathrm{mn}\}. \tag{3.1}$$

Definition 3.1 *A mapping $\mathbf{H} : \Omega \longrightarrow \mathcal{W}$ is said to be* **additive** *if*

$$\mathbf{H}(p \vee q) = \mathbf{H}(p) + \mathbf{H}(q) \quad \text{for all} \quad (p, q) \in (\Omega^2)_{\mathrm{sep}}. \tag{3.2}$$

A function $\mathbf{I} : (\Omega^2)_{\mathrm{sep}} \longrightarrow \mathcal{W}$ is said to be an **interaction** *in Ω if for all $p \in \Omega$ both*

$$\mathbf{I}(\,\cdot\,, p^{\mathrm{rem}}) : \Omega_p \longrightarrow \mathcal{W} \;\text{ and }\; \mathbf{I}(p^{\mathrm{rem}}, \,\cdot\,) : \Omega_p \longrightarrow \mathcal{W}$$

are additive. The **resultant** $\mathrm{Res}_{\mathbf{I}} : \Omega \longrightarrow \mathcal{W}$ *of a given interaction* $\mathbf{I}$ *in* Ω *is defined by*

$$\mathrm{Res}_{\mathbf{I}}(p) := \mathbf{I}(p, p^{\mathrm{rem}}) \quad \text{for all} \quad p \in \Omega. \tag{3.3}$$

We say that a given interaction is **skew** *if*

$$\mathbf{I}(q, p) = -\mathbf{I}(p, q) \quad \text{for all} \quad (p, q) \in (\Omega^2)_{\mathrm{sep}}. \tag{3.4}$$

Remark 3.2 The concept of an interaction is an abstraction. Its values may have the interpretation of forces, torques, or heat transfers. In most of the past literature these cases were treated separately even though much of the underlying mathematics is the same for all. Thus, this abstraction, like most others, is a labor saving device. ▮

Theorem 3.3 *An interaction is skew if and only if its resultant is additive.*

Proof: Let $(p, q) \in (\Omega^2)_{\mathrm{sep}}$ be given, so that $p \wedge q = \mathrm{mn}$. Using some of the rules (2.2)–(2.9), a simple calculation shows that

$$p^{\mathrm{rem}} = q \vee (p \vee q)^{\mathrm{rem}} \qquad \text{and} \qquad \mathrm{mn} = q \wedge (p \vee q)^{\mathrm{rem}}, \tag{3.5}$$

so that $(q, (p \vee q)^{\mathrm{rem}}) \in (\Omega^2)_{\mathrm{sep}}$.

Using the additivity of $\mathbf{I}(p, \cdot) : \Omega_{p^{\mathrm{rem}}} \longrightarrow \mathcal{W}$ it follows that

$$\mathrm{Res}_{\mathbf{I}}(p) = \mathbf{I}(p, p^{\mathrm{rem}}) = \mathbf{I}(p, q) + \mathbf{I}(p, (p \vee q)^{\mathrm{rem}}). \tag{3.6}$$

Interchanging the roles of p and q we find that

$$\mathrm{Res}_{\mathbf{I}}(q) = \mathbf{I}(q, q^{\mathrm{rem}}) = \mathbf{I}(q, p) + \mathbf{I}(q, (q \vee p)^{\mathrm{rem}}). \tag{3.7}$$

Adding (3.6) and (3.7), using the additivity of $\mathbf{I}(\cdot, (p \vee q)^{\mathrm{rem}})$, and then (3.3) with p replaced by $p \vee q$, we obtain

$$\mathrm{Res}_{\mathbf{I}}(p) + \mathrm{Res}_{\mathbf{I}}(q) - \mathrm{Res}_{\mathbf{I}}(p \vee q) = \mathbf{I}(p, q) + \mathbf{I}(q, p), \tag{3.8}$$

from which the assertion follows. ▮

4 Continuous Bodies

As stated in the introduction, mechanics is, roughly, the study of the behavior of physical bodies subject to forces or displacements, and the reciprocal effect of the bodies on their environment. In this section we define the type of physical bodies that we will consider. How these bodies are represented in their environment will be discussed in the next section.

To define a continuous body system, two classes must be specified. One being the class Fr of all subsets of three-dimensional Euclidean spaces that are possible regions that a body system can occupy. Intuitively, the term "body" suggests that the regions it can occupy are connected. We do not assume this but we will use the term "body" rather than "body system" from now on. The other being the class Tp of mappings which are possible changes of placement of a body.

It is useful to take Fr to be the class of **fit regions** introduced by Noll and Virga in [FRBV]. Roughly speaking, a fit region is an open bounded subset of a Euclidean space whose boundary fails to have an exterior normal only at exceptional points. Let a Euclidean space $\mathcal{E}$, with translation space $\mathcal{V}$, be given. We denote by $\mathrm{Fr}\,\mathcal{E}$ the set of all fit regions in $\mathcal{E}$. Let a fit region $\mathcal{A} \in \mathrm{Fr}\,\mathcal{E}$ be given. We denote the set of all points in which there is an exterior normal to $\mathcal{A}$ by $\mathrm{Rby}\mathcal{A}$ and call it the **reduced boundary** of $\mathcal{A}$. Let

$$\mathbf{n}_{\mathcal{A}} : \mathrm{Rby}\mathcal{A} \longrightarrow \mathrm{Usph}\mathcal{V} \tag{4.1}$$

be the mapping that assigns to each point of the reduced boundary the exterior unit normal. Let $\mathbf{C} : \mathcal{A} \longrightarrow \mathrm{Lin}(\mathcal{V}, \mathcal{W})$ be a differentiable mapping that assigns to each point $x \in \mathcal{A}$ a linear mapping from $\mathcal{V}$ to some linear space $\mathcal{W}$. Then the divergence theorem holds (this follows from Theorem 3.36 of [FBV]), namely

$$\int_{\mathrm{Rby}\mathcal{A}} \mathbf{C}\mathbf{n}_{\mathcal{A}} = \int_{\mathcal{A}} \mathrm{div}\mathbf{C}. \tag{4.2}$$

The class Tp consists of all mappings λ with the following properties:

(T$_1$) λ is an invertible mapping whose domain $\mathrm{Dom}\lambda$ and range $\mathrm{Rng}\lambda$ are fit regions in Euclidean spaces $\mathrm{Dsp}\lambda$ and $\mathrm{Rsp}\lambda$, which are called the **domain-space** and **range-space** of λ, respectively.

(T$_2$) There is a C^2-diffeomorphism $\phi : \mathrm{Dsp}\lambda \longrightarrow \mathrm{Rsp}\lambda$ such that $\lambda = \phi|_{\mathrm{Dom}\,\lambda}^{\mathrm{Rng}\,\lambda}$.

The class Tp, whose elements are called **transplacements**, is stable under composition in the sense that for all $\lambda, \gamma \in \mathrm{Tp}$ with $\mathrm{Dom}\,\lambda = \mathrm{Rng}\,\gamma$ we have $\lambda \circ \gamma \in \mathrm{Tp}$. It is also stable under inversion in the sense that if $\lambda \in \mathrm{Tp}$ then $\lambda^{\leftarrow} \in \mathrm{Tp}$.

Assume that a set $\mathcal{B}$ is given. We say that a metric $\delta : \mathcal{B} \times \mathcal{B} \longrightarrow \mathbb{P}$ is a **fit Euclidean metric** on $\mathcal{B}$ if it makes $\mathcal{B}$ isometric to some fit region in some Euclidean space. As shown in Section 6 of Part 2 of [FC], it is then possible to use δ to imbed[2] $\mathcal{B}$ into a Euclidean space $\mathcal{E}_\delta$ constructed from $\mathcal{B}$ using δ. The imbedding imb_δ is invertible with $\mathrm{Dom}\,\mathrm{imb}_\delta = \mathcal{B}$ and $\mathrm{Rng}\,\mathrm{imb}_\delta \in \mathrm{Fr}\,\mathcal{E}_\delta$ such that

$$\delta(X, Y) = \mathrm{dist}(\mathrm{imb}_\delta(X), \mathrm{imb}_\delta(Y)) \quad \text{for all} \quad X, Y \in \mathcal{B}, \tag{4.3}$$

where dist denotes the Euclidean distance in $\mathcal{E}_\delta$. We call $\mathcal{E}_\delta$ the **imbedding space** for δ and imb_δ the **imbedding mapping** for δ.

Definition 4.1 *A* **continuous body** $\mathcal{B}$ *is a set endowed with structure by the specification of a non-empty set* $\mathrm{Conf}\,\mathcal{B}$*, whose elements are called* **configurations** *of* $\mathcal{B}$*, satisfying the following requirements:*

(B$_1$) *Every* $\delta \in \mathrm{Conf}\,\mathcal{B}$ *is a fit Euclidean metric on* $\mathcal{B}$*.*

(B$_2$) *For all* $\delta, \epsilon \in \mathrm{Conf}\,\mathcal{B}$ *the mapping* $\lambda := \mathrm{imb}_\delta \circ \mathrm{imb}_\epsilon^{\leftarrow}$ *is a transplacement, with* $\mathrm{Dsp}\lambda = \mathcal{E}_\epsilon$ *and* $\mathrm{Rsp}\lambda = \mathcal{E}_\delta$*, where* $\mathcal{E}_\epsilon$ *and* $\mathcal{E}_\delta$ *are the imbedding spaces associated with* ϵ *and* δ*, respectively.*

[2] Only the existence of such an imbedding is important here, not the details of its construction.

(B$_3$) *For every* $\delta \in \operatorname{Conf}\mathcal{B}$ *and every transplacement* λ *such that* $\operatorname{Dom}\lambda = \operatorname{Rng}\operatorname{imb}_\delta$*, the function* $\epsilon : \mathcal{B}\times\mathcal{B} \longrightarrow \mathbb{P}$*, defined by*

$$\epsilon(X,Y) := \operatorname{dist}(\lambda(\operatorname{imb}_\delta(X)), \lambda(\operatorname{imb}_\delta(Y)) \quad \text{for all} \quad X, Y \in \mathcal{B}, \tag{4.4}$$

is a fit Euclidean metric that belongs to $\operatorname{Conf}\mathcal{B}$*.*

The elements of $\mathcal{B}$ are called **material points**.

For the rest of Chapter I we assume that a non-empty continuous body $\mathcal{B}$ is given. The imbeddings of $\mathcal{B}$ endow $\mathcal{B}$ with the structure of a three-dimensional C^2-manifold. Thus $\mathcal{B}$ is a topological space and at each material point $X \in \mathcal{B}$ there is a tangent space $\mathcal{T}_X$ which is a three-dimensional linear space. The space $\mathcal{T}_X$ is called the (infinitesimal) **body element** of $\mathcal{B}$ at X since it is the precise mathematical representation of what many engineers refer to as an "infinitesimal element" of the body. Note that the tangent spaces are not inner-product spaces and hence the dual $\mathcal{T}_X^*$ of the tangent space cannot be identified with $\mathcal{T}_X$. However, since $\mathcal{T}_X$ is a finite-dimensional linear space, its second dual $\mathcal{T}_X^{**}$ can be identified with $\mathcal{T}_X$ in a natural way. See Section 22 of [FDS].

Let $\delta \in \operatorname{Conf}\mathcal{B}$ be given. Let $\mathcal{E}_\delta$ denote the corresponding imbedding space, with translation space $\mathcal{V}_\delta$, and let $\operatorname{imb}_\delta$ denote the imbedding mapping for δ. The gradient of $\operatorname{imb}_\delta$ at a material point $X \in \mathcal{B}$,

$$\mathbf{I}_\delta(X) := \nabla_X \operatorname{imb}_\delta \in \operatorname{Lis}(\mathcal{T}_X, \mathcal{V}_\delta), \tag{4.5}$$

is a linear isomorphism from the body element $\mathcal{T}_X$ to the translation space $\mathcal{V}_\delta$ of the imbedding space. It can be used to define

$$\mathbf{G}_\delta(X) := \mathbf{I}_\delta^\top(X)\mathbf{I}_\delta(X) \in \operatorname{Pos}^+(\mathcal{T}_X, \mathcal{T}_X^*), \tag{4.6}$$

which we call the **configuration of the body element** $\mathcal{T}_X$ since it is the localization of the global configuration $\delta \in \operatorname{Conf}\mathcal{B}$.

Remark 4.2 In the literature on differential geometry the mappings $\mathbf{I}_\delta := (X \mapsto \mathbf{I}_\delta(X))$ and $\mathbf{G}_\delta := (X \mapsto \mathbf{G}_\delta(X))$ are cross-sections of appropriately defined fiber bundles. A configuration δ gives $\mathcal{B}$ the structure of a Riemannian manifold. The cross-section $\mathbf{G}_\delta$ is often called the *metric tensor field* for the Euclidean-Riemannian structure defined by δ. ▮

Remark 4.3 The class Tp specified above corresponds to materials without constraints. If one wished to describe materials with constraints then the class Tp would have to be restricted. For example, if one also requires that the transplacements are volume preserving then Tp makes the body incompressible. ▮

Consider the set $\Omega_\mathcal{B}$ defined by

$$\Omega_\mathcal{B} := \{\mathcal{P} \in \operatorname{Sub}\mathcal{B} \mid \operatorname{imb}_{\delta>}(\mathcal{P}) \in \operatorname{Fr} \text{ for some } \delta \in \operatorname{Conf}\mathcal{B}\}. \tag{4.7}$$

It follows from (T$_1$) and (B$_2$) that if $\operatorname{imb}_{\delta>}(\mathcal{P}) \in \operatorname{Fr}$ for some $\delta \in \operatorname{Conf}\mathcal{B}$ then $\operatorname{imb}_{\delta>}(\mathcal{P})$ is a fit region for every configuration δ. If $\mathcal{P}$ is an element of $\Omega_\mathcal{B}$ then the set

$$\operatorname{Conf}\mathcal{P} := \{\delta|_{\mathcal{P}\times\mathcal{P}} \mid \delta \in \operatorname{Conf}\mathcal{B}\} \tag{4.8}$$

endows $\mathcal{P}$ with the structure of a continuous body. For this reason such $\mathcal{P} \in \Omega_{\mathcal{B}}$ are called **parts** or **sub-bodies** of $\mathcal{B}$.

The set $\Omega_{\mathcal{B}}$ is materially ordered, by inclusion in the sense of Definition 2.1 and hence, by Theorem 2.2, it has the structure of a Boolean algebra. We have

$$\mathcal{P} \wedge \mathcal{Q} := \mathcal{P} \cap \mathcal{Q}, \tag{4.9}$$

$$\mathcal{P} \vee \mathcal{Q} := \text{Int Clo}(\mathcal{P} \cup \mathcal{Q}), \tag{4.10}$$

$$\mathcal{P}^{\text{rem}} := \text{Int}(\mathcal{B} \backslash \mathcal{P}). \tag{4.11}$$

The proof of this highly non-trivial, though not surprising, result can be found in [FRBV]. One should think of $\mathcal{P} \wedge \mathcal{Q}$ as the common part of $\mathcal{P}$ and $\mathcal{Q}$, $\mathcal{P} \vee \mathcal{Q}$ as the part obtained by merging $\mathcal{P}$ and $\mathcal{Q}$, and $\mathcal{P}^{\text{rem}}$ as the part of $\mathcal{B}$ left when $\mathcal{P}$ is taken away.

5 Frames of Reference and Placements

When dealing with the behavior of a continuous body in an environment it is useful to employ a frame of reference. It takes very little reflection to realize that it makes no sense to speak of location, and hence of motion etc., except relative to an explicitly or tacitly specified frame of reference. Such frames are represented mathematically by three-dimensional Euclidean spaces. We call Euclidean spaces that represent frames of reference **frame-spaces**.

Definition 5.1 *Let μ be an invertible mapping with* $\text{Dom}\,\mu = \mathcal{B}$ *and* $\mathcal{B}_\mu := \text{Rng}\,\mu \in \text{Fr}$. *We say that μ is a* **placement** *of $\mathcal{B}$ if* $\text{imb}_\delta \circ \mu^{\leftarrow}$ *is a transplacement for every* $\delta \in \text{Conf}\,\mathcal{B}$. *The Euclidean space in which* $\text{Rng}\,\mu$ *is a fit region is called the* **range-space** *of μ and will be denoted by* $\text{Frm}\,\mu$. *We denote the translation space of* $\text{Frm}\,\mu$ *by* $\text{Vfr}\,\mu$. *We denote the set of all placements of $\mathcal{B}$ by* $\text{Pl}\,\mathcal{B}$.

The following facts are easy consequences of Definitions. 4.1 and 5.1:

($\mathbf{P}_1$) For all $\kappa, \gamma \in \text{Pl}\,\mathcal{B}$ we have $\kappa \circ \gamma^{\leftarrow} \in \text{Tp}$.

($\mathbf{P}_2$) For every $\kappa \in \text{Pl}\,\mathcal{B}$ and $\lambda \in \text{Tp}$ such that $\text{Rng}\,\kappa = \text{Dom}\,\lambda$ we have $\lambda \circ \kappa \in \text{Pl}\,\mathcal{B}$.

Note that an imbedding induced by a configuration is a placement, but not every placement is an imbedding. It follows from (P_2) that in Definition 5.1 "every" could be replaced by "some" without changing the meaning.

Let a placement $\mu : \mathcal{B} \longrightarrow \mathcal{B}_\mu \subseteq \text{Frm}\mu$ be given. We then define $\delta_\mu : \mathcal{B} \times \mathcal{B} \longrightarrow \mathbb{P}$ by

$$\delta_\mu(X, Y) := \text{dist}(\mu(X), \mu(Y)) \quad \text{for all} \quad X, Y \in \mathcal{B}, \tag{5.1}$$

where dist is the Euclidean distance in $\text{Frm}\,\mu$. It follows from (B_3) and Definition 5.1 that δ_μ is a Euclidean metric and, in fact, a configuration of $\mathcal{B}$. We call δ_μ the **configuration induced by the placement** μ[3]. Now let a configuration $\delta \in \text{Conf}\mathcal{B}$ be given. Let μ

[3] In [NLFT] a "placement" is called a "configuration". Following what is done in [NTSM] here we use the word "configuration" for the concept used in Definition 4.1.

and μ' be two placements that induce the same configuration δ. Since $\delta = \delta_\mu = \delta_{\mu'}$, it follows from (5.1) that $\mu' \circ \mu^{\leftarrow}$ is a Euclidean isometry. Since there are infinitely many Euclidean isometries with domain $\mathcal{B}_\mu$ there are infinitely many placements that induce the given configuration. In particular, in the case when $\mu' := \mathrm{imb}_\delta$,

$$\alpha := \mathrm{imb}_\delta \circ \mu^{\leftarrow} : \mathcal{B}_\mu \longrightarrow \mathrm{Rng}\,\mathrm{imb}_\delta \tag{5.2}$$

is a Euclidean isometry. Its (constant) gradient

$$\mathbf{Q} := \nabla_x(\mathrm{imb}_\delta \circ \mu^{\leftarrow}) \in \mathrm{Orth}(\mathrm{Vfr}\,\mu, \mathcal{V}_\delta) \quad \text{for all} \;\; x \in \mathcal{B}_\mu \tag{5.3}$$

is an inner-product isomorphism.

Define

$$\mathbf{M}_\mu(X) := \nabla_X \mu \in \mathrm{Lis}(\mathcal{T}_X, \mathrm{Vfr}\,\mu) \qquad \text{for all} \quad X \in \mathcal{B}. \tag{5.4}$$

In view of (4.5), it follows from (5.3) and the chain rule that

$$\mathbf{I}_\delta = \mathbf{Q}\mathbf{M}_\mu, \tag{5.5}$$

and hence, by (4.6), that the configuration of the body element $\mathcal{T}_X$ induced by δ is given by

$$\mathbf{G}_\delta(X) = \mathbf{M}_\mu(X)^\top \mathbf{M}_\mu(X) \in \mathrm{Pos}^+(\mathcal{T}_X, \mathcal{T}_X^*) \quad \text{for all} \;\; X \in \mathcal{B}. \tag{5.6}$$

For a given placement μ we use the notation

$$\mathcal{P}_\mu := \mu_>(\mathcal{P}) \quad \text{for all} \;\; \mathcal{P} \in \Omega_{\mathcal{B}}. \tag{5.7}$$

Given $(\mathcal{P}, \mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\mathrm{sep}}$ we define the **reduced contact** of $(\mathcal{P}, \mathcal{Q})$ in μ by

$$\mathrm{Rct}_\mu(\mathcal{P}, \mathcal{Q}) := \mathrm{Rby}\mathcal{P}_\mu \cap \mathrm{Rby}\mathcal{Q}_\mu. \tag{5.8}$$

Let another placement μ' be given. Let $\mathbf{B} \in \mathrm{Orth}(\mathrm{Vfr}\,\mu, \mathrm{Vfr}\,\mu')$ be an inner-product preserving isomorphism from $\mathrm{Vfr}\,\mu$ to $\mathrm{Vfr}\,\mu'$. Put

$$\lambda := \mu \circ \mu'^{\leftarrow}. \tag{5.9}$$

Define the **local volume change function** $\rho_{\mu',\mu} : \mathcal{B}_{\mu'} \longrightarrow \mathbb{P}^\times$ by

$$\rho_{\mu',\mu}(x) := |\det(\nabla_x \lambda \mathbf{B})| \quad \text{for all} \;\; x \in \mathcal{B}_{\mu'}. \tag{5.10}$$

This definition is independent of which inner-product isomorphism from $\mathrm{Vfr}\,\mu$ to $\mathrm{Vfr}\,\mu'$ is used. To see this let $\mathbf{B}' \in \mathrm{Orth}(\mathrm{Vfr}\,\mu, \mathrm{Vfr}\,\mu')$ be another such isomorphism from $\mathrm{Vfr}\,\mu$ to $\mathrm{Vfr}\,\mu'$. It is clear that $\mathbf{Q} := \mathbf{B}^{-1}\mathbf{B}' \in \mathrm{Orth}(\mathrm{Vfr}\,\mu)$. Hence, since the determinant of an orthogonal lineon is ± 1, we have

$$|\det(\nabla_x \lambda \mathbf{B}')| = |\det(\nabla_x \lambda \mathbf{B}\mathbf{Q})| = |\det(\nabla_x \lambda \mathbf{B})||\det \mathbf{Q}| = |\det(\nabla_x \lambda \mathbf{B})|. \tag{5.11}$$

Since the transplacement λ is of class C^2, $\rho_{\mu',\mu}$ is of class C^1.

Let placements $\mu_1, \mu_2 \in \mathrm{Pl}\mathcal{B}$ of the body be given. Let $X \in \mathcal{B}$ be given and put $x := \mu_1(X)$. Furthermore, put $\mathbf{M}_1 := \nabla_X \mu_1$ and $\mathbf{M}_2 := \nabla_X \mu_2$. Then

$$\begin{aligned} \mathbf{M}_1^\top \mathbf{M}_1 = \mathbf{M}_2^\top \mathbf{M}_2 \;\; &\Longrightarrow \;\; (\mathbf{M}_2\mathbf{M}_1^{-1})^\top (\mathbf{M}_2\mathbf{M}_1^{-1}) = \mathbf{1}_{\mathrm{Vfr}\mu_1} \\ &\Longrightarrow \;\; \mathbf{M}_2\mathbf{M}_1^{-1} \in \mathrm{Orth}(\mathrm{Vfr}\mu_1, \mathrm{Vfr}\mu_2) \\ &\Longrightarrow \;\; |\det(\nabla_x(\mu_2 \circ \mu_1^{\leftarrow})\mathbf{B}| = 1 \\ &\Longrightarrow \;\; \rho_{\mu_1,\mu_2}(x) = 1. \end{aligned} \tag{5.12}$$

This fact will be needed in Section 8.

6 Time-families

In much of the rest of this thesis we assume that a genuine real interval I, called the **time-interval**, is given. Any family $f := (f_t \mid t \in I)$ indexed on I will be called a **time-family**. In some cases, each of the terms f_t of the family belong to a given set $\mathcal{S}$. In this case, the family can be identified with a mapping $f : I \longrightarrow \mathcal{S}$, so that

$$f(t) := f_t \quad \text{for all} \;\; t \in I. \tag{6.1}$$

If $\mathcal{S}$ is a Euclidean space or linear space, it makes sense to consider the case when f is of class C^1 or C^2 and then define the time-families $(f_t^{\bullet} \mid t \in I)$ or $(f_t^{\bullet\bullet} \mid t \in I)$ by[4]

$$f_t^{\bullet} := f^{\bullet}(t) \;\; \text{or} \;\; f_t^{\bullet\bullet} := f^{\bullet\bullet}(t) \quad \text{for all} \;\; t \in I. \tag{6.2}$$

In some cases, one deals with a time-family $(\mathcal{A}_t \mid t \in I)$ of sets and considers a time-family $(g_t \mid t \in I)$ of mappings $g_t : \mathcal{A}_t \longrightarrow \mathcal{S}$ with values in a set $\mathcal{S}$. Putting $\mathcal{M} := \{(X, t) \mid X \in \mathcal{A}_t \text{ and } t \in I\}$, we can then identify the family $(g_t \mid t \in I)$ with the mapping $g : \mathcal{M} \longrightarrow \mathcal{S}$ defined by

$$g(X, t) := g_t(X) \quad \text{for all} \;\; X \in \mathcal{A}_t \text{ and } t \in I. \tag{6.3}$$

Assume now that the terms in the family $(\mathcal{A}_t \mid t \in I)$ are all equal to a fixed set $\mathcal{A}$, so that $\mathcal{M} = \mathcal{A} \times I$ and that $\mathcal{S}$ is a Euclidean space or linear space. Then it makes sense to consider the case when $g(X, \cdot)$ is of class C^1 for all $X \in \mathcal{A}$ and consider the time-family $(g_t^{\bullet} \mid t \in I)$ of mappings defined by

$$g_t^{\bullet}(X) := g(X, \cdot)^{\bullet}(t) \qquad \text{for all} \;\; X \in \mathcal{A} \text{ and } t \in I, \tag{6.4}$$

which we call the **time-derivative** of the time-family $(g_t \mid t \in I)$. The formula (6.4) can also be applied when $\mathcal{M}$ is a subset of $\mathcal{A} \times I$ such that for each $(X, t) \in \mathcal{M}$ there is a neighborhood of t such that $(X, s) \in \mathcal{M}$ for all s in this neighborhood.

In Chapters II and III we will use time-families that are piecewise C^1 in the sense that they are continuous, and have continuous derivatives except at a finite number of jumps. Let $f : I \longrightarrow \mathcal{S}$ be a piecewise C^1 time-family. Let $t \in I$ be a time in which $f^{\bullet}$ has a jump. If t is not the left endpoint of I we denote by

$$f^{\bullet}(t) := \lim_{h \to 0^+} \frac{f(t) - f(t-h)}{h} \tag{6.5}$$

the left-hand derivative of f at t. If t is the left endpoint of I we denote by $f^{\bullet}(t)$ the right-hand derivative of f at t. Thus, the domain of $f^{\bullet}$ can be taken to be all of I.

In general, all equations, involving either mappings or families, are understood to hold value-wise or term-wise.

[4]Given a differentiable mapping f whose domain is an open interval we denote its derivative by $f^{\bullet}$.

7 Motions

This section deals with the kinematics of a continuous body. The presentation here is by no means complete. For a more detailed treatment of kinematics similar in spirit to the treatment below see Chapter III of [ICM].

We assume that a time-interval I and a fixed frame-space $\mathcal{F}$ with translation space $\mathcal{V}$ are given.

Definition 7.1 *A* **motion** *is a C^2 mapping $\bar{\mu} : \mathcal{B} \times I \longrightarrow \mathcal{F}$ such that for all $t \in I$, $\bar{\mu}_t := \bar{\mu}(\cdot, t) \in \mathrm{Pl}\,\mathcal{B}$. Thus, a motion can also be viewed as a time-family of placements in the space $\mathcal{F}$. The* **trajectory** *of the motion $\bar{\mu}$ is the set*

$$\mathcal{M} := \{(\bar{\mu}_t(X), t) \in \mathcal{F} \times I \mid (X, t) \in \mathcal{B} \times I\}. \tag{7.1}$$

A mapping from $\mathcal{B} \times I$ to some linear space will be called a **material field**, *and a mapping from the trajectory $\mathcal{M}$ to some linear space will be called a* **spatial field**.[5]

We assume now that a motion $\bar{\mu}$ is given. A material field can be used to generate a spatial field and vice vera in the following way: Let $\mathcal{W}$ be a linear space and $\Phi : \mathcal{B} \times I \longrightarrow \mathcal{W}$ be a material field. We can define the **associated spatial field** $\Phi_{\mathrm{s}} : \mathcal{M} \longrightarrow \mathcal{W}$ by

$$\Phi_{\mathrm{s}}(x, t) := \Phi(\bar{\mu}_t^{\leftarrow}(x), t) \quad \text{for all} \quad (x, t) \in \mathcal{M}. \tag{7.2}$$

Given a spatial field $\Psi : \mathcal{M} \longrightarrow \mathcal{W}$ we can define the **associated material field** $\Psi_{\mathrm{m}} : \mathcal{B} \times I \longrightarrow \mathcal{W}$ by

$$\Psi_{\mathrm{m}}(X, t) := \Psi(\bar{\mu}_t(X), t) \quad \text{for all} \quad (X, t) \in \mathcal{B} \times I. \tag{7.3}$$

Note that $(\Phi_{\mathrm{s}})_{\mathrm{m}} = \Phi$ and $(\Psi_{\mathrm{m}})_{\mathrm{s}} = \Psi$.

If a material field is continuous, of class C^1, or class C^2, so is the associated spatial field and vice versa. If they are of class C^1, we use the notations

$$\Phi^{\bullet}(X, t) := \Phi(X, \cdot)^{\bullet}(t), \quad \nabla\Phi(X, t) := \nabla(\Phi(\cdot, t))(X) \quad \text{for all} \quad (X, t) \in \mathcal{B} \times I \tag{7.4}$$

and

$$\Psi^{\bullet}(x, t) := \Psi(x, \cdot)^{\bullet}(t), \quad \nabla\Psi(x, t) := \nabla(\Psi(\cdot, t))(x) \quad \text{for all} \quad (x, t) \in \mathcal{M}. \tag{7.5}$$

Assume that Φ and Ψ are of class C^1. Using (6.3) we can then consider the time-families $(\Phi_t \mid t \in I)$ and $(\Psi_t \mid t \in I)$ and their time-derivatives $(\Phi_t^{\bullet} \mid t \in I)$ and $(\Psi_t^{\bullet} \mid t \in I)$. Of course, $\Phi^{\bullet}$ is a continuous material field and $\Psi^{\bullet}$ is a continuous spatial field. They are of class C^1 if the original fields were of class C^2.

The **spatial velocity** $\bar{\mathbf{v}} : \mathcal{B} \times I \longrightarrow \mathcal{V}$ and **spatial acceleration** $\bar{\mathbf{a}} : \mathcal{B} \times I \longrightarrow \mathcal{V}$ are defined by

$$\bar{\mathbf{v}} := (\bar{\mu}^{\bullet})_{\mathrm{s}}, \quad \text{and} \quad \bar{\mathbf{a}} := (\bar{\mu}^{\bullet\bullet})_{\mathrm{s}}. \tag{7.6}$$

[5] In much of the past literature, the terms "Lagrangian" and "Eulerian" have been used instead of "spatial" and "material". This is unfortunate because these terms are non-descripive and historically inaccurate.

We use the following notation for the velocity gradient and its value-wise symmetric part:

$$\bar{\mathbf{L}} := \nabla\bar{\mathbf{v}} : \mathcal{M} \longrightarrow \mathrm{Lin}\mathcal{V}, \tag{7.7}$$

$$\bar{\mathbf{D}} := \frac{1}{2}(\bar{\mathbf{L}} + \bar{\mathbf{L}}^{\top}) : \mathcal{M} \longrightarrow \mathrm{Sym}\mathcal{V}. \tag{7.8}$$

The **material time-derivative** Ψ° of the spatial field Ψ is the spatial field defined by

$$\Psi^{\circ} := ((\Psi_{\mathrm{m}})^{\bullet})_{\mathrm{s}}. \tag{7.9}$$

Using (7.3) and the chain rule, it follows that

$$\Psi^{\circ} = \Psi^{\bullet} + (\nabla\Psi)\bar{\mathbf{v}}. \tag{7.10}$$

Applying (7.10) to the case when Ψ is the spatial velocity, we obtain the following relation between the spatial fields associated with the velocity and acceleration:

$$\bar{\mathbf{a}} = \bar{\mathbf{v}}^{\bullet} + \bar{\mathbf{L}}\bar{\mathbf{v}}. \tag{7.11}$$

We use the notation

$$\bar{\mathbf{M}}(X,t) := \bar{\mathbf{M}}_t(X) := \nabla\bar{\mu}_t(X) \in \mathrm{Lis}(\mathcal{T}_X, \mathcal{V}) \quad \text{for all} \quad (X,t) \in \mathcal{B} \times I, \tag{7.12}$$

and, for each $X \in \mathcal{B}$, we call the mapping $\bar{\mathbf{M}}(X,\cdot) : I \longrightarrow \mathrm{Lis}(\mathcal{T}_X, \mathcal{V})$ the **motion of the body element** $\mathcal{T}_X$ induced by the motion $\bar{\mu}$ of the body.

It is sometimes useful to specify a fixed **reference placement** $\kappa : \mathcal{B} \longrightarrow \mathcal{B}_\kappa$ in the frame-space $\mathcal{F}$ and characterize all other placements μ in $\mathcal{F}$ by their transplacements from κ. Thus, we obtain the **transplacement process** $\bar{\chi} : \mathcal{B}_\kappa \times I \longrightarrow \mathcal{F}$ given by

$$\bar{\chi}(p,t) := \bar{\mu}(\kappa^{\leftarrow}(p),t) \quad \text{for all} \quad (p,t) \in \mathcal{B}_\kappa \times I. \tag{7.13}$$

We use the notation

$$\mathbf{K}(X) := \nabla_X\kappa \qquad \text{for all} \qquad X \in \mathcal{B} \tag{7.14}$$

and call, for each $X \in \mathcal{B}$, $\mathbf{K}(X)$ the **reference placement of the body element** $\mathcal{T}_X$ induced by the reference placement κ of the whole body. The **transplacement gradient process** $\bar{\mathbf{F}} : \mathcal{B}_\kappa \times I \longrightarrow \mathrm{Lis}\ \mathcal{V}$ is given by

$$\bar{\mathbf{F}}(p,t) = \nabla\bar{\chi}(p,t) = \bar{\mathbf{M}}_t(\kappa^{\leftarrow}(p))\mathbf{K}^{-1}(\kappa^{\leftarrow}(p)) \quad \text{for all} \quad (p,t) \in \mathcal{B}_\kappa \times I. \tag{7.15}$$

8 Densities and Contactors

Often times additive mappings and interactions are represented by integrals. This section discusses these representations.

Definition 8.1 *An* **internal part** *is a part $\mathcal{P} \in \Omega_{\mathcal{B}}$ in which for every placement $\mu \in \mathrm{Pl}\,\mathcal{B}$ we have $\mathrm{Clo}\mathcal{P}_\mu \subseteq \mathcal{B}_\mu$. We denote the set of all internal parts by $\Omega_{\mathcal{B}}^{\mathrm{int}}$.*

It is easily seen, using the properties (B_2) and (T_2) in Section 4, that in this definition "every" can be replaced by "some"' without changing the meaning.

We assume now that a linear space $\mathcal{W}$ is given.

Definition 8.2 *An additive mapping* $\mathbf{H} : \Omega_{\mathcal{B}} \longrightarrow \mathcal{W}$ *is said to have* **densities** *if for every* $\mu \in \mathrm{Pl}\,\mathcal{B}$ *there is a continuous mapping* $\mathbf{h}_\mu : \mathcal{B}_\mu \longrightarrow \mathcal{W}$ *such that*

$$\mathbf{H}(\mathcal{P}) = \int_{\mathcal{P}_\mu} \mathbf{h}_\mu \qquad \text{for all } \ \mathcal{P} \in \Omega_{\mathcal{B}}^{\mathrm{int}}. \tag{8.1}$$

We call $\mathbf{h}_\mu$ *the* **density** *of* $\mathbf{H}$ *in the placement* μ.

Let $\mu, \mu' \in \mathrm{Pl}\,\mathcal{B}$ be two placements such that (8.1) holds for μ. Consider the transplacement $\alpha := \mu \circ \mu'^{\leftarrow} : \mathcal{B}_{\mu'} \longrightarrow \mathcal{B}_\mu$. By the Theorem on Transformation of Volume Integrals[6] and (5.10) we have

$$\mathbf{H}(\mathcal{P}) = \int_{\mathcal{P}_\mu} \mathbf{h}_\mu = \int_{\mathcal{P}_{\mu'}} \rho_{\mu',\mu}(\mathbf{h}_\mu \circ \alpha) \quad \text{for all } \ \mathcal{P} \in \Omega_{\mathcal{B}}^{\mathrm{int}}. \tag{8.2}$$

Therefore, it follows that in Definition 8.2, "every" could be replaced by "some" without changing the meaning. Moreover, if $\mathbf{h}_\mu$ is the density of $\mathbf{H}$ in the placement μ, then $\mathbf{h}_{\mu'} := \rho_{\mu',\mu}(\mathbf{h}_\mu \circ \alpha)$ is the density of $\mathbf{H}$ in the placement μ'.

Definition 8.3 *We say that an interaction* $\mathbf{I} : (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \longrightarrow \mathcal{W}$ *has* **contactors** *if, for every placement* μ*, there is a* C^1 *mapping* $\mathbf{C}_\mu : \mathcal{B}_\mu \longrightarrow \mathrm{Lin}(\mathrm{Vfr}\mu, \mathcal{W})$ *such that*

$$\mathbf{I}(\mathcal{P},\mathcal{Q}) = \int_{\mathrm{Rct}_\mu(\mathcal{P},\mathcal{Q})} \mathbf{C}_\mu \mathbf{n}_{\mathcal{P}_\mu} \quad \text{for all } \ (\mathcal{P},\mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \ \text{ with } \ \mathcal{P} \in \Omega_{\mathcal{B}}^{\mathrm{int}}. \tag{8.3}$$

We call $\mathbf{C}_\mu$ *the* **contactor** *of* $\mathbf{I}$ *in the placement* μ.

Let $\mu, \mu' \in \mathrm{Pl}\,\mathcal{B}$ be two placements and assume that (8.3) holds for μ. Consider again the transplacement $\alpha := \mu \circ \mu'^{\leftarrow} : \mathcal{B}_{\mu'} \longrightarrow \mathcal{B}_\mu$. By the Theorem on Transformation of Surface Integrals[7] and (5.10) we have

$$\mathbf{I}(\mathcal{P},\mathcal{Q}) = \int_{\mathrm{Rct}_{\mu'}(\mathcal{P},\mathcal{Q})} \rho_{\mu',\mu}(\mathbf{C}_\mu \circ \alpha)(\nabla\alpha)^{-\top} \mathbf{n}_{\mathcal{P}_{\mu'}} \quad \text{for all } \ (\mathcal{P},\mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \ \text{ with } \ \mathcal{P} \in \Omega_{\mathcal{B}}^{\mathrm{int}}. \tag{8.4}$$

Since $\mathbf{C}_\mu$ is of class C^1, so is $\mathbf{C}_{\mu'} := \rho_{\mu',\mu}(\mathbf{C}_\mu \circ \alpha)(\nabla\alpha)^{-\top}$ and hence is the contactor of $\mathbf{I}$ in the placement μ'. We conclude that in Definition 8.3, "every" could be replaced by "some" without changing the meaning.

[6]See Vol. II of [FDS].

[7]This follows from Theorem 2.91 of [FBV]. A standard proof of the transformation of Volume Integrals can not be used since fit regions are sets of finite perimeter. Thus, a proof using geometric measure theory is needed.

In the case when $\mathcal{Q} := \mathcal{P}^{\rm rem}$, (8.3) reduces to

$$\mathrm{Res}_{\mathbf{I}}(\mathcal{P}) = \int_{\mathrm{Rby}\mathcal{P}_\mu} \mathbf{C}_\mu \mathbf{n}_{\mathcal{P}_\mu} \quad \text{for all} \;\; \mathcal{P} \in \Omega_{\mathcal{B}}^{\rm int}. \tag{8.5}$$

Theorem 8.4 *Given an interaction $\mathbf{I} : (\Omega_{\mathcal{B}})^2_{\rm sep} \longrightarrow \mathcal{W}$ with contactors and an additive mapping $\mathbf{H} : \Omega_{\mathcal{B}} \longrightarrow \mathcal{W}$ with densities, the following three conditions are equivalent:*

1. *We have*
$$\mathrm{Res}_{\mathbf{I}}(\mathcal{P}) + \mathbf{H}(\mathcal{P}) = \mathbf{0} \quad \text{for all} \;\; \mathcal{P} \in \Omega_{\mathcal{B}}^{\rm int}. \tag{8.6}$$
2. *For every placement $\mu \in \mathrm{Pl}\,\mathcal{B}$, we have*
$$\mathrm{div}\mathbf{C}_\mu + \mathbf{h}_\mu = \mathbf{0}, \tag{8.7}$$
where $\mathbf{h}_\mu$ is the density of $\mathbf{H}$ in the placement μ, and $\mathbf{C}_\mu$ is the contactor of $\mathbf{I}$ in the placement μ.
3. *Condition 2 holds with "every" replaced by "some".*

Proof: Assume that (8.6) holds, let $\mu \in \mathrm{Pl}\,\mathcal{B}$ be given, let $\mathbf{h}_\mu$ be the density of $\mathbf{H}$ in the placement μ as characterized by (8.1), and let $\mathbf{C}_\mu$ be the contactor of $\mathbf{I}$ in the placement μ as characterized by (8.3). In view of (8.1) and (8.5), (8.6) is equivalent to

$$\int_{\mathrm{Rby}\mathcal{P}_\mu} \mathbf{C}_\mu \mathbf{n}_{\mathcal{P}_\mu} + \int_{\mathcal{P}_\mu} \mathbf{h}_\mu = \mathbf{0} \quad \text{for all} \;\; \mathcal{P} \in \Omega_{\mathcal{B}}^{\rm int}. \tag{8.8}$$

Since $\mathbf{C}_\mu$ is of class C^1 we can use the divergence theorem (4.2) to show that (8.8) is equivalent to

$$\int_{\mathcal{P}_\mu} (\mathrm{div}\mathbf{C}_\mu + \mathbf{h}_\mu) = \mathbf{0} \quad \text{for all} \;\; \mathcal{P} \in \Omega_{\mathcal{B}}^{\rm int}. \tag{8.9}$$

Since $\mathrm{div}\mathbf{C}_\mu$ and $\mathbf{h}_\mu$ are continuous and (8.9) holds for all interior parts we see that (8.9) is equivalent to

$$\mathrm{div}\mathbf{C}_\mu + \mathbf{h}_\mu = \mathbf{0}. \tag{8.10}$$

Since $\mu \in \mathrm{Pl}\,\mathcal{B}$ was arbitrary this implies that condition 2 is valid. If (8.7) is valid just for some $\mu \in \mathrm{Pl}\,\mathcal{B}$ then the equivalences mentioned show that (8.6) holds. ∎

The proof of the following result is analogous to the proof just presented.

Theorem 8.5 *Given a real valued interaction $I : (\Omega_{\mathcal{B}})^2_{\rm sep} \longrightarrow \mathbb{R}$ with contactors and a real valued additive mapping $H : \Omega_{\mathcal{B}} \longrightarrow \mathbb{R}$ with densities, then the following three conditions are equivalent:*

1. *We have*
$$\mathrm{Res}_I(\mathcal{P}) + H(\mathcal{P}) \geq 0 \quad \text{for all} \quad \mathcal{P} \in \Omega_{\mathcal{B}}^{\rm int}. \tag{8.11}$$

2. *For every placement $\mu \in \mathrm{Pl}\,\mathcal{B}$, we have*

$$\operatorname{div} \mathbf{c}_\mu + h_\mu \geq 0. \tag{8.12}$$

where h_μ is the density of H in the placement μ, and $\mathbf{c}_\mu$ is the contactor of I in the placement μ.

3. *Condition 2 hold with "every" replaced by "some".*

It is often useful to introduce a **reference mass**, which is an additive mapping $m : \Omega_{\mathcal{B}} \longrightarrow \mathbb{P}^\times$ with densities. Given a placement μ and a sub-body $\mathcal{P} \in \Omega_{\mathcal{B}}$ the **mass** of this sub-body is given by

$$m(\mathcal{P}) = \int_{\mathcal{P}_\mu} \rho_\mu, \tag{8.13}$$

where $\rho_\mu : \mathcal{B}_\mu \longrightarrow \mathbb{P}^\times$ is the density of m in the placement μ and is called the **mass-density** of the body in the placement μ. It follows from (8.2) and (5.12) that if μ' is another placement that for $X \in \mathcal{B}$

$$\rho_\mu(\mu(X)) = \rho_{\mu'}(\mu'(X)) \quad \text{when} \quad \mathbf{M}_\mu^\top(X)\mathbf{M}_\mu(X) = \mathbf{M}_{\mu'}^\top(X)\mathbf{M}_{\mu'}(X). \tag{8.14}$$

It follows that for all $X \in \mathcal{B}$ we may define a mapping $\hat{\rho}_X : \mathrm{Pos}^+(\mathcal{T}_X, \mathcal{T}_X^*) \longrightarrow \mathbb{P}^\times$ given by

$$\begin{gathered} \hat{\rho}_X(\mathbf{G}) := \rho_\mu(\mu(X)) \quad \text{for all} \ \ \mathbf{G} \in \mathrm{Pos}^+(\mathcal{T}_X, \mathcal{T}_X^*) \text{ and } \mu \in \mathrm{Pl}\mathcal{B} \\ \text{such that } \mathbf{M}_\mu^\top(X)\mathbf{M}_\mu(X) = \mathbf{G}. \end{gathered} \tag{8.15}$$

We also have

$$\rho_\mu \circ \mu = \rho_{\mu'} \circ \mu' \quad \text{when} \quad \delta_\mu = \delta_{\mu'}. \tag{8.16}$$

From now on we assume that a reference mass m is given.

Theorem 8.6 *Let $\mathbf{H} : \Omega_{\mathcal{B}} \longrightarrow \mathcal{W}$ be an additive mapping with densities. Then there is a mapping $\mathbf{h} : \mathcal{B} \longrightarrow \mathcal{W}$, called the* **specific density** *of $\mathbf{H}$, such that*

$$\mathbf{h} = \frac{\mathbf{h}_\mu}{\rho_\mu} \circ \mu \qquad \text{for all} \ \ \mu \in \mathrm{Pl}\,\mathcal{B}. \tag{8.17}$$

Proof: Let $\mu, \mu' \in \mathrm{Pl}\,\mathcal{B}$ be given. Using (8.2) we have $\mathbf{h}_{\mu'} = \rho_{\mu',\mu}(\mathbf{h}_\mu \circ \mu \circ \mu'^{\leftarrow})$ and $\rho_{\mu'} = \rho_{\mu',\mu}(\rho_\mu \circ \mu \circ \mu'^{\leftarrow})$. It follows that

$$\frac{\mathbf{h}_{\mu'}}{\rho_{\mu'}} \circ \mu' = \frac{\mathbf{h}_\mu \circ \mu \circ \mu'^{\leftarrow}}{\rho_\mu \circ \mu \circ \mu'^{\leftarrow}} \circ \mu' = \frac{\mathbf{h}_\mu}{\rho_\mu} \circ \mu. \tag{8.18}$$

Since the placements μ and μ' were arbitrary, this proves the theorem. ∎

In light of Theorem 8.6 we will use the following notation:

$$\int_{\mathcal{P}} \mathbf{h} \, \mathrm{d}m := \mathbf{H}(\mathcal{P}) = \int_{\mathcal{P}_\mu} \mathbf{h}_\mu = \int_{\mathcal{P}_\mu} \rho_\mu(\mathbf{h} \circ \mu^{\leftarrow}) \quad \text{for all} \ \ \mu \in \mathrm{Pl}\,\mathcal{B} \text{ and } \mathcal{P} \in \Omega_{\mathcal{B}}^{\mathrm{int}}. \tag{8.19}$$

Let $\bar{\mathbf{H}} := (\bar{\mathbf{H}}_t : \Omega_{\mathcal{B}} \longrightarrow \mathcal{W} \mid t \in I)$ be a time-family of additive mappings. As explained in Section 6, this time-family can be identified with a mapping $\bar{\mathbf{H}} : \Omega_{\mathcal{B}} \times I \longrightarrow \mathcal{W}$. We say that $\bar{\mathbf{H}}$ is of class C^1 if the mapping $\bar{\mathbf{H}}(\mathcal{P}, \cdot) : I \longrightarrow \mathcal{W}$ is of class C^1 for all $\mathcal{P} \in \Omega_{\mathcal{B}}$. If this is the case, we can form the time-derivative $\bar{\mathbf{H}}_t^{\bullet}(\mathcal{P}) := \bar{\mathbf{H}}(\mathcal{P}, \cdot)^{\bullet}(t)$ for all $\mathcal{P} \in \Omega_{\mathcal{B}}$ and $t \in I$ of the given family. It is clear that this time-derivative is also a time-family of additive mappings.

We now assume that every mapping in the time-family $\bar{\mathbf{H}}$ has densities in the sense of Definition 8.2. Let $\bar{\mathbf{h}} : \mathcal{B} \times I \longrightarrow \mathcal{W}$ be the mapping such that $\bar{\mathbf{h}}_t := \bar{\mathbf{h}}(\cdot, t)$ is the specific density of $\bar{\mathbf{H}}_t$ for all $t \in I$. If $\bar{\mathbf{h}}$ is of class C^1 then so is $\bar{\mathbf{H}}$ and we have

$$\bar{\mathbf{H}}_t^{\bullet}(\mathcal{P}) = \int_{\mathcal{P}} \bar{\mathbf{h}}_t^{\bullet} \mathrm{d}m = \int_{\mathcal{P}_\mu} \rho_\mu (\bar{\mathbf{h}}_t^{\bullet} \circ \mu^{\leftarrow}) \quad \text{for all } \ t \in I, \ \mu \in \mathrm{Pl}\mathcal{B} \text{ and } \mathcal{P} \in \Omega_{\mathcal{B}}. \tag{8.20}$$

Remark 8.7 In practice the reference mass is usually taken to be the inertial-gravitational mass. However, here we do not assume that this is the case. There could be situations in which it is useful to take the reference mass to be different from the inertial-gravitational mass. For example, one may wish to fix a reference configuration δ_R and define, $m(\mathcal{P})$ to be the volume of the region $\mathrm{imb}_{\delta_R>}(\mathcal{P})$ for every $\mathcal{P} \in \Omega_{\mathcal{B}}$.

In (8.19) the reference mass is, in the language of measure theory, a *measure* on $\mathcal{B}$. Given an additive mapping with density $\mathbf{H}$, the mapping $\mathbf{h}$ in (8.17) is nothing but the density of $\mathbf{H}$ with respect to the measure m. ▮

9 Balance of Forces and Torques

It is often useful to fix a frame-space $\mathcal{F}$, with translation space $\mathcal{V}$, and confine one's attention to placements whose range-space is $\mathcal{F}$. It is then useful to consider force systems with values in $\mathcal{V}$, independent of the choice of placement, as follows:

Definition 9.1 *A* **force system** *in the space* $\mathcal{V}$ *is a pair* $(\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}, \mathbf{F}^{\mathrm{e}}_{\mathcal{V}})$, *where* $\mathbf{F}^{\mathrm{i}}_{\mathcal{V}} : (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \longrightarrow \mathcal{V}$ *is an interaction and* $\mathbf{F}^{\mathrm{e}}_{\mathcal{V}} : \Omega_{\mathcal{B}} \longrightarrow \mathcal{V}$ *is additive. The mapping* $\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}$ *is called the* **internal force system** *in* $\mathcal{V}$ *and* $\mathbf{F}^{\mathrm{e}}_{\mathcal{V}}$ *is called the* **external force system** *in* $\mathcal{V}$.

Let a force system $(\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}, \mathbf{F}^{\mathrm{e}}_{\mathcal{V}})$ in $\mathcal{V}$ be given. The first law of mechanics, called the **Balance of Forces**, says:

$$\mathrm{Res}_{\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}}(\mathcal{P}) + \mathbf{F}^{\mathrm{e}}_{\mathcal{V}}(\mathcal{P}) = \mathbf{0} \quad \text{for all } \ \mathcal{P} \in \Omega_{\mathcal{B}}. \tag{9.1}$$

We say that the system $(\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}, \mathbf{F}^{\mathrm{e}}_{\mathcal{V}})$ is **force-balanced** if (9.1) holds.

Since $\mathbf{F}^{\mathrm{e}}_{\mathcal{V}}$ is additive, the following **Law of Action and Reaction** is an immediate consequence of (9.1) and Theorem 3.3: The internal force system is skew, i.e.,

$$\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}(\mathcal{P}, \mathcal{Q}) = -\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}(\mathcal{Q}, \mathcal{P}) \qquad \text{for all} \quad (\mathcal{P}, \mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\mathrm{sep}}. \tag{9.2}$$

We assume now that $\mathbf{F}^{\mathrm{i}}_{\mathcal{V}}$ has contactors and $\mathbf{F}^{\mathrm{e}}_{\mathcal{V}}$ has densities.

Let μ be a placement of the body in $\mathcal{F}$ and put $\mathcal{B}_\mu := \mu_>(\mathcal{B})$. Let $\mathbf{T}_\mu : \mathcal{B}_\mu \longrightarrow \text{Lin}\mathcal{V}$ denote the contactor for $\mathbf{F}^{\text{i}}_{\mathcal{V}}$ and let $\mathbf{b}_\mu : \mathcal{B}_\mu \longrightarrow \mathcal{V}$ denote the density of $\mathbf{F}^{\text{e}}_{\mathcal{V}}$ in the placement μ. It follows from Theorem 8.4 that (9.1) restricted to internal parts $\mathcal{P}$ is equivalent to

$$\text{div}\mathbf{T}_\mu + \mathbf{b}_\mu = \mathbf{0}. \tag{9.3}$$

Definition 9.2 *A* **torque system** *in the space $\mathcal{V}$ is a pair $(\mathbf{M}^{\text{i}}_{\mathcal{V}}, \mathbf{M}^{\text{e}}_{\mathcal{V}})$, where $\mathbf{M}^{\text{i}}_{\mathcal{V}} : (\Omega_{\mathcal{B}})^2_{\text{sep}} \longrightarrow \text{Skew}\,\mathcal{V}$ is an interaction and $\mathbf{M}^{\text{e}}_{\mathcal{V}} : \Omega_{\mathcal{B}} \longrightarrow \text{Skew}\,\mathcal{V}$ is additive. The mapping $\mathbf{M}^{\text{i}}_{\mathcal{V}}$ is called the* **internal torque system** *and $\mathbf{M}^{\text{e}}_{\mathcal{V}}$ is called the* **external torque system**.

Let $(\mathbf{M}^{\text{i}}_{\mathcal{V}}, \mathbf{M}^{\text{e}}_{\mathcal{V}})$ be a torque system in $\mathcal{V}$. The second law of mechanics is the **Balance of Torques**, which states that

$$\text{Res}_{\mathbf{M}^{\text{i}}_{\mathcal{V}}}(\mathcal{P}) + \mathbf{M}^{\text{e}}_{\mathcal{V}}(\mathcal{P}) = \mathbf{0} \quad \text{for all} \;\; \mathcal{P} \in \Omega_{\mathcal{B}}. \tag{9.4}$$

Again, an immediate consequence of (9.4) and Theorem 3.3 is the following: The internal torque system is skew, i.e.,

$$\mathbf{M}^{\text{i}}_{\mathcal{V}}(\mathcal{P}, \mathcal{Q}) = -\mathbf{M}^{\text{i}}_{\mathcal{V}}(\mathcal{Q}, \mathcal{P}) \qquad \text{for all} \qquad (\mathcal{P}, \mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\text{sep}}. \tag{9.5}$$

Here we will assume that all torques come from forces. Let a force system $(\mathbf{F}^{\text{i}}_{\mathcal{V}}, \mathbf{F}^{\text{e}}_{\mathcal{V}})$, that is force-balanced, be given. When (9.4) is considered only in cases when $\mathcal{P}$ is internal, the assumption that all torques come from forces means that given $q \in \mathcal{E}$, $\mathbf{M}^{\text{i}}_{\mathcal{V}}$ has a contactor $\mathbf{C}_\mu : \mathcal{B}_\mu \longrightarrow \text{Lin}(\mathcal{V}, \text{Skew}\,\mathcal{V})$ and $\mathbf{M}^{\text{e}}_{\mathcal{V}}$ has a density $\mathbf{m}_\mu : \mathcal{B}_\mu \longrightarrow \mathcal{V}$ in a given placement μ, and they are given by

$$\begin{array}{l} \mathbf{C}_\mu(x)\mathbf{u} := (x-q) \wedge \mathbf{T}_\mu(x)\mathbf{u} \\ \mathbf{m}_\mu(x) := (x-q) \wedge \mathbf{b}_\mu(x) \end{array} \qquad \text{for all} \;\; \mathbf{u} \in \mathcal{V} \text{ and } x \in \mathcal{B}_\mu. \tag{9.6}$$

A calculation using the results from Chapter 6 of [FDS] shows that the divergence of $\mathbf{C}_\mu$ is given by

$$\text{div}_x\mathbf{C}_\mu = \mathbf{T}^\top_\mu(x) - \mathbf{T}_\mu(x) + (x-q) \wedge \text{div}_x\mathbf{T}_\mu \quad \text{for all} \;\; x \in \mathcal{B}. \tag{9.7}$$

Using Theorem 8.4, it follows from the balance law (9.4) restricted to internal parts is equivalent to $\text{div}\mathbf{C}_\mu + \mathbf{m}_\mu = \mathbf{0}$. Combining this result with (9.7) and (9.6), we obtain

$$\mathbf{T}^\top_\mu(x) - \mathbf{T}_\mu(x) + (x-q) \wedge (\text{div}_x\mathbf{T}_\mu + \mathbf{b}_\mu(x)) = \mathbf{0} \quad \text{for all} \;\; x \in \mathcal{B}_\mu. \tag{9.8}$$

Using (9.3) we find that the condition

$$\text{Rng}\,\mathbf{T}_\mu \subseteq \text{Sym}\,\mathcal{V} \tag{9.9}$$

is equivalent to balance of torques for internal parts.

We say that the system $(\mathbf{F}^{\text{i}}_{\mathcal{V}}, \mathbf{F}^{\text{e}}_{\mathcal{V}})$ is **torque-balanced** if the system of torques derived from it satisfies (9.4).

Remark 9.3 When $\mathcal{P}$ is not internal, (9.1) and (9.4) must be taken into account when considering what are often called *boundary conditions.* ▮

From here on we will assume that a force system in $\mathcal{V}$ in given for which (9.3) and (9.9) are valid. We adjust the codomain of $\mathbf{T}_\mu$ to Sym $\mathcal{V}$ without change of notation and call $\mathbf{T}_\mu : \mathcal{B}_\mu \longrightarrow \mathrm{Sym}\mathcal{V}$ the **Cauchy stress** of the force system in the placement μ and the mapping $\mathbf{b}_\mu : \mathcal{B}_\mu \longrightarrow \mathcal{V}$ the **external body force** in the placement μ.

As mentioned at the end of Section 7, it is sometimes useful to specify a fixed reference placement $\kappa : \mathcal{B} \longrightarrow \mathcal{B}_\kappa$ in the frame-space $\mathcal{F}$ and characterize all other placements μ in $\mathcal{F}$ by their transplacements

$$\chi := \mu \circ \kappa^{\leftarrow} : \mathcal{B}_\kappa \longrightarrow \mathcal{B}_\mu \tag{9.10}$$

from the reference placement. The **transplacement gradient** $\mathbf{F} : \mathcal{B}_\kappa \longrightarrow \mathrm{Lis}\mathcal{V}$, defined by $\mathbf{F} := \nabla\chi$, can then be used to represent an internal force interaction, whose contactor in the placement μ is the Cauchy stress $\mathbf{T}_\mu$, by a contactor in the reference placement κ. Using (8.4), we see that this contactor is given by

$$\mathbf{T}_{\mathrm{R}} := |\det\mathbf{F}|(\mathbf{T}_\mu \circ \chi)\mathbf{F}^{-\top}. \tag{9.11}$$

$\mathbf{T}_{\mathrm{R}} : \mathcal{B}_\kappa \longrightarrow \mathrm{Lis}\mathcal{V}$ is usually called the **Piola-Kirchhoff stress** (see (43 A.3) of [NLFT]). Note that $\mathbf{T}_{\mathrm{R}}$ does not have symmetric values. Instead, since $\mathbf{T}_\mu$ has symmetric values, it follows from (9.11) that $\mathbf{T}_{\mathrm{R}}$ must satisfy

$$\mathbf{T}_{\mathrm{R}}\mathbf{F}^\top = \mathbf{F}\mathbf{T}_{\mathrm{R}}^\top. \tag{9.12}$$

The transplacement gradient can also be used to represent the external force system whose density in the placement μ is the external body force $\mathbf{b}_\mu$, by a density in the reference placement κ. Using (8.2), we see that this density is given by

$$\mathbf{b}_{\mathrm{R}} := |\det\mathbf{F}|(\mathbf{b}_\mu \circ \chi). \tag{9.13}$$

$\mathbf{b}_{\mathrm{R}} : \mathcal{B}_\kappa \longrightarrow \mathcal{V}$ may be called the **referential external body force** for the placement μ.

Remark 9.4 The description of force systems described so far is similar to the one given in the traditional textbooks, for example in [NLFT] or [ICM]. It has the disadvantage that it involves an external frame-space $\mathcal{F}$, often considered to be an *absolute* space. In Part 1 of [FC], it is argued that such a space is an illusion and an explanation for why this illusion is so widespread is provided.

The *principle of frame-indifference* states that constitutive laws (see Section 13) should not depend on whatever external frame of reference is used to describe them. It will be vacuously satisfied if no external frames of reference are used to state these laws. Therefore, it is useful to describe force systems without using an external frame-space, which we will do below. ▮

Let a configuration $\delta \in \mathrm{Conf}\,\mathcal{B}$ be given. As in Section 4, we denote the imbedding space for δ by $\mathcal{E}_\delta$ and its translation space by $\mathcal{V}_\delta$, and we use the notation of Section 8 for the placement imb_δ and write, for simplicity, δ rather than imb_δ as a subscript.

Definition 9.5 *A* **force system in the configuration** δ *is a pair* $(\mathbf{F}^{\mathrm{i}}_\delta, \mathbf{F}^{\mathrm{e}}_\delta)$ *which is a force system in the space* $\mathcal{V}_\delta$ *in the sense of Definition 9.1.*

Let such a force system $(\mathbf{F}^{\mathrm{i}}_\delta, \mathbf{F}^{\mathrm{e}}_\delta)$ in $\mathcal{V}_\delta$ be given. We assume that the balance of forces and the balance of torques are valid, that $\mathbf{F}^{\mathrm{i}}_\delta$ has contactors and that $\mathbf{F}^{\mathrm{e}}_\delta$ has densities. The results (9.3) and (9.9) remain valid when the subscript δ is used instead of μ, when $\mathbf{T}_\delta$ is interpreted to be the contactor of $\mathbf{F}^{\mathrm{i}}_\delta$ in the placement imb_δ, and when $\mathbf{b}_\delta$ is interpreted to be the density of $\mathbf{F}^{\mathrm{e}}_\delta$ in the placement imb_δ. We may call $\mathbf{T}_\delta$ the **imbedding stress** and $\mathbf{b}_\delta$ the **external imbedding body force** for δ.

Since $\mathbf{I}_\delta(X)$, defined in (4.5), is a linear isomorphism from $\mathcal{T}_X$ to $\mathcal{V}_\delta$, it can be used to transform the mappings $\mathbf{T}_\delta$ and $\mathbf{b}_\delta$, whose codomains involve $\mathcal{V}_\delta$, into mappings whose codomains involve $\mathcal{T}_X$. Thus, we define, for every $X \in \mathcal{B}$, the **intrinsic stress** $\mathbf{S}_\delta$ and the **intrinsic external body force** $\mathbf{d}_\delta$ associated with the configuration δ by

$$\mathbf{S}_\delta(X) := \mathbf{I}_\delta^{-1}(X)\mathbf{T}_\delta(\mathrm{imb}_\delta(X))\mathbf{I}_\delta^{-\top}(X) \in \mathrm{Sym}(\mathcal{T}_X^*, \mathcal{T}_X) \tag{9.14}$$

and

$$\mathbf{d}_\delta(X) := \mathbf{I}_\delta^{-1}(X)\mathbf{b}_\delta(\mathrm{imb}_\delta(X)) \in \mathcal{T}_X \quad \text{for all} \quad X \in \mathcal{B}, \tag{9.15}$$

respectively.

Remark 9.6 An interpretation of the fact that intrinsic stresses are mappings from the dual of a tangent space to the tangent space is in order. The Cauchy stress at a point maps a normal vector into the force per unit area (traction) acting on a surface with said normal. Normal vectors are elements of the dual of a vector space that has been identified with a vector using the inner-product. Since the tangent spaces of the body are not inner-product spaces, a normal on the tangent space is an element of the dual of the tangent space. A force is represented on the body by an element of the tangent space. Thus it makes sense for the intrinsic stress to map elements of the dual of the tangent space to an element of the tangent space. ∎

Remark 9.7 The mappings $\mathbf{S}_\delta := (X \mapsto \mathbf{S}_\delta(X))$ and $\mathbf{d}_\delta := (X \mapsto \mathbf{d}_\delta(X))$ are cross-sections of fiber bundles. This is related to the fact that stresses and forces are what some people call “frame-indifferent” or “objective”. ∎

Let μ be a placement of the body in a fixed frame-space $\mathcal{F}$ as considered in the beginning of this section and put $\mathcal{B}_\mu := \mu_>(\mathcal{B})$. Denote by δ the configuration induced by μ in accord with (5.1). We use $\mathbf{Q}$, as defined in (5.3), to transport the values of the force system in the configuration δ to $\mathcal{V}$ and obtain a force system in the sense of Definition 9.1. Then the balance laws (9.1) and (9.4) hold if and only if corresponding balance laws hold for the force system in the configuration δ. The corresponding Cauchy stress $\mathbf{T}_\mu : \mathcal{B}_\mu \longrightarrow \mathrm{Sym}\mathcal{V}$ and the corresponding external body force $\mathbf{b}_\mu : \mathcal{B}_\mu \longrightarrow \mathcal{V}$ are related to $\mathbf{T}_\delta$ and $\mathbf{b}_\delta$ by

$$\mathbf{T}_\delta \circ \mathrm{imb}_\delta = \mathbf{Q}\mathbf{T}_\mu\mathbf{Q}^\top \circ \mu \quad \text{and} \quad \mathbf{b}_\delta \circ \mathrm{imb}_\delta = \mathbf{Q}\mathbf{b}_\mu \circ \mu, \tag{9.16}$$

respectively. Using (5.4), (5.5), (9.14) and (9.15) we find that the Cauchy stress $\mathbf{T}_\mu : \mathcal{B}_\mu \longrightarrow \mathrm{Sym}\mathcal{V}$ and the external body force $\mathbf{b}_\mu : \mathcal{B}_\mu \longrightarrow \mathcal{V}$ are related to the intrinsic stress and the intrinsic external body force by

$$\mathbf{T}_\mu \circ \mu = \mathbf{M}_\mu \mathbf{S}_\delta \mathbf{M}_\mu^\top \tag{9.17}$$

and

$$\mathbf{b}_\mu \circ \mu = \mathbf{M}_\mu \mathbf{d}_\delta. \tag{9.18}$$

Conversely, if one has the Cauchy stress $\mathbf{T}_\mu$ and external body force $\mathbf{b}_\mu$ in a placement μ, one can obtain the corresponding intrinsic stress and intrinsic external body force by

$$\mathbf{S}_\delta = \mathbf{M}_\mu^{-1}(\mathbf{T}_\mu \circ \mu)\mathbf{M}_\mu^{-\top} \tag{9.19}$$

and

$$\mathbf{d}_\delta = \mathbf{M}_\mu^{-1}(\mathbf{d}_\mu \circ \mu). \tag{9.20}$$

10 Deformation Processes and Mechanical Processes

In Section 8 the motion of a continuous body in a frame of reference was discussed. Here we describe the frame-free counterpart of a motion, namely a deformation process. As before, we assume that a time-interval I is given.

Definition 10.1 *We say that a time-family $(\bar{\delta}_t \mid t \in I)$ of configurations, and the corresponding mapping $\bar{\delta} : I \longrightarrow \mathrm{Conf}\mathcal{B}$ (see (6.1)), is a* **deformation process***.*

Let a deformation process $(\bar{\delta}_t \mid t \in I)$, be given. Since $\bar{\delta}_t$ is a configuration it can be used to construct an imbedding space $\mathcal{E}_t := \mathcal{E}_{\bar{\delta}_t}$[8], with translation space $\mathcal{V}_t$, in which $\mathcal{B}$ is imbedded using the mapping $\mathrm{imb}_t := \mathrm{imb}_{\bar{\delta}(t)} : \mathcal{B} \longrightarrow \mathcal{B}_t := \mathcal{B}_{\bar{\delta}(t)}$. We will denote the mappings defined in (4.5) and (4.6) for the imbedding imb_t by $\bar{\mathbf{I}}_t$ and $\bar{\mathbf{G}}_t$, respectively, i.e.,

$$\bar{\mathbf{I}}_t(X) := \nabla_X \mathrm{imb}_t \in \mathrm{Lis}(\mathcal{T}_X, \mathcal{V}_t)\ , \quad \bar{\mathbf{G}}_t(X) := \bar{\mathbf{I}}_t^\top(X)\bar{\mathbf{I}}_t(X) \in \mathrm{Pos}^+(\mathcal{T}_X, \mathcal{T}_X^*), \tag{10.1}$$

The family $\bar{\mathbf{G}} = (\bar{\mathbf{G}}_t(X) \mid t \in I)$ is called the **deformation process of the body element** $\mathcal{T}_X$ induced by the deformation process $\bar{\delta}$ of the body. We say that the deformation process $\bar{\delta}$ is of class C^1 or of class C^2 if $t \mapsto \bar{\mathbf{G}}_t(X)$ is of class C^1 or of class C^2 for all $X \in \mathcal{B}$, respectively. In this case, we define the family $(\bar{\mathbf{G}}_t^\bullet \mid t \in I)$ or the family $(\bar{\mathbf{G}}_t^{\bullet\bullet} \mid t \in I)$ by

$$\bar{\mathbf{G}}_t^\bullet(X) := (s \mapsto \bar{\mathbf{G}}_s(X))^\bullet(t) \quad \text{or} \quad \bar{\mathbf{G}}_t^{\bullet\bullet}(X) := (s \mapsto \bar{\mathbf{G}}_s(X))^{\bullet\bullet}(t) \quad \text{for all} \quad X \in \mathcal{B}, \tag{10.2}$$

respectively.

We now let a motion $\bar{\mu} = (\bar{\mu}_t \in \mathrm{Pl}\mathcal{B} \mid t \in I)$ in a frame $\mathcal{F}$ as defined in Definition 7.1 be given. We consider the deformation process $\bar{\delta}$ induced by this motion in the sense that, for each $t \in I$, the placement $\bar{\mu}_t$ induces the configuration $\bar{\delta}_t$ as explained in (5.1). Since the placements imb_t and $\bar{\mu}_t$ both induce the same configuration $\bar{\delta}_t$, it follows from (5.6) that

$$\bar{\mathbf{G}} = \bar{\mathbf{M}}^\top \bar{\mathbf{M}}. \tag{10.3}$$

[8]The time-family $(\mathcal{E}_t \mid t \in I)$ of Euclidean spaces could be interpreted as describing what has been called a pre-classical event world in Section 4.1 of [MSSR] and called a neo-classical event world in [LFCM].

It follows from (5.4), (7.7), (7.3) and the chain rule that

$$\bar{\mathbf{M}}^{\bullet} = \bar{\mathbf{L}}_{\mathrm{m}}\bar{\mathbf{M}}. \tag{10.4}$$

Differentiating (10.3) with respect to time, using (10.4), (7.8), and the product rule, we find

$$\bar{\mathbf{G}}^{\bullet} = 2\bar{\mathbf{M}}^{\top}\bar{\mathbf{D}}_{\mathrm{m}}\bar{\mathbf{M}}. \tag{10.5}$$

We define the **mass-density field** $\bar{\rho} : \mathcal{M} \longrightarrow \mathbb{P}^{\times}$ by (recall the notation in (8.13))

$$\bar{\rho}(x,t) := \rho_{\bar{\mu}_t}(x) \quad \text{for all} \;\; (x,t) \in \mathcal{M}. \tag{10.6}$$

Let a reference placement κ be given and set $\bar{\alpha}_t := \bar{\mu}_t \circ \bar{\kappa}^{\leftarrow}$. It follows from (8.13) and (5.10) that

$$\bar{\rho}(p,t_o) = |\det(\nabla_p\bar{\alpha}_t\mathbf{B})|\bar{\rho}(\bar{\alpha}_t(p),t) \quad \text{for all} \;\; t \in I \text{ and } p \in \mathcal{B}_{t_o}. \tag{10.7}$$

By looking at the derivative of this equation with respect to time one can obtain the **Balance of Mass** equation:

$$\bar{\rho}^{\bullet} + \operatorname{div}(\bar{\rho}\bar{\mathbf{v}}) = 0. \tag{10.8}$$

It is easily seen that this derivation does not depend on the choice of κ.

Definition 10.2 *A* **mechanical process** *is a time-family* $((\bar{\delta}_t, \bar{\mathbf{F}}^{\mathrm{i}}_t, \bar{\mathbf{F}}^{\mathrm{e}}_t) \mid t \in I)$ *of triples where* $(\bar{\delta}_t \mid t \in I)$ *is a deformation process and, for every* $t \in I$, $(\bar{\mathbf{F}}^{\mathrm{i}}_t, \bar{\mathbf{F}}^{\mathrm{e}}_t)$ *is a force system in the configuration* $\bar{\delta}_t$*, as defined by Definition 9.5, which is both force-balanced and torque-balanced.* $(\bar{\mathbf{F}}^{\mathrm{e}}_t \mid t \in I)$ *is called an* **external force process** *and* $(\bar{\mathbf{F}}^{\mathrm{i}}_t \mid t \in I)$ *is called an* **internal force process**.

Let a mechanical process $((\bar{\delta}_t, \bar{\mathbf{F}}^{\mathrm{i}}_t, \bar{\mathbf{F}}^{\mathrm{e}}_t) \mid t \in I)$ be given. We assume that for each $t \in I$, $\bar{\mathbf{F}}^{\mathrm{i}}_t$ has contactors and $\bar{\mathbf{F}}^{\mathrm{e}}_t$ has densities. We can then define, using the formulas (9.14) and (9.15), a time-family $\bar{\mathbf{S}} := (\bar{\mathbf{S}}_t := \mathbf{S}_{\bar{\delta}_t} \mid t \in I)$ of intrinsic stresses and a time-family $\bar{\mathbf{d}} := (\bar{\mathbf{d}}_t := \mathbf{d}_{\bar{\delta}_t} \mid t \in I)$ of intrinsic external body forces for the family $\bar{\delta}$. These two families, together with the time-family $\bar{\mathbf{G}}$, describe the given mechanical process, apart from boundary conditions, without using an external frame of reference.

Now let a mechanical process be given such that its deformation process is the one induced by a given motion $\bar{\mu}$. The considerations of Section 9 then apply for every $t \in I$ with μ and δ replaced by $\bar{\mu}_t$ and $\bar{\delta}_t$. The Cauchy stress and the external body force now become time-families and hence are identified with the mappings $\bar{\mathbf{T}} : \mathcal{M} \longrightarrow \operatorname{Sym}\mathcal{V}$ and $\bar{\mathbf{b}} : \mathcal{M} \longrightarrow \mathcal{V}$, respectively. By (9.17) and (9.18) the corresponding intrinsic stress $\bar{\mathbf{S}}$ and external intrinsic body force $\bar{\mathbf{d}}$ are related to $\bar{\mathbf{T}}$ and $\bar{\mathbf{b}}$ by

$$\bar{\mathbf{S}} = \bar{\mathbf{M}}^{-1}\bar{\mathbf{T}}_{\mathrm{m}}\bar{\mathbf{M}}^{-\top} \tag{10.9}$$

and

$$\bar{\mathbf{d}} = \bar{\mathbf{M}}^{-1}\bar{\mathbf{b}}_{\mathrm{m}}. \tag{10.10}$$

The balance law (9.3) becomes

$$\mathrm{div}\bar{\mathbf{T}} + \bar{\mathbf{b}} = \mathbf{0}. \tag{10.11}$$

Using Proposition 2 in Section 67 of [FDS], (7.7), (7.8) and the fact that $\bar{\mathbf{T}}$ has symmetric values, we find that

$$\mathrm{div}(\bar{\mathbf{T}}\bar{\mathbf{v}}) = \mathrm{div}(\bar{\mathbf{T}}) \cdot \bar{\mathbf{v}} + \mathrm{tr}(\bar{\mathbf{T}}\bar{\mathbf{D}}). \tag{10.12}$$

Using (10.12), (10.11) and the divergence theorem (4.2), we conclude that

$$\int_{\mathrm{Rby}\mathcal{P}_{\bar{\mu}_t}} \bar{\mathbf{v}}_t \cdot \bar{\mathbf{T}}_t \mathbf{n}_{\mathrm{Rby}\mathcal{P}_{\bar{\mu}_t}} + \int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\mathbf{v}}_t \cdot \bar{\mathbf{b}}_t = \int_{\mathcal{P}_{\bar{\mu}_t}} \mathrm{tr}(\bar{\mathbf{T}}_t\bar{\mathbf{D}}_t) \quad \text{for all } \ \mathcal{P} \in \Omega_{\mathcal{B}} \text{ and } t \in I. \tag{10.13}$$

The term on the right hand side is the work per unit time, i.e., the power, of the forces acting on the part $\mathcal{P}$. It easily follows from (10.9) and (10.5) that

$$\frac{1}{2}\mathrm{tr}(\bar{\mathbf{S}}_t\bar{\mathbf{G}}_t^\bullet)(X) = \mathrm{tr}(\bar{\mathbf{T}}_t\bar{\mathbf{D}}_t)(\bar{\mu}_t(X)) \quad \text{ for all } \ (X,t) \in \mathcal{B} \times I, \tag{10.14}$$

which means that the material field associated with $\mathrm{tr}(\bar{\mathbf{T}}\bar{\mathbf{D}})$, according to (7.3), is

$$\frac{1}{2}\mathrm{tr}(\bar{\mathbf{S}}\bar{\mathbf{G}}^\bullet) = (\mathrm{tr}(\bar{\mathbf{T}}\bar{\mathbf{D}}))_{\mathrm{m}}. \tag{10.15}$$

Therefore the **power** of the forces acting on the parts of the body is given by

$$\bar{P}_t(\mathcal{P}) := \int_{\mathcal{P}_{\bar{\mu}_t}} \mathrm{tr}(\bar{\mathbf{T}}_t\bar{\mathbf{D}}_t) = \frac{1}{2}\int_{\mathcal{P}_{\mathrm{imb}_t}} \mathrm{tr}(\bar{\mathbf{S}}_t\bar{\mathbf{G}}_t^\bullet) \circ \mathrm{imb}_t^{\leftarrow} \quad \text{ for all } \ \mathcal{P} \in \Omega_{\mathcal{B}} \text{ and } t \in I, \tag{10.16}$$

and depends only on the mechanical process and not on the motion as the left hand side of (10.13) might lead one to believe. It is clear that the power family $(\bar{P}_t \mid t \in I)$ is a time-family of additive mappings determined by the given mechanical process.

Remark 10.3 Noll proved in [MCO] that the balance laws hold if and only if the work done by the forces is frame-indifferent[9]. In view of this fact it is not surprising that the power does not depend on the motion. ▮

11 Energy Balance

Up until now we have only dealt with mechanical phenomena. Here we introduce the concepts of heat and energy and how they are related.

Definition 11.1 *A* **heat transfer system** *is a pair* $(Q^{\mathrm{i}}, Q^{\mathrm{e}})$, *where* $Q^{\mathrm{i}} : (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \longrightarrow \mathbb{R}$ *is an interaction and* $Q^{\mathrm{e}} : \Omega_{\mathcal{B}} \longrightarrow \mathbb{R}$ *is additive. The function* Q^{i} *is called the* **internal heat transfer** *and* Q^{e} *is called the* **external heat transfer***.*

[9]Some people take as a fundamental axiom that the work is frame-indifferent and use this to obtain the balance of forces and balance of torques.

Let $(Q^{\mathrm{i}}, Q^{\mathrm{e}})$ be a heat transfer system. We will assume that Q^{i} has contactors and Q^{e} has densities. Let a placement μ be given. Let us denote the contactor of Q^{i} in this placement by[10]

$$-\,\mathbf{q}_\mu : \mathcal{B}_\mu \longrightarrow \mathrm{Lin}(\mathrm{Vfr}\,\mu, \mathbb{R}) \cong \mathrm{Vfr}\,\mu. \tag{11.1}$$

The mapping $\mathbf{q}_\mu$ is called the **heat flux** in the placement μ.

Let δ be the configuration induced by μ, as in (5.1), and let $\mathbf{q}_\delta : \mathrm{Rng}\,\mathrm{imb}_\delta \longrightarrow \mathcal{V}_\delta$ denote the heat flux of Q^{i} in the placement imb_δ. Since $\mathbf{I}_\delta(X)$, defined in (4.5), is a linear isomorphism from $\mathcal{T}_X$ to $\mathcal{V}_\delta$, it can be used to transform the mapping $\mathbf{q}_\delta$, whose codomain involves $\mathcal{V}_\delta$, into a mapping whose codomain involves $\mathcal{T}_X$. Thus, we define, for every $X \in \mathcal{B}$, the **intrinsic heat flux** $\mathbf{h}_\delta$ associated with the configuration δ by

$$\mathbf{h}_\delta(X) := \mathbf{I}_\delta^{-1}(X)\mathbf{q}_\delta(\mathrm{imb}_\delta(X)) \in \mathcal{T}_X \quad \text{for all} \quad X \in \mathcal{B}. \tag{11.2}$$

Remark 11.2 The mapping $\mathbf{h}_\delta := (X \mapsto \mathbf{h}_\delta(X))$ is a tangent-vector field on $\mathcal{B}$. ▮

As was pointed out in Section 5,

$$\alpha := \mathrm{imb}_\delta \circ \mu^{\leftarrow} : \mathcal{B}_\mu \longrightarrow \mathrm{Rng}\,\mathrm{imb}_\delta \tag{11.3}$$

is an adjusted Euclidean isomorphism. Its (constant) gradient $\mathbf{Q} := \nabla\alpha \in \mathrm{Orth}(\mathrm{Vfr}\mu, \mathcal{V}_\delta)$ is an inner-product preserving isomorphism. We use $\mathbf{Q}$ to transport the values of the heat flux $\mathbf{q}_\delta$ in the configuration δ to $\mathrm{Vfr}\mu$. The corresponding heat flux $\mathbf{q}_\mu : \mathcal{B}_\mu \longrightarrow \mathrm{Vfr}\mu$ is related to $\mathbf{q}_\delta$ by

$$\mathbf{q}_\delta \circ \alpha = \mathbf{Q}\mathbf{q}_\mu. \tag{11.4}$$

Using the definition (5.4) of $\mathbf{M}_\mu$, it follows from (11.2) and (5.5) that the heat flux $\mathbf{q}_\mu$ is related to the intrinsic heat flux $\mathbf{h}_\delta$ by

$$\mathbf{q}_\mu \circ \mu = \mathbf{M}_\mu \mathbf{h}_\delta. \tag{11.5}$$

Definition 11.3 *An* **energetical process** *is a sextuplet* $((\bar{\delta}_t, \bar{\mathbf{F}}^{\mathrm{i}}_t, \bar{\mathbf{F}}^{\mathrm{e}}_t, \bar{Q}^{\mathrm{i}}_t, \bar{Q}^{\mathrm{e}}_t, \bar{E}_t) \mid t \in I)$ *of time-families such that* $((\bar{\delta}_t, \bar{\mathbf{F}}^{\mathrm{i}}_t, \bar{\mathbf{F}}^{\mathrm{e}}_t) \mid t \in I)$ *is a mechanical process,* $((\bar{Q}^{\mathrm{i}}_t, \bar{Q}^{\mathrm{e}}_t) \mid t \in I)$ *is a time-family of heat transfer systems, and* $(\bar{E}_t \mid t \in I)$ *is a differentiable time-family of additive mappings, called the* **internal energy process**.

We say that a given energetical process $((\bar{\delta}_t, \bar{\mathbf{F}}^{\mathrm{i}}_t, \bar{\mathbf{F}}^{\mathrm{e}}_t, \bar{Q}^{\mathrm{i}}_t, \bar{Q}^{\mathrm{e}}_t, \bar{E}_t) \mid t \in I)$ is **energy-balanced** if

$$\bar{E}^{\bullet}_t(\mathcal{P}) = \bar{P}_t(\mathcal{P}) + \mathrm{Res}_{\bar{Q}^{\mathrm{i}}_t}(\mathcal{P}) + \bar{Q}^{\mathrm{e}}_t(\mathcal{P}) \quad \text{for all} \quad \mathcal{P} \in \Omega_{\mathcal{B}} \text{ and } t \in I, \tag{11.6}$$

where $(\bar{P}_t \mid t \in I)$, defined in (10.16), is the power-family determined by the mechanical process $((\bar{\delta}_t, \bar{\mathbf{F}}^{\mathrm{i}}_t, \bar{\mathbf{F}}^{\mathrm{e}}_t) \mid t \in I)$. The formula (11.6) is known as the **Balance of Energy** or **The First Law of Thermodynamics**.

[10]Here we use the standard convention that when $\mathbf{q}_\mu$ is pointing away from the body, the body is losing heat. This is the reason for the minus sign.

From now on we assume that, for all $t \in I$, $\bar{\mathbf{F}}^{\mathrm{e}}_t$, $\bar{Q}^{\mathrm{e}}_t$, and $\bar{E}_t$ have densities, and that $\bar{\mathbf{F}}^{\mathrm{i}}_t$ and $\bar{Q}^{\mathrm{i}}_t$ have contactors. Also, we assume that a reference mass $m : \Omega_{\mathcal{B}} \longrightarrow \mathbb{P}^{\times}$, as described in Section 8, is given. We define the **specific internal energy** $\bar{\epsilon} : \mathcal{B} \times I \longrightarrow \mathbb{R}$ by the condition that $\bar{\epsilon}_t$ is the specific density of $\bar{E}_t$ for all $t \in I$. We assume that $\bar{\epsilon}$ is of class C^1. The **specific external heat transfer**, or **radiation**, $\bar{r} : \mathcal{B} \times I \longrightarrow \mathbb{R}$ is defined by the condition that $\bar{r}_t$ is the specific density of $\bar{Q}^{\mathrm{e}}_t$ for all $t \in I$.

Let a frame-space $\mathcal{F}$, with translation space $\mathcal{V}$, be given. Let $\bar{\mu}$ be a motion in $\mathcal{F}$ such that the placement $\bar{\mu}_t$ induces the configuration $\bar{\delta}_t$ for all $t \in I$. Note that $\bar{r}$, $\bar{\epsilon}$ and $\bar{\epsilon}^{\bullet}$ are material fields. Using the associated spatial fields as defined by (7.2) and also (8.20) and (8.19), we have

$$\bar{E}^{\bullet}_t(\mathcal{P}) = \int_{\mathcal{P}} \bar{\epsilon}^{\bullet}_t \mathrm{d}m = \int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\rho}_t((\bar{\epsilon}^{\bullet})_{\mathrm{s}})_t \qquad \text{and} \qquad \bar{Q}^{\mathrm{e}}_t(\mathcal{P}) = \int_{\mathcal{P}} \bar{r}_t \mathrm{d}m = \int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\rho}_t(\bar{r}_{\mathrm{s}})_t$$
$$\text{for all} \;\; t \in I \text{ and } \mathcal{P} \in \Omega_{\mathcal{B}}. \tag{11.7}$$

Let $\bar{\mathbf{q}}_t$ denote the heat flux in the placement $\bar{\mu}_t$ for all $t \in I$. It can be used to define the spatial field $\bar{\mathbf{q}} : \mathcal{M} \longrightarrow \mathcal{V}$. Using (10.16), and (11.7), we can apply Theorem 8.4 to conclude that (11.6), limited to internal parts, is equivalent to

$$\bar{\rho}(\bar{\epsilon}^{\bullet})_{\mathrm{s}} = \mathrm{tr}(\bar{\mathbf{T}}\bar{\mathbf{D}}) - \mathrm{div}\bar{\mathbf{q}} + \bar{\rho}\bar{r}_{\mathrm{s}}. \tag{11.8}$$

12 Temperature and Entropy

The following theorem says that a heat transfer system together with a temperature can be used to construct an entropy transfer system.

Theorem 12.1 *Let a heat transfer system* $(Q^{\mathrm{i}}, Q^{\mathrm{e}})$*, as defined by Definition 11.1, and a function* $\theta : \mathcal{B} \longrightarrow \mathbb{P}^{\times}$ *of class* C^1*, called the (absolute)* **temperature***, be given, and assume that* Q^{i} *has contactors and* Q^{e} *has densities. Then there is a pair* $(H^{\mathrm{i}}, H^{\mathrm{e}})$*, where* $H^{\mathrm{i}} : (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \longrightarrow \mathbb{R}$ *is an interaction and* $H^{\mathrm{e}} : \Omega_{\mathcal{B}} \longrightarrow \mathbb{R}$ *is an additive mapping such that for every placement* μ *we have*

$$H^{\mathrm{i}}(\mathcal{P}, \mathcal{Q}) = -\int_{\mathrm{Rct}_{\mu}(\mathcal{P},\mathcal{Q})} \frac{\mathbf{q}_{\mu}}{\theta_{\mu}} \cdot \mathbf{n}_{\mathcal{P}_{\mu}} \quad \text{for all} \;\; (\mathcal{P}, \mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \;\; \text{with} \;\; \mathcal{P} \in \Omega^{\mathrm{int}}_{\mathcal{B}} \tag{12.1}$$

and

$$H^{\mathrm{e}}(\mathcal{P}) = \int_{\mathcal{P}_{\mu}} \frac{\rho_{\mu}(r \circ \mu^{\leftarrow})}{\theta_{\mu}} \qquad \text{for all} \;\; \mathcal{P} \in \Omega^{\mathrm{int}}_{\mathcal{B}}, \tag{12.2}$$

where $\mathbf{q}_{\mu}$ *is the heat flux in the placement* μ*,* r *is the specific external heat absorption density in the placement* μ*, and* $\theta_{\mu} := \theta \circ \mu^{\leftarrow}$*. The pair* $(H^{\mathrm{i}}, H^{\mathrm{e}})$ *is called the* **entropy transfer system** *generated by the heat transfer system* $(Q^{\mathrm{i}}, Q^{\mathrm{e}})$ *and the temperature* θ*.* H^{i} *is called the* **internal entropy transfer***, and* H^{e} *is called the* **external entropy transfer***.*

Proof: Let μ and μ' be placements and define an entropy transfer system $(H^{\mathrm{i}}, H^{\mathrm{e}})$ by (12.1) and (12.2) using μ. Put $\alpha := \mu \circ \mu'^{\leftarrow}$. By (8.4) and (12.1) we have

$$H^{\mathrm{i}}(\mathcal{P}, \mathcal{Q}) = -\int_{\mathrm{Rct}_{\mu'}(\mathcal{P},\mathcal{Q})} \frac{\rho_{\mu',\mu}(\nabla\alpha)^{-1}\mathbf{q}_{\mu} \circ \alpha}{\theta_{\mu} \circ \alpha} \cdot \mathbf{n}_{\mathcal{P}_{\mu'}} \quad \text{for all } (\mathcal{P}, \mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \text{ with } \mathcal{P} \in \Omega^{\mathrm{int}}_{\mathcal{B}}. \tag{12.3}$$

From the discussion following (8.4) we know that the contactor $\mathbf{q}_{\mu'}$ of Q^{i} in the placement μ' is given by $\mathbf{q}_{\mu'} = \rho_{\mu',\mu}(\nabla\alpha)^{-1}\mathbf{q}_{\mu} \circ \alpha$ and, since $\mu = \alpha \circ \mu'$, we have $\theta_{\mu} \circ \alpha = \theta_{\mu'}$. Thus (12.3) becomes

$$H^{\mathrm{i}}(\mathcal{P}, \mathcal{Q}) = -\int_{\mathrm{Rct}_{\mu'}(\mathcal{P},\mathcal{Q})} \frac{\mathbf{q}_{\mu'}}{\theta_{\mu'}} \cdot \mathbf{n}_{\mathcal{P}_{\mu'}} \quad \text{for all } (\mathcal{P}, \mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \text{ with } \mathcal{P} \in \Omega^{\mathrm{int}}_{\mathcal{B}}. \tag{12.4}$$

Equation (12.4) shows that H^{i} would be the same if it were defined in terms of the placement μ'. Since μ and μ' were arbitrary placements, the definition of H^{i} does not depend on the placement.

The proof that the definition of H^{e} does not depend on the placement is analogous to the one just given for H^{i} except the change of placement formula for densities (8.2) is used.

∎

Let a placement μ and a temperature θ be given. By definition, we have $\theta_{\mu} \circ \mu = \theta$. Taking the gradient of this equation at $X \in \mathcal{B}$, using the chain rule, and (5.4), we obtain

$$\boldsymbol{\gamma}(X) := \nabla_X \theta = (\nabla_{\mu(X)}\theta_{\mu})\nabla_X \mu = (\nabla_{\mu(X)}\theta_{\mu})\mathbf{M}_{\mu}(X) \in \mathcal{T}^*_X \quad \text{for all } X \in \mathcal{B}. \tag{12.5}$$

Let δ denote the configuration associated with μ, as defined in (5.1). Then (12.5) and (11.5) can be used to obtain

$$\boldsymbol{\gamma}(X)\mathbf{h}_{\delta}(X) = (\nabla_{\mu(X)}\theta_{\mu}) \cdot \mathbf{q}_{\mu}(\mu(X)) \quad \text{for all } X \in \mathcal{B}. \tag{12.6}$$

Definition 12.2 *Let a time-family* $((\bar{Q}^{\mathrm{i}}_t, \bar{Q}^{\mathrm{e}}_t) \mid t \in I)$ *of heat transfer systems and a time-family of temperatures* $(\bar{\theta}_t \mid t \in I)$ *be given, and let* $((\bar{H}^{\mathrm{i}}_t, \bar{H}^{\mathrm{e}}_t) \mid t \in I)$ *be the resulting entropy transfer system as described in Theorem 12.1. Let* $(\bar{N}_t : \Omega_{\mathcal{B}} \longrightarrow \mathbb{R} \mid t \in I)$ *be a differentiable time-family of additive mappings, called the* **internal entropy**. *We say that the family* $((\bar{H}^{\mathrm{i}}_t, \bar{H}^{\mathrm{e}}_t, \bar{N}_t) \mid t \in I)$ *is a* **dissipative entropical process** *if*

$$\bar{N}^{\bullet}_t(\mathcal{P}) \geq \mathrm{Res}_{\bar{H}^{\mathrm{i}}_t}(\mathcal{P}) + \bar{H}^{\mathrm{e}}_t(\mathcal{P}) \quad \text{for all } \mathcal{P} \in \Omega^{\mathrm{int}}_{\mathcal{B}} \text{ and } t \in I. \tag{12.7}$$

Note that the time-family of temperatures can be identified with a function $\bar{\theta} : \mathcal{B} \times I \longrightarrow \mathbb{P}^{\times}$. From now on we assume that, for all $t \in I$, $\bar{N}_t$ has densities. As in the previous section, we assume that a reference mass $m : \Omega_{\mathcal{B}} \longrightarrow \mathbb{P}^{\times}$, as described in Section 8, is given. Let $\bar{\eta} : \mathcal{B} \times I \longrightarrow \mathbb{R}$ be the mapping defined by the condition that $\bar{\eta}_t$ is the specific density of $\bar{N}_t$ for all $t \in I$. We call this mapping the **specific entropy** and we will assume that it is of class C^1.

Now let a motion $\bar{\mu}$ in a given frame-space $\mathcal{F}$, with translation space $\mathcal{V}$, be given. Note that $\bar{\theta}$ is a material field. The spatial field $\bar{\theta}_{\mathrm{s}}$ associated with $\bar{\theta}$, according to (7.2), satisfies

$$(\bar{\theta}_{\mathrm{s}})_t := \bar{\theta}(t)_{\bar{\mu}(t)} \quad \text{for all } t \in I. \tag{12.8}$$

If we let $\bar{\mathbf{q}}_t$ denote the heat flux in the placement $\bar{\mu}_t$ it follows from (12.1) and (12.2) that

$$\bar{H}^{\mathrm{i}}_t(\mathcal{P},\mathcal{Q}) = -\int_{\mathrm{Rct}_{\bar{\mu}_t}(\mathcal{P},\mathcal{Q})} \frac{\bar{\mathbf{q}}_t}{(\bar{\theta}_{\mathrm{s}})_t} \cdot \mathbf{n}_{\mathcal{P}_{\bar{\mu}(t)}} \quad \text{for all } \ (\mathcal{P},\mathcal{Q}) \in (\Omega_{\mathcal{B}})^2_{\mathrm{sep}} \ \text{ with } \ \mathcal{P} \in \Omega^{\mathrm{int}}_{\mathcal{B}} \tag{12.9}$$

and

$$\bar{H}^{\mathrm{e}}_t(\mathcal{P}) = \int_{\mathcal{P}_{\bar{\mu}_t}} \frac{\bar{\rho}_t(\bar{r}_{\mathrm{s}})_t}{(\bar{\theta}_{\mathrm{s}})_t} \qquad \text{for all } \ \mathcal{P} \in \Omega^{\mathrm{int}}_{\mathcal{B}}. \tag{12.10}$$

Using a line of reasoning analogous to the one that led from (11.6) to (11.8) in the previous section and using Theorem 8.5 and (12.8), we find that (12.7) is equivalent to

$$\bar{\rho}(\bar{\eta}^{\bullet})_{\mathrm{s}} \geq \frac{\bar{\rho}\bar{r}_{\mathrm{s}}}{\bar{\theta}_{\mathrm{s}}} - \operatorname{div}\left(\frac{\bar{\mathbf{q}}}{\bar{\theta}_{\mathrm{s}}}\right). \tag{12.11}$$

Hence, using Proposition 1 in Section 67 of [FDS], we obtain

$$\bar{\rho}(\bar{\eta}^{\bullet})_{\mathrm{s}} \geq \frac{\bar{\rho}\bar{r}_{\mathrm{s}}}{\bar{\theta}_{\mathrm{s}}} - \frac{1}{\bar{\theta}_{\mathrm{s}}}\operatorname{div}\bar{\mathbf{q}} + \frac{1}{\bar{\theta}_{\mathrm{s}}^2}(\nabla\bar{\theta}_{\mathrm{s}}) \cdot \bar{\mathbf{q}}. \tag{12.12}$$

Definition 12.3 *A* **dynamical process** *is an octuple*

$$(\bar{\delta}, \bar{\theta}, \bar{\mathbf{F}}^{\mathrm{i}}, \bar{N}, \bar{Q}^{\mathrm{i}}, \bar{E}, \bar{\mathbf{F}}^{\mathrm{e}}, \bar{Q}^{\mathrm{e}}) \tag{12.13}$$

such that $((\bar{\delta}_t, \bar{\mathbf{F}}^{\mathrm{i}}_t, \bar{\mathbf{F}}^{\mathrm{e}}_t, \bar{Q}^{\mathrm{i}}_t, \bar{Q}^{\mathrm{e}}_t, \bar{E}_t) \mid t \in I)$ *is an energy-balanced energetical process as defined by Definition 11.3,* $\bar{\theta}$ *is a temperature process and* $\bar{N}$ *an internal entropy, as defined in Definition 12.2.*

We assume that such a dynamical process is given and that all the density assumptions made before are satisfied, so that both (11.8) and (12.12) are valid.

By multiplying both sides of the inequality (12.12) by $\bar{\theta}_{\mathrm{s}}$ and using (11.8) to eliminate $\bar{\rho}\bar{r}_{\mathrm{s}} - \operatorname{div}\bar{\mathbf{q}}$ we obtain

$$\bar{\rho}(\bar{\theta}\bar{\eta}^{\bullet} - \bar{\epsilon}^{\bullet})_{\mathrm{s}} + \operatorname{tr}(\bar{\mathbf{T}}\bar{\mathbf{D}}) - \frac{1}{\bar{\theta}_{\mathrm{s}}}\nabla\bar{\theta}_{\mathrm{s}} \cdot \bar{\mathbf{q}} \geq 0. \tag{12.14}$$

Given $X \in \mathcal{B}$ and $t \in I$, let $\bar{\mathbf{h}}_t(X) \in \mathcal{T}_X$ denote the intrinsic heat flux at X in the configuration $\bar{\delta}_t$ and put

$$\bar{\boldsymbol{\gamma}}_t(X) := \nabla_X\bar{\theta}_t \in \mathcal{T}^*_X. \tag{12.15}$$

It follows from (12.6) that

$$(\bar{\boldsymbol{\gamma}}_t\bar{\mathbf{h}}_t)(X) = \bar{\boldsymbol{\gamma}}_t(X)\bar{\mathbf{h}}_t(X) = ((\nabla\bar{\theta}_{\mathrm{s}})_t \cdot \bar{\mathbf{q}}_t)(\bar{\mu}_t(X)) \in \mathbb{R} \quad \text{ for all } \ t \in I \text{ and } X \in B. \tag{12.16}$$

If we replace the left side of (12.14) by its associated material field, using (10.15) and (12.16), we obtain

$$\bar{\rho}_{\mathrm{m}}(\bar{\theta}\bar{\eta}^{\bullet} - \bar{\epsilon}^{\bullet}) + \frac{1}{2}\operatorname{tr}(\bar{\mathbf{S}}\bar{\mathbf{G}}^{\bullet}) - \frac{1}{\bar{\theta}}\bar{\boldsymbol{\gamma}}\bar{\mathbf{h}} \geq 0. \tag{12.17}$$

By (8.16), $\bar{\rho}_{\mathrm{m}}(X,t) = \rho_{\bar{\mu}_t}(\bar{\mu}_t(X)) = \rho_{\mathrm{imb}_t}(\mathrm{imb}_t(X))$ for all $(X,t) \in \mathcal{B} \times I$ and so $\bar{\rho}_{\mathrm{m}}$ only depends on the deformation process and not the motion. Thus, (12.17) does not involve any external frames of reference. Also, (12.17) does not depend on the particular choice of a reference mass. If one uses a different reference mass then the value of $\bar{\rho}_{\mathrm{m}}(X,t)$ would change by a strictly positive factor but both $\bar{\eta}^{\bullet}(X,t)$ and $\bar{\epsilon}^{\bullet}(X,t)$ would change by the reciprocal of this same factor, for all $(X,t) \in \mathcal{B} \times I$. Thus, the left side of (12.17) would remain the same.

13 Constitutive Laws and The Second Law

Definition 13.1 *A* **thermodeformation process** *is a pair*

$$(\bar{\delta}, \bar{\theta}) \tag{13.1}$$

in which $\bar{\delta} : I \longrightarrow \mathrm{Conf}\ \mathcal{B}$ *is a deformation process, as defined in Definition 10.1, and* $\bar{\theta} : \mathcal{B} \times I \longrightarrow \mathbb{P}^{\times}$ *is a temperature process, which can be identified with a time-family of temperatures.*

A **response process** *is an quadruple*

$$(\bar{\mathbf{F}}^{\mathrm{i}}, \bar{N}, \bar{Q}^{\mathrm{i}}, \bar{E}) \tag{13.2}$$

where $\bar{\mathbf{F}}^{\mathrm{i}}$ *is an internal force system process,* $\bar{N}$ *is an internal entropy, as defined in Definition 12.2,* $\bar{Q}^{\mathrm{i}}$ *is a internal heat transfer process, defined in Definition 11.1, and* $\bar{E}$ *is an internal energy process, defined in Definition 11.3.*

A **thermomechanical process** *is a sextuple*

$$(\bar{\delta}, \bar{\theta}, \bar{\mathbf{F}}^{\mathrm{i}}, \bar{Q}^{\mathrm{i}}, \bar{E}, \bar{N}), \tag{13.3}$$

where $(\bar{\delta}, \bar{\theta})$ *is a thermodeformation process and* $(\bar{\mathbf{F}}^{\mathrm{i}}, \bar{N}, \bar{Q}^{\mathrm{i}}, \bar{E})$ *is a response process.*

Note that every thermomechanical process can be used to generate a dynamical process by using the balance of forces to *define* the external force system process and the balance of energy to *define* the external heat transfer process needed to produce the dynamical process.

Constitutive laws are used to describe the internal properties of a system and the internal interactions between its parts. In the framework presented here this means that given a set of constitutive laws each deformation process can be used to generate a response process and hence a thermomechanical process. All thermomechanical processes generated in this way are called **admissible processes** with respect to the given set of constitutive laws. The act of constructing admissible thermomechanical processes from a given set of constitutive laws will be carried out explicitly in Chapters III and IV.

We are now in a position to state the final fundamental law of thermomechanics.

Second Law of Thermodynamics: Given a set of constitutive laws, every admissible thermomechanical process must satisfy the reduced dissipation inequality (12.17).

This law is a restriction on the set of constitutive laws, *not* on the class of thermodeformation processes a body can under go. There is an enormous amount of literature on this subject. See, for example, [CNP], [TMM] or [TMM2]. The restrictions found using this law are more easily expressed if one introduces the following concept.

Definition 13.2 *The* **specific free energy** *is a* C^1 *mapping* $\bar{\psi} : \mathcal{B} \times I \longrightarrow \mathbb{R}$ *defined by*

$$\bar{\psi} := \bar{\epsilon} - \bar{\theta}\bar{\eta}. \tag{13.4}$$

The **internal free energy process** $\bar{E}^{\mathrm{f}} : \Omega_{\mathcal{B}}^{\mathrm{int}} \times I \longrightarrow \mathbb{R}$ *associated with the specific free energy* $\bar{\psi}$ *is a differentiable time-family of additive mappings with densities defined by*

$$\bar{E}_t^{\mathrm{f}}(\mathcal{P}) := \int_{\mathcal{P}} \bar{\psi}_t \mathrm{d}m \quad \text{for all } \; t \in I \text{ and } \mathcal{P} \in \Omega_{\mathcal{B}}^{\mathrm{int}}. \tag{13.5}$$

Using (12.17) and the time derivative of (13.4) we obtain the **reduced dissipation inequality**

$$-\bar{\rho}_{\mathrm{m}}(\bar{\psi}^{\bullet} + \bar{\theta}\bar{\eta}^{\bullet}) + \frac{1}{2}\mathrm{tr}(\bar{\mathbf{S}}\bar{\mathbf{G}}^{\bullet}) - \frac{1}{\bar{\theta}}\bar{\gamma}\bar{\mathbf{h}} \geq 0. \tag{13.6}$$

Constitutive laws can change from point to point and are local[11] in the sense that at a material point X they should only involve arbitrarily small neighborhoods of X in $\mathcal{B}$. We say that the body $\mathcal{B}$ consists of a **simple material** if the constitutive laws for every point $X \in \mathcal{B}$ involve only the body element $\mathcal{T}_X$. Most material properties of real materials are covered by the theory of simple materials. These materials will be discussed in detail in the next chapter.

Remark 13.3 A lot of work has been done on the foundations of the second law. The best, and most accessible, treatment of the foundations of the second law can be found in the book *A First Course in the Mathematical Foundations of Thermodynamics* [FCFT] by Owen. In it Owen formulates a version of the second law involving cycles that does not use the concept of entropy. Using this formulation one can prove the existence of entropies and show that an inequality of the form (12.17) holds. ▮

14 External Influences

External influences specify how the environment influences the behavior of the body. The description of these external influences depend on the choice of an external frame of reference. Perhaps the most important of these external influences are boundary conditions. Another important external influence is **inertia**. The total external body force density can be written as a sum

$$\bar{\mathbf{b}} = \bar{\mathbf{b}}_{\mathrm{ni}} + \bar{\mathbf{b}}_{\mathrm{i}}, \tag{14.1}$$

where $\bar{\mathbf{b}}_{\mathrm{ni}}$ denotes the external body force density that comes from non-inertial forces, and $\bar{\mathbf{b}}_{\mathrm{i}}$ is the inertial body force density.

[11] In [NLFT] this was called the *principle of local action*.

When an inertial frame of reference is used, then $\bar{\mathbf{b}}_{\mathrm{i}}$ is given by

$$\bar{\mathbf{b}}_{\mathrm{i}} = -\bar{\rho}\bar{\mathbf{a}}, \tag{14.2}$$

where $\bar{\rho}$ gives the inertial mass density[12] at each point of the trajectory. However, if a non-inertial frame of reference is used, then the inertial body force density is given by the more complicated formula

$$\bar{\mathbf{b}}_{\mathrm{i}} = -\bar{\rho}(\bar{\mathbf{u}}^{\bullet\bullet} + 2\bar{\mathbf{A}}\bar{\mathbf{u}}^{\bullet} + (\bar{\mathbf{A}}^{\bullet} - \bar{\mathbf{A}}^2)\bar{\mathbf{u}}) \tag{14.3}$$

where $\bar{\mathbf{u}}$ is a mapping whose value gives the position vector of a material point relative to a reference point which is at rest in some inertial frame $\mathcal{F}$. The mapping $\bar{\mathbf{A}}$, whose range consists of skew lineons, describes the motion of the non-inertial frame relative to the inertial frame $\mathcal{F}$. The second and fourth terms in the above formula are called the *Coriolis* force and *centrifugal* force, respectively. (See Part 2, Section 3, of [FC].)

Remark 14.1 Substituting (14.1) into the second term on the left of (10.13) one obtains

$$\int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\mathbf{v}}_t \cdot \bar{\mathbf{b}}_t = \int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\mathbf{v}}_t \cdot (\bar{\mathbf{b}}_{\mathrm{ni}})_t + \int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\mathbf{v}}_t \cdot (\bar{\mathbf{b}}_{\mathrm{i}})_t. \tag{14.4}$$

When one is using an inertial frame of reference (14.2) holds and the second term on the right of (14.4) is given by

$$\int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\mathbf{v}}_t \cdot (\bar{\mathbf{b}}_{\mathrm{i}})_t = -\left(\int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\rho}_t |\bar{\mathbf{v}}_t|^2\right)^{\bullet}. \tag{14.5}$$

The term $\int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\rho}_t |\bar{\mathbf{v}}_t|^2$ is called the *kinetic energy*. Substituting (14.5) into (10.13) one obtains

$$\int_{\mathrm{Rby}\mathcal{P}_{\bar{\mu}_t}} \bar{\mathbf{v}}_t \cdot \bar{\mathbf{T}}_t \mathbf{n}_{\mathrm{Rby}\mathcal{P}_{\bar{\mu}_t}} + \int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\mathbf{v}}_t \cdot (\bar{\mathbf{b}}_{\mathrm{ni}})_t = \int_{\mathcal{P}_{\bar{\mu}_t}} \mathrm{tr}(\bar{\mathbf{T}}_t \bar{\mathbf{D}}_t) + \left(\int_{\mathcal{P}_{\bar{\mu}_t}} \bar{\rho}_t |\bar{\mathbf{v}}_t|^2\right)^{\bullet} \tag{14.6}$$

for all $\mathcal{P} \in \Omega_{\mathcal{B}}$ and $t \in I$. In the literature on continuum mechanics it is often implicitly assumed that the frame of reference being used is inertial so the formula (14.6) is valid. However, when the frame of reference is not inertial then (14.6) is not valid. ▮

Constitutive laws can be specified using an external frame of reference. These laws would give the Cauchy stress $\bar{\mathbf{T}}$, spatial description of the specific free energy $\bar{\psi}_{\mathrm{s}}$, heat flux $\bar{\mathbf{q}}$ and spatial description of the specific entropy $\bar{\eta}_{\mathrm{s}}$ in terms of a transplacement gradient process $\bar{\mathbf{F}}$ and the spatial description of a temperature process $\bar{\theta}_{\mathrm{s}}$. Such constitutive laws would implicitly depend on the frame being used. Such dependence should be ruled out using the **Principle of Material Frame-Indifference**[13]. It states:

[12] This mass density must be the *inertial* mass density. The mass density introduced in Section 8 may or may not of been the inertial mass density.

[13] See Section 4 of Part 2 of [FC]

> Constitutive laws should not depend on whatever external frame of reference is used to describe them.

Traditionally one would specify a constitutive law in some frame of reference and then have to go though the effort of finding what restrictions are placed on this law by the principle of material frame-indifference. For some constitutive laws this can take a considerable amount of work[14]. The way to eliminate this work is to formulate constitutive laws without using any external frames of reference. This can be done by specifying constitutive laws for the intrinsic stress, intrinsic heat flux, specific free energy and specific entropy in terms of a thermodeformation process since these do not depend on the choice of a frame of reference. This superior method was used by Noll in [NTSM] and [FFFE] and will be used here as well.

[14]One can see this done in virtually all of the older literature that discuss constitutive laws. See, for example, [CNP], [MTCM], [NLFT] and [ICM].

Chapter II

Simple Materials

15 Introduction

In the previous chapter we laid down the basic concepts of thermomechanics. In Section 13 we briefly discussed the role of constitutive laws. This was done in a general, but informal, way. This part will describe in detail how to formulate the constitutive laws for simple materials. Simple materials are defined by the condition that the constitutive laws at each point $X \in \mathcal{B}$ only involve the body element (tangent space) $\mathcal{T}_X$ at that point. For this reason one can forget about the rest of the body and just work with the body element when discussing the constitutive laws for simple materials.

When this is done the (global) processes given in Definition 13.1 are not needed, only their localization to the body element are required for simple materials. Given a thermodeformation process $(\bar{\delta}, \bar{\theta})$, it can be localized to obtain a deformation process of the body element (see (10.2))

$$(\bar{\mathbf{G}}_t(X) \mid t \in I). \tag{15.1}$$

For the temperature process what comes into play is the temperature of the body element and the temperature gradient across the element (see (12.15)):

$$(\bar{\theta}_t(X) \mid t \in I) \quad \text{and} \quad (\bar{\boldsymbol{\gamma}}_t(X) \mid t \in I). \tag{15.2}$$

Thus the thermodeformation process induces the process

$$((\bar{\mathbf{G}}_t(X), \bar{\theta}_t(X), \boldsymbol{\gamma}_t(X)) \mid t \in I) \tag{15.3}$$

on the body element. We will call such processes *condition processes.* In a similar way a response process $(\bar{\mathbf{F}}^{\mathrm{i}}, \bar{N}, \bar{Q}^{\mathrm{i}}, \bar{E})$ can be used to generate a local response process:

$$((\bar{\mathbf{S}}_t(X), \bar{\eta}_t(X), \bar{\mathbf{h}}_t(X), \bar{\psi}_t(X)) \mid t \in I). \tag{15.4}$$

Here we are using the specific free energy instead of the specific internal energy (see Definition 13.2).

In general, given a set of constitutive laws, the (global) thermodeformation process maybe be needed to generate a response process. However, for simple materials only the process in (15.3) is required. Since constitutive laws are local and simple materials only involve the tangent space structure the material point X will be dropped in the following sections. When doing so time-families will be considered as mappings whose domain is a subset of $\mathbb{R}$.

The treatment of simple materials presented here is based on the work of Noll in *A New Mathematical Theory of Simple Materials* [NTSM]. In [NTSM] Noll devised a general framework for formulating frame-free constitutive laws for simple materials based on the concept of a state space[1]. An element σ of the state space Σ represents a possible *state* that the body element can be in. A state describes everything about the element: its configuration, its stress, and so on.

This framework was designed to overcome the defects in the old theory of simple materials developed by Noll in *A Mathematical Theory of the Mechanical Behavior of Continuous Media* [OTSM]. The old theory is based on the concept of a *history*. It was assumed that if you knew the entire history of the body element then you could determine the current stress, entropy, heat flux and free energy of the element. The new theory was successful in the sense that it was able to describe materials that the old theory could not. For example, materials that exhibited plastic behavior, see [TIM1] and [TIM2], and materials that exhibit plugs in viscometric flows, see [PVF] and Part 4 of [FC], could be formulated within the new theory. In [PVF] Noll and I proved the existence of plugs in various viscometric flows of simple semi-liquids (incompressible semi-fluids) and found explicit procedures for determining the sizes of the plugs.

Noll's framework in [NTSM] only dealt with mechanical phenomena. The treatment presented here expands his work to include thermal effects. In this expanded framework each state of a body element is associated with a condition, which consists of a configuration, a temperature and a temperature gradient, and a response, which consists of an intrinsic stress, specific entropy, intrinsic heat flux and specific free energy. Many of the results in this chapter are straightforward generalizations of results in [NTSM] however, some of the results dealing with symmetry are not. This is due to the fact that in the treatment presented here the lineon minus the identity, i.e., the central inversion, need not be in the symmetry group while in the purely mechanical theory minus the identity is always in the symmetry group. This allows for the possibility for materials with thermal chirality.

16 Simple Thermomechanical Elements

Let a set $\mathcal{A}$ be given. A mapping of the form

$$\mathbf{P} : [0, d_{\mathbf{P}}] \longrightarrow \mathcal{A}, \quad d_{\mathbf{P}} \in \mathbb{P}$$

[1]It has been found that the concept of a state space is the appropriate framework for the foundations of mechanics and thermodynamics. See, for example, [FCFT], [MFT], and the references cited in Section 4.1 of [MTCM].

is called a **process** with values in $\mathcal{A}$. The number $d_{\mathbf{P}}$ is called the **duration** of the process. The **initial value** $\mathbf{P}^i$ and the **final value** $\mathbf{P}^f$ of the process $\mathbf{P}$ are given by

$$\mathbf{P}^i := \mathbf{P}(0) \quad \text{and} \quad \mathbf{P}^f := \mathbf{P}(d_{\mathbf{P}}).$$

Given $\mathbf{C} \in \mathcal{A}$ and $d \in \mathbb{P}$ one can construct a process $\mathbf{C}_{(d)} : [0, d] \longrightarrow \mathcal{A}$, called the **freeze of duration d**, defined by

$$\mathbf{C}_{(d)}(t) := \mathbf{C} \quad \text{for all} \quad t \in [0, d]. \tag{16.1}$$

Let $\mathbf{P}$ be a process and $t_1, t_2 \in [0, d_{\mathbf{P}}]$, with $t_1 \leq t_2$, be given. One can define the **segment** $\mathbf{P}_{[t_1,t_2]} : [0, t_2 - t_1] \longrightarrow \mathcal{A}$ of $\mathbf{P}$ by

$$\mathbf{P}_{[t_1,t_2]}(s) := \mathbf{P}(t_1 + s) \quad \text{for all} \quad s \in [0, t_2 - t_1].$$

Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be processes such that $\mathbf{P}_1^f = \mathbf{P}_2^i$. One can define a new process $\mathbf{P}_1 * \mathbf{P}_2$, called the **continuation of $\mathbf{P}_1$ with $\mathbf{P}_2$**, by

$$(\mathbf{P}_1 * \mathbf{P}_2)(t) := \begin{cases} \mathbf{P}_1(t) & \text{if} \quad t \in [0, d_{\mathbf{P}_1}] \\ \mathbf{P}_2(t - d_{\mathbf{P}_1}) & \text{if} \quad t \in [d_{\mathbf{P}_1}, d_{\mathbf{P}_1} + d_{\mathbf{P}_2}]. \end{cases} \tag{16.2}$$

Let an open subset $\mathcal{U}$ of some linear space and a closed interval I of $\mathbb{R}$ be given. We denote by

$$\mathrm{pwC}^1(I, \mathcal{U}) \tag{16.3}$$

the set of all mappings from I to $\mathcal{U}$ that are piecewise C^1 (see Section 6). Note that when $\mathcal{U}$ is a linear space $\mathrm{pwC}^1(I, \mathcal{U})$ has the structure of an infinite-dimensional linear space.

The concept of a body element introduced in Chapter I is expanded here to include more structure.

Definition 16.1 *A* **body element** *is a triple* $(\mathcal{T}, \mathcal{C}, \Pi)$, *where* $\mathcal{T}$ *is a three-dimensional linear space,* $\mathcal{C}$ *is an open connected subset of* $\mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*) \times \mathbb{P}^\times \times \mathcal{T}^*$ *and* Π *is a collection of processes with values in* $\mathcal{C}$ *with the following properties:*

(P1) *Every freeze at* $\mathbf{C} \in \mathcal{C}$ *belongs to* Π.

(P2) *If* $\mathbf{P}$ *is in* Π *then every segment of* $\mathbf{P}$ *is in* Π.

(P3) *If* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are in* Π *such that* $\mathbf{P}_1^f = \mathbf{P}_2^i$ *then* $\mathbf{P}_1 * \mathbf{P}_2$ *is in* Π.

(P4) *Given any two elements* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *in* $\mathcal{C}$ *there is a process* $\mathbf{P} \in \Pi$ *such that* $\mathbf{P}^i = \mathbf{C}_1$ *and* $\mathbf{P}^f = \mathbf{C}_2$.

(P5) *For all* $d \in \mathbb{P}$, *the set of all processes of duration* d, *denoted by*

$$\Pi^d := \{\mathbf{P} \in \Pi \mid d_{\mathbf{P}} = d\}, \tag{16.4}$$

is a subset of $\mathrm{pwC}^1([0, d], \mathcal{C})$.

The elements of $\mathcal{C}$ are called **conditions** of the body element and the elements of Π are called **condition processes**. Given a condition $\mathbf{C} = (\mathbf{G}, \theta, \boldsymbol{\gamma}) \in \mathcal{C}$, $\mathbf{G}$ is called the **configuration** , θ the **temperature** and $\boldsymbol{\gamma}$ the **temperature gradient**. Put

$$\mathcal{C}' := \mathrm{Lsp}\mathcal{C} = \mathrm{Sym}(\mathcal{T}, \mathcal{T}^*) \times \mathbb{R} \times \mathcal{T}^*. \tag{16.5}$$

The component mappings of $\mathbf{P} \in \Pi$ are denoted by $\bar{\mathbf{G}} : [0, d_{\mathbf{P}}] \longrightarrow \mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*)$, $\bar{\theta} : [0, d_{\mathbf{P}}] \longrightarrow \mathbb{P}^\times$ and $\bar{\boldsymbol{\gamma}} : [0, d_{\mathbf{P}}] \longrightarrow \mathcal{T}^*$, so that $\mathrm{Rng}(\bar{\mathbf{G}}, \bar{\theta}, \bar{\boldsymbol{\gamma}}) \subseteq \mathcal{C}$ and

$$\mathbf{P} = (\bar{\mathbf{G}}, \bar{\theta}, \bar{\boldsymbol{\gamma}})|^{\mathcal{C}}. \tag{16.6}$$

Given a body element, one can specify a mass density function $\hat{\rho} : \mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*) \longrightarrow \mathbb{P}^\times$ that assigns to each configuration a mass density, as mentioned before (see (8.15)).

Remark 16.2 It follows from (P3) that in general not all condition processes are of class C^1. Namely, if there are condition processes $\mathbf{P}_1$ and $\mathbf{P}_2$ of class C^1 in which $\mathbf{P}_1^f = \mathbf{P}_2^i$ but $\mathbf{P}_1^{\bullet f} \neq \mathbf{P}_2^{\bullet i}$ then $\mathbf{P}_1 * \mathbf{P}_2$ is a condition process by (P3) that is not differentiable. One could alter (P3) so that the continuation of $\mathbf{P}_1$ by $\mathbf{P}_2$ is only required to be a condition process if $\mathbf{P}_1^{\bullet f} = \mathbf{P}_2^{\bullet i}$. This alteration would make this framework inapplicable to shock waves. The treatment of such waves will not be considered here but should be investigated in the future. ▮

Put

$$\mathcal{R} := \mathrm{Sym}(\mathcal{T}^*, \mathcal{T}) \times \mathbb{R} \times \mathcal{T} \times \mathbb{R} \tag{16.7}$$

and call its elements possible **responses** of the body element. Given a response $(\mathbf{S}, \eta, \mathbf{h}, \psi) \in \mathcal{R}$, $\mathbf{S}$ is the **intrinsic stress**, η the **specific entropy**, $\mathbf{h}$ the **intrinsic heat flux** and ψ the **specific free energy**.

Definition 16.3 *A* **thermomechanical element** *is a septuple* $(\mathcal{T}, \mathcal{C}, \Pi, \Sigma, \hat{\mathbf{C}}, \hat{\mathbf{R}}, \hat{e})$ *whose terms are described as follows:*

(T1) $(\mathcal{T}, \mathcal{C}, \Pi)$ *is a body element.*

(T2) Σ *is a set, called the* **state space** *of the element, whose elements are called* **states**.

(T3) *A* **condition mapping** $\hat{\mathbf{C}} : \Sigma \longrightarrow \mathcal{C}$ *which assigns to each state a condition.*

(T4) *A* **response mapping** $\hat{\mathbf{R}} : \Sigma \longrightarrow \mathcal{R}$ *which assigns to each state a response.*

(T5) *An* **evolution mapping** $\hat{e} : (\Sigma \times \Pi)_{\mathrm{fit}} \longrightarrow \Sigma$*, where*

$$(\Sigma \times \Pi)_{\mathrm{fit}} := \{(\sigma, \mathbf{P}) \in \Sigma \times \Pi \mid \mathbf{P}^i = \hat{\mathbf{C}}(\sigma)\}.$$

Roughly speaking, $\hat{e}(\sigma, \mathbf{P})$ is the state reached by subjecting the state σ to the "thought experiment" represented by the condition process $\mathbf{P}$. We define the **result mapping** $\tilde{\mathbf{R}} : (\Sigma \times \Pi)_{\mathrm{fit}} \longrightarrow \mathcal{R}$ by

$$\tilde{\mathbf{R}} := \hat{\mathbf{R}} \circ \hat{e}. \tag{16.8}$$

We denote the components of $\hat{\mathbf{R}}$ by $(\hat{\mathbf{S}}, \hat{\eta}, \hat{\mathbf{h}}, \hat{\psi})$ so that

$$\hat{\mathbf{R}} = (\hat{\mathbf{S}}, \hat{\eta}, \hat{\mathbf{h}}, \hat{\psi}) \tag{16.9}$$

and the components of $\tilde{\mathbf{R}}$ by $(\tilde{\mathbf{S}}, \tilde{\eta}, \tilde{\mathbf{h}}, \tilde{\psi})$ so that

$$\tilde{\mathbf{R}} = (\tilde{\mathbf{S}}, \tilde{\eta}, \tilde{\mathbf{h}}, \tilde{\psi}). \tag{16.10}$$

We will need to introduce notation that will be used throughout the next two chapters. Let $\mathbf{C} \in \mathcal{C}$ and $d \in \mathbb{P}$ be given.

- Define the **C-section** by

$$\Sigma_{\mathbf{C}} := \{\sigma \in \Sigma \mid \hat{\mathbf{C}}(\sigma) = \mathbf{C}\}. \tag{16.11}$$

  This set consists of all states that have the condition $\mathbf{C}$. Note that Σ is the disjoint union of all $\mathbf{C}$-sections.

- The set of all condition processes that start with the condition $\mathbf{C}$ will be denoted by

$$\Pi_{\mathbf{C}} := \{\mathbf{P} \in \Pi \mid \mathbf{P}^i = \mathbf{C}\}. \tag{16.12}$$

- The set of all condition processes that start at the condition $\mathbf{C}$ and have duration d will be denoted by

$$\Pi^d_{\mathbf{C}} := \Pi_{\mathbf{C}} \cap \Pi^d. \tag{16.13}$$

- Given a state $\sigma \in \Sigma_{\mathbf{C}}$ we define the mapping $\tilde{\mathbf{R}}_d(\sigma, \cdot) : \Pi^d_{\mathbf{C}} \longrightarrow \mathcal{R}$ by

$$\tilde{\mathbf{R}}_d(\sigma, \cdot)(\mathbf{P}) := \tilde{\mathbf{R}}(\sigma, \mathbf{P}) \quad \text{for all} \quad \mathbf{P} \in \Pi^d_{\mathbf{C}}. \tag{16.14}$$

Sometimes we will refer to a thermomechanical element $(\mathcal{T}, \mathcal{C}, \Pi, \Sigma, \hat{\mathbf{C}}, \hat{\mathbf{R}}, \hat{e})$ by just the linear space $\mathcal{T}$.

An **unconstrained element** is one in which the set of condition processes is as large as possible, i.e., $\Pi^d = \mathrm{pwC}^1([0, d], \mathcal{C})$ for all $d \in \mathbb{P}$.

Remark 16.4 Consider a continuous body $\mathcal{B}$ and let $X \in \mathcal{B}$ be given. Assume the tangent space $\mathcal{T}_X$ has the structure of a thermomechanical element. If the body $\mathcal{B}$ is subjected to internal constraints this will affect the set of condition processes Π. If the body is unconstrained then the set of condition processes is as large as possible, i.e., $\Pi^d = \mathrm{pwC}^1([0, d], \mathcal{C})$ for all $d \in \mathbb{P}$. If the body is constrained then for all $d \in \mathbb{P}^\times$, Π^d would be a suitable proper subset of $\mathrm{pwC}^1([0, d], \mathcal{C})$. The definition of a thermomechanical element works whether or not there are constraints. However, when there are constraints the stress component of the response mapping $\hat{\mathbf{S}}$ must be interpreted as the *intrinsic extra stress* in the element. See Section 9 of [NTSM] for more about internal constraints within this framework. ▮

The ingredients of a thermomechanical element are subject to several axioms, six in total. The first three will be given here while the remaining three will be introduced in the following sections.

Axiom 1 For all $(\sigma, \mathbf{P}) \in (\Sigma \times \Pi)_{\mathrm{fit}}$, we have

$$\hat{e}(\sigma, \mathbf{P}) \in \Sigma_{\mathbf{P}^f}, \quad \text{i.e.} \quad \hat{\mathbf{C}}(\hat{e}(\sigma, \mathbf{P})) = \mathbf{P}^f. \tag{16.15}$$

This axiom implies that the mapping $\hat{\mathbf{C}}$ is surjective. To see this let $\mathbf{C} \in \mathcal{C}$ be given and let σ be an arbitrary state. From (P4) of Definition 16.1 there is a condition process $\mathbf{P}$ such that $\mathbf{P}^i = \hat{\mathbf{C}}(\sigma)$ and $\mathbf{P}^f = \mathbf{C}$. Axiom 1 says that the condition of the state $\hat{e}(\sigma, \mathbf{P})$ is $\mathbf{C}$. Since $\mathbf{C} \in \mathcal{C}$ was arbitrary, $\hat{\mathbf{C}}$ is surjective.

Axiom 2 If $\mathbf{P}_1, \mathbf{P}_2 \in \Pi$, $\sigma \in \Sigma_{\mathbf{P}_1^i}$ and $\mathbf{P}_1^f = \mathbf{P}_2^i$, then

$$\hat{e}(\sigma, \mathbf{P}_1 * \mathbf{P}_2) = \hat{e}(\hat{e}(\sigma, \mathbf{P}_1), \mathbf{P}_2). \tag{16.16}$$

The following axiom says, roughly, that there must be some way to distinguish between two different states.

Axiom 3 For all $\mathbf{C} \in \mathcal{C}$, if $\sigma_1, \sigma_2 \in \Sigma_{\mathbf{C}}$ and $\tilde{\mathbf{R}}(\sigma_1, \mathbf{P}) = \tilde{\mathbf{R}}(\sigma_2, \mathbf{P})$ for all $\mathbf{P} \in \Pi_{\mathbf{C}}$, then $\sigma_1 = \sigma_2$.

This axiom is very powerful and is the only way to show that two states are the same.

Proposition 16.5 *Let $\sigma \in \Sigma$ be given. Then we have*

$$\hat{e}(\sigma, \hat{\mathbf{C}}(\sigma)_{(0)}) = \sigma. \tag{16.17}$$

Proof: Let $\mathbf{P} \in \Pi_{\hat{\mathbf{C}}(\sigma)}$ be given. Note that $\mathbf{P} = \mathbf{P}^i_{(0)} * \mathbf{P}$. Applying (16.16) with $\mathbf{P}_1 := \mathbf{P}^i_{(0)}$ and $\mathbf{P}_2 := \mathbf{P}$ we obtain

$$\hat{e}(\sigma, \mathbf{P}) = \hat{e}(\hat{e}(\sigma, \hat{\mathbf{C}}(\sigma)_{(0)}), \mathbf{P}). \tag{16.18}$$

Applying $\hat{\mathbf{R}}$ to this equations yields

$$\tilde{\mathbf{R}}(\sigma, \mathbf{P}) = \tilde{\mathbf{R}}(\hat{e}(\sigma, \hat{\mathbf{C}}(\sigma)_{(0)}), \mathbf{P}). \tag{16.19}$$

Since $\mathbf{P} \in \Pi_{\hat{\mathbf{C}}(\sigma)}$ was arbitrary, using Axiom 3, we obtain the desired result. ∎

Motivated by this last result we will sometimes use the following notation:

$$\Pi^\times := \{\mathbf{P} \in \Pi \mid d_{\mathbf{P}} \in \mathbb{P}^\times\} \qquad \text{and} \qquad (\Sigma \times \Pi^\times)_{\text{fit}} := \{(\sigma, \mathbf{P}) \in \Sigma \times \Pi^\times \mid \mathbf{P}^i = \hat{\mathbf{C}}(\sigma)\}. \tag{16.20}$$

17 Material Isomorphisms and Material Symmetry

Let thermomechanical elements $(\mathcal{T}_1, \mathcal{C}_1, \Pi_1, \Sigma_1, \hat{\mathbf{C}}_1, \hat{\mathbf{R}}_1, \hat{e}_1)$ and $(\mathcal{T}_2, \mathcal{C}_2, \Pi_2, \Sigma_2, \hat{\mathbf{C}}_2, \hat{\mathbf{R}}_2, \hat{e}_2)$ be given. A linear isomorphism $\mathbf{A} \in \text{Lis}(\mathcal{T}_1, \mathcal{T}_2)$ induces two mappings:

- $\mathbf{A}_{\mathcal{C}} : \mathcal{C}_1 \longrightarrow \text{Pos}^+(\mathcal{T}_2, \mathcal{T}_2^*) \times \mathbb{P}^\times \times \mathcal{T}_2^*$, defined by

$$\mathbf{A}_{\mathcal{C}}(\mathbf{G}, \theta, \boldsymbol{\gamma}) := (\mathbf{A}^{-\top}\mathbf{G}\mathbf{A}^{-1}, \theta, \mathbf{A}^{-\top}\boldsymbol{\gamma}) \quad \text{for all} \quad (\mathbf{G}, \theta, \boldsymbol{\gamma}) \in \mathcal{C}_1. \tag{17.1}$$

- $\mathbf{A}_{\mathcal{R}} : \mathcal{R}_1 \longrightarrow \mathcal{R}_2$, defined by

$$\mathbf{A}_{\mathcal{R}}(\mathbf{S}, \eta, \mathbf{h}, \psi) := (\mathbf{A}\mathbf{S}\mathbf{A}^\top, \eta, \mathbf{A}\mathbf{h}, \psi) \quad \text{for all} \quad (\mathbf{S}, \eta, \mathbf{h}, \psi) \in \mathcal{R}_1. \tag{17.2}$$

Definition 17.1 *A mapping $\mathbf{A} \in \text{Lis}(\mathcal{T}_1, \mathcal{T}_2)$ is a* **material isomorphism** *of the two thermomechanical elements if there is an invertible mapping*

$$\iota_{\mathbf{A}} : \Sigma_1 \longrightarrow \Sigma_2 \tag{17.3}$$

such that the following conditions hold:

(MI1) $\mathbf{A}_{\mathcal{C}>}(\mathcal{C}_1) = \mathcal{C}_2$.
(MI2) *The mapping* $\mathbf{P} \longmapsto \mathbf{A}_{\mathcal{C}} \circ \mathbf{P}$ *from* Π_1 *to* Π_2 *is invertible.*
(MI3) $\mathbf{A}_{\mathcal{C}} \circ \hat{\mathbf{C}}_1 = \hat{\mathbf{C}}_2 \circ \iota_{\mathbf{A}}$.
(MI4) $\mathbf{A}_{\mathcal{R}} \circ \hat{\mathbf{R}}_1 = \hat{\mathbf{R}}_2 \circ \iota_{\mathbf{A}}$.
(MI5) *For all* $(\sigma, \mathbf{P}) \in (\Sigma_1 \times \Pi_1)_{\mathrm{fit}}$,

$$\iota_{\mathbf{A}}(\hat{e}_1(\sigma, \mathbf{P})) = \hat{e}_2(\iota_{\mathbf{A}}(\sigma), \mathbf{A}_{\mathcal{C}} \circ \mathbf{P}). \tag{17.4}$$

In this case, the two elements are said to be **materially isomorphic**.

Proposition 17.2 *The mapping* $\iota_{\mathbf{A}}$ *in (17.3) is uniquely determined by the material isomorphism* $\mathbf{A}$.

Proof: Let $\sigma \in \Sigma_1$ be given. Applying $\hat{\mathbf{R}}_2$ to (17.4) and using (MI4) we obtain

$$\mathbf{A}_{\mathcal{R}}(\hat{\mathbf{R}}_1(\hat{e}_1(\sigma, \mathbf{P}))) = \hat{\mathbf{R}}_2(\hat{e}_2(\iota_{\mathbf{A}}(\sigma), \mathbf{A}_{\mathcal{C}} \circ \mathbf{P})) \quad \text{for all } \mathbf{P} \in \Pi_1 \text{ such that } \hat{\mathbf{C}}_1(\sigma) = \mathbf{P}^i. \tag{17.5}$$

and hence

$$\mathbf{A}_{\mathcal{R}}(\hat{\mathbf{R}}_1(\hat{e}_1(\sigma, \mathbf{A}_{\mathcal{C}}^{\leftarrow} \circ \mathbf{P}))) = \hat{\mathbf{R}}_2(\hat{e}_2(\iota_{\mathbf{A}}(\sigma), \mathbf{P})) \quad \text{for all } \mathbf{P} \in \Pi_2 \text{ such that } \hat{\mathbf{C}}_1(\sigma) = \mathbf{A}_{\mathcal{C}}(\mathbf{P}^i). \tag{17.6}$$

Thus if $\iota_{\mathbf{A}}$ and $\iota'_{\mathbf{A}}$ are both bijections satisfying (MI1)–(MI5), we have

$$\hat{\mathbf{R}}_2(\hat{e}_2(\iota_{\mathbf{A}}(\sigma), \mathbf{P})) = \hat{\mathbf{R}}_2(\hat{e}_2(\iota'_{\mathbf{A}}(\sigma), \mathbf{P})) \quad \text{for all } \mathbf{P} \in \Pi_2 \text{ such that } \hat{\mathbf{C}}_1(\sigma) = \mathbf{A}_{\mathcal{C}}(\mathbf{P}^i). \tag{17.7}$$

By Axiom 3 this implies that $\iota_{\mathbf{A}}(\sigma) = \iota'_{\mathbf{A}}(\sigma)$. Since $\sigma \in \Sigma_1$ was arbitrary, we obtain $\iota_{\mathbf{A}} = \iota'_{\mathbf{A}}$. ▮

If thermomechanical elements $(\mathcal{T}_1, \mathcal{C}_1, \Pi_1, \Sigma_1, \hat{\mathbf{C}}_1, \hat{\mathbf{R}}_1, \hat{e}_1)$ and $(\mathcal{T}_2, \mathcal{C}_2, \Pi_2, \Sigma_2, \hat{\mathbf{C}}_2, \hat{\mathbf{R}}_2, \hat{e}_2)$ are materially isomorphic we say that they *consist of the same material*. More precisely, a **material** is an equivalence class of material elements, the equivalence being material isomorphy.

Remark 17.3 Let $\mathcal{B}$ be a continuous body, as defined in Section 4, and assume that at every material point $X \in \mathcal{B}$ the tangent space $\mathcal{T}_X$ has the structure of a thermomechanical element. Such a body is called a (simple) **material body**. If all of the thermomechanical element structures on a material body are isomorphic then we say that the body is **materially uniform**. Given a material body, if there is a configuration $\delta \in \mathrm{Conf}\mathcal{B}$ of the body such that (recall (4.5))

$$\mathbf{I}_\delta(Y)^{-1}\mathbf{I}_\delta(X) \in \mathrm{Lis}(\mathcal{T}_X, \mathcal{T}_Y)$$

is a material isomorphism for all $X, Y \in \mathcal{B}$ then we say the body is **homogeneous**. A material body can be materially uniform without being homogeneous. The theory of continuous distributions of dislocations is nothing but the analysis of the deviation from homogeneity for materially uniform bodies. See [MUBI] for earlier versions of the definitions given in this remark. ▮

If the elements coincide, $\mathcal{T} := \mathcal{T}_1 = \mathcal{T}_2$ etc., then an $\mathbf{A} \in \mathrm{Unim}(\mathcal{T})$ satisfying (MI1)–(MI5) is called a **symmetry** of the element. Denote the set of all symmetries of $\mathcal{T}$ by $\mathfrak{G} \subseteq \mathrm{Unim}\mathcal{T}$. It is not difficult to check that $\mathfrak{G}$ is a subgroup of $\mathrm{Unim}\mathcal{T}$.

Remark 17.4 In the past the assumption that symmetries must be unimodular has been based on the intuitive idea that a symmetry should not change the density of the element. It is worthwhile to make this idea precise. Although it is not stated explicitly in the definition of a thermomechanical element, each element has with it a mass-density function $\hat{\rho}$ (see 8.15). Let a symmetry $\mathbf{A} \in \mathrm{Lis}\mathcal{T}$ of an element $\mathcal{T}$ be given and let us not assume that it is unimodular. Then $\mathbf{A}$ would have to satisfy an invariance relation for $\hat{\rho}$ similar to the invariance relations involving the other ingredients of the element, namely

$$\hat{\rho}(\mathbf{A}^\top \mathbf{G}\mathbf{A}) = \hat{\rho}(\mathbf{G}) \quad \text{for all} \quad \mathbf{G} \in \mathcal{G} \tag{17.8}$$

where $\mathcal{G}$ is the projection of $\mathcal{C}$ onto $\mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*)$. It turns out that if $\mathbf{A}$ satisfies (17.8) for even one $\mathbf{G}$ then $\mathbf{A}$ must be unimodular.

To see this let $\mathbf{G} \in \mathcal{G}$ be given. Let $\mathcal{B}$ be a continuous body as defined in Section 4 and assume that at $X \in \mathcal{B}$ the tangent space $\mathcal{T}$ at X has the structure of a thermomechanical element. By (8.15) we can find placements $\mu, \mu_1 \in \mathrm{Pl}\mathcal{B}$ such that by putting $\mathbf{M} := \mathbf{M}_\mu(X)$ and $\mathbf{M}_1 := \mathbf{M}_{\mu_1}(X)$ we have

$$\begin{array}{c} \mathbf{G} = \mathbf{M}^\top \mathbf{M} \text{ and so } \hat{\rho}(\mathbf{G}) = \rho_\mu(\mu(X)) \\ \mathbf{A}^\top \mathbf{G}\mathbf{A} = \mathbf{M}_1^\top \mathbf{M}_1 \text{ and so } \hat{\rho}(\mathbf{A}^\top \mathbf{G}\mathbf{A}) = \rho_{\mu'}(\mu'(X)). \end{array} \tag{17.9}$$

Put $\lambda := \mu \circ \mu_1^{\leftarrow} : \mathrm{Frm}\mu_1 \longrightarrow \mathrm{Frm}\mu$. By (8.2) we have $\rho_{\mu_1}(\mu_1(X)) = \rho_{\mu_1,\mu}(\mu_1(X))\rho_\mu(\mu(X))$ and thus by (17.9) and (17.8) we know $\rho_{\mu_1,\mu}(\mu_1(X)) = 1$. Using (5.10) and the definition of λ we have

$$1 = |\det(\nabla_{\mu_1(X)}\lambda\mathbf{B})| = |\det(\mathbf{M}\mathbf{M}_1^{-1}\mathbf{B})|. \tag{17.10}$$

From $(17.9)_2$ we know $\mathbf{M}_1^{-1} = \mathbf{A}^{-1}\mathbf{G}^{-1}\mathbf{A}^{-\top}\mathbf{M}_1^\top$ and so using some properties of the determinant we can obtain

$$\begin{array}{rcl} 1 & = & |\det(\mathbf{M}\mathbf{A}^{-1}\mathbf{G}^{-1}\mathbf{A}^{-\top}\mathbf{M}_1^\top \mathbf{B})| \\ & = & |\det(\mathbf{M}\mathbf{G}^{-1}\mathbf{M}_1^\top \mathbf{B})||\det(\mathbf{A}^{-1}\mathbf{G}^{-1}\mathbf{A}^{-\top}\mathbf{G})| \\ & = & |\det(\mathbf{M}\mathbf{G}^{-1}\mathbf{M}_1^\top \mathbf{B})||\det(\mathbf{A}^{-1})||\det(\mathbf{A}^{-\top})| \\ & = & |\det(\mathbf{M}^{-\top}\mathbf{G}\mathbf{M}_1^{-1}\mathbf{B})|^{-1}|\det\mathbf{A}|^{-2}. \end{array} \tag{17.11}$$

By $(17.9)_1$ $\mathbf{M}^{-\top}\mathbf{G} = \mathbf{M}$ and so $|\det(\mathbf{M}^{-\top}\mathbf{G}\mathbf{M}_1^{-1}\mathbf{B})| = |\det(\mathbf{M}\mathbf{M}_1^{-1}\mathbf{B})| = 1$ by (17.10). This together with (17.11) says that $\mathbf{A} \in \mathrm{Unim}\mathcal{T}$. ∎

The following result is an immediate consequence of Proposition 17.2.

Proposition 17.5 *There is a unique homomorphism*

$$\iota : \mathfrak{G} \longrightarrow \mathrm{Perm}\Sigma \tag{17.12}$$

such that for all $\mathbf{A} \in \mathfrak{G}$ *the value* $\iota_\mathbf{A} := \iota(\mathbf{A})$ *has the following properties:*

(S1) $\mathbf{A}_{\mathcal{C}} \circ \hat{\mathbf{C}} = \hat{\mathbf{C}} \circ \iota_{\mathbf{A}}$.
(S2) $\mathbf{A}_{\mathcal{R}} \circ \hat{\mathbf{R}} = \hat{\mathbf{R}} \circ \iota_{\mathbf{A}}$.
(S3) *For all* $(\sigma, \mathbf{P}) \in (\Sigma \times \Pi)_{\text{fit}}$,

$$\iota_{\mathbf{A}}(\hat{e}(\sigma, \mathbf{P})) = \hat{e}(\iota_{\mathbf{A}}(\sigma), \mathbf{A}_{\mathcal{C}} \circ \mathbf{P}). \tag{17.13}$$

Definition 17.6 *If* $\mathfrak{G}$ *is the full unimodular group then the element is called a* **semi-fluid**.

Definition 17.7 *Given* $\sigma \in \Sigma$ *the* **symmetry group** *of* σ, *denoted by*

$$\mathfrak{G}_{\sigma} := \{\mathbf{A} \in \mathfrak{G} \mid \iota_{\mathbf{A}}(\sigma) = \sigma\}, \tag{17.14}$$

is the set of all symmetries that leave σ *fixed.*

Of course, $\mathfrak{G}_{\sigma}$ is a subgroup of $\mathfrak{G}$. Put

$$\text{Fix}(\boldsymbol{\gamma}) := \{\mathbf{A} \in \text{Lis}(\mathcal{T}) \mid \mathbf{A}^{\top}\boldsymbol{\gamma} = \boldsymbol{\gamma}\} \quad \text{for all} \quad \boldsymbol{\gamma} \in \mathcal{T}^*, \tag{17.15}$$

$$\text{Orth}(\mathbf{G}) := \{\mathbf{A} \in \text{Lis}(\mathcal{T}) \mid \mathbf{A}^{\top}\mathbf{G}\mathbf{A} = \mathbf{G}\} \quad \text{for all} \quad \mathbf{G} \in \text{Pos}^{+}(\mathcal{T}, \mathcal{T}^*). \tag{17.16}$$

Proposition 17.8 *Let* $\sigma \in \Sigma$ *be given. Then*

$$\mathfrak{G}_{\sigma} \subseteq \mathfrak{G} \cap \text{Orth}(\hat{\mathbf{G}}(\sigma)) \cap \text{Fix}(\hat{\boldsymbol{\gamma}}(\sigma)). \tag{17.17}$$

Proof: Let $\mathbf{A} \in \mathfrak{G}_{\sigma}$ be given. It follows from (S1), (17.15) and (17.1) that $\hat{\mathbf{G}}(\sigma) = \mathbf{A}^{\top}\hat{\mathbf{G}}(\sigma)\mathbf{A}$ and $\hat{\boldsymbol{\gamma}}(\sigma) = \mathbf{A}^{\top}\hat{\boldsymbol{\gamma}}(\sigma)$. The result immediately follows. ∎

The state σ is said to be an **isotropic state** if

$$\mathfrak{G}_{\sigma} = \text{Orth}(\hat{\mathbf{G}}(\sigma)). \tag{17.18}$$

An element is said to be an **isotropic element** if it has isotropic states. It follows from Proposition 17.8 that a state is isotropic only if the temperature gradient across the element is zero.

Motivated by Proposition 17.8 we consider the following concept: A state σ is said to be a **thermally axisymmetric state**[2] if

$$\mathfrak{G}_{\sigma} = \text{Orth}(\hat{\mathbf{G}}(\sigma)) \cap \text{Fix}(\hat{\boldsymbol{\gamma}}(\sigma)). \tag{17.19}$$

An element is said to be a **thermally axisymmetric element** if it has thermaly axisymmetric states.

Proposition 17.9 *Let* $\sigma \in \Sigma$ *be given. Put* $\mathbf{S} := \hat{\mathbf{S}}(\sigma)$, $\mathbf{G} := \hat{\mathbf{G}}(\sigma)$, $\boldsymbol{\gamma} := \hat{\boldsymbol{\gamma}}(\sigma)$ *and* $\mathbf{h} := \hat{\mathbf{h}}(\sigma)$. *Then*

$$\mathbf{ASG} = \mathbf{SGA} \quad \text{and} \quad \mathbf{Ah} = \mathbf{h} \quad \text{for all} \quad \mathbf{A} \in \mathfrak{G}_{\sigma}. \tag{17.20}$$

[2]The term "transversly isotropic" has been used for what I call "axisymmetric". The adjective "thermally" is added because the asymmetry is due to thermal effects, namely the temperature gradient.

If the state is thermally axisymmetric then there are numbers $p, \alpha, \beta \in \mathbb{R}$ such that

$$\mathbf{S} = -p\mathbf{G}^{-1} + \alpha(\mathbf{G}^{-1}\boldsymbol{\gamma} \otimes \mathbf{G}^{-1}\boldsymbol{\gamma}) \quad \text{and} \quad \mathbf{h} = \beta\mathbf{G}^{-1}\boldsymbol{\gamma}. \tag{17.21}$$

If the state is isotropic then

$$\mathbf{S} = -p\mathbf{G}^{-1} \quad \text{and} \quad \mathbf{h} = \mathbf{0}. \tag{17.22}$$

Proof: (17.20) is a consequence of (17.1), (MI4) and (17.14).

Suppose the state σ is thermally axisymmetric. Since $\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*)$ is fixed in this proof we will use it to endow $\mathcal{T}$ with the structure of an inner-product space. We will use the standard notation for inner-product spaces, namely $\mathbf{u} \cdot \mathbf{v} := (\mathbf{G}\mathbf{u})\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{T}$. Also, put

$$\mathbf{g} := \mathbf{G}^{-1}\boldsymbol{\gamma} \in \mathcal{T} \text{ and } \mathbf{T} := \mathbf{S}\mathbf{G} \in \mathrm{Sym}(\mathcal{T}). \tag{17.23}$$

Since σ is thermally axisymmetric $(17.20)_2$ with (17.19) says that

$$\mathbf{A}\mathbf{h} = \mathbf{h} \quad \text{for all} \quad \mathbf{A} \in \mathrm{Orth}(\mathbf{G}) \cap \mathrm{Fix}(\boldsymbol{\gamma}). \tag{17.24}$$

Notice by (17.23) and (17.15), $\mathrm{Fix}(\boldsymbol{\gamma}) = \{\mathbf{A} \in \mathrm{Lin}(\mathcal{T}) \mid \mathbf{A}\mathbf{g} = \mathbf{g}\}$. Put

$$\mathcal{U} := \{\mathbf{g}\}^\perp = \{\mathbf{v} \in \mathcal{T} \mid (\mathbf{G}\mathbf{v})\mathbf{g} = 0\}. \tag{17.25}$$

Since $\mathcal{U} \cap \mathbb{R}\mathbf{g} = \{\mathbf{0}\}$, the Characterization of Regular Subspaces from [FDS] says that $\mathcal{U}$ and $\mathbb{R}\mathbf{g}$ are supplementary in $\mathcal{T}$ and there is a $\beta \in \mathbb{R}$ and a $\mathbf{u} \in \mathcal{U}$ such that

$$\mathbf{h} = \mathbf{u} + \beta\mathbf{g}. \tag{17.26}$$

Define $\mathbf{A} \in \mathrm{Lin}(\mathcal{T})$ by

$$\mathbf{A}\mathbf{v} := \begin{cases} -\mathbf{v} & \text{if} \quad \mathbf{v} \in \mathcal{U} \\ \ \ \mathbf{v} & \text{if} \quad \mathbf{v} \in \mathbb{R}\mathbf{g}. \end{cases} \tag{17.27}$$

One can see that $\mathbf{A} \in \mathrm{Orth}(\mathbf{G}) \cap \mathrm{Fix}(\boldsymbol{\gamma})$ and so (17.24) and (17.26) show that $\mathbf{u} = \mathbf{0}$. Thus, $\mathbf{h}$ has the desired form.

To show $(17.21)_1$ must hold start by putting $(17.20)_1$ together with (17.19) with the notation (17.23) to obtain

$$\mathbf{A}\mathbf{T} = \mathbf{T}\mathbf{A} \quad \text{for all} \quad \mathbf{A} \in \mathrm{Orth}(\mathbf{G}) \cap \mathrm{Fix}(\boldsymbol{\gamma}). \tag{17.28}$$

Define $\mathcal{U}$ as in (17.25). It follows from (17.28) that

$$\mathbf{Q}(\mathbf{T}|_{\mathcal{U}}) = (\mathbf{T}|_{\mathcal{U}})\mathbf{Q} \quad \text{for all} \quad \mathbf{Q} \in \mathrm{Orth}(\mathbf{G}|_{\mathcal{U}}). \tag{17.29}$$

This is possible only if there is an $\xi \in \mathbb{R}$ such that

$$\mathbf{T}|_{\mathcal{U}} = \xi\mathbf{1}_{\mathcal{U}}. \tag{17.30}$$

Using (17.28) we also know that

$$\mathbf{Q}\mathbf{T}\mathbf{g} = \mathbf{T}\mathbf{Q}\mathbf{g} = \mathbf{T}\mathbf{g} \quad \text{for all} \quad \mathbf{Q} \in \mathrm{Orth}(\mathbf{G}) \cap \mathrm{Fix}(\boldsymbol{\gamma}). \tag{17.31}$$

It follows from the proof in the previous paragraph, with $\mathbf{h} := \mathbf{Tg}$, that there is a $\zeta \in \mathbb{R}$ such that $\mathbf{Tg} = \zeta\mathbf{g}$. Hence

$$\mathbf{T}|_{\mathbb{R}\mathbf{g}} = \frac{\zeta}{\mathbf{g}\cdot\mathbf{g}}\mathbf{g}\otimes\mathbf{g}. \tag{17.32}$$

Since $\mathcal{U}$ and $\mathbb{R}\mathbf{g}$ and supplementary in $\mathcal{T}$, (17.30) and (17.32) together imply that there are $p, \alpha \in \mathbb{R}$ such that

$$\mathbf{T} = -p\mathbf{1}_{\mathcal{T}} + \alpha\mathbf{g}\otimes\mathbf{g}.$$

Using (17.23) we see that this gives us the desired representation for $\mathbf{S}$.

Since isotropic states are thermally axisymmetric and $\boldsymbol{\gamma} = \mathbf{0}$ for the isotropic case (17.21) follows immediately. ∎

Remark 17.10 The symmetry group for a thermally axisymmetric state σ has the following interpretation. One can use the configuration $\mathbf{G} := \hat{\mathbf{G}}(\sigma)$ of that state to temporarily endow the body element with the structure of an inner-product space. Then the temperature gradient $\boldsymbol{\gamma} := \hat{\boldsymbol{\gamma}}(\sigma)$ of the state can be associated with an element of $\mathcal{T}$, namely $\mathbf{g} := \mathbf{G}^{-1}\boldsymbol{\gamma} \in \mathcal{T}$. Then the symmetry group of σ is generated by all rotations about the axis determined by $\mathbf{g}$ and all reflections about a plane that contains $\mathbf{g}$.

The case of thermally axisymmetric states is given special consideration here because this concept arises naturally in this context. However, there are many other types of symmetry one might want to consider. A state σ is **properly thermally axisymmetric** if $\mathfrak{G}_\sigma = \mathrm{Orth}^+(\hat{\mathbf{G}}(\sigma)) \cap \mathrm{Fix}(\hat{\boldsymbol{\gamma}}(\sigma))$ or **hemitropic** if $\mathfrak{G}_\sigma = \mathrm{Orth}^+(\hat{\mathbf{G}}(\sigma))$. ∎

Since the mapping $\mathbf{A} \longmapsto \iota_{\mathbf{A}}$ is a homomorphism, if $\mathbf{A}, \mathbf{B} \in \mathfrak{G}$, then

$$\iota_{\mathbf{A}^{-1}} = \iota_{\mathbf{A}}^{\leftarrow} \quad \text{and} \quad \iota_{\mathbf{AB}} = \iota_{\mathbf{A}} \circ \iota_{\mathbf{B}}. \tag{17.33}$$

Given $\sigma \in \Sigma$, we define the **reduced state** Ω_σ of σ by

$$\Omega_\sigma := \{\iota_{\mathbf{A}}(\sigma) \mid \mathbf{A} \in \mathfrak{G}\}. \tag{17.34}$$

Note that the reduced states partition the state space Σ.

This next result says that the symmetry groups of two states that determine the same reduced state are conjugate and hence isomorphic.

Proposition 17.11 *Let $\sigma \in \Sigma$ and $\mathbf{A} \in \mathfrak{G}$ be given. Then*

$$\mathfrak{G}_{\iota_{\mathbf{A}}(\sigma)} = \mathbf{A}\mathfrak{G}_\sigma\mathbf{A}^{-1}. \tag{17.35}$$

Proof: Let $\mathbf{B} \in \mathfrak{G}_\sigma$ be given. Since $\mathfrak{G}$ is a group, $\mathbf{ABA}^{-1} \in \mathfrak{G}$ and so $\iota_{\mathbf{ABA}^{-1}}$ is meaningful. By (17.33) and (17.14) we have

$$\iota_{\mathbf{ABA}^{-1}}(\iota_{\mathbf{A}}(\sigma)) = (\iota_{\mathbf{A}} \circ \iota_{\mathbf{B}} \circ \iota_{\mathbf{A}}^{-1} \circ \iota_{\mathbf{A}})(\sigma) = \iota_{\mathbf{A}}(\iota_{\mathbf{B}}(\sigma)) = \iota_{\mathbf{A}}(\sigma)$$

and hence $\mathbf{ABA}^{-1} \in \mathfrak{G}_{\iota_{\mathbf{A}}(\sigma)}$. Since $\mathbf{B} \in \mathfrak{G}_\sigma$ we arbitrary this shows $\mathbf{A}\mathfrak{G}_\sigma\mathbf{A}^{-1} \subseteq \mathfrak{G}_{\iota_{\mathbf{A}}(\sigma)}$.

Now let $\mathbf{B} \in \mathfrak{G}_{\iota_{\mathbf{A}}(\sigma)}$ be given. Then by (17.33) and (17.14) again we have

$$\iota_{\mathbf{A}^{-1}\mathbf{B}\mathbf{A}}(\sigma) = \iota_{\mathbf{A}^{-1}}(\iota_{\mathbf{B}}(\iota_{\mathbf{A}}(\sigma))) = \iota_{\mathbf{A}^{-1}}(\iota_{\mathbf{A}}(\sigma)) = \sigma.$$

Thus, $\mathbf{A}^{-1}\mathbf{B}\mathbf{A} \in \mathfrak{G}_\sigma$ and hence $\mathbf{B} \in \mathbf{A}\mathfrak{G}_\sigma\mathbf{A}^{-1}$. Since $\mathbf{B} \in \mathfrak{G}_{\iota_{\mathbf{A}}(\sigma)}$ was arbitrary we conclude that $\mathfrak{G}_{\iota_{\mathbf{A}}(\sigma)} \subseteq \mathbf{A}\mathfrak{G}_\sigma\mathbf{A}^{-1}$.

The result follows immediately. ∎

18 The Basic Topology on the State Space

Let a thermomechanical element $\mathcal{T}$ with result mapping $\tilde{\mathbf{R}}$, see (16.8), be given.

Definition 18.1 *For every* $\mathbf{C} \in \mathcal{C}$ *the* **basic uniformity** *on the* $\mathbf{C}$*-section* $\Sigma_{\mathbf{C}}$*, as defined by (16.11), is the coarsest uniformity on* $\Sigma_{\mathbf{C}}$ *that makes the mappings*

$$\tilde{\mathbf{R}}(\cdot, \mathbf{P}) : \Sigma_{\mathbf{C}} \longrightarrow \mathcal{R} \quad \text{for all} \quad \mathbf{P} \in \Pi_{\mathbf{C}}$$

uniformly continuous. The **basic topology** *on* $\Sigma = \cup_{\mathbf{C}\in\mathcal{C}}\Sigma_{\mathbf{C}}$ *is the disjoint union topology*[3] *induced by the topologies on the* $\mathbf{C}$*-sections induced by the basic uniformities.*

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are thermomechanical elements with state spaces Σ_1 and Σ_2 and $\mathbf{A} : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ is a material isomorphism then the bijection $\iota_{\mathbf{A}} : \Sigma_1 \longrightarrow \Sigma_2$ is a homeomorphism with respect to the basic topologies on Σ_1 and Σ_2. This is a consequence of the general fact that an isomorphism preserves all structures induced by the given structure.

The following result, which follows from the above definition, is a special case of Proposition 4 of Section 2, Chapter II, in [TG], characterizes the baisc uniformities on the $\mathbf{C}$-sections.

Proposition 18.2 *A mapping* $\phi : \mathcal{U} \longrightarrow \Sigma_{\mathbf{C}}$ *from a uniform space* $\mathcal{U}$ *into a* $\mathbf{C}$*-section* $\Sigma_{\mathbf{C}}$ *is uniformly continuous if and only if the mappings*

$$\tilde{\mathbf{R}}(\phi(\cdot), \mathbf{P}) : \mathcal{U} \longrightarrow \mathcal{R} \quad \text{for all} \quad \mathbf{P} \in \Pi_{\mathbf{C}}$$

are uniformly continuous.

This result is used to prove the following result.

Proposition 18.3 *Let* $\mathbf{P} \in \Pi$ *be given. The mapping*

$$\hat{e}(\cdot, \mathbf{P}) : \Sigma_{\mathbf{P}^i} \longrightarrow \Sigma_{\mathbf{P}^f} \tag{18.1}$$

is uniformly continuous.

[3]Let $(X_i \mid i \in I)$ be a disjoint family of topological spaces. Put $X := \cup_{i\in I}X_i$. The disjoint union topology on X is the finest topology on X such that the canonical injections $\mathrm{inj}_i : X_i \longrightarrow X$, $i \in I$, are all continuous.

Proof: Let $\mathbf{P}' \in \Pi_{\mathbf{P}^f}$ be given. From (16.16) and (16.8) we have

$$\tilde{\mathbf{R}}(\hat{e}(\sigma, \mathbf{P}), \mathbf{P}') = \tilde{\mathbf{R}}(\sigma, \mathbf{P} * \mathbf{P}') \quad \text{for all} \quad \sigma \in \Sigma_{\mathbf{P}^i}. \tag{18.2}$$

Thus, since $\hat{\mathbf{R}}(\cdot, \mathbf{P} * \mathbf{P}')$ is uniformly continuous by definition of the basic uniformity on $\Sigma_{\mathbf{P}^i}$ we know $\tilde{\mathbf{R}}(\hat{e}(\cdot, \mathbf{P}), \mathbf{P}')$ is uniformly continuous. Since $\mathbf{P}' \in \Pi_{\mathbf{P}^f}$ was arbitrary an application of Proposition 18.2 proves the result. ▮

The following result, a special case of Proposition 10 of Section 8, Chapter I, of [TG] characterizes the basic topology on Σ.

Proposition 18.4 *A net $(i \mapsto \sigma_i)$ in Σ converges to $\sigma \in \Sigma$ if and only if*

1. *σ_i belongs eventually to some fixed section $\Sigma_{\mathbf{C}}$,*
2. *$\sigma \in \Sigma_{\mathbf{C}}$ and*
3. *$\lim_i \tilde{\mathbf{R}}(\sigma_i, \mathbf{P}) = \tilde{\mathbf{R}}(\sigma, \mathbf{P})$ for all $\mathbf{P} \in \Pi_{\mathbf{C}}$.*

Proposition 18.5 *The basic topology on Σ is Hausdorff, i.e., nets in Σ cannot have more then one limit.*

Proof: Let σ_1 and σ_2 both be limits of a net $(i \mapsto \sigma_i)$. Then by the previous proposition we have $\hat{\mathbf{C}}(\sigma_1) = \hat{\mathbf{C}}(\sigma_2) =: \mathbf{C}$ and $\tilde{\mathbf{R}}(\sigma_1, \mathbf{P}) = \tilde{\mathbf{R}}(\sigma_2, \mathbf{P})$ for all $\mathbf{P} \in \Pi_{\mathbf{C}}$. By Axiom 3 it follows that $\sigma_1 = \sigma_2$. ▮

The **C**-sections, equipped with their basic uniformities, are both open and closed but need not be complete. One can always consider the completion of these sections and then extend the mappings $\hat{\mathbf{R}}$, $\hat{\mathbf{C}}$ and $\hat{e}(\cdot, \mathbf{P})$, $\mathbf{P} \in \Pi$, so that they are uniformly continuous on the completed **C**-sections. By doing this one would obtain a new thermomechanical element structure. However, this new thermomechanical element would not be physically different from the original one. To avoid dealing with these different, but physically equivalent, structures we assume the following:

Axiom 4 *The sections $\Sigma_{\mathbf{C}}$, $\mathbf{C} \in \mathcal{C}$, are complete with respect to their natural uniformities.*

19 Relaxed States and Accessibility

Axiom 5 *For all $\sigma \in \Sigma$, the limit*

$$\lim_{t \to \infty} \hat{e}(\sigma, \hat{\mathbf{C}}(\sigma)_{(t)}) =: \hat{\lambda}(\sigma) \tag{19.1}$$

exists.

The mapping $\hat{\lambda} : \Sigma \longrightarrow \Sigma$ defined in (19.1) is called the **state relaxation mapping**. The elements of the set

$$\Sigma_{\text{rel}} := \hat{\lambda}_{>}(\Sigma) \tag{19.2}$$

are called **relaxed states**. It is clear that $\hat{\mathbf{C}}(\sigma) = \hat{\mathbf{C}}(\hat{\lambda}(\sigma))$.

The following result is analogous to Proposition 12.1 in [NTSM].

Proposition 19.1 *Let $\lambda \in \Sigma$ and $\mathbf{C} := \hat{\mathbf{C}}(\lambda)$ be given. The following four conditions are equivalent:*
(i) $\lambda \in \Sigma_{\text{rel}}$.
(ii) *For all $t \in \mathbb{P}^\times$, $\hat{e}(\lambda, \mathbf{C}_{(t)}) = \lambda$.*
(iii) *For some $t_o \in \mathbb{P}^\times$, $\hat{e}(\lambda, \mathbf{C}_{(t_o)}) = \lambda$.*
(iv) $\hat{\lambda}(\lambda) = \lambda$.

Proof: First note that by the definition of a freeze and a continuation, see (16.1) and (16.2), we have

$$\mathbf{C}_{(r+t)} = \mathbf{C}_{(r)} * \mathbf{C}_{(t)} \quad \text{for all} \quad s, t \in \mathbb{P}^\times. \tag{19.3}$$

For every $\sigma \in \Sigma_{\mathbf{C}}$, it follows from (16.16) and (19.3) that

$$\hat{e}(\sigma, \mathbf{C}_{(t+r)}) = \hat{e}(\hat{e}(\sigma, \mathbf{C}_{(r)}), \mathbf{C}_{(t)}). \tag{19.4}$$

To prove (i)$\Longrightarrow$(ii) let $\lambda \in \Sigma_{\text{rel}}$ be given. There is a $\sigma \in \Sigma$ such that $\hat{\lambda}(\sigma) = \lambda$. Taking the limit $r \to \infty$ in (19.4) and using Proposition 18.3 we obtain

$$\lambda = \hat{\lambda}(\sigma) = \hat{e}(\hat{\lambda}(\sigma), \mathbf{C}_{(t)}) = \hat{e}(\lambda, \mathbf{C}_{(t)}) \tag{19.5}$$

which proves condition (ii).

The implication (ii)$\Longrightarrow$(iii) is trivial.

Now assume that (iii) holds. Apply (19.4) with $\sigma := \lambda$, $r := t_o$ and $t := (n-1)t_o$, with $n \in \mathbb{N}$ to obtain

$$\hat{e}(\lambda, \mathbf{C}_{(nt_o)}) = \hat{e}(\hat{e}(\lambda, \mathbf{C}_{(t_o)}), \mathbf{C}_{((n-1)t_o)}) = \hat{e}(\lambda, \mathbf{C}_{((n-1)t_o)}). \tag{19.6}$$

Since $\lambda = \hat{e}(\lambda, \mathbf{C}_{(0)})$ by Proposition 16.5, we see that induction over n yields $\lambda = \hat{e}(\lambda, \mathbf{C}_{(nt_o)})$ for all $n \in \mathbb{N}$. Taking the limit $n \to \infty$ we obtain $\lambda = \hat{\lambda}(\lambda)$.

The implication (iv)$\Longrightarrow$(i) is trivial. ∎

This result says that a state is relaxed if and only if freezing it at its condition does not change the state. Furthermore, it says that a state frozen in its condition cannot return to that state unless the state is relaxed.

Proposition 19.2 *If every relaxed state is thermally axisymmetric then the element is a semi-fluid.*

Proof: Let $\mathcal{G}$ denote the projection of $\mathcal{C}$ onto $\text{Pos}^+(\mathcal{T}, \mathcal{T}^*)$, in the sense that[4]

$$\mathcal{G} = \{\mathbf{G} \in \text{Pos}^+(\mathcal{T}, \mathcal{T}^*) \mid (\mathbf{G}, \theta, \boldsymbol{\gamma}) \in \mathcal{C} \text{ for some } \theta \in \mathbb{P}^\times, \boldsymbol{\gamma} \in \mathcal{T}^*\}.$$

Since $\mathcal{C}$ is open in $\text{Pos}^+(\mathcal{T}, \mathcal{T}^*) \times \mathbb{P}^\times \times \mathcal{T}^*$, $\mathcal{G}$ is open in $\text{Pos}^+(\mathcal{T}, \mathcal{T}^*)$. Let $\mathbf{G} \in \mathcal{G}$ be given. Recall the following fact about a three dimensional linear space $\mathcal{T}$ with inner-project $\mathbf{G}$: Let A be a subgroup of $\text{Orth}(\mathbf{G})$. If

$$\text{Fix}(\boldsymbol{\gamma}_1) \cup \text{Fix}(\boldsymbol{\gamma}_2) \subseteq A \ \text{ for linearly independent } \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \mathcal{T}^*, \tag{19.7}$$

[4]This projection can also be written using evaluation mappings: $\mathcal{G} = \text{ev}_{1>}(\mathcal{C})$. See (04.9) of [FDS].

then $A = \mathrm{Orth}(\mathbf{G})$.

Assume that every relaxed state is thermally axisymmetric. Since every relaxed state is thermally axisymmetric, $\hat{\mathbf{C}}$ is surjective (by (16.15)), and for every condition $\mathbf{C}$ there is a relaxed state λ such that $\hat{\mathbf{C}}(\lambda) = \mathbf{C}$ we have

$$\mathrm{Orth}(\mathbf{G}) \cap \mathrm{Fix}(\boldsymbol{\gamma}) \subseteq \mathfrak{G} \quad \text{for all } \ (\mathbf{G}, \theta, \boldsymbol{\gamma}) \in \mathcal{C}. \tag{19.8}$$

Let $\mathbf{G} \in \mathcal{G}$ be given. Since $\mathcal{C}$ is open there are $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \mathcal{T}^*$, with $\boldsymbol{\gamma}_1$ not in $\mathrm{Lsp}\{\boldsymbol{\gamma}_2\}$, and $\theta_1, \theta_2 \in \mathbb{P}^\times$ such that $(\mathbf{G}, \theta_1, \boldsymbol{\gamma}_1), (\mathbf{G}, \theta_2, \boldsymbol{\gamma}_2) \in \mathcal{C}$. Using this fact with the fact stated in the previous paragraph, (19.8) implies that

$$\mathrm{Orth}(\mathbf{G}) \subseteq \mathfrak{G}.$$

Since $\mathcal{G}$ is open there is a $\mathbf{G}' \in \mathcal{G}$ not in $\mathrm{Lsp}\{\mathbf{G}\}$. The same reasoning as above leads us to the conclusion that

$$\mathrm{Orth}(\mathbf{G}') \subseteq \mathfrak{G}.$$

By the maximality of the orthogonal group in the unimodular group, see [MOU], it follows that $\mathrm{Unim}(\mathcal{T}) \subseteq \mathfrak{G}$. ▮

Proposition 19.3 *Let $\sigma \in \Sigma$ be given. Then*

$$\mathfrak{G}_\sigma \subseteq \mathfrak{G}_{\hat{\lambda}(\sigma)}. \tag{19.9}$$

Namely, relaxation can only enlarge the symmetry group of a state.

Proof: Put $\mathbf{C} := \hat{\mathbf{C}}(\sigma) = \hat{\mathbf{C}}(\hat{\lambda}(\sigma))$ and let $\mathbf{A} \in \mathfrak{G}_\sigma$ be given. By (S1) we have $\mathbf{A}_{\mathcal{C}}(\mathbf{C}) = \mathbf{C}$ and hence, for all $t \in \mathbb{P}$, $\mathbf{A}_{\mathcal{C}} \circ \mathbf{C}_{(t)} = \mathbf{C}_{(t)}$. Thus (S3), with $\mathbf{P} := \mathbf{C}_{(t)}$, implies

$$\iota_{\mathbf{A}}(\hat{e}(\sigma, \mathbf{C}_{(t)})) = \hat{e}(\sigma, \mathbf{C}_{(t)}). \tag{19.10}$$

Taking the limit as $t \to \infty$ and using (19.1) we obtain $\iota_{\mathbf{A}}(\hat{\lambda}(\sigma)) = \hat{\lambda}(\sigma)$ and hence $\mathbf{A} \in \mathfrak{G}_{\hat{\lambda}(\sigma)}$. Since $\mathbf{A} \in \mathfrak{G}_\sigma$ was arbitrary, the result holds. ▮

The following is an immediate result of the above proposition and the definition of a thermally axisymmetric element.

Corollary 19.4 *A thermomechanical element is thermally axisymmetric if and only if it has a thermally axisymmetric relaxed state.*

For all $\sigma \in \Sigma$ put

$$\Sigma_\sigma := \mathrm{Clo}\{\hat{e}(\sigma, \mathbf{P}) \mid \mathbf{P} \in \Pi_{\hat{\mathbf{C}}(\sigma)}\}. \tag{19.11}$$

Definition 19.5 A state $\tau \in \Sigma$ is said to be **accessible** from $\sigma \in \Sigma$ if $\tau \in \Sigma_\sigma$ and is said to be **strictly accessible** from $\sigma \in \Sigma$ if there is a condition process $\mathbf{P} \in \Pi_{\hat{\mathbf{C}}(\sigma)}$ such that $\tau = \hat{e}(\sigma, \mathbf{P})$.

Proposition 19.6 *Let $\sigma \in \Sigma$ be given. If $\tau \in \Sigma_\sigma$, then $\Sigma_\tau \subseteq \Sigma_\sigma$.*

Proof: Let $\tau \in \Sigma_\sigma$ be given. Also, let $\mu \in \Sigma_\tau$ and a neighborhood $\mathcal{U}_\mu$ of μ be given such that $\mathcal{U}_\mu \subseteq \Sigma_{\hat{\mathbf{C}}(\mu)}$. By the definition of Σ_τ there is a condition process $\mathbf{P}_1 \in \Pi_{\hat{\mathbf{C}}(\tau)}$ such that $\hat{e}(\tau, \mathbf{P}_1) \in \mathcal{U}_\mu$. By (16.15) we have $\mathbf{P}_1^f = \hat{\mathbf{C}}(\mu)$. By the continuity of $\hat{e}(\cdot, \mathbf{P}_1) : \Sigma_{\hat{\mathbf{C}}(\tau)} \longrightarrow \Sigma_{\hat{\mathbf{C}}(\mu)}$ there is a neighborhood $\mathcal{U}_\tau$ of τ in $\Sigma_{\hat{\mathbf{C}}(\tau)}$ such that $\hat{e}(\cdot, \mathbf{P}_1)_>(\mathcal{U}_\tau) \subseteq \mathcal{U}_\mu$. The fact that $\tau \in \Sigma_\sigma$ guarantees that there is a condition process $\mathbf{P}_2 \in \Pi_{\hat{\mathbf{C}}(\sigma)}$ such that $\hat{e}(\sigma, \mathbf{P}_2) \in \mathcal{U}_\tau$. Hence we have $\hat{e}(\hat{e}(\sigma, \mathbf{P}_2), \mathbf{P}_1) = \hat{e}(\sigma, \mathbf{P}_2 * \mathbf{P}_1) \in \mathcal{U}_\mu$. Since the neighborhood $\mathcal{U}_\mu$ of μ in $\Sigma_{\hat{\mathbf{C}}(\mu)}$ was arbitrary, $\mu \in \Sigma_\sigma$. Since $\mu \in \Sigma_\tau$ was arbitrary, we have $\Sigma_\tau \subseteq \Sigma_\sigma$. ▮

The following result follows immediately from the previous proposition and (19.1).

Proposition 19.7 *For every $\sigma \in \Sigma$ and $\tau \in \Sigma_\sigma$ we have $\hat{\lambda}(\tau) \in \Sigma_\sigma$.*

This last axiom limits the size of the state space.

Axiom 6 *There is at least one $\lambda_o \in \Sigma_{\text{rel}}$ such that $\Sigma_{\lambda_o} = \Sigma$.*

If the septuple $(\mathcal{T}, \mathcal{C}, \Pi, \Sigma, \hat{\mathbf{C}}, \hat{\mathbf{R}}, \hat{e})$ satisfies Axioms 1–5 but not Axiom 6, one can use the following procedure to define thermomechanical element structures on $\mathcal{T}$: Select $\lambda_o \in \Sigma_{\text{rel}}$ arbitrarily and put $\Sigma' := \Sigma_{\lambda_o}$. Put $\hat{\mathbf{C}}' := \hat{\mathbf{C}}|_{\Sigma'}$, $\hat{\mathbf{R}}' := \hat{\mathbf{R}}|_{\Sigma'}$ and $\hat{e}' := \hat{e}|_{(\Sigma' \times \Pi)_{\text{fit}}}$. It is not hard to verify that the septuple $(\mathcal{T}, \mathcal{C}, \Pi, \Sigma', \hat{\mathbf{C}}', \hat{\mathbf{R}}', \hat{e}')$ is a thermomechanical element and it satisfies Axiom 6. This new element depends on the initial choice of λ_o and so the original element may give rise to many different thermomechanical element structures on $\mathcal{T}$.

Remark 19.8 Axiom 6 states that every state is accessible from a given relaxed state. There is a good reason for not requiring the stronger condition that every state be strictly accessible from a given relaxed state. To obtain a viscometric flow for a semi-fluid starting from rest (a relaxed state) it may be necessary to wait an infinite amount of time. Thus the state of the material corresponding to being in a viscometric flow is accessible from the rest state but need not be strictly accessible. ▮

20 Semi-elastic Elements

The following definition is useful.

Definition 20.1 *We say a thermomechanical element is* **semi-elastic** *if the mapping $\hat{\mathbf{C}}|_{\Sigma_{\text{rel}}} : \Sigma_{\text{rel}} \longrightarrow \mathcal{C}$ is invertible. We say an element is* **elastic** *if it is semi-elastic and every state is relaxed.*

It follows from (19.1) and (16.15) that the mapping $\hat{\mathbf{C}}|_{\Sigma_{\text{rel}}}$ is always surjective. Thus a thermomechanical element is semi-elastic if and only if $\hat{\mathbf{C}}|_{\Sigma_{\text{rel}}}$ is injective.

Remark 20.2 Noll proved in [NTSM], see Theorem 16.1, that, assuming minor technical assumptions, the states for a semi-elastic element can be identified with histories and thus are simple materials in the old sense of [OTSM]. ∎

For the rest of this section we assume that a semi-elastic thermomechanical element is given.

Proposition 20.3 *If λ is a relaxed state, then $\mathfrak{G}_\lambda = \mathfrak{G} \cap \mathrm{Orth}(\hat{\mathbf{G}}(\lambda)) \cap \mathrm{Fix}(\hat{\boldsymbol{\gamma}}(\lambda))$.*

Proof: Let $\lambda \in \Sigma_{\mathrm{rel}}$ and $\mathbf{A} \in \mathfrak{G} \cap \mathrm{Orth}(\hat{\mathbf{G}}(\lambda)) \cap \mathrm{Fix}(\hat{\boldsymbol{\gamma}}(\lambda))$ be given. Using (S1) we have

$$\begin{aligned}\hat{\mathbf{C}}(\lambda) &= (\hat{\mathbf{G}}(\lambda), \hat{\theta}(\lambda), \hat{\boldsymbol{\gamma}}(\lambda)) \\ &= (\mathbf{A}^{-\top}\hat{\mathbf{G}}(\lambda)\mathbf{A}^{-1}, \hat{\theta}(\lambda), \mathbf{A}^{-\top}\hat{\boldsymbol{\gamma}}(\lambda)) \\ &= \mathbf{A}_{\mathcal{C}}(\hat{\mathbf{C}}(\lambda)) \\ &= \hat{\mathbf{C}}(\iota_{\mathbf{A}}(\lambda)).\end{aligned}$$

Since the element is semi-elastic, $\hat{\mathbf{C}}|_{\Sigma_{\mathrm{rel}}}$ is injective and so it follows that $\lambda = \iota_{\mathbf{A}}(\lambda)$ and hence $\mathbf{A} \in \mathfrak{G}_\lambda$. Since $\mathbf{A} \in \mathfrak{G} \cap \mathrm{Orth}(\hat{\mathbf{G}}(\lambda)) \cap \mathrm{Fix}(\hat{\boldsymbol{\gamma}}(\lambda))$ was arbitrary we have $\mathfrak{G} \cap \mathrm{Orth}(\hat{\mathbf{G}}(\lambda)) \cap \mathrm{Fix}(\hat{\boldsymbol{\gamma}}(\lambda)) \subseteq \mathfrak{G}_\lambda$.

It follows from Proposition 17.8 that $\mathfrak{G}_\lambda \subseteq \mathfrak{G} \cap \mathrm{Orth}(\hat{\mathbf{G}}(\lambda)) \cap \mathrm{Fix}(\hat{\boldsymbol{\gamma}}(\lambda))$ and so the result holds. ∎

Definition 20.4 *An element is said to be* **fluid** *if it is both semi-elastic and a semi-fluid.*

Corollary 20.5 *If the element is fluid then every relaxed state is thermally axisymmetric.*

Proposition 20.6 *In a semi-elastic element every state is accessible from every other state, i.e., $\Sigma_\sigma = \Sigma$ for all $\sigma \in \Sigma$.*

Proof: Let λ_o denote the state guaranteed by Axiom 6 and let $\sigma \in \Sigma$ be given. It suffices to show that $\Sigma_{\lambda_o} \subseteq \Sigma_\sigma$. It follows from Proposition 19.6 that this will follow if it is shown that $\lambda_o \in \Sigma_\sigma$. Find a condition process $\mathbf{P}$ such that $\mathbf{P}^i = \hat{\mathbf{C}}(\sigma)$ and $\mathbf{P}^f = \hat{\mathbf{C}}(\lambda_o)$. Such a processes exists by (P4). Then $\hat{e}(\sigma, \mathbf{P}) \in \Sigma_\sigma$ and hence by Proposition 19.7 $\hat{\lambda}(\hat{e}(\sigma, \mathbf{P})) \in \Sigma_\sigma$. Since $\hat{\mathbf{C}}(\hat{\lambda}(\hat{e}(\sigma, \mathbf{P}))) = \hat{\mathbf{C}}(\lambda_o)$ and for semi-elastic elements $\hat{\mathbf{C}}|_{\Sigma_{\mathrm{rel}}}$ is injective it follows that $\lambda_o = \hat{\lambda}(\hat{e}(\sigma, \mathbf{P})) \in \Sigma_\sigma$. Since $\sigma \in \Sigma$ was arbitrary, this proves the claim. ∎

Of course, since the above results hold for semi-elastic elements they also hold for elastic elements. For elastic elements the mapping $\hat{\mathbf{C}} : \Sigma \longrightarrow \mathcal{C}$ is a bijection and can be used to identify the state space with $\mathcal{C}$. However, $\mathcal{C}$, when viewed as the state space with the basic topology, has the discrete topology. This topology is too weak to capture the smoothness assumptions that are usually made on constitutive laws in elasticity. It will be shown in Chapter III that by introducing *memory* into the thermomechanical element a more suitable topology on the state space can be generated.

21 The Second Law of Thermodynamics

In this section we lay the foundations for using the second law of thermodynamics, as discussed in Section 13, for simple materials.

Let $(\sigma, \mathbf{P}) \in (\Sigma \times \Pi^{\times})_{\text{fit}}$ be given. One can define the **response process** $\mathbf{P}_{\text{r}} : [0, d_{\mathbf{P}}] \longrightarrow \mathcal{R}$ induced by $\mathbf{P}$ and σ by

$$\mathbf{P}_{\text{r}}(t) := \tilde{\mathbf{R}}(\sigma, \mathbf{P}_{[0,t]}) \quad \text{for all} \quad t \in [0, d_{\mathbf{P}}].$$

We denote the components of $\mathbf{P}_{\text{r}}$ by

$$\mathbf{P}_{\text{r}} = (\bar{\mathbf{S}}, \bar{\eta}, \bar{\mathbf{h}}, \bar{\psi}). \tag{21.1}$$

Now define the **admissible process** induced by $\mathbf{P}$ and σ by

$$\mathbf{P}_{\text{a}} : [0, d_{\mathbf{P}}] \longrightarrow \mathcal{C} \times \mathcal{R} \quad \text{by} \quad \mathbf{P}_{\text{a}} := (\mathbf{P}, \mathbf{P}_r). \tag{21.2}$$

Using the notation $\bar{\rho} := \hat{\rho} \circ \bar{\mathbf{G}}$ (see (8.15)), we say that the admissible process satisfies the **reduced dissipation inequality** (see (13.6)) if

$$-\bar{\rho}(\bar{\theta}^{\bullet}\bar{\eta} + \bar{\psi}^{\bullet}) + \frac{1}{2}\text{tr}(\bar{\mathbf{S}}\bar{\mathbf{G}}^{\bullet}) - \bar{\Upsilon} \geq 0, \quad \text{where } \bar{\Upsilon} := \frac{1}{\bar{\theta}}\bar{\boldsymbol{\gamma}}\bar{\mathbf{h}}. \tag{21.3}$$

We now assume the following holds:

Second Law of Thermodynamics: *For every $(\sigma, \mathbf{P}) \in (\Sigma \times \Pi^{\times})_{\text{fit}}$ the admissible process $\mathbf{P}_{\text{a}}$, as defined in (21.2), satisfies (21.3).*

It is useful to introduce the function $\tilde{\Upsilon} : (\Sigma \times \Pi)_{\text{fit}} \longrightarrow \mathbb{R}$ defined by

$$\tilde{\Upsilon}(\sigma, \mathbf{P}) := \frac{1}{\bar{\theta}^f}\tilde{\boldsymbol{\gamma}}(\sigma, \mathbf{P})\tilde{\mathbf{h}}(\sigma, \mathbf{P}) \quad \text{for all} \quad (\Sigma \times \Pi)_{\text{fit}}. \tag{21.4}$$

Let $\bar{\psi}_{(\sigma,\mathbf{P})} : [0, d_{\mathbf{P}}] \longrightarrow \mathbb{R}$ denote the specific free energy component of the response process generated by $(\sigma, \mathbf{P})$, i.e.,

$$\bar{\psi}_{(\sigma,\mathbf{P})}(t) := \tilde{\psi}(\sigma, \mathbf{P}_{[0,t]}) \quad \text{for all} \quad t \in [0, d_{\mathbf{P}}]. \tag{21.5}$$

Using this notation, one can see that the Second Law of Thermodynamics is equivalent to

$$-\hat{\rho}(\bar{\mathbf{G}}^f)(\bar{\theta}^{\bullet f}\tilde{\eta}(\sigma, \mathbf{P}) + \bar{\psi}^{\bullet f}_{(\sigma,\mathbf{P})}) + \frac{1}{2}\text{tr}(\tilde{\mathbf{S}}(\sigma, \mathbf{P})\bar{\mathbf{G}}^{\bullet f}) - \tilde{\Upsilon}(\sigma, \mathbf{P}) \geq 0 \quad \text{for all} \quad (\sigma, \mathbf{P}) \in (\Sigma \times \Pi^{\times})_{\text{fit}}. \tag{21.6}$$

Chapter III

Memory

22 Introduction

In 1963 Coleman and Noll first implemented, in [CNP], the interpretation of the second law of thermodynamics as a restriction on the form of constitutive laws[1]. The process of finding the restrictions placed on constitutive laws by the second law became so standard it was dubbed the "Coleman–Noll procedure" and has been applied to many classes of constitutive relations. This procedure was carried out in the context of the old theory of simple materials, as defined in [OTSM], by Coleman in the fundamental paper [TMM]. This procedure was also carried out for materials of grade two by Gurtin in [TMM3] in a paper which, in Gurtin's own words, "lays bare the quite simple concepts which are the essence of the subject."

In the framework presented here the second law of thermodynamics restricts the form of the result mapping $\tilde{\mathbf{R}}$. In all of the past literature the Coleman–Noll procedure requires various smoothness assumptions on the result mapping. For these assumptions to be made, a topology on the set of condition processes is necessary[2]. In this part it is discussed how such a topology can be introduced and how imposing this topology on the set of condition processes induces a topology on the state space that is different from the basic topology introduced in Section 18.

For some materials the response $\tilde{\mathbf{R}}(\sigma, \mathbf{P})$ of a body element in the state σ subjected to the condition process $\mathbf{P}$ may depend in a very complicated way on all of the values of $\mathbf{P}$ while the response for other materials may only depend on the final value of $\mathbf{P}$. In the latter case one can think of the material forgetting what has happened to it in the past. Thus, it makes sense to describe this kind of dependence by using the word *memory*. Within the state space framework memory can be modeled by making various smoothness assumptions on the result mapping. By assuming that for each state σ the mapping $\tilde{\mathbf{R}}(\sigma, \cdot)$ is uniformly continuous with respect to different uniformities on the set of condition processes one is able to model different types of memory. Once a uniformity on

[1]For the history on this interpretation see [TOT].

[2]In the old theory of simple materials instead of working with states and condition processes, only the concept of a *history* was used. Thus, in the older literature various topologies on the set of histories were proposed.

the set of condition processes is specified one can generate, in a natural way, a uniformity on the state space. This is explained in Section 23.

An interesting special type of memory is that of fading memory[3] in which, intuitively, the material element remembers what has happened to it in the recent past more then what happened to it in the distant past. Materials with fading memory have been studied in great detail, see [TMM] and [TMFM] for example. It is shown in Section 24 that a thermomechanical element with fading memory must be semi-elastic.

In Section 25 it is investigated what can be said about materials with fading memory if one assumes that the second law of thermodynamics holds. The Coleman–Noll procedure is carried out for materials with fading memory. To carry out this procedure certain differentiability type assumptions must be made on the result mapping (see Assumptions 1 and 2 in Section 25). By carrying out this procedure I was able to obtain results similar to those found in [TMM]. For example, the stress and entropy components of the result mapping are completely determined by the free energy component of the result mapping.

In 1960 Coleman and Noll proved, in [ATCM], a theorem within the framework of the old theory of simple materials that roughly says that[4] for materials with fading memory the response of a slowed down history can be approximated by a mapping that only depends on the final value of the history and by a flat (affine) mapping that only depends on the final value of the derivative of the history. This theorem, which will be referred to as the Retardation Theorem , explained why the constitutive law for linearly viscous fluids can be used to describe the behavior of many fluids in the limit of slow motions.

The Retardation Theorem had two defects, however. The first is that since histories were used, the concepts of state and process were confused. It makes sense to talk about slowing down a process but one cannot slow down a state. Ideally one would like to have a result that says that the response of a state being subjected to a process can be approximated, in a certain sense, when the process is slowed down. Assuming this first defect could be overcome, there is still the problem that given a state and a process one may need to slow down the process quite a bit before the response can be approximated. It would be more satisfying if given a state and a process one could immediately determine that, if the speed of the process were small enough, the desired approximation holds. Theorem 26.8 in Section 26 corrects these two defects. A version of the Retardation Theorem, Theorem 26.11, is given as an easy consequence to this theorem.

23 Simple Thermomechanical Elements with Memory

Let a thermomechanical element $(\mathcal{T}, \mathcal{C}, \Pi, \Sigma, \hat{\mathbf{C}}, \hat{\mathbf{R}}, \hat{e})$ and a norm ν on $\mathcal{T}$ be given. This norm induces the dual norm ν^* on $\mathcal{T}^*$ defined by

$$\nu^*(\boldsymbol{\gamma}) := \sup_{\mathbf{v}\in\mathcal{T}^\times} \frac{|\boldsymbol{\gamma}\mathbf{v}|}{\nu(\mathbf{v})} \quad \text{for all} \quad \boldsymbol{\gamma} \in \mathcal{T}^*. \tag{23.1}$$

[3]The theory of materials with fading memory was first formulated in [ATCM] in the framework presented in [OTSM].

[4]What is stated here is only a special case of their result.

This in turn can be used to define an operator norm $||\cdot||_{\nu^*,\nu}$ on $\mathrm{Lin}(\mathcal{T}^*,\mathcal{T})$ by

$$||\mathbf{L}||_{\nu^*,\nu} := \sup_{\boldsymbol{\lambda}\in\mathcal{T}^{*\times}} \frac{\nu(\mathbf{L}\boldsymbol{\lambda})}{\nu^*(\boldsymbol{\lambda})} \quad \text{for all} \;\; \mathbf{L}\in\mathrm{Lin}(\mathcal{T}^*,\mathcal{T}). \tag{23.2}$$

In a similar way one can define a norm $||\cdot||_{\nu,\nu^*}$ on $\mathrm{Lin}(\mathcal{T},\mathcal{T}^*)$. Define a norm $\nu_{\mathcal{C}'}$ on $\mathcal{C}'$, see (16.5), by

$$\nu_{\mathcal{C}'}((\mathbf{G}',\theta',\boldsymbol{\gamma}')) := ||\mathbf{G}'||_{\nu,\nu^*} + |\theta'| + \nu^*(\boldsymbol{\gamma}') \quad \text{for all} \;\; (\mathbf{G}',\theta',\boldsymbol{\gamma}')\in\mathcal{C}' \tag{23.3}$$

and a norm $\nu_{\mathcal{R}}$ on $\mathcal{R}$ by

$$\nu_{\mathcal{R}}((\mathbf{S},\eta,\mathbf{h},\psi)) := ||\mathbf{S}||_{\nu^*,\nu} + |\eta| + \nu(\mathbf{h}) + |\psi| \quad \text{for all} \;\; (\mathbf{S},\eta,\mathbf{h},\psi)\in\mathcal{R}. \tag{23.4}$$

Let $d\in\mathbb{P}$ and a semi-norm $\nu_d : \mathrm{pwC}^1([0,d],\mathcal{C}') \longrightarrow \mathbb{P}$ be given. This semi-norm can be used to generate a uniformity on $\mathrm{pwC}^1([0,d],\mathcal{C}')$ in a natural way, see [GT]. Since $\mathcal{C}\subseteq\mathcal{C}'$ we may view $\mathrm{pwC}^1([0,d],\mathcal{C})$, and hence Π^d (see (16.4)), as a subset of $\mathrm{pwC}^1([0,d],\mathcal{C}')$. Thus, Π^d can be equipped with the relative uniformity coming from $\mathrm{pwC}^1([0,d],\mathcal{C}')$ induced by ν_d.

Let $(\nu_d : \mathrm{pwC}^1([0,d],\mathcal{C}') \longrightarrow \mathbb{P} \mid d\in\mathbb{P})$ be a family of semi-norms. As described in the previous paragraph, for all $d\in\mathbb{P}$, ν_d can be used to generate a uniformity on Π^d. Since Π is the disjoint union of Π^d, $d\in\mathbb{P}$, the uniformities on Π^d can generate a uniformity on Π. Namely, we consider the finest uniformity on Π such that all the mappings in the family of canonical injections $(\mathrm{inj}_d : \Pi^d \longrightarrow \Pi \mid d\in\mathbb{P})$ are uniformly equicontinuous. It follows that the collection of sets of the form

$$\mathcal{O}_\epsilon := \{(\mathbf{P}_1,\mathbf{P}_2)\in\Pi\times\Pi \mid d_{\mathbf{P}_1} = d_{\mathbf{P}_2} =: d \text{ and } \nu_d(\mathbf{P}_1-\mathbf{P}_2)<\epsilon\} \quad \text{for all} \;\; \epsilon\in\mathbb{P} \tag{23.5}$$

form a basis for this uniformity on Π.

Definition 23.1 *A* **thermomechanical element with memory** *is a thermomechanical element* $(\mathcal{T},\mathcal{C},\Pi,\Sigma,\hat{\mathbf{C}},\hat{\mathbf{R}},\hat{e})$ *together with a family of semi-norms* $(\nu_d : \mathrm{pwC}^1([0,d],\mathcal{C}') \longrightarrow \mathbb{P} \mid d\in\mathbb{P})$ *such that for all* $\sigma\in\Sigma$ *the mapping*

$$\tilde{\mathbf{R}}(\sigma,\cdot) : \Pi_{\hat{\mathbf{C}}(\sigma)} \longrightarrow \mathcal{R} \tag{23.6}$$

is uniformly continuous with respect to the uniformity on Π *induced by the family of semi-norms.*

The set of condition processes of a thermomechanical element with memory will always be considered to be endowed with the uniformity whose base is described in (23.5). This uniformity, together with the result mapping $\tilde{\mathbf{R}}$, can be used to generate a topology on the state space that is different from the basic topology.

To simplify the notation, define the mapping $\nu_\Pi : \Pi \longrightarrow \mathbb{P}$ by

$$\nu_\Pi(\mathbf{P}) := \nu_{d_\mathbf{P}}(\mathbf{P}) \quad \text{for all} \;\; \mathbf{P}\in\Pi. \tag{23.7}$$

Definition 23.2 *The* **memory uniformity** *on* Σ *is the coarsest uniformity on* Σ *that makes the result mapping* $\tilde{\mathbf{R}} : (\Sigma \times \Pi)_{\rm fit} \longrightarrow \mathcal{R}$ *and the condition mapping* $\hat{\mathbf{C}} : \Sigma \longrightarrow \mathcal{C}$ *uniformly continuous. The* **memory topology** *on* Σ *is the topology induced by the memory uniformity.*

Let $\mathrm{Unif}(\Sigma)$, which is a collection of subsets of $\Sigma \times \Sigma$, denote the memory uniformity. It follows from (23.5) that this uniformity is characterized by the following property: for all $\epsilon \in \mathbb{P}^\times$ there is a $\mathcal{U} \in \mathrm{Unif}(\Sigma)$ and a $\delta \in \mathbb{P}^\times$ such that

$$\begin{gathered} \text{for all } (\sigma_1, \mathbf{P}_1), (\sigma_2, \mathbf{P}_2) \in (\Sigma \times \Pi)_{\rm fit} \\ \text{with } (\sigma_1, \sigma_2) \in \mathcal{U},\ d_{\mathbf{P}_1} = d_{\mathbf{P}_2} \text{ and } \nu_\Pi(\mathbf{P}_1 - \mathbf{P}_2) < \delta \\ \text{we have } \nu_\mathcal{R}(\tilde{\mathbf{R}}(\sigma_1, \mathbf{P}_1) - \tilde{\mathbf{R}}(\sigma_2, \mathbf{P}_2)) < \epsilon \text{ and } \nu_{\mathcal{C}'}(\hat{\mathbf{C}}(\sigma_1) - \hat{\mathbf{C}}(\sigma_2)) < \epsilon. \end{gathered} \tag{23.8}$$

Let $\mathcal{U} \in \mathrm{Unif}(\Sigma)$ and $\sigma \in \Sigma$ be given. We will use the notation

$$\mathcal{U}[\sigma] := \{\sigma' \in \Sigma \mid (\sigma, \sigma') \in \mathcal{U}\}. \tag{23.9}$$

The sets of the form $\mathcal{U}[\sigma]$ for all $\mathcal{U} \in \mathrm{Unif}(\Sigma)$ form a neighborhood base for σ with respect to the memory topology.

Remark 23.3 Physically, this uniformity can be interpreted in the following way. Think of the elements of Σ as representing various "states" that a system can be in and think of the elements of Π as possible "thought experiments" that one can perform on the system. Consider two different but "similar" states of the system, say σ_1 and σ_2, and two different but "similar" thought experiments, say $\mathbf{P}_1$ and $\mathbf{P}_2$, that are compatible with the given states of the system, meaning $\hat{\mathbf{C}}(\sigma_1) = \mathbf{P}_1^f$ and $\hat{\mathbf{C}}(\sigma_2) = \mathbf{P}_2^f$. Then the result $\tilde{\mathbf{R}}(\sigma_1, \mathbf{P}_1)$ of $\mathbf{P}_1$ on σ_1 and the result $\tilde{\mathbf{R}}(\sigma_2, \mathbf{P}_2)$ of $\mathbf{P}_2$ on σ_2 are "similar". Also, the conditions of σ_1 and σ_2 are similar. ▮

Just as in the case of the basic topology, in the memory topology limits are unique.

Proposition 23.4 *The memory topology makes* Σ *a Hausdorff space.*

Proof: Let $\sigma_o, \sigma \in \Sigma$ be given and assume that σ is in every neighborhood of σ_o. It suffices to show that $\sigma = \sigma_o$.

Let $\epsilon \in \mathbb{P}^\times$ be given. Since $\hat{\mathbf{C}}$ is continuous at σ_o and σ is in every neighborhood of σ_o we have $\nu_{\mathcal{C}'}(\hat{\mathbf{C}}(\sigma) - \hat{\mathbf{C}}(\sigma_o)) < \epsilon$. Since $\epsilon \in \mathbb{P}^\times$ was arbitrary we conclude that $\hat{\mathbf{C}}(\sigma) = \hat{\mathbf{C}}(\sigma_o) =: \mathbf{C}$.

Let $\epsilon \in \mathbb{P}^\times$ and $\mathbf{P} \in \Pi_\mathbf{C}$ be given. By the continuity of $\tilde{\mathbf{R}}(\cdot, \mathbf{P}) : \Sigma_\mathbf{C} \longrightarrow \mathcal{R}$ at σ_o and the fact that σ is in every neighborhood of σ_o we know $\nu_\mathcal{R}(\tilde{\mathbf{R}}(\sigma, \mathbf{P}) - \tilde{\mathbf{R}}(\sigma_o, \mathbf{P})) < \epsilon$. Since $\epsilon \in \mathbb{P}^\times$ was arbitrary we conclude that $\tilde{\mathbf{R}}(\sigma, \mathbf{P}) = \tilde{\mathbf{R}}(\sigma_o, \mathbf{P})$. Since $\mathbf{P} \in \Pi$ was arbitrary it follows from Axiom 3 that $\sigma = \sigma_o$. ▮

The following result describes the relationship between the basic topology discussed in Section 18 and the memory topology.

Proposition 23.5 *The relativization of the memory uniformity to a* $\mathbf{C}$*-section is finer than the basic uniformity on that* $\mathbf{C}$*-section. More specifically, if* $\mathcal{A}$ *is an entourage from the basic uniformity on a* $\mathbf{C}$*-section* $\Sigma_{\mathbf{C}}$ *then there is a* $\mathcal{U} \in \mathrm{Unif}(\Sigma)$ *such that* $(\Sigma_{\mathbf{C}} \times \Sigma_{\mathbf{C}}) \cap \mathcal{U} \subseteq \mathcal{A}$.

Proof: Let $\mathbf{C} \in \mathcal{C}$ be given. Note that a basis for the basic uniformity on $\Sigma_{\mathbf{C}}$ is the collection of all sets of the form

$$\{(\sigma_1, \sigma_2) \in \Sigma_{\mathbf{C}} \times \Sigma_{\mathbf{C}} \mid \nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\sigma_1, \mathbf{P}_i) - \tilde{\mathbf{R}}(\sigma_2, \mathbf{P}_i)) < \epsilon_1 \text{ for all } i \in I\}, \tag{23.10}$$

where $\mathbf{P}_i \in \Pi_{\mathbf{C}}$ and $\epsilon_i \in \mathbb{P}^{\times}$ for all $i \in I$ where I is a finite index set.

Let an entourage $\mathcal{A}$ from the basic uniformity on $\Sigma_{\mathbf{C}}$ be given. We may assume that $\mathcal{A}$ is of the form given in (23.10). Put $\epsilon := \min_{i \in I} \epsilon_i$. Find $\mathcal{U} \in \mathrm{Unif}(\Sigma)$ such that (23.8) holds with this ϵ. By the choice of $\mathcal{U}$ one can show that $(\Sigma_{\mathbf{C}} \times \Sigma_{\mathbf{C}}) \cap \mathcal{U} \subseteq \mathcal{A}$. ∎

Motivated by this last result we modify Axiom 6 so that it is in terms of the memory topology.

Axiom 6'. *There is at least one* $\lambda_o \in \Sigma_{\mathrm{rel}}$ *such that for all* $\sigma \in \Sigma$ *and* $\mathcal{U} \in \mathrm{Unif}(\Sigma)$ *there is a condition process* $\mathbf{P} \in \Sigma_{\hat{\mathbf{C}}(\lambda_o)}$ *such that*

$$\hat{e}(\lambda_o, \mathbf{P}) \in \mathcal{U}[\sigma]. \tag{23.11}$$

When one considers an elastic element the usefulness of the memory topology over the basic topology becomes clear. Suppose we have an elastic element $\mathcal{T}$. As was mentioned earlier, when $\mathcal{C}$ is equipped with the topology inherited from the basic topology on the state space it has the discrete topology. This topology is not practical. To see this suppose our element $\mathcal{T}$ is associated with a material point in a bar of steel. As one slowly bends the bar the condition (here, just the configuration) of the element will change (assuming it does not lie on the neutral plane). If the basic topology, rather then the memory topology, is used there is nothing preventing the stress of the element during this process to fluctuate radically. This kind of discontinuity in the stress should not be allowed. To avoid this problem we can use an appropriate memory topology on the state space. The response mapping is continuous with respect to the memory topology and thus by smoothly changing the condition the stress in the element will change smoothly as well. The appropriate memory to model elastic elements is given by the following result:

Theorem 23.6 *A thermomechanical element* $\mathcal{T}$ *with memory generated by the family of semi-norms*

$$\nu_d(\mathbf{Y}) := \nu_{\mathcal{C}'}(\mathbf{Y}^f) \quad \text{for all } \ \mathbf{Y} \in \mathrm{pwC}^1([0, d], \mathcal{C}') \text{ and } d \in \mathbb{P} \tag{23.12}$$

is elastic.

Proof: To see this let $\mathbf{C} \in \mathcal{C}$, $\sigma \in \Sigma_{\mathbf{C}}$ and $\lambda \in \Sigma_{\mathbf{C}} \cap \Sigma_{\mathrm{rel}}$ be given. It suffices to show that $\sigma = \lambda$. Let $\epsilon \in \mathbb{P}^{\times}$ be given. Find $\mathcal{U} \in \mathrm{Unif}(\Sigma)$ and $\delta \in \mathbb{P}^{\times}$ such that (23.8) holds. By Axiom 6' we can find $\mathbf{P}_\sigma, \mathbf{P}_\lambda \in \Sigma_{\hat{\mathbf{C}}(\lambda_o)}$ such that

$$\hat{e}(\lambda_o, \mathbf{P}_\sigma) \in \mathcal{U}[\sigma] \cap \Sigma_{\mathbf{C}} \quad \text{and} \quad \hat{e}(\lambda_o, \mathbf{P}_\lambda) \in \mathcal{U}[\lambda] \cap \Sigma_{\mathbf{C}}. \tag{23.13}$$

Since λ_o is relaxed we may assume that $d_{\mathbf{P}_\sigma} = d_{\mathbf{P}_\lambda} =: d$. To see this assume that $d_{\mathbf{P}_\sigma} > d_{\mathbf{P}_\lambda}$. If this were the case we could replace $\mathbf{P}_\lambda$ with $\hat{\mathbf{C}}(\lambda_o)_{(d_{\mathbf{P}_\sigma} - d_{\mathbf{P}_\lambda})} * \mathbf{P}_\lambda$. This process has the same duration as $\mathbf{P}_\sigma$. We assume that the processes were initially chosen so that they have the same duration.

Let $\mathbf{P} \in \Sigma_{\mathbf{C}}$ be given. Since $\nu_\Pi(\mathbf{P} - \mathbf{P}) = 0 \leq \delta$, by (23.13) and the choice of $\mathcal{U}$ we have

$$\begin{aligned} \nu_{\mathcal{R}}\left(\tilde{\mathbf{R}}(\sigma, \mathbf{P}) - \tilde{\mathbf{R}}(\lambda_o, \mathbf{P}_\sigma * \mathbf{P})\right) &= \nu_{\mathcal{R}}\left(\tilde{\mathbf{R}}(\sigma, \mathbf{P}) - \tilde{\mathbf{R}}(\hat{e}(\lambda_o, \mathbf{P}_\sigma), \mathbf{P})\right) < \epsilon, \\ \nu_{\mathcal{R}}\left(\tilde{\mathbf{R}}(\lambda_o, \mathbf{P}_\lambda * \mathbf{P}) - \tilde{\mathbf{R}}(\lambda, \mathbf{P})\right) &= \nu_{\mathcal{R}}\left(\tilde{\mathbf{R}}(\hat{e}(\lambda_o, \mathbf{P}_\lambda), \mathbf{P}) - \tilde{\mathbf{R}}(\lambda, \mathbf{P})\right) < \epsilon. \end{aligned} \tag{23.14}$$

From (23.12) we know $\nu_\Pi(\mathbf{P}_\lambda * \mathbf{C}_{(s)} * \mathbf{P} - \mathbf{P}_\sigma * \mathbf{C}_{(s)} * \mathbf{P}) = 0 < \delta$ and so by (23.8) we obtain

$$\nu_{\mathcal{R}}\left(\tilde{\mathbf{R}}(\lambda_o, \mathbf{P}_\lambda * \mathbf{C}_{(s)} * \mathbf{P}) - \tilde{\mathbf{R}}(\lambda_o, \mathbf{P}_\sigma * \mathbf{C}_{(s)} * \mathbf{P})\right) < \epsilon. \tag{23.15}$$

Thus we have

$$\begin{aligned} \nu_{\mathcal{R}}\left(\tilde{\mathbf{R}}(\sigma, \mathbf{P}) - \tilde{\mathbf{R}}(\lambda, \mathbf{P})\right) &\leq \nu_{\mathcal{R}}\left(\tilde{\mathbf{R}}(\sigma, \mathbf{P}) - \tilde{\mathbf{R}}(\lambda_o, \mathbf{P}_\sigma * \mathbf{P})\right) \\ &\quad + \nu_{\mathcal{R}}\left(\tilde{\mathbf{R}}(\lambda_o, \mathbf{P}_\sigma * \mathbf{P}) - \tilde{\mathbf{R}}(\lambda_o, \mathbf{P}_\lambda * \mathbf{P})\right) \\ &\quad + \nu_{\mathcal{R}}\left(\tilde{\mathbf{R}}(\lambda_o, \mathbf{P}_\lambda * \mathbf{P}) - \tilde{\mathbf{R}}(\lambda, \mathbf{P})\right) \\ &< \epsilon + \epsilon + \epsilon = 3\epsilon. \end{aligned}$$

Since $\epsilon \in \mathbb{P}^\times$ was arbitrary $\tilde{\mathbf{R}}(\sigma, \mathbf{P}) = \tilde{\mathbf{R}}(\lambda, \mathbf{P})$. Since $\mathbf{P} \in \Pi_{\mathbf{C}}$ was arbitrary it follows from Axiom 3 that $\sigma = \lambda$. ▮

24 Fading Memory

Materials with fading memory have been studied in great detail in the literature; see, for example, [ATCM], [TMM] and [TMFM]. This section shows how one can describe materials with this kind of memory in the framework introduced in the previous section. An important ingredient in the theory of fading memory is that of an influence function.

Definition 24.1 *An* **influence function** *of order $r \in \mathbb{P}^\times$ is a function $h : \mathbb{P} \longrightarrow \mathbb{P}^\times$ with the following properties:*

(IF1) *h is continuous.*

(IF2) *$h(0) = 1$.*

(IF3) *h is antitone.*

(IF4) *There is an $\xi \in \mathbb{P}^\times$ such that*

$$\sup\left\{\frac{s^r h(s)}{t^r h(t)} \;\middle|\; s, t \in \mathbb{P}^\times \text{ with } \xi \leq t \leq s\right\} =: K < \infty.$$

Remark 24.2 The origins of using an influence function to model materials with fading memory goes back to [ATCM]. There Coleman and Noll assumed that the influence function satisfies condition (IF1) and that for all $\zeta \in \mathbb{P}^\times$, there is a $M_\zeta \in \mathbb{P}^\times$ such that

$$\sup_{s\in\mathbb{P}+\zeta} \frac{h(s/\alpha)}{\alpha^r h(s)} \le M_\zeta \quad \text{for all} \quad \alpha \in]0,1]. \tag{24.1}$$

One can show that this condition is equivalent to (IF4)[5]. In [TMM] Coleman added the additional assumption (IF3). I have added the assumption (IF2) to normalize the influence function and simplify some of the following calculations. (IF2) is not necessary to obtain the results. ∎

It follows from (IF4) that

$$\sup_{t\in\mathbb{P}} t^r h(t) < \infty. \tag{24.2}$$

Let an influence function h of order $r > 2$ be given. It follows from (24.2) that $\sup_{t\in\mathbb{P}} th(t)$ and $\sup_{t\in\mathbb{P}} t^2h(t)$ are both finite. Put

$$M := \max\{1, \sup_{t\in\mathbb{P}} th(t), \sup_{t\in\mathbb{P}} t^2h(t)\} < \infty. \tag{24.3}$$

Definition 24.3 *A* **thermomechanical element with fading memory** *is a thermomechanical element with memory whose family of semi-norms* $(\nu_d : \mathrm{pwC}^1([0,d],\mathcal{C}') \longrightarrow \mathbb{P} \mid d \in \mathbb{P})$ *is given by*

$$\nu_d(\mathbf{Y}) := \sup_{t\in[0,d]} h(t)\nu_{\mathcal{C}'}(\mathbf{Y}(d-t)) \quad \text{for all} \quad \mathbf{Y} \in \mathrm{pwC}^1([0,d],\mathcal{C}') \quad \text{and} \quad d \in \mathbb{P} \tag{24.4}$$

Furthermore, the element is unconstrained, in the sense that $\Pi^d = \mathrm{pwC}^1([0,d],\mathcal{C})$ *for all* $d \in \mathbb{P}$.

Note that the mappings ν_d given in (24.4) are norms.

Theorem 24.4 *A thermomechanical element with fading memory whose state space is equipped with the memory uniformity is semi-elastic.*

Proof: This proof uses the same idea as the proof of Theorem 23.6. Let $\mathbf{C} \in \mathcal{C}$, $\sigma \in \Sigma_\mathbf{C}$ and $\lambda \in \Sigma_\mathbf{C} \cap \Sigma_{\mathrm{rel}}$ be given. If suffices to show that

$$\lambda = \hat{\lambda}(\sigma) = \lim_{s\to\infty} \hat{e}(\sigma, \mathbf{C}_{(s)}). \tag{24.5}$$

Let $\epsilon \in \mathbb{P}^\times$ and $\mathbf{P} \in \Pi_\mathbf{C}$ be given. Find $\delta \in \mathbb{P}^\times$ and $\mathcal{U} \in \mathrm{Unif}\Sigma$ such that (23.8) holds. By Axiom 6' there are condition processes $\mathbf{P}_\sigma, \mathbf{P}_\lambda \in \Sigma_{\hat{\mathbf{C}}(\lambda_o)}$ such that

$$\hat{e}(\lambda_o, \mathbf{P}_\sigma) \in \mathcal{U}[\sigma] \cap \Sigma_\mathbf{C} \quad \text{and} \quad \hat{e}(\lambda_o, \mathbf{P}_\lambda) \in \mathcal{U}[\lambda] \cap \Sigma_\mathbf{C}. \tag{24.6}$$

[5]The insight to use the easier to understand condition (IF4) is due to Juan Schäffer.

Since $\nu_\Pi(\mathbf{C}_{(s)} * \mathbf{P} - \mathbf{C}_{(s)} * \mathbf{P}) = 0 \leq \delta$ for all $s \in \mathbb{P}$, by (24.6), (23.8) and the choice of $\mathcal{U}$ we have

$$\begin{aligned}\nu_\mathcal{R}&\left(\tilde{\mathbf{R}}(\sigma, \mathbf{C}_{(s)} * \mathbf{P}) - \tilde{\mathbf{R}}(\lambda_o, \mathbf{P}_\sigma * \mathbf{C}_{(s)} * \mathbf{P})\right)\\ &= \nu_\mathcal{R}\left(\tilde{\mathbf{R}}(\sigma, \mathbf{C}_{(s)} * \mathbf{P}) - \tilde{\mathbf{R}}(\hat{e}(\lambda_o, \mathbf{P}_\sigma), \mathbf{C}_{(s)} * \mathbf{P})\right) < \epsilon\end{aligned} \tag{24.7}$$

$$\begin{aligned}\nu_\mathcal{R}&\left(\tilde{\mathbf{R}}(\lambda_o, \mathbf{P}_\lambda * \mathbf{C}_{(s)} * \mathbf{P}) - \tilde{\mathbf{R}}(\lambda, \mathbf{C}_{(s)} * \mathbf{P})\right)\\ &= \nu_\mathcal{R}\left(\tilde{\mathbf{R}}(\hat{e}(\lambda_o, \mathbf{P}_\lambda), \mathbf{C}_{(s)} * \mathbf{P}) - \tilde{\mathbf{R}}(\lambda, \mathbf{C}_{(s)} * \mathbf{P})\right) < \epsilon.\end{aligned} \tag{24.8}$$

Since λ_o is relaxed we may assume that $d_{\mathbf{P}_\sigma} = d_{\mathbf{P}_\lambda} =: d$. To see this assume that $d_{\mathbf{P}_\sigma} > d_{\mathbf{P}_\lambda}$. If this were the case we could replace $\mathbf{P}_\lambda$ with $\hat{\mathbf{C}}(\lambda_o)_{(d_{\mathbf{P}_\sigma} - d_{\mathbf{P}_\lambda})} * \mathbf{P}_\lambda$. This process has the same duration as $\mathbf{P}_\sigma$. We assume that the processes were initially chosen so that they have the same duration.

Notice that by (24.4) and (24.3) we have

$$\begin{aligned}\lim_{s\to\infty} &\nu_\Pi\left(\mathbf{P}_\lambda * \mathbf{C}_{(s)} * \mathbf{P} - \mathbf{P}_\sigma * \mathbf{C}_{(s)} * \mathbf{P}\right)\\ &= \lim_{s\to\infty} \sup_{t\in[s+d_\mathbf{P}, d+s+d_\mathbf{P}]} h(t)\nu_{\mathcal{C}'}\left(\mathbf{P}_\lambda(s + d_\mathbf{P} + d - t) - \mathbf{P}_\sigma(s + d_\mathbf{P} + d - t)\right)\\ &= \lim_{s\to\infty} \sup_{t\in[0,d]} h(s + d_\mathbf{P} + t)\nu_{\mathcal{C}'}\left(\mathbf{P}_\lambda(d - t) - \mathbf{P}_\sigma(d - t)\right)\\ &\leq \left(\sup_{t\in[0,d]} \nu_{\mathcal{C}'}\left(\mathbf{P}_\lambda(d - t) - \mathbf{P}_\sigma(d - t)\right)\right) \lim_{s\to\infty} h(s + d_\mathbf{P}) = 0\end{aligned}$$

and so there is a $\bar{s} \in \mathbb{P}$ such that for all $s \in \bar{s} + \mathbb{P}^\times$

$$\nu_\Pi\left(\mathbf{P}_\lambda * \mathbf{C}_{(s)} * \mathbf{P} - \mathbf{P}_\sigma * \mathbf{C}_{(s)} * \mathbf{P}\right) < \delta. \tag{24.9}$$

Thus by the uniform continuity of $\tilde{\mathbf{R}}(\lambda_o, \cdot)$, for all $s \in \bar{s} + \mathbb{P}^\times$ we have

$$\nu_\mathcal{R}\left(\tilde{\mathbf{R}}(\lambda_o, \mathbf{P}_\lambda * \mathbf{C}_{(s)} * \mathbf{P}) - \tilde{\mathbf{R}}(\lambda_o, \mathbf{P} * \mathbf{C}_{(s)} * \mathbf{P})\right) < \epsilon. \tag{24.10}$$

Putting this together with (24.7) and (24.8), for all $s \in \bar{s} + \mathbb{P}^\times$ we have

$$\begin{aligned}\nu_\mathcal{R}\left(\tilde{\mathbf{R}}(\sigma, \mathbf{C}_{(s)} * \mathbf{P}) - \tilde{\mathbf{R}}(\lambda, \mathbf{C}_{(s)} * \mathbf{P})\right) &\leq \nu_\mathcal{R}\left(\tilde{\mathbf{R}}(\sigma, \mathbf{C}_{(s)} * \mathbf{P}) - \tilde{\mathbf{R}}(\lambda_o, \mathbf{P}_\sigma * \mathbf{C}_{(s)} * \mathbf{P})\right)\\ &\quad + \nu_\mathcal{R}\left(\tilde{\mathbf{R}}(\lambda_o, \mathbf{P}_\sigma * \mathbf{C}_{(s)} * \mathbf{P}) - \tilde{\mathbf{R}}(\lambda_o, \mathbf{P}_\lambda * \mathbf{C}_{(s)} * \mathbf{P})\right)\\ &\quad + \nu_\mathcal{R}\left(\tilde{\mathbf{R}}(\lambda_o, \mathbf{P}_\lambda * \mathbf{C}_{(s)} * \mathbf{P}) - \tilde{\mathbf{R}}(\lambda, \mathbf{C}_{(s)} * \mathbf{P})\right)\\ &< \epsilon + \epsilon + \epsilon = 3\epsilon.\end{aligned}$$

Since $\epsilon \in \mathbb{P}^\times$ was arbitrary we have

$$\lim_{s\to\infty} \tilde{\mathbf{R}}(\sigma, \mathbf{C}_{(s)} * \mathbf{P}) = \lim_{s\to\infty} \tilde{\mathbf{R}}(\lambda, \mathbf{C}_{(s)} * \mathbf{P}) = \tilde{\mathbf{R}}(\lambda, \mathbf{P}). \tag{24.11}$$

Since the mapping $\tilde{\mathbf{R}}(\cdot, \mathbf{P})$ is continuous we also have

$$\begin{aligned}\lim_{s\to\infty} \tilde{\mathbf{R}}(\sigma, \mathbf{C}_{(s)} * \mathbf{P}) &= \lim_{s\to\infty} \tilde{\mathbf{R}}(\hat{e}(\sigma, \mathbf{C}_{(s)}), \mathbf{P}) \\ &= \tilde{\mathbf{R}}(\lim_{s\to\infty} \hat{e}(\sigma, \mathbf{C}_{(s)}), \mathbf{P}) \\ &= \tilde{\mathbf{R}}(\hat{\lambda}(\sigma), \mathbf{P}).\end{aligned}$$

Putting this together with (24.11) and the fact that $\mathbf{P} \in \Pi_{\mathbf{C}}$ was arbitrary, Axiom 3 says that (24.5) holds. ▮

Remark 24.5 It follows from Remark 20.2 that this result says if an element has fading memory then it fits within the old theory of simple materials. ▮

25 Consequences of the Second Law

Here we investigate what restrictions the second law of thermodynamics places on the result mapping for a thermomechanical element with fading memory. To find these restrictions three assumptions are made. These assumptions are based on those found in [TMM2][6].

Let $d \in \mathbb{P}^\times$ and $h \in [0, d]$ be given. We can define a mapping $L_h : \Pi^d \longrightarrow \Pi^d$ by

$$L_h(\mathbf{P}) = \mathbf{P}_{[0,d-h]} * \mathbf{P}(d-h)_{(h)} \quad \text{for all} \quad \mathbf{P} \in \Pi_d. \tag{25.1}$$

This first assumption is a kind of differentiability assumption.

Assumption 1 *There is a mapping* $\partial\tilde{\psi} : (\Sigma \times \Pi^\times)_{\text{fit}} \longrightarrow \mathcal{C}'^*$ *such that for all* $(\sigma, \mathbf{P}) \in (\Sigma \times \Pi^\times)_{\text{fit}}$ *we have*

$$\lim_{h\to 0^+} \frac{1}{h}(\tilde{\psi}(\sigma, \mathbf{P}) - \tilde{\psi}(\sigma, L_h(\mathbf{P}))) = \partial\tilde{\psi}(\sigma, \mathbf{P})\mathbf{P}^{\bullet f}. \tag{25.2}$$

Since $\mathcal{C}'$ is the product of linear spaces (see (16.5)) every element of $\mathcal{C}'^* = \text{Lin}(\mathcal{C}', \mathbb{R})$ can be identified with a 1×3 matrix whose terms are linear mappings themselves. Thus we can represent $\partial\tilde{\psi}$ in the form

$$\partial\tilde{\psi}(\sigma, \mathbf{P})\mathbf{C}' = \partial_{(1)}\tilde{\psi}(\sigma, \mathbf{P})\mathbf{G}' + \partial_{(2)}\tilde{\psi}(\sigma, \mathbf{P})\theta' + \partial_{(3)}\tilde{\psi}(\sigma, \mathbf{P})\boldsymbol{\gamma}' \tag{25.3}$$

for all $(\sigma, \mathbf{P}) \in (\Sigma \times \Pi^\times)_{\text{fit}}$ and $\mathbf{C}' = (\mathbf{G}', \theta', \boldsymbol{\gamma}') \in \mathbf{C}'$. For example

$$\partial_{(1)}\tilde{\psi}(\sigma, \mathbf{P}) \in \text{Sym}(\mathcal{T}, \mathcal{T}^*)^* \quad \text{for all} \quad (\sigma, \mathbf{P}) \in (\Sigma \times \Pi^\times)_{\text{fit}}.$$

Assumption 2 *Let* $(\sigma, \mathbf{P}) \in (\Sigma \times \Pi^\times)_{\text{fit}}$ *be given. Define* $\bar{\psi}_{(\sigma,\mathbf{P})}$ *as in (21.5). Then* $\bar{\psi}_{(\sigma,\mathbf{P})} \in \text{pwC}^1([0, d_{\mathbf{P}}], \mathbb{R})$ *and* $\bar{\psi}^{\bullet}_{(\sigma,\mathbf{P})}$ *has jumps at exactly the same points as* $\mathbf{P}^\bullet$.

[6] One main difference between what is assumed here and what is assumed in [TMM2] is that Coleman and Owen take the existence of the limit in (25.4) as a basic assumption and then prove the contents of Assumption 2.

This assumption says that the mapping $\bar{\psi}_{(\sigma,\mathbf{P})}$ inherits the smoothness of $\mathbf{P}$.

Lemma 25.1 *Let $(\sigma,\mathbf{P}) \in (\Sigma \times \Pi^{\times})_{\rm fit}$ be given and define $\bar{\psi}_{(\sigma,\mathbf{P})}$ as in (21.5). Then the limit*

$$d\tilde{\psi}(\sigma,\mathbf{P}) := \lim_{h\to 0^+} \frac{1}{h}\left[\tilde{\psi}(\sigma, L_h\mathbf{P}) - \tilde{\psi}(\sigma, \mathbf{P}_{[0,d_{\mathbf{P}}-h]})\right] \tag{25.4}$$

exists and we have

$$\bar{\psi}^{\bullet f}_{(\sigma,\mathbf{P})} = \partial\tilde{\psi}(\sigma,\mathbf{P})\mathbf{P}^{\bullet f} + d\tilde{\psi}(\sigma,\mathbf{P}). \tag{25.5}$$

Proof: By (6.5) and Assumption 2 we have

$$\begin{aligned}\bar{\psi}^{\bullet f}_{(\sigma,\mathbf{P})} - \partial\tilde{\psi}(\sigma,\mathbf{P})\mathbf{P}^{\bullet f} &= \lim_{h\to 0} \frac{1}{h}\left[\tilde{\psi}(\sigma,\mathbf{P}) - \tilde{\psi}(\sigma, \mathbf{P}_{[0,d_{\mathbf{P}}-h]})\right] \\ &\quad - \lim_{h\to 0^+} \frac{1}{h}\left[\tilde{\psi}(\sigma,\mathbf{P}) - \tilde{\psi}(\sigma, L_h\mathbf{P})\right] \\ &= \lim_{h\to 0^+} \frac{1}{h}\left[\tilde{\psi}(\sigma, L_h\mathbf{P}) - \tilde{\psi}(\sigma, \mathbf{P}_{[0,d_{\mathbf{P}}-h]})\right] = d\tilde{\psi}(\sigma,\mathbf{P}).\end{aligned}$$

∎

Using (25.5), (25.3) and (21.6) we find that the second law holds if and only if

$$\begin{aligned}-\hat{\rho}(\bar{\mathbf{G}}^f)\left(\tilde{\eta}(\sigma,\mathbf{P}) + \partial_{(2)}\tilde{\psi}(\sigma,\mathbf{P})\right)\bar{\theta}^{\bullet f} + \tfrac{1}{2}\mathrm{tr}\left[\left(\tilde{\mathbf{S}}(\sigma,\mathbf{P}) - 2\hat{\rho}(\bar{\mathbf{G}}^f)\partial_{(1)}\tilde{\psi}(\sigma,\mathbf{P})\right)\bar{\mathbf{G}}^{\bullet f}\right] \\ +\partial_{(3)}\tilde{\psi}(\sigma,\mathbf{P})\bar{\boldsymbol{\gamma}}^{\bullet f} + d\tilde{\psi}(\sigma,\mathbf{P}) - \tilde{\Upsilon}(\sigma,\mathbf{P}) \geq 0 \quad \text{for all} \ \ (\sigma,\mathbf{P}) \in (\Sigma\times\Pi^{\times})_{\rm fit}.\end{aligned} \tag{25.6}$$

Proposition 25.2 *Let $d \in \mathbb{P}^{\times}$ and $\mathbf{C}' \in \mathcal{C}'$ be given. Define a family of mappings $(\Phi_h^{\mathbf{C}'} : [0,d] \longrightarrow \mathcal{C}' \mid h \in]0,d])$ by*

$$\Phi_h^{\mathbf{C}'}(t) := \begin{cases} 0 & \text{if} \quad t \in [0, d-h] \\ \frac{1}{h}(d-t-h)(d-t)\mathbf{C}' & \text{if} \quad t \in [d-h, d]. \end{cases} \tag{25.7}$$

Then

$$\lim_{h\to 0^+} \nu_d(\Phi_h^{\mathbf{C}'}) = 0.$$

Proof: Let $h \in]0,d]$ be given. Then by (24.4)

$$\begin{aligned}\nu_{\Pi}(\Phi_h^{\mathbf{C}'}) &= \sup_{t\in[0,d]} h(t)\nu_{\mathcal{C}'}(\Phi_h^{\mathbf{C}'}(d-t)) \\ &= \sup_{t\in[0,h]} \frac{1}{h}|(t-h)t| \ \nu_{\mathcal{C}'}(\mathbf{C}')dt\end{aligned}$$

Since the maximum of the function $|(\iota - h)\iota|$, restricted to $[0,h]$, is $h^2/4$, we have

$$\sup_{t\in[0,h]} \frac{1}{h}|(t-h)t| \ \nu_{\mathcal{C}'}(\mathbf{C}')dt \leq \frac{h}{4}\nu_{\mathcal{C}'}(\mathbf{C}').$$

Taking the limit as $h \to 0^+$ proves the desired result. ∎

Assumption 3 *For all $\sigma \in \Sigma$, the mappings $\partial\tilde{\psi}(\sigma,\cdot)$ and $d\tilde{\psi}(\sigma,\cdot)$ are continuous.*

It follows from Proposition 25.2 and Assumption 3 that if $F : (\Sigma \times \Pi^\times)_{\text{fit}} \longrightarrow \mathcal{W}$, $\mathcal{W}$ being a linear space, is one of the mappings listed in Assumption 3 or the mapping $\tilde{\mathbf{R}}$ and $(\Phi_h^{\mathbf{C}'} : [0,d] \longrightarrow \mathcal{C}' \mid h \in]0,d])$ is the family defined in (25.7) then

$$\lim_{h\to 0^+} F(\sigma, \mathbf{P} + \Phi_h^{\mathbf{C}'}) = F(\sigma,\mathbf{P}) \quad \text{for all} \ \ (\sigma,\mathbf{P}) \in (\Sigma \times \Pi^\times)_{\text{fit}}. \tag{25.8}$$

Notice that although $\mathbf{P} + \Phi_h^{\mathbf{C}'}$ may not be a condition process for all $h \in]0,d]$, it is a condition process for small $h \in]0,d]$ since $\mathcal{C}$ is open in $\mathcal{C}'$. Thus, the limit in (25.8) makes sense.

Theorem 25.3 *The second law of thermodynamics holds if and only if*

1. $\tilde{\mathbf{S}}(\sigma,\mathbf{P}) = 2\hat{\rho}(\bar{\mathbf{G}}^f)\partial_{(1)}\tilde{\psi}(\sigma,\mathbf{P})$,
2. $\tilde{\eta}(\sigma,\mathbf{P}) = -\partial_{(2)}\tilde{\psi}(\sigma,\mathbf{P})$,
3. $\partial_{(3)}\tilde{\psi}(\sigma,\mathbf{P}) = \mathbf{0}$,
4. $d\tilde{\psi}(\sigma,\mathbf{P}) - \tilde{\Upsilon}(\sigma,\mathbf{P}) \geq 0$

for all $(\sigma,\mathbf{P}) \in (\Sigma \times \Pi^\times)_{\text{fit}}$.

Proof: Assume that the second law of thermodynamics holds. Let $(\sigma,\mathbf{P}) \in (\Sigma \times \Pi^\times)_{\text{fit}}$ be given and put $d := d_{\mathbf{P}}$. Let $\mathbf{C}' = (\mathbf{G}', \theta', \boldsymbol{\gamma}') \in \mathcal{C}'$ be given and define the family $(\Phi_h^{\mathbf{C}'} \mid h \in]0,d])$ as in (25.7). Notice that

$$(\mathbf{P} + \Phi_h^{\mathbf{C}'})^f = \mathbf{P}^f \qquad \text{and} \qquad (\mathbf{P} + \Phi_h^{\mathbf{C}'})^{\bullet f} = \mathbf{P}^{\bullet f} + \mathbf{C}'$$

for all $h \in]0,d]$. Thus (25.6) with $\mathbf{P}$ replaced with $\mathbf{P} + \Phi_h^{\mathbf{C}'}$ gives

$$\begin{aligned} -\hat{\rho}(\bar{\mathbf{G}}^f)\left(\tilde{\eta}(\sigma,\mathbf{P}+\Phi_h^{\mathbf{C}'}) + \partial_{(2)}\tilde{\psi}(\sigma,\mathbf{P}+\Phi_h^{\mathbf{C}'})\right)(\bar{\theta}^{\bullet f} + \theta') + \partial_{(3)}\tilde{\psi}(\sigma,\mathbf{P}+\Phi_h^{\mathbf{C}'})(\bar{\boldsymbol{\gamma}}^{\bullet f} + \boldsymbol{\gamma}') \\ +\tfrac{1}{2}\mathrm{tr}\left[\left(\tilde{\mathbf{S}}(\sigma,\mathbf{P}+\Phi_h^{\mathbf{C}'}) - 2\hat{\rho}(\bar{\mathbf{G}}^f)\partial_{(1)}^f\tilde{\psi}(\sigma,\mathbf{P}+\Phi_h^{\mathbf{C}'})\right)(\bar{\mathbf{G}}^{\bullet f} + \mathbf{G}')\right] \\ +d\tilde{\psi}(\sigma,\mathbf{P}+\Phi_h^{\mathbf{C}'}) - \tilde{\Upsilon}(\sigma,\mathbf{P}+\Phi_h^{\mathbf{C}'}) \geq 0 \quad \text{for all small } h \in]0,d]. \end{aligned} \tag{25.9}$$

By "small h" we mean those h such that $\mathbf{P} + \Phi_h^{\mathbf{C}'} \in \Pi$. Taking the limit $h \longrightarrow 0^+$ and using Assumption 3 together with Proposition 25.2 and the fact that $\mathbf{C}' \in \mathcal{C}'$ was arbitrary yields

$$\begin{aligned} -\hat{\rho}(\bar{\mathbf{G}}^f)\left(\tilde{\eta}(\sigma,\mathbf{P}) + \partial_{(2)}\tilde{\psi}(\sigma,\mathbf{P})\right)(\bar{\theta}^{\bullet f} + \theta') + \partial_{(3)}\tilde{\psi}(\sigma,\mathbf{P})(\bar{\boldsymbol{\gamma}}^{\bullet f} + \boldsymbol{\gamma}') \\ +\tfrac{1}{2}\mathrm{tr}\left[\left(\tilde{\mathbf{S}}(\sigma,\mathbf{P}) - 2\hat{\rho}(\bar{\mathbf{G}}^f)\partial_{(1)}\tilde{\psi}(\sigma,\mathbf{P})\right)(\bar{\mathbf{G}}^{\bullet f} + \mathbf{G}')\right] \\ +d\tilde{\psi}(\sigma,\mathbf{P}) - \tilde{\Upsilon}(\sigma,\mathbf{P}) \geq 0 \quad \text{for all } \ \mathbf{C}' \in \mathcal{C}'. \end{aligned} \tag{25.10}$$

Since this must hold for all $\mathbf{C}' = (\mathbf{G}', \theta', \boldsymbol{\gamma}') \in \mathcal{C}'$ it follows that

$$\tilde{\mathbf{S}}(\sigma,\mathbf{P}) = 2\hat{\rho}(\bar{\mathbf{G}}^f)\partial_{(1)}^f\tilde{\psi}(\sigma,\mathbf{P}), \quad \tilde{\eta}(\sigma,\mathbf{P}) = -\partial_{(2)}^f\tilde{\psi}(\sigma,\mathbf{P}) \ \text{ and } \ \partial_{(3)}^f\tilde{\psi}(\sigma,\mathbf{P}) = \mathbf{0}. \tag{25.11}$$

Using these equalities, (25.10) yields

$$d\tilde{\psi}(\sigma,\mathbf{P}) - \tilde{\Upsilon}(\sigma,\mathbf{P}) \geq 0. \tag{25.12}$$

Since $(\sigma, \mathbf{P}) \in (\Sigma \times \Pi^\times)_{\text{fit}}$ were arbitrary, (25.11) and (25.12) show that items 1–4 of the theorem hold.

The converse of the theorem follows easily. ∎

26 Approximation Theorem

Let an influence function h of order $r > 2$, as defined by Definition 24.1, be given. Let $d \in \mathbb{P}^\times$ be given. Using (IF4) with $s := d/\alpha$ and $t := \xi$ we find that

$$\frac{h(d/\alpha)}{\alpha} = \frac{(d/\alpha)^r h(d/\alpha)}{\xi^r h(\xi)} \cdot \frac{\alpha^{r-1}\xi^r h(\xi)}{d^r} \leq K \cdot \frac{\xi^r h(\xi)}{d^r}\alpha^{r-1} \quad \text{for all} \quad \alpha \in]0, d/\xi]$$

and thus

$$\lim_{\alpha \to 0^+} \frac{1}{\alpha} h(d/\alpha) = 0. \tag{26.1}$$

Since $\sup_{t \in \mathbb{P}} th(t)$ is finite (see (24.3)), we may define a mapping $\bar{h} : \mathbb{P} \longrightarrow \mathbb{P}$ by

$$\bar{h}(x) := \sup_{t \in [x, \infty[} th(t) \quad \text{for all} \quad x \in \mathbb{P}. \tag{26.2}$$

It follows from (26.1) that $th(t) \to 0$ as $t \to \infty$. Thus, by (26.2)

$$\lim_{x \to \infty} \bar{h}(x) = 0. \tag{26.3}$$

The following result may seem odd to the reader at first glance but plays a crucial role is the proofs of Theorem 26.8 and Theorem 26.11.

Proposition 26.1 *Let $\epsilon \in \mathbb{P}^\times$ and $N \in \mathbb{P}$ be given. Then there is a mapping $T : \mathbb{P}^\times \longrightarrow \mathbb{P}$ such that for all $s \in \mathbb{P}^\times$ and $d \in T(s) + \mathbb{P}^\times$ we have*
(a) $h(d) \leq s\epsilon/N$
(b) $\bar{h}(d) \leq \epsilon$.
Furthermore, T can be choosen in such a way that for fixed $d, s \in \mathbb{P}^\times$ there is an $\eta \in \mathbb{P}^\times$ such that

$$\frac{d}{\alpha} \geq T(\alpha s) \quad \text{for all} \quad \alpha \in]0, \eta]. \tag{26.4}$$

Proof: Let $s \in \mathbb{P}^\times$ be given. Since $\lim_{t \to \infty} h(t) = 0$, by (24.2), and (26.3) there is a $d \in \mathbb{P}^\times$ such that (a) and (b) hold. Notice that if (a) and (b) hold for d then, since h and $\bar{h}$ are antitone, (a) and (b) hold with d replaced by d', for all $d' \in d + \mathbb{P}$. Hence, if we define $T(s)$ to be the smallest element of $\mathbb{P}$ such that

$$h(T(s)) \leq s\epsilon/N \quad \text{and} \quad \bar{h}(T(s)) \leq \epsilon. \tag{26.5}$$

T satisfies the first part of the proposition.

To prove that this T satisfies the "Furthermore" part of the proposition, let $d, s \in \mathbb{P}^\times$ be given. It follows from (26.1) and (26.3) that there is an $\eta \in \mathbb{P}^\times$ such that

$$h(d/\alpha) \leq \alpha s \epsilon / N \quad \text{and} \quad \bar{h}(d/\alpha) \leq \epsilon \quad \text{for all} \quad \alpha \in]0, \eta].$$

By the definition of T we have

$$h(T(\alpha s)) \leq \alpha s \epsilon / N \quad \text{and} \quad \bar{h}(T(\alpha s)) \leq \epsilon$$

and since $T(s)$ is the smallest number in $\mathbb{P}$ such that (26.5) holds, it follows that $d/\alpha \geq T(\alpha s)$ for all $\alpha \in]0, \eta]$. ∎

Let a thermomechanical element $(\mathcal{T}, \mathcal{C}, \Pi, \Sigma, \hat{\mathbf{C}}, \hat{\mathbf{R}}, \hat{e})$ with fading memory, as defined in Definition 23.1, be given. Let $d \in \mathbb{P}$ be given. Recall (16.5). We will use the topology on $\mathrm{pwC}^1([0, d], \mathcal{C}')$ induced by the norm ν_d (see (24.4)). Recall $\Pi^d = \mathrm{pwC}^1([0, d], \mathcal{C})$ which we consider to be a subset of $\mathrm{pwC}^1([0, d], \mathcal{C}')$. Π^d is an open subset of the infinite-dimensional linear space $\mathrm{pwC}^1([0, d], \mathcal{C}')$. For all $\mathbf{C} \in \mathcal{C}$ the set $\Pi^d_{\mathbf{C}}$ is an open subset of a flat in Π^d whose translation space is

$$\mathrm{pwC}^1_0([0, d], \mathcal{C}') := \{\mathbf{F} \in \mathrm{pwC}^1([0, d], \mathcal{C}') \mid \mathbf{F}(0) = \mathbf{0}\}. \tag{26.6}$$

Let $d \in \mathbb{P}$ and $s \in \mathbb{P}$ be given. Define the mapping

$$T^d_s : \mathrm{pwC}^1_0([0, d], \mathcal{C}') \longrightarrow \mathrm{pwC}^1_0([0, d+s], \mathcal{C}') \tag{26.7}$$

by $T^d_s(\mathbf{F}) := \mathbf{0}_{(s)} * \mathbf{F}$ for all $\mathbf{F} \in \mathrm{pwC}^1_0([0, d], \mathcal{C}')$, i.e.,

$$T^d_s(\mathbf{F})(t) = \begin{cases} \mathbf{0} & \text{if} \quad t \in [0, s] \\ \mathbf{F}(t-s) & \text{if} \quad t \in [s, d+s] \end{cases} \quad \text{for all} \quad \mathbf{F} \in \mathrm{pwC}^1_0([0, d], \mathcal{C}'). \tag{26.8}$$

It is easy to check that T^d_s is linear.

It will be useful for us to connect $\mathrm{pwC}^1_0([0, d], \mathcal{C}')$ with a suitable subset of $\mathrm{pwC}^1(\mathbb{P}, \mathcal{C}')$ in the following way: Define the mapping $U_d : \mathrm{pwC}^1_0([0, d], \mathcal{C}') \longrightarrow \mathrm{pwC}^1(\mathbb{P}, \mathcal{C}')$ by

$$U_d(\mathbf{F})(t) := \begin{cases} \mathbf{F}(d-t) & \text{if} \quad t \in [0, d] \\ \mathbf{0} & \text{if} \quad t \in [d, \infty[\end{cases} \quad \text{for all} \quad \mathbf{F} \in \mathrm{pwC}^1_0([0, d], \mathcal{C}'). \tag{26.9}$$

It is easy to see that the mapping U_d is injective and hence $U_d|^{\mathrm{Rng}}$ is invertible.

Proposition 26.2 *Let $d_1, d_2 \in \mathbb{P}$ be given such that $d_1 \leq d_2$. Also, let $\mathbf{F}_1 \in \mathrm{pwC}^1_0([0, d_1], \mathcal{C}')$ and $\mathbf{F}_2 \in \mathrm{pwC}^1_0([0, d_2], \mathcal{C}')$ be given. Suppose $U_{d_1}(\mathbf{F}_1) = U_{d_2}(\mathbf{F}_2)$. Then $T^{d_1}_{d_2 - d_1}(\mathbf{F}_1) = \mathbf{F}_2$.*

Proof: By definition

$$U_{d_1}(\mathbf{F}_1)(t) = \begin{cases} \mathbf{F}_1(d_1 - t) & \text{if} \quad t \in [0, d_1] \\ \mathbf{0} & \text{if} \quad t \in [d_1, \infty[\end{cases} \tag{26.10}$$

and

$$U_{d_2}(\mathbf{F}_2)(t) = \begin{cases} \mathbf{F}_2(d_2 - t) & \text{if} \quad t \in [0, d_2] \\ \mathbf{0} & \text{if} \quad t \in [d_2, \infty[\end{cases} . \tag{26.11}$$

Since $d_1 \leq d_2$ and $U_{d_1}(\mathbf{F}_1) = U_{d_2}(\mathbf{F}_2)$ it follows that

$$\begin{array}{rl} \mathbf{F}_1(d_1 - t) = \mathbf{F}_2(d_2 - t) & \text{for} \quad t \in [0, d_1] \\ \mathbf{F}_2(d_2 - t) = \mathbf{0} & \text{for} \quad t \in [d_1, d_2] \end{array} \tag{26.12}$$

and thus

$$\begin{array}{rl} \mathbf{F}_1(t - d_2 + d_1) = \mathbf{F}_2(t) & \text{for} \quad t \in [d_2 - d_1, d_2] \\ \mathbf{F}_2(t) = \mathbf{0} & \text{for} \quad t \in [0, d_2 - d_1]. \end{array} \tag{26.13}$$

It follows immediately that $T^{d_1}_{d_2 - d_1}(\mathbf{F}_1) = \mathbf{F}_2$. ∎

Let $d \in \mathbb{P}$ be given. The freeze, or constant mapping, $\mathbf{0}_{(d)} \in \mathrm{pwC}^1([0, d], \mathcal{C}')$ will be identified with the zero $\mathbf{0}$ of $\mathcal{C}'$. Thus we will write

$$\mathbf{0} = \mathbf{0}_{(d)} \in \mathrm{pwC}^1([0, d], \mathcal{C}') \quad \text{for all} \;\; d \in \mathbb{P}. \tag{26.14}$$

Let $\mathrm{pwC}^1_{\mathrm{c}}(\mathbb{P}, \mathcal{C}')$ denote the set of all elements of $\mathrm{pwC}^1(\mathbb{P}, \mathcal{C}')$ that have compact support, i.e., whose support is a closed and bounded subset of $\mathbb{P}$. Note that $\mathrm{pwC}^1_{\mathrm{c}}(\mathbb{P}, \mathcal{C}')$ is an infinite-dimensional linear space. It is not hard to show that

$$\mathrm{pwC}^1_{\mathrm{c}}(\mathbb{P}, \mathcal{C}') = \bigcup_{d \in \mathbb{P}} U_{d>}(\mathrm{pwC}^1_0([0, d], \mathcal{C}')). \tag{26.15}$$

Define a mapping $\overleftarrow{U} : \mathrm{pwC}^1_{\mathrm{c}}(\mathbb{P}, \mathcal{C}') \longrightarrow \bigcup_{d \in \mathbb{P}} \mathrm{pwC}^1_0([0, d], \mathcal{C}')$ by the following procedure: Let $\mathbf{H} \in \mathrm{pwC}^1_{\mathrm{c}}(\mathbb{P}, \mathcal{C}')$ by given. Put $d := \sup \operatorname{supt} \mathbf{H}$. Define $\overleftarrow{U}(\mathbf{H}) := (U_d|^{\mathrm{Rng}})^{\leftarrow}(\mathbf{H})$. Note that, using (26.14), $\overleftarrow{U}(\mathbf{0}) = \mathbf{0}$. A short calculation shows that

$$\overleftarrow{U}(\alpha \mathbf{H}) = \alpha \overleftarrow{U}(\mathbf{H}) \quad \text{for all} \;\; \alpha \in \mathbb{R}, \; \mathbf{H} \in \mathrm{pwC}^1_{\mathrm{c}}(\mathbb{P}, \mathcal{C}'). \tag{26.16}$$

When $\alpha = 0$ one must use the identification (26.14).

The space $\mathrm{pwC}^1(\mathbb{P}, \mathcal{C}')$ with the mapping $\nu_\infty : \mathrm{pwC}^1(\mathbb{P}, \mathcal{C}') \longrightarrow \mathbb{P}$ defined by

$$\nu_\infty(\mathbf{H}) := \sup_{t \in \mathbb{P}} h(t) \nu_{\mathcal{C}'}(\mathbf{H}(t)) \quad \text{for all} \;\; \mathbf{H} \in \mathrm{pwC}^1(\mathbb{P}, \mathcal{C}') \tag{26.17}$$

is easily seen to be a normed space. Let $d \in \mathbb{P}$ be given. Recall (23.7). Using the definition of U_d we have

$$\nu_\Pi(\mathbf{F}) = \nu_\infty(U_d(\mathbf{F})) \quad \text{for all} \;\; \mathbf{F} \in \mathrm{pwC}^1_0([0, d], \mathcal{C}'). \tag{26.18}$$

Assumption 4 *For all $\lambda \in \Sigma_{\mathrm{rel}}$ the family $(\tilde{\mathbf{R}}_d(\lambda, \cdot) : \Pi^d_{\hat{\mathbf{C}}(\lambda)} \longrightarrow \mathcal{R} \mid d \in \mathbb{P}^\times)$ (see (16.14)) is* **equidifferentiable** *at $(\hat{\mathbf{C}}(\lambda)_{(d)} \mid d \in \mathbb{P}^\times)$ in the sense that for all $d \in \mathbb{P}^\times$ there is a* *continuous linear mapping* $\nabla_\lambda \tilde{\mathbf{R}}_d \in \mathrm{Lin}(\mathrm{pwC}_0^1([0,d],\mathcal{C}'),\mathcal{R})$ *such that for all* $\epsilon \in \mathbb{P}^\times$ *there is a* $\delta \in \mathbb{P}^\times$ *such that*

$$\frac{\nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\lambda,\mathbf{P}) - \hat{\mathbf{R}}(\lambda) - \nabla_\lambda \tilde{\mathbf{R}}_{d_{\mathbf{P}}}(\mathbf{P} - \hat{\mathbf{C}}(\lambda)_{(d_{\mathbf{P}})}))}{\nu_\Pi(\mathbf{P} - \hat{\mathbf{C}}(\lambda)_{(d_{\mathbf{P}})})} < \epsilon \tag{26.19}$$

$$\text{for all } \ \mathbf{P} \in \Pi_{\hat{\mathbf{C}}(\lambda)} \ \text{ with } \ 0 < \nu_\Pi(\mathbf{P} - \hat{\mathbf{C}}(\lambda)_{(d_{\mathbf{P}})}) < \delta.$$

Furthermore,

$$\sup_{d \in \mathbb{P}^\times} ||\nabla_\lambda \tilde{\mathbf{R}}_d||_{\nu_d,\nu_{\mathcal{R}}} =: R < \infty. \tag{26.20}$$

It follows from Assumption 4 that given $\lambda \in \Sigma_{\mathrm{rel}}$ and $d \in \mathbb{P}^\times$ the mapping $\tilde{\mathbf{R}}(\lambda,\cdot)$ is differentiable at $\hat{\mathbf{C}}(\lambda)_{(d)}$ and its gradient at this point is $\nabla_\lambda \tilde{\mathbf{R}}_d$.

Proposition 26.3 *Let* $\lambda \in \Sigma_{\mathrm{rel}}$, $d \in \mathbb{P}^\times$ *and* $s \in \mathbb{P}$ *be given. Then*

$$\nabla_\lambda \tilde{\mathbf{R}}_d = \nabla_\lambda \tilde{\mathbf{R}}_{d+s} \circ T_s^d. \tag{26.21}$$

Proof: Put $\mathbf{C} := \hat{\mathbf{C}}(\lambda)$ and let $\epsilon \in \mathbb{P}^\times$ be given. By the differentiability of $\tilde{\mathbf{R}}_{d+s}(\lambda,\cdot)$ at $\mathbf{C}_{(d+s)}$ there is a $\delta \in \mathbb{P}^\times$ such that for all $\mathbf{P} \in \Pi_{\mathbf{C}}^{d+s}$ with $0 < \nu_\Pi(\mathbf{P} - \mathbf{C}_{(d+s)}) < \delta$ we have

$$\frac{\nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\lambda,\mathbf{P}) - \hat{\mathbf{R}}(\lambda) - \nabla_\lambda \tilde{\mathbf{R}}_{d+s}(\mathbf{P} - \mathbf{C}_{(d+s)})}{\nu_\Pi(\mathbf{P} - \mathbf{C}_{(d+s)}))} < \epsilon. \tag{26.22}$$

Since λ is a relaxed state, $\tilde{\mathbf{R}}(\lambda,\mathbf{P}) = \tilde{\mathbf{R}}(\lambda,\mathbf{C}_{(s)} * \mathbf{P})$ for all $\mathbf{P} \in \Pi_{\mathbf{C}}$ by (16.16) and (iii) of Proposition 19.1. Also, a short calculation shows that

$$T_s^d(\mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}})}) = \mathbf{C}_{(s)} * \mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}}+s)} \tag{26.23}$$

$$\nu_\Pi(\mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}})}) = \nu_\Pi(\mathbf{C}_{(s)} * \mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}}+s)}) \tag{26.24}$$

for all $\mathbf{P} \in \Pi_{\mathbf{C}}$. Thus, for all $\mathbf{P} \in \Pi_{\mathbf{C}}^d$ with $0 < \nu_\Pi(\mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}})}) = \nu_\Pi(\mathbf{C}_{(s)} * \mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}}+s)}) < \delta$ we have

$$\frac{\nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\lambda,\mathbf{P}) - \hat{\mathbf{R}}(\lambda) - (\nabla_\lambda \tilde{\mathbf{R}}_{d+s} \circ T_s^d)(\mathbf{P} - \mathbf{C}_{(d)}))}{\nu_\Pi(\mathbf{P} - \mathbf{C}_{(d)})} = \tag{26.25}$$

$$\frac{\nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\lambda,\mathbf{C}_{(s)} * \mathbf{P}) - \hat{\mathbf{R}}(\lambda) - \nabla_\lambda \tilde{\mathbf{R}}_{d+s}(\mathbf{C}_{(s)} * \mathbf{P} - \mathbf{C}_{(d+s)})}{\nu_\Pi(\mathbf{C}_{(s)} * \mathbf{P} - \mathbf{C}_{(d+s)})} < \epsilon. \tag{26.26}$$

Since $\epsilon \in \mathbb{P}^\times$ was arbitrary and the gradients are unique, we conclude that $\nabla_\lambda \tilde{\mathbf{R}}_{d+s} \circ T_s^d = \nabla_\lambda \tilde{\mathbf{R}}_d$. ∎

Theorem 26.4 *For all* $\lambda \in \Sigma_{rel}$ *there is a continuous linear mapping*

$$\nabla_\lambda \tilde{\mathbf{R}} \in \mathrm{Lin}(\mathrm{pwC}^1(\mathbb{P},\mathcal{C}'),\mathcal{R})$$

such that

$$\nabla_\lambda \tilde{\mathbf{R}}(U_d(\mathbf{F})) = \nabla_\lambda \tilde{\mathbf{R}}_d(\mathbf{F}) \quad \text{for all} \;\; \mathbf{F} \in \mathrm{pwC}^1_0([0,d], \mathcal{C}') \text{ and } d \in \mathbb{P}^\times \tag{26.27}$$

and

$$\sup_{\mathbf{H} \in \mathrm{pwC}^1(\mathbb{P},\mathcal{R})^\times} \frac{\nu_\mathcal{R}(\nabla_\lambda \tilde{\mathbf{R}}(\mathbf{H}))}{\nu_\infty(\mathbf{H})} =: ||\nabla_\lambda \tilde{\mathbf{R}}||_{\nu_\infty, \nu_\mathcal{R}} \leq R. \tag{26.28}$$

Proof: Define a mapping $\mathbf{B} : \mathrm{pwC}^1_\mathrm{c}(\mathbb{P}, \mathcal{C}') \longrightarrow \mathcal{R}$ by

$$\mathbf{B}(\mathbf{H}) := (\nabla_\lambda \tilde{\mathbf{R}}_{d_{\overleftarrow{U}(\mathbf{H})}} \circ \overleftarrow{U})(\mathbf{H}) \quad \text{for all} \;\; \mathbf{H} \in \mathrm{pwC}^1_\mathrm{c}(\mathbb{P}, \mathcal{C}') \tag{26.29}$$

We will show that $\mathbf{B}$ is linear.

Let $\alpha \in \mathbb{R}$ and $\mathbf{H} \in \mathrm{pwC}^1_\mathrm{c}(\mathbb{P}, \mathcal{C}')$ be given. By (26.16) we have

$$\mathbf{B}(\alpha\mathbf{H}) = \nabla_\lambda \tilde{\mathbf{R}}_{d_{\overleftarrow{U}(\mathbf{H})}}(\overleftarrow{U}(\alpha\mathbf{H})) = \nabla_\lambda \tilde{\mathbf{R}}_{d_{\overleftarrow{U}(\mathbf{H})}}(\alpha\overleftarrow{U}(\mathbf{H})) = \alpha(\nabla_\lambda \tilde{\mathbf{R}}_{d_{\overleftarrow{U}(\mathbf{H})}} \circ \overleftarrow{U})(\mathbf{H}) = \alpha\mathbf{B}(\mathbf{H}). \tag{26.30}$$

Since $\mathbf{H} \in \mathrm{pwC}^1_\mathrm{c}(\mathbb{P}, \mathcal{C}')$ and $\alpha \in \mathbb{R}$ were arbitrary, $\mathbf{B}$ is homogeneous.

Now let $\mathbf{H}_1, \mathbf{H}_2 \in \mathrm{pwC}^1_\mathrm{c}(\mathbb{P}, \mathcal{C}')$ be given. Put $\mathbf{F}_1 := \overleftarrow{U}(\mathbf{H}_1)$, $\mathbf{F}_2 := \overleftarrow{U}(\mathbf{H}_2)$, $d_1 := d_{\mathbf{F}_1}$ and $d_2 := d_{\mathbf{F}_2}$. Assume $d_1 \leq d_2$. By the definition (26.29) of $\mathbf{B}$ and Proposition 26.3

$$\mathbf{B}(\mathbf{H}_1) + \mathbf{B}(\mathbf{H}_2) = \nabla_\lambda \tilde{\mathbf{R}}_{d_1}\mathbf{F}_1 + \nabla_\lambda \tilde{\mathbf{R}}_{d_2}\mathbf{F}_2 = \nabla_\lambda \tilde{\mathbf{R}}_{d_2}(T^{d_1}_{d_2 - d_1}(\mathbf{F}_1) + \mathbf{F}_2) \tag{26.31}$$

A straightforward, but tedious, calculation shows that

$$U_{d_2}(T^{d_1}_{d_2-d_1}(\mathbf{F}_1) + \mathbf{F}_2) = \mathbf{H}_1 + \mathbf{H}_2. \tag{26.32}$$

Put $\mathbf{F} := \overleftarrow{U}(\mathbf{H}_1 + \mathbf{H}_2)$ so that $U_{d_\mathbf{F}}(\mathbf{F}) = \mathbf{H}_1 + \mathbf{H}_2$ (when $\mathbf{H}_1 = -\mathbf{H}_2$ the identification (26.14) must be used here). By Propositions 26.2 and 26.3 we have

$$\nabla_\lambda \tilde{\mathbf{R}}_{d_2}(T^{d_1}_{d_2-d_1}(\mathbf{F}_1) + \mathbf{F}_2) = \nabla_\lambda \tilde{\mathbf{R}}_{d_\mathbf{F}}\mathbf{F}. \tag{26.33}$$

Since $\mathbf{B}(\mathbf{H}_1 + \mathbf{H}_2) = \nabla_\lambda \tilde{\mathbf{R}}_{d_\mathbf{F}}\mathbf{F}$ by (26.29), we conclude that

$$\mathbf{B}(\mathbf{H}_1 + \mathbf{H}_2) = \mathbf{B}(\mathbf{H}_1) + \mathbf{B}(\mathbf{H}_2). \tag{26.34}$$

Since $\mathbf{H}_1, \mathbf{H}_2 \in \mathrm{pwC}^1_c(\mathbb{P}, \mathcal{C}')$ were arbitrary, $\mathbf{B}$ is additive and thus linear.

To see that $\mathbf{B}$ is continuous let $\mathbf{H} \in \mathrm{pwC}^1_\mathrm{c}(\mathbb{P}, \mathcal{C}')^\times$ be given. Put $\mathbf{F} := \overleftarrow{U}(\mathbf{H})$ and $d := d_\mathbf{F}$. By (26.20) and (26.18) we have

$$\frac{\nu_\mathcal{R}(\mathbf{B}(\mathbf{H}))}{\nu_\infty(\mathbf{H})} = \frac{\nu_\mathcal{R}(\nabla_\lambda \tilde{\mathbf{R}}_d\mathbf{F})}{\nu_d(\mathbf{F})} \leq ||\nabla_\lambda \tilde{\mathbf{R}}_d||_{\nu_d, \nu_\mathcal{R}} \leq R < \infty. \tag{26.35}$$

Taking the supremum over all such $\mathbf{H} \in \mathrm{pwC}^1_\mathrm{c}(\mathbb{P}, \mathcal{C}')^\times$ shows that $\mathbf{B}$ is bounded. A result from functional analysis then says that $\mathbf{B}$ is continuous.

Using the norm (24.4), it can be shown that $\mathrm{pwC}^1_c(\mathbb{P},\mathcal{C}')$ is dense in $\mathrm{pwC}^1(\mathbb{P},\mathcal{C}')$. Thus, there is exactly one continuous element $\nabla_\lambda\tilde{\mathbf{R}} \in \mathrm{Lin}(\mathrm{pwC}^1_c(\mathbb{P},\mathcal{C}'),\mathcal{R})$ such that $\nabla_\lambda\tilde{\mathbf{R}}(\mathbf{H}) = \mathbf{B}(\mathbf{H})$ for all $\mathbf{H} \in \mathrm{pwC}^1_c(\mathbb{P},\mathcal{C}')$ and whose operator norm is bounded by R. ▮

It follows from (26.28) that

$$\nu_{\mathcal{R}}(\nabla_\lambda\tilde{\mathbf{R}}(\mathbf{H})) \le R\nu_\infty(\mathbf{H}) \quad \text{for all} \;\; \mathbf{H} \in \mathrm{pwC}^1(\mathbb{P},\mathcal{C}') \tag{26.36}$$

It follows from (26.27) and Assumption 4 that, given $\lambda \in \Sigma_{\mathrm{rel}}$, for all $\epsilon \in \mathbb{P}^\times$ there is a $\delta \in \mathbb{P}^\times$ such that

$$\frac{\nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\lambda,\mathbf{P}) - \hat{\mathbf{R}}(\lambda) - (\nabla_\lambda\tilde{\mathbf{R}} \circ U_{d_\mathbf{P}})(\mathbf{P} - \hat{\mathbf{C}}(\lambda)_{(d_\mathbf{P})}))}{\nu_\Pi(\mathbf{P} - \hat{\mathbf{C}}(\lambda)_{(d_\mathbf{P})})} < \epsilon \tag{26.37}$$
$$\text{for all} \;\; \mathbf{P} \in \Pi_\mathbf{C} \;\; \text{with} \;\; 0 < \nu_\Pi(\mathbf{P} - \hat{\mathbf{C}}(\lambda)_{(d_\mathbf{P})}) < \delta.$$

The following three lemmas are needed for Theorem 26.8 below.

Lemma 26.5 *Let $\mathbf{C} \in \mathcal{C}$, $d \in \mathbb{P}^\times$ and $\mathbf{P}_o \in \Pi^d_\mathbf{C}$ be given. Put*

$$N := \sup_{t\in[0,d]} \nu_{\mathcal{C}'}(\mathbf{P}_o(t) - \mathbf{C}). \tag{26.38}$$

We have

$$\nu_\infty\left(U_{d_\mathbf{P}+d}(\mathbf{P}_o * \mathbf{P} - \mathbf{C}_{(d_\mathbf{P}+d)}) - U_{d_\mathbf{P}}(\mathbf{P} - \mathbf{C}_{(d)})\right) \le Nh(d_\mathbf{P}) \tag{26.39}$$

for all $\mathbf{P} \in \Pi_{\mathbf{P}_o^f}$ with $\mathbf{P}^f = \mathbf{C}$.

Proof: Let $\mathbf{P} \in \Pi_{\mathbf{P}_o^f}$ with $\mathbf{P}^f = \mathbf{C}$ be given. Using (26.9) we have

$$\left[U_{d_\mathbf{P}+d}(\mathbf{P}_o * \mathbf{P} - \mathbf{C}_{(d_\mathbf{P}+d)}) - U_{d_\mathbf{P}}(\mathbf{P} - \mathbf{C}_{(d)})\right](t)$$
$$:= \begin{cases} \mathbf{0} & \text{if} \quad t \in [0, d_\mathbf{P}] \\ \mathbf{P}_o(d_\mathbf{P} + d - t) - \mathbf{C} & \text{if} \quad t \in [d_\mathbf{P}, d_\mathbf{P} + d] \\ \mathbf{0} & \text{if} \quad t \in [d_\mathbf{P} + d, \infty[\end{cases}$$

and thus by (26.17), (26.38) and (IF3) we have

$$\begin{aligned} \nu_\infty&\left(U_{d_\mathbf{P}+d}(\mathbf{P}_o * \mathbf{P} - \mathbf{C}_{(d_\mathbf{P}+d)}) - U_{d_\mathbf{P}}(\mathbf{P} - \mathbf{C}_{(d)})\right) \\ &= \sup\nolimits_{t\in[d_\mathbf{P},d_\mathbf{P}+d]} h(t)\nu_{\mathcal{C}'}(\mathbf{P}_o(d_\mathbf{P} + d - t) - \mathbf{C}) \\ &= \sup\nolimits_{s\in[0,d]} h(s + d_\mathbf{P})\nu_{\mathcal{C}'}(\mathbf{P}_o(d - s) - \mathbf{C}) \\ &\le N \sup\nolimits_{s\in[0,d]} h(s + d_\mathbf{P}) \\ &\le Nh(d_\mathbf{P}). \end{aligned}$$

▮

Recall (16.20). Let $\mathbf{P} \in \Pi^\times$ be given. Put

$$\mathrm{Spd}(\mathbf{P}) := \sup_{t\in[0,d]} \nu_{\mathcal{C}'}(\mathbf{P}^\bullet(t)) \tag{26.40}$$

and call Spd($\mathbf{P}$) the **speed** of $\mathbf{P}$. Furthermore, if $\mathbf{P}$ is of class C^2 put

$$\mathrm{Acc}(\mathbf{P}) := \sup_{t\in[0,d]} \nu_{\mathcal{C}'}(\mathbf{P}^{\bullet\bullet}(t)) \tag{26.41}$$

and call Acc($\mathbf{P}$) the **acceleration** of $\mathbf{P}$.

Lemma 26.6 *Let $\mathbf{C} \in \mathcal{C}$ and $\mathbf{P} \in \Pi^\times$ be given such that $\mathbf{P}^f = \mathbf{C}$. Assume that $\mathbf{P}$ is of class C^2. Put $\mathbf{C}' := \mathbf{P}^{\bullet f}$. Recall (24.3). Then*

$$\nu_\infty(U_{d_\mathbf{P}}(\mathbf{P} - \mathbf{P}_{(d_\mathbf{P})}) + \mathbf{C}'\iota|_\mathbb{P}) \leq \max\{\mathrm{Spd}(\mathbf{P})\bar{h}(d_\mathbf{P}), \mathrm{Acc}(\mathbf{P})M\}. \tag{26.42}$$

Proof: A calculation using (24.4) and (23.7) shows that

$$\nu_\infty(U_{d_\mathbf{P}}(\mathbf{P}-\mathbf{C}_{(d_\mathbf{P})})+\mathbf{C}'\iota|_\mathbb{P}) = \max\{ \sup_{t\in[0,d_\mathbf{P}]} h(t)\nu_{\mathcal{C}'}(\mathbf{P}(d_\mathbf{P}-t)-\mathbf{C}+\mathbf{C}'t)\, , \sup_{t\in[d_\mathbf{P},\infty[} th(t)\nu_{\mathcal{C}'}(\mathbf{C}')\ \} \tag{26.43}$$

By (26.40) and (26.2) we have

$$\sup_{t\in[d_\mathbf{P},\infty[} th(t)\nu_{\mathcal{C}'}(\mathbf{C}') \leq \bar{h}(d_\mathbf{P})\mathrm{Spd}(\mathbf{P}). \tag{26.44}$$

Let $t \in [0, d_\mathbf{P}]$ be given. We have

$$\begin{aligned} \nu_{\mathcal{C}'}(\mathbf{P}(d_\mathbf{P} - t) - \mathbf{C} + \mathbf{C}'t) &= \nu_{\mathcal{C}'}(\textstyle\int_{d_\mathbf{P}-t}^{d_\mathbf{P}}(s - t - d_\mathbf{P})\mathbf{P}^{\bullet\bullet}(s)ds) \\ &\leq t^2\mathrm{Acc}(\mathbf{P}). \end{aligned} \tag{26.45}$$

Since $t \in [0, d_\mathbf{P}]$ was arbitrary it follows from (24.3) that

$$\sup_{t\in[0,d_\mathbf{P}]} h(t)\nu_{\mathcal{C}'}(\mathbf{P}(d_\mathbf{P} - t) - \mathbf{C} + \mathbf{C}'t) \leq \mathrm{Acc}(\mathbf{P}) \sup_{t\in[0,d_\mathbf{P}]} t^2h(t) \leq \mathrm{Acc}(\mathbf{P})M. \tag{26.46}$$

The result follows from (26.43), (26.44) and (26.46). ∎

Lemma 26.7 *Let $\mathbf{C} \in \mathcal{C}$, $d \in \mathbb{P}^\times$ and $\mathbf{P}_o \in \Pi^d_\mathbf{C}$ be given. Define N as in (26.38) and M as in (24.3). Then*

$$\nu_\Pi(\mathbf{P}_o * \mathbf{P} - \mathbf{C}_{(d+d_\mathbf{P})}) \leq \max\{\mathrm{Spd}(\mathbf{P})M, h(d_\mathbf{P})N\}, \tag{26.47}$$

for all $\mathbf{P} \in \Pi_{\mathbf{P}_o^f}$ with $\mathbf{P}^f = \mathbf{C}$.

Proof: Let $\mathbf{P} \in \Pi_{\mathbf{P}_o^f}$ with $\mathbf{P}^f = \mathbf{C}$ be given. A calculation using (26.17) and (26.9) shows that

$$\begin{aligned} \nu_\Pi(\mathbf{P}_o * \mathbf{P} - \mathbf{C}_{(d+d_\mathbf{P})}) \leq \max\{&\sup\nolimits_{t\in[0,d_\mathbf{P}]} h(t)\nu_{\mathcal{C}'}(\mathbf{P}(d_\mathbf{P} - t) - \mathbf{C}), \\ &\sup\nolimits_{t\in[d_\mathbf{P},d_\mathbf{P}+d]} h(t)\nu_\mathcal{C}(\mathbf{P}_o(d_\mathbf{P} + d - t) - \mathbf{C})\}. \end{aligned} \tag{26.48}$$

Using (26.40) and (24.3) we have

$$\begin{aligned} \sup\nolimits_{t\in[0,d_\mathbf{P}]} h(t)\nu_{\mathcal{C}'}(\mathbf{P}(d_\mathbf{P} - t) - \mathbf{C}) &= \sup\nolimits_{t\in[0,d_\mathbf{P}]} h(t)\nu_{\mathcal{C}'}(\textstyle\int_{d_\mathbf{P}-t}^{d_\mathbf{P}} \mathbf{P}^\bullet(s)ds) \\ &\leq \sup\nolimits_{t\in[0,d_\mathbf{P}]} th(t)\mathrm{Spd}(\mathbf{P}) \\ &\leq \mathrm{Spd}(\mathbf{P})M. \end{aligned} \tag{26.49}$$

Since h is antitone, and using (26.38) we have

$$\sup_{t\in[d_{\mathbf{P}},d_{\mathbf{P}}+d]} h(t)\nu_{\mathcal{C}}(\mathbf{P}_o(d_{\mathbf{P}}+d-t)-\mathbf{C}) \;\leq\; h(d_{\mathbf{P}})N. \tag{26.50}$$

The result follows from (26.48), (26.49) and (26.50). ∎

Since the element is semi-elastic (see Theorem 24.4) every condition is associated with exactly one relaxed state, i.e., $\hat{\mathbf{C}}|_{\Sigma_{\rm rel}} : \Sigma_{\rm rel} \longrightarrow \mathcal{C}$ is invertible (see Definition 20.1). Let

$$\bar{\lambda} : \mathcal{C} \longrightarrow \Sigma_{\rm rel}$$

denote the inverse of $\hat{\mathbf{C}}|_{\Sigma_{\rm rel}} : \Sigma_{\rm rel} \longrightarrow \mathcal{C}$.

Theorem 26.8 *Under Assumption 4, there is a mapping $\mathbf{L} : \mathcal{C} \longrightarrow \mathrm{Lin}(\mathcal{C}', \mathcal{R})$ with the following property: Let $\mathbf{C} \in \mathcal{C}$ be given and put $\lambda := \bar{\lambda}(\mathbf{C}) \in \Sigma_{\rm rel}$. Also let $\sigma \in \Sigma$, strictly accessible from λ, be given. For all $\epsilon \in]0,1]$ there is a mapping $T : \mathbb{P}^\times \longrightarrow \mathbb{P}$ and a $\delta \in \mathbb{P}^\times$ such that for all $\mathbf{P} \in \Pi_{\hat{\mathbf{C}}(\sigma)}$ of class C^2 satisfying*

$$\mathbf{P}^f = \mathbf{C}, \quad \mathrm{Acc}(\mathbf{P}) < \mathrm{Spd}(\mathbf{P})\epsilon < \delta\epsilon \quad \text{and} \quad d_{\mathbf{P}} \geq T(\mathrm{Spd}(\mathbf{P})), \tag{26.51}$$

we have

$$\nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\sigma,\mathbf{P}) - \hat{\mathbf{R}}(\lambda) - \mathbf{L}(\mathbf{C})\mathbf{P}^{\bullet f}) \leq \mathrm{Spd}(\mathbf{P})\epsilon. \tag{26.52}$$

Proof: Define $\mathbf{L} : \mathcal{C} \longrightarrow \mathrm{Lin}(\mathcal{C}', \mathcal{R})$ by

$$\mathbf{L}(\mathbf{C})\mathbf{C}' := -\nabla_{\bar{\lambda}(\mathbf{C})}\tilde{\mathbf{R}}(\mathbf{C}'\iota|_{\mathbb{P}}) \quad \text{for all} \;\; \mathbf{C} \in \mathcal{C}, \;\; \mathbf{C}' \in \mathcal{C}'. \tag{26.53}$$

Since σ is strictly accessible from λ we may choose a $\mathbf{P}_o \in \Pi_{\mathbf{C}}$ such that $\sigma = \hat{e}(\lambda, \mathbf{P}_o)$. Put $d := d_{\mathbf{P}_o}$ and define N as in (26.38). Let $\epsilon \in]0,1]$ be given. Choose $\delta' \in \mathbb{P}^\times$ such that (26.19) holds with ϵ, and with δ replaced by δ'. Put

$$\delta := \delta'/M \tag{26.54}$$

Choose $T : \mathbb{P}^\times \longrightarrow \mathbb{P}$ so that Proposition 26.1 holds.

Let $\mathbf{P} \in \Pi_{\hat{\mathbf{C}}(\sigma)}$ be given such that (26.51) holds. Put $\mathbf{C}' := \mathbf{P}^{\bullet f}$. It follows from Proposition 26.1, (26.4), with $s := \mathrm{Spd}(\mathbf{P})$, that

$$h(d_{\mathbf{P}}) \leq \mathrm{Spd}(\mathbf{P})\epsilon/N \quad \text{and} \quad \bar{h}(d_{\mathbf{P}}) \leq \epsilon. \tag{26.55}$$

By Lemma 26.7, $(26.55)_1$, the fact that $\epsilon \leq 1$, (24.3), $(26.51)_2$ and (26.54) we have

$$\begin{aligned} \nu_{\Pi}(\mathbf{P}_o * \mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}}+d)}) &\leq \max\{\mathrm{Spd}(\mathbf{P})M, h(d_{\mathbf{P}})N\} \\ &\leq M\mathrm{Spd}(\mathbf{P}) \\ &\leq M\delta \\ &\leq \delta' \end{aligned} \tag{26.56}$$

Thus (26.37) holds with $\mathbf{P}$ replaced by $\mathbf{P}_o * \mathbf{P}$.

Since $\sigma = \hat{e}(\lambda, \mathbf{P}_o)$, by (16.16), (26.53) and the triangle inequality we have

$$\nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\sigma,\mathbf{P}) - \hat{\mathbf{R}}(\lambda) - \mathbf{L}(\mathbf{C})\mathbf{C}') = \nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\lambda,\mathbf{P}_o * \mathbf{P}) - \hat{\mathbf{R}}(\lambda) + \nabla_\lambda\tilde{\mathbf{R}}(\mathbf{C}'\iota|_{\mathbb{P}})) \tag{26.57}$$
$$\leq \nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\lambda,\mathbf{P}_o * \mathbf{P}) - \hat{\mathbf{R}}(\lambda) - (\nabla_\lambda\tilde{\mathbf{R}} \circ U_{d_{\mathbf{P}}+d})(\mathbf{P}_o * \mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}}+d)})) \tag{26.58}$$
$$+\nu_{\mathcal{R}}((\nabla_\lambda\tilde{\mathbf{R}} \circ U_{d_{\mathbf{P}}+d})(\mathbf{P}_o * \mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}}+d)}) - (\nabla_\lambda\tilde{\mathbf{R}} \circ U_{d_{\mathbf{P}}})(\mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}})})) \tag{26.59}$$
$$+\nu_{\mathcal{R}}((\nabla_\lambda\tilde{\mathbf{R}} \circ U_{d_{\mathbf{P}}})(\mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}})}) + \nabla_\lambda\tilde{\mathbf{R}}(\mathbf{C}'\iota|_{\mathbb{P}})) \tag{26.60}$$

Looking at (26.58), using (26.37) with $\mathbf{P} := \mathbf{P}_o * \mathbf{P}$ and (26.56), we have

$$\begin{aligned}\nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\sigma,\mathbf{P}) - \hat{\mathbf{R}}(\lambda) - (\nabla_\lambda\tilde{\mathbf{R}} \circ U_{d_{\mathbf{P}}+d})(\mathbf{P}_o * \mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}}+d)})) &\leq \nu_\Pi(\mathbf{P}_o * \mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}}+d)})\epsilon \\ &\leq M\mathrm{Spd}(\mathbf{P})\epsilon.\end{aligned} \tag{26.61}$$

Looking at (26.59), using (26.36), Lemma 26.5 and $(26.55)_1$, we have

$$\begin{aligned}&\nu_{\mathcal{R}}((\nabla_\lambda\tilde{\mathbf{R}} \circ U_{d_{\mathbf{P}}+d})(\mathbf{P}_o * \mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}}+d)}) - (\nabla_\lambda\tilde{\mathbf{R}} \circ U_{d_{\mathbf{P}}})(\mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}})})) \\ &\qquad \leq R\nu_\infty\big(U_{d_{\mathbf{P}}+d}(\mathbf{P}_o * \mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}}+d)}) - U_{d_{\mathbf{P}}}(\mathbf{P} - \mathbf{C}_{(d)})\big) \\ &\qquad \leq RNh(d_{\mathbf{P}}) \\ &\qquad \leq R\mathrm{Spd}(\mathbf{P})\epsilon.\end{aligned} \tag{26.62}$$

Looking at (26.60), using (26.36), Lemma 26.6, $(26.55)_2$ and (26.51), we have

$$\begin{aligned}\nu_{\mathcal{R}}((\nabla_\lambda\tilde{\mathbf{R}} \circ U_{d_{\mathbf{P}}})(\mathbf{P} - \mathbf{C}_{(d_{\mathbf{P}})}) + \nabla_\lambda\tilde{\mathbf{R}}(\mathbf{C}'\iota|_{\mathbb{P}})) &\leq R\nu_\infty(U_{d_{\mathbf{P}}}(\mathbf{P} - \mathbf{P}_{(d_{\mathbf{P}})}) + \mathbf{C}'\iota|_{\mathbb{P}}) \\ &\leq R\max\{\mathrm{Spd}(\mathbf{P})\bar{h}(d_{\mathbf{P}}), \mathrm{Acc}(\mathbf{P})M\} \\ &\leq RM\mathrm{Spd}(\mathbf{P})\epsilon\end{aligned} \tag{26.63}$$

Putting (26.61)-(26.63) together with (26.57)-(26.60) we obtain

$$\nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\sigma,\mathbf{P}) - \hat{\mathbf{R}}(\lambda) - \mathbf{L}(\mathbf{C})\mathbf{C}') \leq (M + R + RM)\mathrm{Spd}(\mathbf{P})\epsilon. \tag{26.64}$$

Since $\mathbf{P} \in \Pi_{\hat{\mathbf{C}}(\sigma)}$ satisfying (26.51) was arbitrary and $\epsilon \in \mathbb{P}^\times$ was arbitrary the result holds. ▮

Note that from $(26.51)_2$ it follows that the above theorem does not apply to processes with zero speed. It follows from (26.40) that processes with zero speed are freezes. In the case of a freeze we have the following result.

Proposition 26.9 *Let $\mathbf{C} \in \mathcal{C}$ be given and put $\lambda := \bar{\lambda}(\mathbf{C}) \in \Sigma_{\mathrm{rel}}$. Also let $\sigma \in \Sigma$, strictly accessible from λ, be given. Then*

$$\lim_{t\to\infty} t\nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\sigma,\mathbf{C}_{(t)}) - \hat{\mathbf{R}}(\lambda)) = 0. \tag{26.65}$$

Proof: Since σ is strictly accessible from λ there is a $\mathbf{P}_o \in \Pi_{\mathbf{C}}$ such that $\sigma = \hat{e}(\lambda, \mathbf{P}_o)$. Put $d := d_{\mathbf{P}_o}$ and define N as in (26.38).

Notice that if $N = 0$ then $\mathbf{P}_o$ is a freeze. In this case, since λ is relaxed, by (iv) of Proposition 19.1,

$$\tilde{\mathbf{R}}(\sigma,\mathbf{C}_{(t)}) = \tilde{\mathbf{R}}(\hat{e}(\lambda,\mathbf{C}_{(d)}),\mathbf{C}_{(t)}) = \tilde{\mathbf{R}}(\lambda,\mathbf{C}_{(d+t)}) \quad \text{for all} \quad t \in \mathbb{P}.$$

Thus (26.65) holds.

Assume that $N \neq 0$. Let $\epsilon \in \mathbb{P}^\times$ be given. Choose $\delta \in \mathbb{P}^\times$ such that (26.19) holds. Since $\lim_{t\to\infty} h(t) = \lim_{t\to\infty} \bar{h}(t) = 0$, by (24.2) and (26.3), there is a $\hat{t} \in \mathbb{P}$ such that for all $t \in \mathbb{P} + \hat{t}$ we have

$$h(t) < \frac{\delta}{N} \quad \text{and} \quad \bar{h}(t) < \frac{\epsilon}{(\epsilon + R)N}. \tag{26.66}$$

Let $t \in \mathbb{P} + \hat{t}$ be given. By (26.38), (24.4), $(26.66)_1$ and Lemma 26.7 with $\mathbf{P} := \mathbf{C}_{(t)}$, we have

$$0 < h(d+t)N \leq \nu_\Pi(\mathbf{P}_o * \mathbf{C}_{(t)} - \mathbf{C}_{(d+t)}) \leq h(t)N < \delta$$

and so (26.19) can be applied with $\mathbf{P}$ replaced by $\mathbf{P}_o * \mathbf{C}_{(t)}$. By the triangle inequality, (26.19), (26.20), Lemma 26.7, (26.2) and $(26.66)_2$ we have

$$\begin{aligned} t\nu_\mathcal{R}(\tilde{\mathbf{R}}(\lambda, \mathbf{P}_o * \mathbf{C}_{(t)}) - \hat{\mathbf{R}}(\lambda)) &\leq t\nu_\mathcal{R}(\tilde{\mathbf{R}}(\lambda, \mathbf{P}_o * \mathbf{C}_{(t)}) - \hat{\mathbf{R}}(\lambda) - \nabla_\lambda \tilde{\mathbf{R}}_{d+t}(\mathbf{P}_o * \mathbf{P} - \mathbf{C}_{(d+t)})) \\ &\quad + t\nu_\mathcal{R}(\nabla_\lambda \tilde{\mathbf{R}}_{d+t}(\mathbf{P}_o * \mathbf{P} - \mathbf{C}_{(d+t)})) \\ &\leq t(\epsilon + R)\nu_\Pi(\mathbf{P}_o * \mathbf{P} - \mathbf{C}_{(d+t)}) \\ &\leq t(\epsilon + R)h(t)N \\ &\leq \bar{h}(t)(1 + R)\epsilon \\ &< \epsilon. \end{aligned}$$

Since $\epsilon \in \mathbb{P}^\times$ was arbitrary, the result holds. ▮

Remark 26.10 Using (19.1) and the basic topology on Σ, we have

$$\lim_{t\to\infty} \nu_\mathcal{R}(\tilde{\mathbf{R}}(\sigma, \mathbf{C}_{(t)}) - \tilde{\mathbf{R}}(\hat{\lambda}(\sigma))) = 0. \tag{26.67}$$

The difference between (26.67) and (26.65) is the factor t. It states that the convergence in (26.67) to 0 is faster then the convergence of $\frac{1}{t}$ to 0. This stronger convergence depends on Assumption 4. ▮

Let $\alpha \in]0, 1]$ and $\mathbf{P} \in \Pi^\times$ be given. The **retardation of P by** α is the mapping $\Gamma_\alpha \mathbf{P} : [0, d_\mathbf{P}/\alpha] \longrightarrow \mathcal{C}$ defined by

$$(\Gamma_\alpha \mathbf{P})(t) := \mathbf{P}(\alpha t) \quad \text{for all} \;\; t \in [0, d_\mathbf{P}/\alpha]. \tag{26.68}$$

Note that $\Gamma_\alpha \mathbf{P} \in \Pi^{d_\mathbf{P}/\alpha}_{\mathbf{P}^i}$. It is easy to see that

$$(\Gamma_\alpha \mathbf{P})^{\bullet f} = \alpha \mathbf{P}^{\bullet f}, \quad \mathrm{Spd}(\Gamma_\alpha \mathbf{P}) = \alpha \mathrm{Spd}(\mathbf{P}) \quad \text{and} \quad \mathrm{Acc}(\Gamma_\alpha \mathbf{P}) = \alpha^2 \mathrm{Acc}(\mathbf{P}). \tag{26.69}$$

Theorem 26.11 *Let* $\mathbf{L} : \mathcal{C} \longrightarrow \mathrm{Lin}(\mathcal{C}', \mathcal{R})$ *be the mapping guaranteed by Theorem 26.8. Let* $\mathbf{P} \in \Pi^\times$ *of class* C^2 *and* $\sigma \in \Sigma_{\mathbf{P}^i}$ *strictly accessible from* $\lambda := \bar{\lambda}(\mathbf{P}^f)$ *be given. Then*

$$\lim_{\alpha\to 0^+} \frac{1}{\alpha} \nu_\mathcal{R}(\tilde{\mathbf{R}}(\sigma, \Gamma_\alpha \mathbf{P}) - \hat{\mathbf{R}}(\lambda) - \alpha \mathbf{L}(\mathbf{P}^f)\mathbf{P}^{\bullet f}) = 0. \tag{26.70}$$

Proof: Since σ is strictly accessible from λ there is a $\mathbf{P}_o \in \Pi_{\mathbf{C}}$ such that $\sigma = \hat{e}(\lambda, \mathbf{P}_o)$. Put $d := d_{\mathbf{P}_o}$ and define N as in (26.38). There are two case distinctions.

Case 1: $\text{Spd}(\mathbf{P}) \neq 0$. Let $\epsilon \in]0,1]$ be given. Choose $\delta \in \mathbb{P}^\times$ and $T : \mathbb{P}^\times \longrightarrow \mathbb{P}$ guaranteed by Theorem 26.8. Also, choose η so that (26.4) holds. Put

$$\zeta := \begin{cases} \min\{\text{Spd}(\mathbf{P})\epsilon/\text{Acc}(\mathbf{P}), \eta, \delta/\text{Spd}(\mathbf{P})\} & \text{if} \quad \text{Acc}(\mathbf{P}) \neq 0 \\ \min\{\eta, \delta/\text{Spd}(\mathbf{P})\} & \text{if} \quad \text{Acc}(\mathbf{P}) = 0 \end{cases} \tag{26.71}$$

Then for all $\alpha \in]0, \zeta[$ we have

$$\alpha^2 \text{Acc}(\mathbf{P}) < \alpha \text{Spd}(\mathbf{P})\epsilon < \delta\epsilon \quad \text{and} \quad \frac{d_{\mathbf{P}}}{\alpha} \geq T(\alpha \text{Spd}(\mathbf{P}))$$

and thus, by (26.69), $\Gamma_\alpha \mathbf{P}$ satisfies (26.51). Therefore, it follows from Theorem 26.8 that

$$\nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\sigma, \Gamma_\alpha \mathbf{P}) - \hat{\mathbf{R}}(\lambda) - \mathbf{L}(\mathbf{P}^f)\alpha \mathbf{P}^{\bullet f}) \leq \alpha \text{Spd}(\mathbf{P})\epsilon.$$

Since $\epsilon \in]0,1]$ was arbitrary, (26.70) holds.

Case 2: $\text{Spd}(\mathbf{P}) = 0$. In this case $\mathbf{P}$ is a freeze, i.e., $\mathbf{P} = \mathbf{C}_{(d_{\mathbf{P}})}$, and hence $\mathbf{P}^\bullet = \mathbf{0}$. Thus, Proposition 26.9 can be used to obtain

$$\begin{aligned} \lim_{\alpha \to 0^+} \tfrac{1}{\alpha} \nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\sigma, \Gamma_\alpha \mathbf{P}) - \hat{\mathbf{R}}(\lambda) - \alpha \mathbf{L}(\mathbf{P}^f)\mathbf{P}^{\bullet f}) &= \lim_{\alpha \to 0^+} \tfrac{1}{\alpha} \nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\sigma, \mathbf{C}_{(d_{\mathbf{P}}/\alpha)}) - \hat{\mathbf{R}}(\lambda)) \\ &= d_{\mathbf{P}} \lim_{\alpha \to 0^+} \tfrac{d_{\mathbf{P}}}{\alpha} \nu_{\mathcal{R}}(\tilde{\mathbf{R}}(\sigma, \mathbf{C}_{(d_{\mathbf{P}}/\alpha)}) - \hat{\mathbf{R}}(\lambda)) \\ &= 0. \end{aligned}$$

∎

Remark 26.12 Given a condition process $\mathbf{P}$ of class C^2, both Theorem 26.8 and Theorem 26.11 involve using states that are strictly accessible from the relaxed state $\lambda := \bar{\lambda}(\mathbf{P}^f)$. It follows from (19.11) and Proposition 20.6 that the set of all states strictly accessible from λ is dense in Σ. Thus, this is not a very restrictive condition. ∎

Chapter IV

Thermoelasto-viscous Materials

27 Introduction

Thermoelastic materials are those in which the constitutive laws governing the stress, free energy, entropy and heat flux depend on the present values of the transplacement gradient, temperature and temperature gradient. One can find the Coleman–Noll procedure carried out for such materials in [LTE]. Coleman and Mizel in [ECES] studied the case when one assumes that the constitutive laws also depend on the velocity gradient. There has been other work in which constitutive laws also allow for the dependence on the rate of change of temperature, see the work of Bogy and Naghdi in [HCWP] for example. In this chapter we consider constitutive laws that depend not only on the present values of the transplacement gradient, temperature, temperature gradient, velocity gradient and the rate of change of temperature but also on the rate of change of the temperature gradient. Such dependence can be expected to be important in situations where the temperature and temperature gradient change rapidly, as in quenching. I refer to materials with such a dependence as *thermoelasto-viscous materials*.[1]

What really distinguishes the work here from other papers on the Coleman–Noll procedure is that it is modeled on the concepts introduced in *A New Mathematical Theory of Simple Materials* by Noll [NTSM]. In this new theory of simple materials constitutive laws can be specified without the use of an external frame of reference. More specifically, we will use the framework introduced in Chapters II and III.

Section 28 defines a thermoelastic material and carries out the Coleman–Noll procedure for this material. Here one obtains the same stress and entropy relations that can be found in [LTE]. This section can be viewed as a continuation of the the paper *A Frame-Free Formulation of Elasticity* by Noll [FFFE] which discusses how elasticity can be put into the framework described in [NTSM].

Section 29 generalizes the results of Section 28 by carrying out the Coleman–Noll procedure for thermoelasto-viscous materials. The concept of a thermoelasto-viscous material

[1] There is a huge literature devoted to studying the restriction of constitutive laws using the Coleman–Noll procedure. The references cited in these first two paragraphs, as well as those mentioned in the introduction to Chapter III, should by no means be considered exhaustive.

came out of applying the *principle of equipresence* to the set of constitutive laws proposed in [ECES]. The principle of equipresence, introduced in Section 293, part η, of *The Classical Field Theories of Mechanics* by Truesdell and Toupin [CFT], states that "*A variable present as an independent variable in one constitutive equation should be present in all.*" In this chapter the principle of equipresence is taken to be even more inclusive since I reason that if one constitutive law depends on the rate of change of one independent variable it should also depend on the rates of change of all independent variables. On a more practical note, it is useful to develop such a general theory and then, depending on which rates are important, consider a special case of this theory.

Section 30 investigates what form the constitutive laws of a thermoelasto-viscous material take when the material is assumed to be a fluid.

When dealing with the behavior of a continuous body in an environment, a frame of reference is needed because the environment is described in terms of such a frame. Section 31 shows how the frame-free constitutive laws introduced in Section 29 can be represented relative to a frame of reference. Section 32 gives the governing equations obtained by substituting the constitutive laws into the balance laws. For example, we obtain a generalization of the Navier-Stokes equation that takes into account thermal phenomena.

In Section 33 a result that is needed in Section 29 is given whose proof can be found in [TEVM].

Let a body element $\mathcal{T}$ be given. The following result, which will be needed later, is easy to prove and follows from the fact that $\mathcal{C}$ is an open subset of $\mathrm{Pos}^+(\mathcal{T},\mathcal{T}^*)\times\mathbb{P}^\times\times\mathcal{T}^*$.

Proposition 27.1 *For all $\mathbf{C}\in\mathcal{C}$ and $\mathbf{C}',\mathbf{C}''\in\mathcal{C}'$ there is a condition process $\mathbf{P}\in\Pi^\times$ such that $(\mathbf{P}^f,\mathbf{P}^{\bullet f},\mathbf{P}^{\bullet\bullet f})=(\mathbf{C},\mathbf{C}',\mathbf{C}'')$.*

28 Thermoelastic Materials

Before we tackle the thermoelasto-viscous case we will analyze the consequences of the second law for thermoelastic elements.

Let an elastic thermomechanical element $\mathcal{T}$ (see Definition 20.1) be given. We will use $\hat{\mathbf{C}}$, which is a bijection, to identify the state space with the set of conditions $\mathcal{C}$. Thus the domain of the response mapping can be taken to be $\mathcal{C}$:

$$\hat{\mathbf{R}}:\mathcal{C}\longrightarrow\mathcal{R}. \tag{28.1}$$

For an elastic element the evolution of a state σ subjected to a process $\mathbf{P}$ only depends on the final value $\mathbf{P}^f$ of the process, i.e.,

$$\hat{e}(\sigma,\mathbf{P})=\mathbf{P}^f.$$

For this reason the evolution mapping, for all practical purposes, can be ignored in this case and only the response mapping plays a prominent role. We will use the mapping $\hat{\Upsilon}:\mathcal{C}\longrightarrow\mathbb{R}$, which is analogous to (21.4), defined by

$$\hat{\Upsilon}(\mathbf{G},\theta,\boldsymbol{\gamma}):=\frac{1}{\theta}\boldsymbol{\gamma}\hat{\mathbf{h}}(\mathbf{G},\theta,\boldsymbol{\gamma})\quad\text{for all}\quad(\mathbf{G},\theta,\boldsymbol{\gamma})\in\mathcal{C}. \tag{28.2}$$

Definition 28.1 *A* **thermoelastic element** *is an unconstrained thermomechanical element $\mathcal{T}$ with the following properties:*

(TE1) *The element is elastic.*

(TE2) *The response mapping $\hat{\mathbf{R}} : \mathcal{C} \longrightarrow \mathcal{R}$ is of class C^1.*

(TE3) *There is a $\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*)$ and a $\theta \in \mathbb{P}^\times$ such that $(\mathbf{G}, \theta, \mathbf{0}) \in \mathcal{C}$.*

A body $\mathcal{B}$ is called a **thermoelastic material** *if for every $X \in \mathcal{B}$, $\mathcal{T}_X$ has the structure of a thermoelastic element.*

It is not hard to show that a thermoelastic element satisfies Axioms 1–6 from Chapter II.

Recall the definitions and notations of a response process and thermomechanical process given in Section 21. Let $(\sigma, \mathbf{P}) \in (\Sigma \times \Pi^\times)_{\mathrm{fit}}$ be given. We will drop the subscript on $\bar{\psi}_{(\sigma,\mathbf{P})}$ to avoid clutter. Using the chain rule on the free energy component of the response process one obtains

$$\bar{\psi}^{\bullet f} = \mathrm{tr}((\nabla_{(1)}\hat{\psi}(\mathbf{P}^f)\bar{\mathbf{G}}^{\bullet}) + \nabla_{(2)}\hat{\psi}(\mathbf{P}^f)\bar{\theta}^{\bullet f} + \nabla_{(3)}\hat{\psi}(\mathbf{P}^f)\bar{\boldsymbol{\gamma}}^{\bullet f}. \tag{28.3}$$

Substituting (28.3) into (21.6) yields

$$\begin{aligned} \mathrm{tr}\left[\left(\tfrac{1}{2}\hat{\mathbf{S}}(\mathbf{P}^f) - \hat{\rho}(\bar{\mathbf{G}}^f)\nabla_{(1)}\hat{\psi}(\mathbf{P}^f)\right)\bar{\mathbf{G}}^{\bullet f}\right] - \hat{\rho}(\bar{\mathbf{G}}^f)\left(\nabla_{(2)}\hat{\psi}(\mathbf{P}^f) + \hat{\eta}(\mathbf{P}^f)\right)\bar{\theta}^{\bullet f} \\ -\hat{\rho}(\bar{\mathbf{G}}^f)(\nabla_{(3)}\hat{\psi}(\mathbf{P}^f))\bar{\boldsymbol{\gamma}}^{\bullet f} - \hat{\Upsilon}(\mathbf{P}^f) \geq 0. \end{aligned} \tag{28.4}$$

Since the condition process $\mathbf{P}$ was arbitrary, (28.4) must hold for all condition processes. Thus, by Proposition 27.1 and (28.2) we obtain

$$\begin{aligned} \mathrm{tr}\left[\left(\tfrac{1}{2}\hat{\mathbf{S}}(\mathbf{C}) - \hat{\rho}(\mathbf{G})\nabla_{(1)}\hat{\psi}(\mathbf{C})\right)\mathbf{G}'\right] - \hat{\rho}(\mathbf{G})\left(\nabla_{(2)}\hat{\psi}(\mathbf{C}) + \hat{\eta}(\mathbf{C})\right)\theta' \\ -\hat{\rho}(\mathbf{G})(\nabla_{(3)}\hat{\psi}(\mathbf{C}))\boldsymbol{\gamma}' - \hat{\Upsilon}(\mathbf{C}) \geq 0 \end{aligned} \tag{28.5}$$

for all $(\mathbf{G}, \theta, \boldsymbol{\gamma}) = \mathbf{C} \in \mathcal{C}$ and $(\mathbf{G}', \theta', \boldsymbol{\gamma}') \in \mathcal{C}'$.

To simplify matters we will reduce this inequality to an equivalent polynomial inequality. Define several mappings from $\mathcal{C} \times \mathcal{C}'$ to $\mathbb{R}$ by

$$\begin{aligned} \hat{\mathsf{A}}(\mathbf{C}, \mathbf{C}') &:= \mathrm{tr}\left[\left(\tfrac{1}{2}\hat{\mathbf{S}}(\mathbf{C}) - \hat{\rho}(\mathbf{G})\nabla_{(1)}\hat{\psi}(\mathbf{C})\right)\mathbf{G}'\right], \\ \hat{\mathsf{B}}(\mathbf{C}, \mathbf{C}') &:= -\hat{\rho}(\mathbf{G})(\nabla_{(2)}\hat{\psi}(\mathbf{C}) + \hat{\eta}(\mathbf{C}))\theta', \\ \hat{\mathsf{C}}(\mathbf{C}, \mathbf{C}') &:= \hat{\rho}(\mathbf{G})(\nabla_{(3)}\hat{\psi}(\mathbf{C}))\boldsymbol{\gamma}', \end{aligned} \tag{28.6}$$

for all $(\mathbf{G}, \theta, \boldsymbol{\gamma}) = \mathbf{C} \in \mathcal{C}$ and $(\mathbf{G}', \theta', \boldsymbol{\gamma}') = \mathbf{C}' \in \mathcal{C}'$. It is clear that given $x, y, z \in \mathbb{R}$, (28.5) remains valid when $\mathbf{G}'$, θ' and $\boldsymbol{\gamma}'$ are replaced by $x\mathbf{G}'$, $y\theta'$ and $z\boldsymbol{\gamma}'$, respectively. Thus, one can see that (28.5) is equivalent to

$$\hat{\mathsf{A}}(\mathbf{C}, \mathbf{C}')x + \hat{\mathsf{B}}(\mathbf{C}, \mathbf{C}')y + \hat{\mathsf{C}}(\mathbf{C}, \mathbf{C}')z - \hat{\Upsilon}(\mathbf{C}) \geq 0 \quad \text{for all} \quad x, y, z \in \mathbb{R} \quad \text{and} \quad (\mathbf{C}, \mathbf{C}') \in \mathcal{C} \times \mathcal{C}'.$$

Using Proposition 33.1 one finds that this is equivalent to

$$\hat{\mathsf{A}}(\mathbf{C}, \mathbf{C}') = \hat{\mathsf{B}}(\mathbf{C}, \mathbf{C}') = \hat{\mathsf{C}}(\mathbf{C}, \mathbf{C}') = 0 \quad \text{and} \quad \hat{\Upsilon}(\mathbf{C}) \leq 0 \quad \text{for all} \quad (\mathbf{C}, \mathbf{C}') \in \mathcal{C} \times \mathcal{C}'. \tag{28.7}$$

Let us take a closer look at the condition $\hat{\Upsilon}(\mathbf{C}) \leq 0$ for all $\mathbf{C} \in \mathcal{C}$. In view of (28.2), it states that

$$\boldsymbol{\gamma}\hat{\mathbf{h}}(\mathbf{G}, \theta, \boldsymbol{\gamma}) \leq 0 \quad \text{for all} \quad (\mathbf{G}, \theta, \boldsymbol{\gamma}) \in \mathcal{C}. \tag{28.8}$$

Let $(\mathbf{G}, \theta, \boldsymbol{\gamma}) = \mathbf{C} \in \mathcal{C}$ be given. (28.8) remains valid if $\boldsymbol{\gamma}$ is replaced by $\alpha\boldsymbol{\gamma}$ with $\alpha \in \mathbb{P}^{\times}$, i.e., $\alpha\boldsymbol{\gamma}\hat{\mathbf{h}}(\mathbf{G}, \theta, \alpha\boldsymbol{\gamma}) \leq 0$ and hence $\boldsymbol{\gamma}\hat{\mathbf{h}}(\mathbf{G}, \theta, \alpha\boldsymbol{\gamma}) \leq 0$ for all $\alpha \in \mathbb{P}^{\times}$. Taking the limit as α goes to zero and using the continuity of $\hat{\mathbf{h}}$ gives $\boldsymbol{\gamma}\hat{\mathbf{h}}(\mathbf{G}, \theta, \mathbf{0}) \leq 0$. By replacing $\boldsymbol{\gamma}$ with $-\boldsymbol{\gamma}$ in the above argument we also have $\boldsymbol{\gamma}\hat{\mathbf{h}}(\mathbf{G}, \theta, \mathbf{0}) \geq 0$[2]. Define

$$\hat{\mathbf{h}}_{\mathrm{p}} := \hat{\mathbf{h}}(\bullet, \bullet, \mathbf{0}) \quad \text{and} \quad \hat{\mathbf{h}}_{\mathrm{T}}(\mathbf{C}) := \hat{\mathbf{h}}(\mathbf{C}) - \hat{\mathbf{h}}_{\mathrm{p}}(\mathbf{G}, \theta).$$

We call $\hat{\mathbf{h}}_{\mathrm{p}}$ the **purely elastic** part of $\hat{\mathbf{h}}$. Then, since $\mathbf{C} \in \mathcal{C}$ was arbitrary, we have shown that (28.8) implies $\hat{\mathbf{h}}_{\mathrm{p}} = \mathbf{0}$ and $\boldsymbol{\gamma}\hat{\mathbf{h}}_{\mathrm{T}}(\mathbf{C}) \leq 0$ for all $(\mathbf{G}, \theta, \boldsymbol{\gamma}) = \mathbf{C} \in \mathcal{C}$. It is clear that the converse holds. The condition $\hat{\mathbf{h}}_{\mathrm{p}} = \mathbf{0}$ says that in the absence of a temperature gradient there is no heat flux.

By plugging the abbreviations (28.6) into $(28.7)_1$ and from the discussion of the last paragraph we have the following theorem.

Theorem 28.2 *For a thermoelastic element the Second Law of Thermodynamics holds if and only if the following conditions are satisfied:*
(T1.1) $\nabla_{(3)}\hat{\psi} = \mathbf{0}$.
(T1.2) $\hat{\mathbf{S}} = 2\hat{\rho}\nabla_{(1)}\hat{\psi}$.
(T1.3) $\hat{\eta} = -\nabla_{(2)}\hat{\psi}$.
(T1.4) $\hat{\mathbf{h}}_{\mathrm{p}} = \mathbf{0}$.
(T1.5) $\boldsymbol{\gamma}\hat{\mathbf{h}}_{\mathrm{T}}(\mathbf{G}, \theta, \boldsymbol{\gamma}) \leq 0$ *for all* $(\mathbf{G}, \theta, \boldsymbol{\gamma}) \in \mathcal{C}$.

Remark 28.3 In [FFFE] condition (T1.2) appears as an axiom, but it is claimed that it can be proven using the second law of thermodynamics. This theorem validates that claim. ∎

29 Thermoelasto-viscous Materials

Definition 29.1 *A* **thermoelasto-viscous element** *is an unconstrained thermomechanical element $\mathcal{T}$ with the following properties:*
(EV1) *There are C^1 mappings $\hat{\mathbf{R}}_{\mathrm{e}} : \mathcal{C} \longrightarrow \mathcal{R}$ and $\hat{\mathbf{R}}_{\mathrm{v}} : \mathcal{C} \longrightarrow \mathrm{Lin}(\mathcal{C}', \mathcal{R})$. Define $\hat{\mathbf{R}} : \mathcal{C} \times \mathcal{C}' \longrightarrow \mathcal{R}$ by*

$$\hat{\mathbf{R}}(\mathbf{C}, \mathbf{C}') = \hat{\mathbf{R}}_{\mathrm{e}}(\mathbf{C}) + \hat{\mathbf{R}}_{\mathrm{v}}(\mathbf{C})\mathbf{C}' \quad \text{for all} \quad (\mathbf{C}, \mathbf{C}') \in \mathcal{C} \times \mathcal{C}'. \tag{29.1}$$

Put $\mathcal{U}_{\mathbf{C}} := \mathrm{Null}(\hat{\mathbf{R}}_{\mathrm{v}}(\mathbf{C}))$ and let

$$\mathbf{N}_{\mathbf{C}} \in \mathrm{Lin}(\mathcal{C}', \mathcal{C}' \backslash \mathcal{U}_{\mathbf{C}})$$

denote the quotient mapping from $\mathcal{C}'$ to the quotient space $\mathcal{C}' \backslash \mathcal{U}_{\mathbf{C}}$.

[2]Without condition (TE3) of the definition of a thermoelastic element this would not make any sense.

(EV2) $\Sigma = \{(\mathbf{C}, \mathbf{N_C C'}) \mid \mathbf{C} \in \mathcal{C}, \mathbf{C}' \in \mathcal{C}'\} = \bigcup_{\mathbf{C}\in\mathcal{C}}(\{\mathbf{C}\} \times \mathcal{C}' \backslash \mathcal{U}_{\mathbf{C}})$.

(EV3) *The condition mapping* $\hat{\mathbf{C}} : \Sigma \longrightarrow \mathcal{C}$ *is given by*

$$\hat{\mathbf{C}}((\mathbf{C}, \mathbf{N_C C'})) = \mathbf{C} \quad \text{for all} \quad (\mathbf{C}, \mathbf{C}') \in \mathcal{C} \times \mathcal{C}'.$$

(EV4) *The response mapping* $\hat{\mathbf{R}}_\Sigma : \Sigma \longrightarrow \mathcal{R}$ *is given by*

$$\hat{\mathbf{R}}_\Sigma((\mathbf{C}, \mathbf{N_C C'})) = \hat{\mathbf{R}}(\mathbf{C}, \mathbf{C}') \quad \text{for all} \quad (\mathbf{C}, \mathbf{C}') \in \mathcal{C} \times \mathcal{C}'. \tag{29.2}$$

(EV5) *The evolution mapping* $\hat{e} : (\Sigma \times \Pi)_{\text{fit}} \longrightarrow \mathcal{R}$ *is given by*

$$\hat{e}((\mathbf{C}, \mathbf{N_C C'}), \mathbf{P}) := \begin{cases} (\mathbf{P}^f, \mathbf{N}_{\mathbf{P}^f}\mathbf{P}^{\bullet f}) & \text{if} \quad ((\mathbf{C}, \mathbf{N_C C'}), \mathbf{P}) \in (\Sigma \times \Pi^\times)_{\text{fit}} \\ (\mathbf{C}, \mathbf{N_C C'}) & \text{if} \quad (\mathbf{C}, \mathbf{N_C C'}) \in \Sigma \quad \text{and} \quad \mathbf{P} = \mathbf{C}_{(0)}. \end{cases} \tag{29.3}$$

(EV6) *There is a* $\mathbf{G} \in \text{Pos}^+(\mathcal{T}, \mathcal{T}^*)$ *and a* $\theta \in \mathbb{P}^\times$ *such that* $(\mathbf{G}, \theta, \mathbf{0}) \in \mathcal{C}$.

A body $\mathcal{B}$ *is called a* **thermoelasto-viscous material** *if for every* $X \in \mathcal{B}$, $\mathcal{T}_X$ *has the structure of a thermoelasto-viscous element.*

A little effort shows that Axioms 1–6 hold for thermoelasto-viscous elements. Let us take a moment to check that Axiom 3 holds. By the definition of Σ, if $\sigma \in \Sigma$ then there is a $(\mathbf{C}, \mathbf{C}') \in \mathcal{C} \times \mathcal{C}'$ such that $\sigma = (\mathbf{C}, \mathbf{N_C C'})$. Thus, we can and will write $(\mathbf{C}, \mathbf{N_C C'}) \in \Sigma$ with the understanding that $(\mathbf{C}, \mathbf{C}') \in \mathcal{C} \times \mathcal{C}'$. Let $(\mathbf{C}, \mathbf{N_C C}'_1), (\mathbf{C}, \mathbf{N_C C}'_2) \in \Sigma$ be given. Then

$$\begin{aligned} \hat{\mathbf{R}}_\Sigma((\mathbf{C}, \mathbf{N_C C}'_1)) = \hat{\mathbf{R}}_\Sigma((\mathbf{C}, \mathbf{N_C C}'_2)) &\iff \hat{\mathbf{R}}_{\text{e}}(\mathbf{C}) + \hat{\mathbf{R}}_{\text{v}}(\mathbf{C})\mathbf{C}'_1 = \hat{\mathbf{R}}_{\text{e}}(\mathbf{C}) + \hat{\mathbf{R}}_{\text{v}}(\mathbf{C})\mathbf{C}'_2 \\ &\iff \hat{\mathbf{R}}_{\text{v}}(\mathbf{C})(\mathbf{C}'_1 - \mathbf{C}'_2) = \mathbf{0} \\ &\iff \mathbf{C}_1 - \mathbf{C}'_2 \in \mathcal{U}_{\mathbf{C}} \\ &\iff \mathbf{N_C C}_1 = \mathbf{N_C C}'_2 \\ &\iff (\mathbf{C}, \mathbf{N_C C}'_1) = (\mathbf{C}, \mathbf{N_C C}'_2). \end{aligned} \tag{29.4}$$

It follows from this, (29.3), (29.2) and (16.8) that we have

$$\tilde{\mathbf{R}}(\sigma, \mathbf{P}) = \hat{\mathbf{R}}(\mathbf{P}^f, \mathbf{P}^{\bullet f}) \quad \text{for all} \quad (\sigma, \mathbf{P}) \in (\Sigma \times \Pi^\times)_{\text{fit}}. \tag{29.5}$$

Using (29.5), (29.3) and (29.2) we see that Axiom 3 holds.

It follows from (29.5) and (29.2) that only the mapping $\hat{\mathbf{R}}$ plays a role, not $\hat{\mathbf{R}}_\Sigma$. For this reason we will also call the mapping $\hat{\mathbf{R}}$ the response mapping. The mappings $\hat{\mathbf{R}}_{\text{e}} : \mathcal{C} \longrightarrow \mathcal{R}$ and $\hat{\mathbf{R}}_{\text{v}} : \mathcal{C} \longrightarrow \text{Lin}(\mathcal{C}', \mathcal{R})$ are called the **elastic part** and the **viscous part** of the response mapping, respectively. One can consider the components of $\hat{\mathbf{R}}$ as was done in (16.9). Since $\mathcal{R}$ is the product of three linear spaces we can consider the components of the elastic part of the response mapping and so $\hat{\mathbf{R}}_{\text{e}}$ can be written as

$$\hat{\mathbf{R}}_{\text{e}} = (\hat{\mathbf{S}}_{\text{e}}, \hat{\eta}_{\text{e}}, \hat{\mathbf{h}}_{\text{e}}, \hat{\psi}_{\text{e}}). \tag{29.6}$$

The components of the viscous part will be denoted in an analogous way. Let $\mathcal{H}$ denote the projection of $\mathcal{C}$ onto $\text{Pos}^+(\mathcal{T}, \mathcal{T}^*) \times \mathbb{P}^\times$. Because of (EV5) we can define the **purely**

elastic part $\hat{\mathbf{R}}_{\mathrm{p}} : \mathcal{H} \longrightarrow \mathcal{R}$ of $\hat{\mathbf{R}}$ by $\hat{\mathbf{R}}_{\mathrm{p}} := \hat{\mathbf{R}}_{\mathrm{e}}(\bullet, \bullet, \mathbf{0})$ and by looking at the components of this mapping $\hat{\mathbf{R}}_{\mathrm{p}}$ can be written as

$$\hat{\mathbf{R}}_{\mathrm{p}} = (\hat{\mathbf{S}}_{\mathrm{p}}, \hat{\eta}_{\mathrm{p}}, \hat{\mathbf{h}}_{\mathrm{p}}, \hat{\psi}_{\mathrm{p}}). \tag{29.7}$$

Since $\mathcal{C}'$ and $\mathcal{R}$ are the product of linear spaces (see (16.5) and (16.7)), every element of $\mathrm{Lin}(\mathcal{C}', \mathcal{R})$ can be identified with a 4×3 matrix whose terms are linear mappings themselves. Hence, we can represent $\hat{\mathbf{R}}_{\mathrm{v}}$ by

$$\hat{\mathbf{R}}_{\mathrm{v}}(\mathbf{C}) \begin{pmatrix} \mathbf{G}' \\ \theta' \\ \boldsymbol{\gamma}' \end{pmatrix} = \begin{bmatrix} \hat{\mathbf{S}}_{\mathrm{vG}}(\mathbf{C}) & \hat{\mathbf{S}}_{\mathrm{v}\theta}(\mathbf{C}) & \hat{\mathbf{S}}_{\mathrm{v}\boldsymbol{\gamma}}(\mathbf{C}) \\ \hat{\eta}_{\mathrm{vG}}(\mathbf{C}) & \hat{\eta}_{\mathrm{v}\theta}(\mathbf{C}) & \hat{\eta}_{\mathrm{v}\boldsymbol{\gamma}}(\mathbf{C}) \\ \hat{\mathbf{h}}_{\mathrm{vG}}(\mathbf{C}) & \hat{\mathbf{h}}_{\mathrm{v}\theta}(\mathbf{C}) & \hat{\mathbf{h}}_{\mathrm{v}\boldsymbol{\gamma}}(\mathbf{C}) \\ \hat{\psi}_{\mathrm{vG}}(\mathbf{C}) & \hat{\psi}_{\mathrm{v}\theta}(\mathbf{C}) & \hat{\psi}_{\mathrm{v}\boldsymbol{\gamma}}(\mathbf{C}) \end{bmatrix} \begin{pmatrix} \mathbf{G}' \\ \theta' \\ \boldsymbol{\gamma}' \end{pmatrix} \tag{29.8}$$

for all $\mathbf{C} \in \mathcal{C}$ and $(\mathbf{G}', \theta', \boldsymbol{\gamma}') \in \mathcal{C}'$. The intrinsic stress component $\hat{\mathbf{S}}$ of the response mapping $\hat{\mathbf{R}}$ is given by, using (29.1), (29.6) and (29.8)

$$\hat{\mathbf{S}}(\mathbf{C}, \mathbf{C}') = \hat{\mathbf{S}}_{\mathrm{e}}(\mathbf{C}) + \hat{\mathbf{S}}_{\mathrm{vG}}(\mathbf{C})\mathbf{G}' + \hat{\mathbf{S}}_{\mathrm{v}\theta}(\mathbf{C})\theta' + \hat{\mathbf{S}}_{\mathrm{v}\gamma}(\mathbf{C})\boldsymbol{\gamma}' \quad \text{for all } \mathbf{C} \in \mathcal{C} \text{ and } (\mathbf{G}', \theta', \boldsymbol{\gamma}') = \mathbf{C}' \in \mathcal{C}' \tag{29.9}$$

where

$$\begin{aligned} &\hat{\mathbf{S}}_{\mathrm{vG}}(\mathbf{C}) \in \mathrm{Lin}(\mathrm{Sym}(\mathcal{T}, \mathcal{T}^*), \mathrm{Sym}(\mathcal{T}^*, \mathcal{T})) \\ &\hat{\mathbf{S}}_{\mathrm{v}\theta}(\mathbf{C}) \in \mathrm{Sym}(\mathcal{T}^*, \mathcal{T}) \\ &\hat{\mathbf{S}}_{\mathrm{v}\boldsymbol{\gamma}}(\mathbf{C}) \in \mathrm{Lin}(\mathcal{T}^*, \mathrm{Sym}(\mathcal{T}^*, \mathcal{T})) \end{aligned} \tag{29.10}$$

for every $\mathbf{C} \in \mathcal{C}$. The other components of $\hat{\mathbf{R}}$ have similar forms.

Historically the term $\hat{\mathbf{S}}_{\mathrm{vG}}(\mathbf{C})$ has had a monopoly on the word "viscosity". Here it seems that the word viscosity could be used to describe any term that depends on the rate of change of condition. Hence, every term in the matrix in (29.8) is a type of viscosity. The term $\hat{\mathbf{S}}_{\mathrm{vG}}(\mathbf{C})$ can be referred to by a more descriptive phrase, namely, the *intrinsic stress viscosity caused by the rate of change of configuration*. Other terms appearing in (29.8) can be given similar names. For example, the term $\hat{\mathbf{h}}_{\mathrm{v}\gamma}(\mathbf{C})$ can be referred to as the *intrinsic heat flux viscosity caused by the rate of change of the temperature gradient*.

Define the mapping $\hat{\Upsilon} : \mathcal{C} \times \mathcal{C}' \longrightarrow \mathbb{R}$ by

$$\hat{\Upsilon}(\mathbf{C}, \mathbf{C}') := \frac{1}{\theta}\boldsymbol{\gamma}\hat{\mathbf{h}}(\mathbf{C}, \mathbf{C}') \quad \text{for all } (\mathbf{G}, \theta, \boldsymbol{\gamma}) = \mathbf{C} \in \mathcal{C} \quad \text{and} \quad \mathbf{C}' \in \mathcal{C}'. \tag{29.11}$$

By using the different parts of the intrinsic heat flux, $\hat{\mathbf{h}}_{\mathrm{e}}, \hat{\mathbf{h}}_{\mathrm{vG}}$ etc., we can define analogous different parts of $\hat{\Upsilon}$. For example, $\hat{\Upsilon}_{\mathrm{vG}}(\mathbf{C})\mathbf{G}' := \frac{1}{\theta}\boldsymbol{\gamma}\hat{\mathbf{h}}_{\mathrm{vG}}(\mathbf{C})\mathbf{G}'$.

Remark 29.2 When the theory of viscous fluids was originally studied by Stokes in [TIFF] he limited his analysis by stating that he was "neglecting squares". In modern language, his theory is a linear theory. It follows from Theorem 26.8 and the Retardation Theorem that materials with fading memory can be approximated by a linear theory when the material is subjected to a "slow" process. Thus, it seems appropriate to give special attention to this linear case. ▮

We now investigate what restrictions the second law of thermodynamics places on $\hat{\mathbf{R}}$. Recall the notation introduced in Section 21. Let $(\sigma, \mathbf{P}) \in (\Sigma \times \Pi^{\times})_{\mathrm{fit}}$, with $\mathbf{P}$ of class C^2, be given. By (29.5) we have

$$\bar{\psi}^{f}_{(\sigma,\mathbf{P})} = \hat{\psi}(\mathbf{P}^{f}, \mathbf{P}^{\bullet f}) \quad \text{for all} \quad t \in [0, d_{\mathbf{P}}]. \tag{29.12}$$

Using the chain rule, one obtains

$$\begin{aligned} \bar{\psi}^{\bullet f}_{(\sigma,\mathbf{P})} = \operatorname{tr}\left[\nabla_{(1)}\hat{\psi}(\mathbf{P}^{f}, \mathbf{P}^{\bullet f})\bar{\mathbf{G}}^{\bullet f}\right] + \nabla_{(2)}\hat{\psi}(\mathbf{P}^{f}, \mathbf{P}^{\bullet f})\bar{\theta}^{\bullet f} + \nabla_{(3)}\hat{\psi}(\mathbf{P}^{f}, \mathbf{P}^{\bullet f})\bar{\boldsymbol{\gamma}}^{\bullet f} \\ +\nabla_{(4)}\hat{\psi}(\mathbf{P}^{f}, \mathbf{P}^{\bullet f})\bar{\mathbf{G}}^{\bullet\bullet f} + \nabla_{(5)}\hat{\psi}(\mathbf{P}^{f}, \mathbf{P}^{\bullet f})\bar{\theta}^{\bullet\bullet f} + \nabla_{(6)}\hat{\psi}(\mathbf{P}^{f}, \mathbf{P}^{\bullet f})\bar{\boldsymbol{\gamma}}^{\bullet\bullet f}. \end{aligned} \tag{29.13}$$

Plugging (29.13) into (21.6) and using (29.5), yields

$$\begin{aligned} \operatorname{tr}\left[\left(\tfrac{1}{2}\hat{\mathbf{S}}(\mathbf{P}^{f}, \mathbf{P}^{\bullet f}) - \hat{\rho}(\bar{\mathbf{G}}^{f})\nabla_{(1)}\hat{\psi}(\mathbf{P}^{f}, \mathbf{P}^{\bullet f})\right)\bar{\mathbf{G}}^{\bullet f}\right] \\ -\hat{\rho}(\bar{\mathbf{G}}^{f})\left(\nabla_{(2)}\hat{\psi}(\mathbf{P}^{f}, \mathbf{P}^{\bullet f}) + \hat{\eta}(\mathbf{P}^{f}, \mathbf{P}^{\bullet f})\right)\bar{\theta}^{\bullet} \\ -\bar{\rho}(\nabla_{(3)}\hat{\psi} \circ (\mathbf{P}^{f}, \mathbf{P}^{\bullet f}))\bar{\boldsymbol{\gamma}}^{\bullet f} - \hat{\rho}(\bar{\mathbf{G}}^{f})(\nabla_{(4)}\hat{\psi}(\mathbf{P}, \mathbf{P}^{\bullet})\bar{\mathbf{G}}^{\bullet\bullet f} - \hat{\rho}(\bar{\mathbf{G}}^{f})(\nabla_{(5)}\hat{\psi}(\mathbf{P}^{f}, \mathbf{P}^{\bullet f}))\bar{\theta}^{\bullet\bullet f} \\ -\hat{\rho}(\bar{\mathbf{G}}^{f})(\nabla_{(6)}\hat{\psi}(\mathbf{P}^{f}, \mathbf{P}^{\bullet f}))\bar{\boldsymbol{\gamma}}^{\bullet\bullet f} - \hat{\Upsilon}(\mathbf{P}^{f}, \mathbf{P}^{\bullet f}) \geq 0. \end{aligned} \tag{29.14}$$

Since the condition process $\mathbf{P}$ was arbitrary, (29.14) must hold for all condition processes. Thus, by Proposition 27.1 and (29.11) we obtain

$$\begin{aligned} \operatorname{tr}\left[\tfrac{1}{2}\hat{\mathbf{S}}(\mathbf{C}, \mathbf{C}')\mathbf{G}' - \hat{\rho}(\mathbf{G})\nabla_{(1)}\hat{\psi}(\mathbf{C}, \mathbf{C}')\mathbf{G}'\right] - \hat{\rho}(\mathbf{G})\left(\hat{\eta}(\mathbf{C}, \mathbf{C}') + \nabla_{(2)}\hat{\psi}(\mathbf{C}, \mathbf{C}')\right)\theta' \\ -\hat{\rho}(\mathbf{G})\nabla_{(3)}\hat{\psi}(\mathbf{C}, \mathbf{C}')\boldsymbol{\gamma}' - \hat{\rho}(\mathbf{G})\nabla_{(4)}\hat{\psi}(\mathbf{C}, \mathbf{C}')\mathbf{G}'' - \hat{\rho}(\mathbf{G})\nabla_{(5)}\hat{\psi}(\mathbf{C}, \mathbf{C}')\theta'' \\ -\hat{\rho}(\mathbf{G})\nabla_{(6)}\hat{\psi}(\mathbf{C}, \mathbf{C}')\boldsymbol{\gamma}'' - \hat{\Upsilon}(\mathbf{C}, \mathbf{C}') \geq 0 \end{aligned} \tag{29.15}$$

for all $\mathbf{C} \in \mathcal{C}$, $(\mathbf{G}', \theta', \boldsymbol{\gamma}') = \mathbf{C}' \in \mathcal{C}'$ and $(\mathbf{G}'', \theta'', \boldsymbol{\gamma}'') \in \mathcal{C}'$.

Let $(\mathbf{C}, \mathbf{C}') \in \mathcal{C} \times \mathcal{C}'$ be given and set $\theta'' = 0$ and $\boldsymbol{\gamma}'' = \mathbf{0}$. There is a mapping $\mathbf{f} : \mathcal{C} \times \mathcal{C}' \longrightarrow \mathbb{R}$ such that (29.15) can be written in the form

$$-\hat{\rho}(\mathbf{G})\nabla_{(4)}\hat{\psi}(\mathbf{C}, \mathbf{C}')\mathbf{G}'' + \mathbf{f}(\mathbf{C}, \mathbf{C}') \geq 0 \quad \text{for all} \quad \mathbf{G}'' \in \operatorname{Sym}(\mathcal{T}, \mathcal{T}^{*}).$$

This is possible only if $\nabla_{(4)}\hat{\psi}(\mathbf{C}, \mathbf{C}') = \mathbf{0}$. In a similar way it can be shown that $\nabla_{(5)}\hat{\psi}(\mathbf{C}, \mathbf{C}') = 0$ and $\nabla_{(6)}\hat{\psi}(\mathbf{C}, \mathbf{C}') = \mathbf{0}$. Since $(\mathbf{C}, \mathbf{C}') \in \mathcal{C} \times \mathcal{C}'$ was arbitrary, the specific free energy cannot depend on the rate of change of the process, i.e.

$$\hat{\psi}_{\mathrm{v}} = 0. \tag{29.16}$$

Using these reductions (29.15) now reads

$$\begin{aligned} \operatorname{tr}\left[\tfrac{1}{2}\hat{\mathbf{S}}(\mathbf{C}, \mathbf{C}')\mathbf{G}' - \hat{\rho}(\mathbf{G})\nabla_{(1)}\hat{\psi}(\mathbf{C}, \mathbf{C}')\mathbf{G}'\right] - \hat{\rho}(\mathbf{G})\left(\hat{\eta}(\mathbf{C}, \mathbf{C}') + \nabla_{(2)}\hat{\psi}(\mathbf{C}, \mathbf{C}')\right)\theta' \\ -\hat{\rho}(\mathbf{G})\nabla_{(3)}\hat{\psi}(\mathbf{C}, \mathbf{C}')\boldsymbol{\gamma}' - \hat{\Upsilon}(\mathbf{C}, \mathbf{C}') \geq 0 \end{aligned} \tag{29.17}$$

for all $(\mathbf{G},\theta,\boldsymbol{\gamma})=\mathbf{C}\in\mathcal{C}$ and $(\mathbf{G}',\theta',\boldsymbol{\gamma}')=\mathbf{C}'\in\mathcal{C}'$. As in the last section, we will reduce this inequality to a polynomial inequality. Define several mappings from $\mathcal{C}\times\mathcal{C}'$ to $\mathbb{R}$ by

$$
\begin{aligned}
&\hat{\mathsf{A}}(\mathbf{C},\mathbf{C}') := \tfrac{1}{2}\mathrm{tr}\left[(\hat{\mathbf{S}}_{\mathrm{vG}}(\mathbf{C})\mathbf{G}')\mathbf{G}'\right]\\
&\hat{\mathsf{B}}(\mathbf{C},\mathbf{C}') := -\hat{\rho}(\mathbf{G})\hat{\eta}_{\mathrm{v}\theta}(\mathbf{C})(\theta')^2\\
&\hat{\mathsf{C}}(\mathbf{C},\mathbf{C}') := \tfrac{1}{2}\mathrm{tr}\left[\hat{\mathbf{S}}_{\mathrm{v}\theta}(\mathbf{C})\theta'\mathbf{G}'\right]-\hat{\rho}(\mathbf{G})\hat{\eta}_{\mathrm{vG}}(\mathbf{C})\mathbf{G}'\theta'\\
&\hat{\mathsf{D}}(\mathbf{C},\mathbf{C}') := \tfrac{1}{2}\mathrm{tr}\left[(\hat{\mathbf{S}}_{\mathrm{v}\gamma}(\mathbf{C})\boldsymbol{\gamma}')\mathbf{G}'\right]\\
&\hat{\mathsf{E}}(\mathbf{C},\mathbf{C}') := -\hat{\rho}(\mathbf{G})\hat{\eta}_{\mathrm{v}\gamma}(\mathbf{C})\boldsymbol{\gamma}'\theta'\\
&\hat{\mathsf{F}}(\mathbf{C},\mathbf{C}') := \mathrm{tr}\left[\left(\tfrac{1}{2}\hat{\mathbf{S}}_{\mathrm{e}}(\mathbf{C})-\hat{\rho}(\mathbf{G})\nabla_{(1)}\hat{\psi}_{\mathrm{e}}(\mathbf{C})\right)\mathbf{G}'\right]-\hat{\Upsilon}_{\mathrm{vG}}(\mathbf{C})\mathbf{G}'\\
&\hat{\mathsf{G}}(\mathbf{C},\mathbf{C}') := -\hat{\rho}(\mathbf{G})\left(\hat{\eta}_{\mathrm{e}}(\mathbf{C})+\nabla_{(2)}\hat{\psi}_{\mathrm{e}}(\mathbf{C})\right)\theta'-\hat{\Upsilon}_{\mathrm{v}\theta}(\mathbf{C})\theta'\\
&\hat{\mathsf{H}}(\mathbf{C},\mathbf{C}') := -\hat{\rho}(\mathbf{G})\nabla_{(3)}\hat{\psi}_{\mathrm{e}}(\mathbf{C})\boldsymbol{\gamma}'-\hat{\Upsilon}_{\mathrm{v}\gamma}(\mathbf{C})\boldsymbol{\gamma}'\\
&\hat{\mathsf{I}}(\mathbf{C},\mathbf{C}') := -\hat{\Upsilon}_{\mathrm{e}}(\mathbf{C})
\end{aligned}
\tag{29.18}
$$

for all $(\mathbf{G},\theta,\boldsymbol{\gamma})=\mathbf{C}\in\mathcal{C}$ and $(\mathbf{G}',\theta',\boldsymbol{\gamma}')=\mathbf{C}'\in\mathcal{C}'$. It is clear that given $x,y,z\in\mathbb{R}$, (29.17) remains valid when $\mathbf{G}'$, θ' and $\boldsymbol{\gamma}'$ are replaced by $x\mathbf{G}'$, $y\theta'$ and $z\boldsymbol{\gamma}'$, respectively. By inserting the parts of the components of the response mappings given in (29.6) and (29.8) into (29.17) one can see that (29.17) is equivalent to

$$\hat{\mathsf{A}}x^2+\hat{\mathsf{B}}y^2+\hat{\mathsf{C}}xy+\hat{\mathsf{D}}xz+\hat{\mathsf{E}}yz+\hat{\mathsf{F}}x+\hat{\mathsf{G}}y+\hat{\mathsf{H}}z+\hat{\mathsf{I}}\geq 0 \quad \text{for all} \quad x,y,z\in\mathbb{R}, \tag{29.19}$$

value-wise with respect to $\mathcal{C}\times\mathcal{C}'$. Using Proposition 33.1 from Section 33 and observing (29.16) we obtain the following theorem.

Theorem 29.3 *A thermoelasto-viscous element satisfies the second law of thermodynamics if and only if the following conditions are hold:*

(T2.1) $\hat{\psi}_{\mathrm{v}}=0$.
(T2.2) $\hat{\mathsf{H}}=\hat{\mathsf{D}}=\hat{\mathsf{E}}=0$.
(T2.3) $\hat{\mathsf{A}}\geq 0,\ \hat{\mathsf{B}}\geq 0,\ \hat{\mathsf{I}}\geq 0$.
(T2.4) $4\hat{\mathsf{A}}\hat{\mathsf{B}}\geq\hat{\mathsf{C}}^2$.
(T2.5) $4\hat{\mathsf{A}}\hat{\mathsf{I}}\geq\hat{\mathsf{F}}^2$.
(T2.6) $4\hat{\mathsf{B}}\hat{\mathsf{I}}\geq\hat{\mathsf{G}}^2$.
(T2.7) $(4\hat{\mathsf{A}}\hat{\mathsf{B}}-\hat{\mathsf{C}}^2)\hat{\mathsf{I}}\geq\hat{\mathsf{A}}\hat{\mathsf{G}}^2+\hat{\mathsf{B}}\hat{\mathsf{F}}^2-\hat{\mathsf{C}}\hat{\mathsf{F}}\hat{\mathsf{G}}$.

This theorem invites several comments.

1. The equations $\hat{\mathsf{D}}=0$ and $\hat{\mathsf{E}}=0$ mean that the intrinsic stress and the specific entropy do not depend on the rate of change of the temperature gradient, i.e., $\hat{\mathbf{S}}_{\mathrm{v}\gamma}=\mathbf{0}$ and $\hat{\eta}_{\mathrm{v}\gamma}=\mathbf{0}$.
2. The equation $\hat{\mathsf{H}}=0$ means that the specific free energy determines how the intrinsic heat flux depends on the rate of change of the temperature gradient, i.e., $\hat{\rho}(\mathbf{G})\nabla_{(3)}\hat{\psi}_{\mathrm{e}}(\mathbf{C})=-\hat{\Upsilon}_{\mathrm{v}\gamma}(\mathbf{C})$ for all $(\mathbf{G},\theta,\boldsymbol{\gamma})=\mathbf{C}\in\mathcal{C}$.

3. Let us consider what the inequalities in the above theorem tell us when there is no temperature gradient across the element, i.e. $\boldsymbol{\gamma} = \mathbf{0}$. In this case, by (29.18) and (29.11), $\hat{\mathsf{I}} = 0$. Hence by (T2.6), $\hat{\mathsf{G}} = 0$. This means that, using (29.7),

$$\nabla_{(2)}\hat{\psi}_{\mathrm{p}} = -\hat{\eta}_{\mathrm{p}},$$

which is refereed to as the *entropy-relation.* Also by (T2.5) we have $\hat{\mathsf{F}} = 0$. This can be used to obtain

$$\hat{\mathbf{S}}_{\mathrm{p}} = 2\hat{\rho}\nabla_{(1)}\hat{\psi}_{\mathrm{p}},$$

which is equivalent to (5.4) in [CNP] and is refereed to as the *stress-relation.*

4. The inequality $(\mathrm{T}2.3)_1$ says that

$$\mathrm{tr}\left[(\hat{\mathbf{S}}_{\mathrm{vG}}(\mathbf{C})\mathbf{G}')\mathbf{G}'\right] \geq 0 \quad \text{for all} \;\; \mathbf{C} \in \mathcal{C} \text{ and } \mathbf{G}' \in \mathrm{Sym}(\mathcal{T},\mathcal{T}^*),$$

i.e., $\hat{\mathbf{S}}_{\mathrm{vG}}(\mathbf{C})$, when viewed as a bilinear form on $\mathrm{Sym}(\mathcal{T},\mathcal{T}^*)$, is positive symmetric for all $\mathbf{C} \in \mathcal{C}$. This agrees with equation (5.7) of [CNP].

5. The inequality $(\mathrm{T}2.3)_2$ says that

$$\hat{\eta}_{\mathrm{v}\theta}(\mathbf{G},\theta,\boldsymbol{\gamma}) \leq 0 \quad \text{for all} \;\; (\mathbf{G},\theta,\boldsymbol{\gamma}) \in \mathcal{C}.$$

6. The inequality $(\mathrm{T}2.3)_3$ says that

$$\hat{\Upsilon}_{\mathrm{e}}(\mathbf{C}) = \frac{1}{\theta}\boldsymbol{\gamma}\hat{\mathbf{h}}_{\mathrm{e}}(\mathbf{C}) \leq 0 \quad \text{for all} \;\; (\mathbf{G},\theta,\boldsymbol{\gamma}) = \mathbf{C} \in \mathcal{C}.$$

This is the same as (28.8) and hence is equivalent to $\hat{\mathbf{h}}_{\mathrm{p}} = \mathbf{0}$ and $\boldsymbol{\gamma}\hat{\mathbf{h}}_{\mathrm{T}}(\mathbf{C}) \leq 0$ for all $(\mathbf{G},\theta,\boldsymbol{\gamma}) \in \mathcal{C}$.

Remark 29.4 At this time it may help the reader if we reconcile the results of this section with those found in [CNP]. The constitutive laws specified in (29.1) defines a material that contains one proposed in [CNP] as a special case. It is for this reason that the restrictions found above are weaker then those in [CNP]. To see that this is the case, suppose that all of the terms in the matrix (29.8) are zero except $\hat{\mathbf{S}}_{\mathrm{vG}}$ and all of the constitutive laws, except the one for the intrinsic heat flux, do not depend on the temperature gradient. Then $\hat{\mathbf{R}}$ defines the same set of constitutive laws as in [CNP], except here they are frame-free. In this case (29.18) reduces to:

$$\begin{aligned}
&\hat{\mathsf{A}}(\mathbf{C},\mathbf{C}') = \tfrac{1}{2}\mathrm{tr}\left[(\hat{\mathbf{S}}_{\mathrm{vG}}(\mathbf{C})\mathbf{G}')\mathbf{G}'\right]\\
&\hat{\mathsf{B}}(\mathbf{C},\mathbf{C}') = \hat{\mathsf{C}}(\mathbf{C},\mathbf{C}') = \hat{\mathsf{D}}(\mathbf{C},\mathbf{C}') = \hat{\mathsf{E}}(\mathbf{C},\mathbf{C}') = 0\\
&\hat{\mathsf{F}}(\mathbf{C},\mathbf{C}') = \mathrm{tr}\left[\left(\tfrac{1}{2}\hat{\mathbf{S}}_{\mathrm{e}}(\mathbf{C}) - \hat{\rho}(\mathbf{G})\nabla_{(1)}\hat{\psi}_{\mathrm{e}}(\mathbf{C})\right)\mathbf{G}'\right]\\
&\hat{\mathsf{G}}(\mathbf{C},\mathbf{C}') = -\hat{\rho}(\mathbf{G})\left(\hat{\eta}_{\mathrm{e}}(\mathbf{C}) + \nabla_{(2)}\hat{\psi}_{\mathrm{e}}(\mathbf{C})\right)\theta'\\
&\hat{\mathsf{H}}(\mathbf{C},\mathbf{C}') = -\hat{\rho}(\mathbf{G})\nabla_{(3)}\hat{\psi}_{\mathrm{e}}(\mathbf{C})\boldsymbol{\gamma}'\\
&\hat{\mathsf{I}}(\mathbf{C},\mathbf{C}') = -\hat{\Upsilon}_{\mathrm{e}}(\mathbf{C}).
\end{aligned} \tag{29.20}$$

Then (T2.5), when $\boldsymbol{\gamma} = \mathbf{0}$, says $\hat{\mathsf{F}} = 0$ and so the stress-relation holds and (T2.5) says $\hat{\mathsf{G}} = 0$ so the entropy-relation holds. Also, $\hat{\mathsf{A}} \geq 0$ is equivalent to (5.7) from [CNP] and $\hat{\mathsf{I}} \geq 0$ is the same as (6.2) from [CNP]. Thus, the above theorem yields the same results given in [CNP]. A similar analysis can be done to reduce the results of this section to those of the previous section. ∎

Remark 29.5 Notice that for a thermoelasto-viscous element the elastic part of the intrinsic stress depends on the temperature gradient, while in the thermoelastic case this dependence is ruled out by the second law. Thus, even in a process in which the configuration, temperature and temperature gradient are held fixed, a thermoelasto-viscous material in general has a different stress from a thermoelastic material. This is because processes which are not constant must be considered in order to eliminate the dependence on the temperature gradient in the thermoelastic case. When one tries to use the same argument in the thermoelasto-viscous case the viscous terms prevent one from obtaining the same result. ∎

30 Thermoelasto-viscous Fluids

Let $\mathcal{T}$ be a thermoelasto-viscous element. Let a material symmetry $\mathbf{A} \in \mathfrak{G}$ be given. In this case the corresponding permutation on the state space of this element, see (17.3), can be given explicitly. Define a mapping $\mathbf{A}_{\mathcal{C}'} : \mathcal{C}' \longrightarrow \mathcal{C}'$ by

$$\mathbf{A}_{\mathcal{C}'}(\mathbf{G}', \theta', \boldsymbol{\gamma}') := (\mathbf{A}^{-\top}\mathbf{G}'\mathbf{A}^{-1}, \theta', \mathbf{A}^{-\top}\boldsymbol{\gamma}') \quad \text{for all} \quad (\mathbf{G}', \theta', \boldsymbol{\gamma}') \in \mathcal{C}'. \tag{30.1}$$

Proposition 30.1 *Let a symmetry $\mathbf{A} \in \mathfrak{G}$ of a thermoelasto-viscous element (see Section 17) be given. Then $\iota_{\mathbf{A}}$ is given by*

$$\iota_{\mathbf{A}}(\mathbf{C}, \mathbf{N}_{\mathbf{C}}\mathbf{C}') = (\mathbf{A}_{\mathcal{C}}\mathbf{C}, \mathbf{N}_{\mathbf{A}_{\mathcal{C}}\mathbf{C}}(\mathbf{A}_{\mathcal{C}'}\mathbf{C}')) \quad \text{for all} \quad (\mathbf{C}, \mathbf{N}_{\mathbf{C}}\mathbf{C}') \in \Sigma. \tag{30.2}$$

Proof: Let $(\mathbf{C}, \mathbf{N}_{\mathbf{C}}\mathbf{C}') \in \Sigma$ be given. Choose $\mathbf{P} \in \Pi^{\times}$ such that

$$(\mathbf{C}, \mathbf{N}_{\mathbf{C}}\mathbf{C}') = (\mathbf{P}^{f}, \mathbf{N}_{\mathbf{C}}\mathbf{P}^{\bullet f}).$$

It follows from this choice of $\mathbf{P}$ that $(\mathbf{A}_{\mathcal{C}} \circ \mathbf{P})^{f} = \mathbf{A}_{\mathcal{C}}\mathbf{C}$ and $(\mathbf{A}_{\mathcal{C}} \circ \mathbf{P})^{\bullet f} = \mathbf{A}_{\mathcal{C}'}\mathbf{C}'$. Let $\sigma \in \Sigma_{\mathbf{C}}$ be given. By (17.13) and (29.3) we have

$$\begin{aligned} \iota_{\mathbf{A}}((\mathbf{C}, \mathbf{N}_{\mathbf{C}}\mathbf{C}')) &= \iota_{\mathbf{A}}(\hat{e}(\sigma, \mathbf{P})) \\ &= \hat{e}(\iota(\sigma), \mathbf{A}_{\mathcal{C}} \circ \mathbf{P}) \\ &= ((\mathbf{A}_{\mathcal{C}} \circ \mathbf{P})^{f}, (\mathbf{A}_{\mathcal{C}} \circ \mathbf{P})^{\bullet f}) \\ &= (\mathbf{A}_{\mathcal{C}}\mathbf{C}, \mathbf{N}_{\mathbf{A}_{\mathcal{C}}\mathbf{C}}(\mathbf{A}_{\mathcal{C}'}\mathbf{C}')). \end{aligned}$$

Since $(\mathbf{C}, \mathbf{N}_{\mathbf{C}}\mathbf{C}') \in \Sigma$ was arbitrary, (30.2) holds. ∎

Suppose we have a thermoelasto-viscous element which is fluid so that $\mathfrak{G} = \text{Unim}\mathcal{T}$. Notice that the first component of (S2) from Proposition 17.5 says that $\hat{\mathbf{S}}$ must satisfy

$$\mathbf{A}\hat{\mathbf{S}}(\mathbf{G}, \theta, \boldsymbol{\gamma}, \mathbf{G}', \theta', \boldsymbol{\gamma}')\mathbf{A}^{\top} = \hat{\mathbf{S}}(\mathbf{A}^{-\top}\mathbf{G}\mathbf{A}^{-1}, \theta, \mathbf{A}^{-\top}\boldsymbol{\gamma}, \mathbf{A}^{-\top}\mathbf{G}'\mathbf{A}^{-1}, \theta', \mathbf{A}^{-\top}\boldsymbol{\gamma}') \tag{30.3}$$

for all $(\mathbf{G},\theta,\boldsymbol{\gamma},\mathbf{G}',\theta',\boldsymbol{\gamma}')\in\mathcal{C}\times\mathcal{C}'$ and $\mathbf{A}\in\mathrm{Unim}\,\mathcal{T}$.

Let us investigate what restrictions (30.3) places on $\hat{\mathbf{S}}$. Start by setting $\mathbf{G}',\theta'$ and $\boldsymbol{\gamma}'$ to their respective zeros in (30.3) and use (29.9) to obtain

$$\mathbf{A}\hat{\mathbf{S}}_{\mathrm{e}}(\mathbf{G},\theta,\boldsymbol{\gamma})\mathbf{A}^{\top}=\hat{\mathbf{S}}_{\mathrm{e}}(\mathbf{A}^{-\top}\mathbf{G}\mathbf{A}^{-1},\theta,\mathbf{A}^{-\top}\boldsymbol{\gamma})\quad\text{for all }\ (\mathbf{G},\theta,\boldsymbol{\gamma})\in\mathcal{C}\text{ and }\mathbf{A}\in\mathrm{Unim}\,\mathcal{T}.\tag{30.4}$$

Using the fact that $\mathbf{A}^{\top}\mathbf{G}=\mathbf{G}\mathbf{A}^{-1}$ for all $\mathbf{A}\in\mathrm{Orth}\,\mathbf{G}$ and $\mathbf{G}\in\mathrm{Pos}^{+}(\mathcal{T},\mathcal{T}^{*})$ we see that (30.4) implies that a necessary, but not sufficient, condition on $\hat{\mathbf{S}}_{\mathrm{e}}$ is

$$\mathbf{A}\hat{\mathbf{S}}_{\mathrm{e}}(\mathbf{G},\theta,\boldsymbol{\gamma})\mathbf{G}\mathbf{A}^{-1}=\hat{\mathbf{S}}_{\mathrm{e}}(\mathbf{G},\theta,\mathbf{A}^{-\top}\boldsymbol{\gamma})\mathbf{G}\quad\text{for all }\ (\mathbf{G},\theta,\boldsymbol{\gamma})\in\mathcal{C}\text{ and }\mathbf{A}\in\mathrm{Orth}(\mathbf{G}).\tag{30.5}$$

Let $(\mathbf{G},\theta)\in\mathcal{H}$ (recall that $\mathcal{H}$ is the projection of $\mathcal{C}$ onto $\mathrm{Pos}^{+}(\mathcal{T},\mathcal{T}^{*})\times\mathbb{P}^{\times}$) be given and define $\mathbf{U}:\mathcal{T}\longrightarrow\mathrm{Sym}(\mathcal{T})$ by

$$\mathbf{U}(\mathbf{v}):=\hat{\mathbf{S}}_{\mathrm{e}}(\mathbf{G},\theta,\mathbf{G}\mathbf{v})\mathbf{G}\quad\text{for all }\ \mathbf{v}\in\mathcal{T}.\tag{30.6}$$

By using (30.5) one can show that $\mathbf{U}$ is isotropic with respect to the inner-product induced by $\mathbf{G}$. Rivlin and Ericksen showed in [SDIM] that isotropic mappings of this form have the representation

$$\mathbf{U}(\mathbf{v})=\hat{\mathrm{s}}_{\mathrm{e}1}(\tilde{\mathbf{C}})\mathbf{1}_{\mathcal{T}}+\hat{\mathrm{s}}_{\mathrm{e}2}(\tilde{\mathbf{C}})\mathbf{v}\otimes\mathbf{G}\mathbf{v}\quad\text{for all }\ \mathbf{v}\in\mathcal{T},\ \text{where }\tilde{\mathbf{C}}=(\mathbf{G},\theta,(\mathbf{G}\mathbf{v})\mathbf{v}).$$

Using (30.6) and the fact that $(\mathbf{G},\theta)\in\mathcal{H}$ was arbitrary we find

$$\hat{\mathbf{S}}_{\mathrm{e}}(\mathbf{G},\theta,\boldsymbol{\gamma})=\hat{\mathrm{s}}_{\mathrm{e}1}(\tilde{\mathbf{C}})\mathbf{G}^{-1}+\hat{\mathrm{s}}_{\mathrm{e}2}(\tilde{\mathbf{C}})(\mathbf{G}^{-1}\boldsymbol{\gamma}\otimes\mathbf{G}^{-1}\boldsymbol{\gamma}),\ \text{where }\tilde{\mathbf{C}}=(\mathbf{G},\theta,\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma}))\tag{30.7}$$

for all $(\mathbf{G},\theta,\boldsymbol{\gamma})\in\mathcal{C}$. The coefficient functions $\hat{\mathrm{s}}_{\mathrm{e}1},\hat{\mathrm{s}}_{\mathrm{e}2}:\mathcal{C}\longrightarrow\mathbb{R}$ depend on the temperature gradient $\boldsymbol{\gamma}$ only through $\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma})$, which is the square of its magnitude with respect to the inner product $\mathbf{G}$. A short calculation shows that this formula for $\hat{\mathbf{S}}_{\mathrm{e}}$ satisfies (30.4) if and only if the coefficients $\hat{e}_{\mathrm{e}1}$, $\hat{e}_{\mathrm{e}2}$ satisfy a condition of the form

$$\hat{f}(\mathbf{G},\theta,\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma}))=\hat{f}(\mathbf{A}^{-\top}\mathbf{G}\mathbf{A}^{-1},\theta,\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma}))\quad\text{for all }\ \mathbf{A}\in\mathrm{Unim}\,\mathcal{T}\text{ and }(\mathbf{G},\theta,\boldsymbol{\gamma})\in\mathcal{C}.\tag{30.8}$$

It is known, see [FFFE], that this implies that the coefficients only depend on the configuration through the mass density, i.e.

$$\hat{\mathbf{S}}_{\mathrm{e}}(\mathbf{G},\theta,\boldsymbol{\gamma})=\hat{\mathrm{s}}_{\mathrm{e}1}(\tilde{c})\mathbf{G}^{-1}+\hat{\mathrm{s}}_{\mathrm{e}2}(\tilde{c})(\mathbf{G}^{-1}\boldsymbol{\gamma}\otimes\mathbf{G}^{-1}\boldsymbol{\gamma}),\quad\text{where}\quad\tilde{c}=(\hat{\rho}(\mathbf{G}),\theta,\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma})).\tag{30.9}$$

Here the coefficients are functions from $\tilde{\mathcal{H}}\times\mathbb{P}$ to $\mathbb{R}$, where $\tilde{\mathcal{H}}$ is an open connected subset of $\mathbb{P}^{\times}\times\mathbb{P}^{\times}$. However, we use the same notation for the functions as before.

Now set θ' and $\boldsymbol{\gamma}'$ to zero in (30.3), with (29.9), to obtain

$$\mathbf{A}\hat{\mathbf{S}}_{\mathrm{vG}}(\mathbf{G},\theta,\boldsymbol{\gamma})\mathbf{G}'\mathbf{A}^{\top}=\hat{\mathbf{S}}_{\mathrm{vG}}(\mathbf{A}^{-\top}\mathbf{G}\mathbf{A}^{-1},\theta,\mathbf{A}^{-\top}\boldsymbol{\gamma})(\mathbf{A}^{-\top}\mathbf{G}'\mathbf{A}^{-1})\tag{30.10}$$

for all $(\mathbf{G},\theta,\boldsymbol{\gamma})\in\mathcal{C}$, $\mathbf{G}'\in\mathrm{Sym}(\mathcal{T}^{*},\mathcal{T})$ and $\mathbf{A}\in\mathfrak{G}$. This condition implies that $\hat{\mathbf{S}}_{\mathrm{vG}}$ must satisfy

$$\mathbf{A}\hat{\mathbf{S}}_{\mathrm{vG}}(\mathbf{G},\theta,\boldsymbol{\gamma})(\mathbf{G}')\mathbf{G}\mathbf{A}^{-1}=\hat{\mathbf{S}}_{\mathrm{vG}}(\mathbf{G},\theta,\mathbf{A}^{-\top}\boldsymbol{\gamma})(\mathbf{A}^{-\top}\mathbf{G}'\mathbf{A}^{-1})\mathbf{G}\tag{30.11}$$

for all $(\mathbf{G},\theta,\boldsymbol{\gamma})\in\mathcal{C}$, $\mathbf{G}'\in\mathrm{Sym}(\mathcal{T},\mathcal{T}^*)$ and $\mathbf{A}\in\mathrm{Orth}(\mathbf{G})$. Let $(\mathbf{G},\theta)\in\mathcal{H}$ be given and define $\mathbf{V}:\mathcal{T}\times\mathrm{Sym}(\mathcal{T})\longrightarrow\mathrm{Sym}(\mathcal{T})$ by

$$\mathbf{V}(\mathbf{v},\mathbf{B}):=\hat{\mathbf{S}}_{\mathrm{vG}}(\mathbf{G},\theta,\mathbf{G}\mathbf{v})(\mathbf{G}\mathbf{B})\mathbf{G}\quad\text{for all }\ \mathbf{v}\in\mathcal{T}\text{ and }\mathbf{B}\in\mathrm{Sym}(\mathcal{T}).\tag{30.12}$$

Using (30.11) one can show that $\mathbf{V}$ is an isotropic mapping with respect to the inner-product induced by $\mathbf{G}$ and is linear in its second variable. It was also shown in [SDIM] that such an isotropic mapping has the form

$$\begin{aligned}\mathbf{V}(\mathbf{v},\mathbf{B})=\;&\hat{\mathrm{s}}_{\mathrm{vG1}}(\tilde{\mathbf{C}})\mathbf{B}+\hat{\mathrm{s}}_{\mathrm{vG2}}(\tilde{\mathbf{C}})\mathrm{tr}(\mathbf{B})\mathbf{1}_{\mathcal{T}}+\hat{\mathrm{s}}_{\mathrm{vG3}}(\tilde{\mathbf{C}})\mathrm{tr}(\mathbf{B})\mathbf{v}\otimes\mathbf{G}\mathbf{v}\\&+\hat{\mathrm{s}}_{\mathrm{vG4}}(\tilde{\mathbf{C}})\left[(\mathbf{B}\mathbf{v})\otimes\mathbf{G}\mathbf{v}+\mathbf{v}\otimes(\mathbf{G}\mathbf{v})\mathbf{B}\right],\end{aligned}$$

where $\tilde{\mathbf{C}}=(\mathbf{G},\theta,\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma}))$, for all $\mathbf{v}\in\mathcal{T}$ and $\mathbf{B}\in\mathrm{Sym}(\mathcal{T})$. Using this representation, the fact that $(\mathbf{G},\theta)\in\mathcal{H}$ was arbitrary and (30.12) we find that

$$\begin{aligned}\hat{\mathbf{S}}_{\mathrm{vG}}(\mathbf{G},\theta,\boldsymbol{\gamma})\mathbf{G}'=\;&\hat{\mathrm{s}}_{\mathrm{vG1}}(\tilde{\mathbf{C}})\mathbf{G}^{-1}\mathbf{G}'\mathbf{G}^{-1}+\hat{\mathrm{s}}_{\mathrm{vG2}}(\tilde{\mathbf{C}})\mathrm{tr}(\mathbf{G}^{-1}\mathbf{G}')\mathbf{G}^{-1}\\&+\hat{\mathrm{s}}_{\mathrm{vG3}}(\tilde{\mathbf{C}})\mathrm{tr}(\mathbf{G}^{-1}\mathbf{G}')(\mathbf{G}^{-1}\boldsymbol{\gamma}\otimes\mathbf{G}^{-1}\boldsymbol{\gamma})\\&+\hat{\mathrm{s}}_{\mathrm{vG4}}(\tilde{\mathbf{C}})\left[\mathbf{G}^{-1}\mathbf{G}'(\mathbf{G}^{-1}\boldsymbol{\gamma}\otimes\mathbf{G}^{-1}\boldsymbol{\gamma})+(\mathbf{G}^{-1}\boldsymbol{\gamma}\otimes\mathbf{G}^{-1}\boldsymbol{\gamma})\mathbf{G}'\mathbf{G}^{-1}\right],\end{aligned}\tag{30.13}$$

where $\tilde{\mathbf{C}}=(\mathbf{G},\theta,\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma}))$, for all $(\mathbf{G},\theta,\boldsymbol{\gamma})\in\mathcal{C}$ and $\mathbf{G}'\in\mathrm{Sym}(\mathcal{T},\mathcal{T}^*)$. As before, we find that this form of $\hat{\mathbf{S}}_{\mathrm{vG}}$ satisfies (30.10) exactly when the coefficients satisfy a condition of the form (30.8) and hence only depend on the configuration only through the mass density.

Using arguments similar to the ones given above, it can be shown that $\hat{\mathbf{S}}_{\mathrm{v}\theta}$ (recall $\hat{\mathbf{S}}_{\mathrm{v}\gamma}=\mathbf{0}$) is given by

$$\hat{\mathbf{S}}_{\mathrm{v}\theta}(\mathbf{G},\theta,\boldsymbol{\gamma})=\hat{\mathrm{s}}_{\mathrm{v}\theta1}(\tilde{c})\mathbf{G}^{-1}+\hat{\mathrm{s}}_{\mathrm{v}\theta2}(\tilde{c})(\mathbf{G}^{-1}\boldsymbol{\gamma}\otimes\mathbf{G}^{-1}\boldsymbol{\gamma}),\quad\text{where}\quad\tilde{c}=(\hat{\rho}(\mathbf{G}),\theta,\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma})),\tag{30.14}$$

for all $(\mathbf{G},\theta,\boldsymbol{\gamma})\in\mathcal{C}$ and $\theta'\in\mathbb{R}$. In a similar fashion one can also find the terms making up $\hat{\mathbf{h}}$ and $\hat{\eta}$ are given by

$$\begin{aligned}&\hat{\mathbf{h}}_{\mathrm{e}}(\mathbf{G},\theta,\boldsymbol{\gamma})=\hat{\mathrm{h}}_{\mathrm{e}}(\tilde{c})\mathbf{G}^{-1}\boldsymbol{\gamma}\\&\hat{\mathbf{h}}_{\mathrm{vG}}(\mathbf{G},\theta,\boldsymbol{\gamma})\mathbf{G}'=\left(\hat{\mathrm{h}}_{\mathrm{vG1}}(\tilde{c})\mathrm{tr}(\mathbf{G}^{-1}\mathbf{G}')1_{\mathcal{T}}+\hat{\mathrm{h}}_{\mathrm{vG2}}(\tilde{c})\mathbf{G}^{-1}\mathbf{G}'\right)\mathbf{G}^{-1}\boldsymbol{\gamma}\\&\hat{\mathbf{h}}_{\mathrm{v}\theta}(\mathbf{G},\theta,\boldsymbol{\gamma})\theta'=\hat{\mathrm{h}}_{\mathrm{v}\theta}(\tilde{c})\theta'\mathbf{G}^{-1}\boldsymbol{\gamma}\\&\hat{\mathbf{h}}_{\mathrm{v}\gamma}(\mathbf{G},\theta,\boldsymbol{\gamma})\boldsymbol{\gamma}'=\hat{\mathrm{h}}_{\mathrm{v}\gamma1}(\tilde{c})\mathbf{G}^{-1}\boldsymbol{\gamma}'+\hat{\mathrm{h}}_{\mathrm{v}\gamma2}(\tilde{c})\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma}')\mathbf{G}^{-1}\boldsymbol{\gamma}\end{aligned}\tag{30.15}$$

$$\begin{aligned}&\hat{\eta}_{\mathrm{e}}(\mathbf{G},\theta,\boldsymbol{\gamma})=\hat{e}_{\tilde{e}}(\tilde{c})\\&\hat{\eta}_{\mathrm{vG}}(\mathbf{G},\theta,\boldsymbol{\gamma})\mathbf{G}'=\hat{e}_{\mathrm{vG1}}(\tilde{c})\boldsymbol{\gamma}(\mathbf{G}^{-1}\mathbf{G}'\mathbf{G}^{-1}\boldsymbol{\gamma})+\hat{e}_{\mathrm{vG2}}(\tilde{c})\mathrm{tr}(\mathbf{G}^{-1}\mathbf{G}')\\&\hat{\eta}_{\mathrm{v}\theta}(\mathbf{G},\theta,\boldsymbol{\gamma})\theta'=\hat{e}_{\tilde{\mathrm{v}\theta}}(\tilde{c})\theta'\end{aligned}\tag{30.16}$$

where $\tilde{c}=(\hat{\rho}(\mathbf{G}),\theta,\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma}))$ for all $(\mathbf{G},\theta,\boldsymbol{\gamma})\in\mathcal{C}$ and $(\mathbf{G}',\theta',\boldsymbol{\gamma}')\in\mathcal{C}'$.

One also finds that the mappings $\hat{\psi}_{\mathrm{e}}$, $\nabla_{(1)}\hat{\psi}_{\mathrm{e}}$ and $\nabla_{(2)}\hat{\psi}_{\mathrm{e}}$ satisfy isotropic conditions and so can be shown to have the forms

$$\begin{aligned}&\hat{\psi}_{\mathrm{e}}(\mathbf{G},\theta,\boldsymbol{\gamma})=\hat{\psi}_{\tilde{e}}(\tilde{c})\\&\nabla_{(1)}\hat{\psi}_{\mathrm{e}}(\mathbf{G},\theta,\boldsymbol{\gamma})=\hat{\psi}_{\mathrm{e}1}(\tilde{c})\mathbf{G}^{-1}+\hat{\psi}_{\mathrm{e}2}(\tilde{c})(\mathbf{G}^{-1}\boldsymbol{\gamma}\otimes\mathbf{G}^{-1}\boldsymbol{\gamma})\\&\nabla_{(2)}\hat{\psi}_{\mathrm{e}}(\mathbf{G},\theta,\boldsymbol{\gamma})=\hat{\psi}_{\tilde{2e}}(\tilde{c})\end{aligned}\tag{30.17}$$

where $\tilde{c} = (\hat{\rho}(\mathbf{G}), \theta, \boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma}))$, for all $(\mathbf{G}, \theta, \boldsymbol{\gamma}) \in \mathcal{C}$.

The conditions (T2.3)-(T2.8) place restrictions on the coefficients listed above. By using standard arguments in the theory of quadratic forms one finds the following restrictions on the coefficient functions given above, identified by the condition of Theorem 29.3 they come from.

For all $(\mathbf{G}, \theta, \boldsymbol{\gamma}) \in \mathcal{C}$, with $\tilde{c} := (\hat{\rho}(\mathbf{G}), \theta, \boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma}))$,

(T2.3) Put $\alpha(\tilde{c}) := \hat{\mathrm{s}}_{\mathrm{vG1}}(\tilde{c}) + \hat{\mathrm{s}}_{\mathrm{vG2}}(\tilde{c}) + \boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma})\hat{\mathrm{s}}_{\mathrm{vG3}}(\tilde{c}) + \boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma})\hat{\mathrm{s}}_{\mathrm{vG4}}(\tilde{c})$,
$\beta(\tilde{c}) := 2(\hat{\mathrm{s}}_{\mathrm{vG1}}(\tilde{c}) + 2\hat{\mathrm{s}}_{\mathrm{vG2}})(\tilde{c})$,
$\gamma(\tilde{c}) := 2(2\hat{\mathrm{s}}_{\mathrm{vG2}}(\tilde{c}) + \boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma})\hat{\mathrm{s}}_{\mathrm{vG3}}(\tilde{c}))$,
$\delta(\tilde{c}) := 2\hat{\mathrm{s}}_{\mathrm{vG1}}(\tilde{c}) + \boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma})\hat{\mathrm{s}}_{\mathrm{vG4}}(\tilde{c})$.
Then $\alpha(\tilde{c}) \geq 0$, $\beta(\tilde{c}) \geq 0$, $\delta(\tilde{c}) \geq 0$, $\hat{\mathrm{s}}_{\mathrm{vG1}}(\tilde{c}) \geq 0$, $\hat{\eta}_{\mathrm{v}\tilde{\theta}}(\tilde{c}) \leq 0$, $\hat{\mathrm{h}}_{\mathrm{e}}(\tilde{c}) \leq 0$ and $4\alpha(\tilde{c})\beta(\tilde{c}) \geq \gamma(\tilde{c})^2$.

(T2.4) Put $\epsilon(\tilde{c}) := \frac{1}{2}\hat{\mathrm{s}}_{\mathrm{v}\theta1}(\tilde{c}) + \frac{1}{2}\hat{\mathrm{s}}_{\mathrm{v}\theta2}(\tilde{c}) - 2\hat{\rho}(\mathbf{G})\hat{\eta}_{\mathrm{vG1}}(\tilde{c})\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma}) - 2\hat{\rho}(\mathbf{G})\hat{\eta}_{\mathrm{vG2}}(\tilde{c})$,
$\zeta(\tilde{c}) := \hat{\mathrm{s}}_{\mathrm{v}\theta1}(\tilde{c}) - 4\hat{\rho}(\mathbf{G})\hat{\eta}_{\mathrm{vG2}}(\tilde{c})$.
Then $2\hat{\rho}(\mathbf{G})\hat{\eta}_{\mathrm{v}\tilde{\theta}}(\tilde{c})\alpha(\tilde{c}) + \epsilon(\tilde{c})^2 \leq 0$, $2\hat{\rho}(\mathbf{G})\hat{\eta}_{\mathrm{v}\tilde{\theta}}(\tilde{c})\beta(\tilde{c}) + \zeta(\tilde{c})^2 \leq 0$ and
$4(2\hat{\rho}(\mathbf{G})\hat{\eta}_{\mathrm{v}\tilde{\theta}}(\tilde{c})\alpha(\tilde{c}) + \epsilon(\tilde{c})^2)(2\hat{\rho}(\mathbf{G})\hat{\eta}_{\mathrm{v}\tilde{\theta}}(\tilde{c})\beta(\tilde{c}) + \zeta(\tilde{c})^2) \geq (2\hat{\rho}(\mathbf{G})\hat{\eta}_{\mathrm{v}\tilde{\theta}}(\tilde{c})\gamma(\tilde{c}) - \epsilon(\tilde{c})\zeta(\tilde{c}))^2$.

(T2.5) Put $\iota(\tilde{c}) := \hat{\rho}(\mathbf{G})(\hat{\eta}_{\tilde{e}}(\tilde{c}) + \hat{\psi}_{\tilde{2e}}(\tilde{c})) + \frac{1}{\theta}\hat{\mathrm{h}}_{\mathrm{v}\theta}(\tilde{c})\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma})$.
Then $4\hat{\rho}(\mathbf{G})\hat{\eta}_{\mathrm{v}\tilde{\theta}}(\tilde{c})\hat{\mathrm{h}}_{\mathrm{e}}(\tilde{c})\frac{\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma})}{\theta} \geq \iota(\tilde{c})^2$.

(T2.6) Put $\lambda(\tilde{c}) := \frac{1}{2}\hat{\mathrm{s}}_{\mathrm{e1}}(\tilde{c}) - 2\hat{\rho}(\mathbf{G})\hat{\psi}_{\mathrm{e1}}(\tilde{c}) - \frac{2\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma})}{\theta}\hat{\mathrm{h}}_{\mathrm{vG1}}(\tilde{c})$,
$\xi(\tilde{c}) := (\frac{1}{2}\hat{\mathrm{s}}_{\mathrm{e2}}(\tilde{c}) - 2\hat{\rho}(\mathbf{G})\hat{\psi}_{\mathrm{e2}}(\tilde{c}) - \frac{2}{\theta}\hat{\mathrm{h}}_{\mathrm{vG2}}(\tilde{c}))\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma})$,
$\sigma(\tilde{c}) := \frac{1}{\theta}\hat{\mathrm{h}}_{\mathrm{e}}(\tilde{c})\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma})$.
Then $2\sigma(\tilde{c})\alpha(\tilde{c}) + (\lambda(\tilde{c}) + \xi(\tilde{c}))^2 \leq 0$, $2\sigma(\tilde{c})\hat{\mathrm{h}}_{\mathrm{e}}(\tilde{c})\beta(\tilde{c}) + 4\lambda(\tilde{c})^2 \leq 0$ and

$$4\left(2\sigma(\tilde{c})\alpha(\tilde{c}) + (\lambda(\tilde{c}) + \xi(\tilde{c}))^2\right)\left(2\sigma(\tilde{c})\beta(\tilde{c}) + 4\lambda(\tilde{c})^2\right) \geq \left(4(\lambda(\tilde{c}) + \xi(\tilde{c}))\lambda(\tilde{c}) - 2\sigma(\tilde{c})\gamma(\tilde{c})\right)^2.$$

(T2.7) Put $\tau(\tilde{c}) := 2\hat{\eta}_{\mathrm{v}\tilde{\theta}}(\tilde{c})\sigma(\tilde{c})\alpha(\tilde{c}) + \lambda(\tilde{c})\epsilon(\tilde{c})(\lambda(\tilde{c}) + \xi(\tilde{c})) - \lambda(\tilde{c})^2\alpha(\tilde{c}) + \hat{\rho}(\mathbf{G})\hat{\eta}_{\mathrm{v}\tilde{\theta}}(\tilde{c})(\lambda(\tilde{c})$
$+\xi(\tilde{c}))^2 + \sigma(\tilde{c})\epsilon(\tilde{c})^2$,
$\phi(\tilde{c}) := 2\hat{\eta}_{\mathrm{v}\tilde{\theta}}(\tilde{c})\sigma(\tilde{c})\beta(\tilde{c}) - 8\hat{\rho}(\mathbf{G})\lambda(\tilde{c})^2\hat{\eta}_{\mathrm{vG2}}(\tilde{c}) - \lambda(\tilde{c})^2\beta(\tilde{c}) + 4\hat{\rho}(\mathbf{G})\hat{\eta}_{\mathrm{v}\tilde{\theta}}(\tilde{c})\lambda(\tilde{c})^2 + \zeta(\tilde{c})^2\sigma(\tilde{c})$,
$\chi(\tilde{c}) := 2\hat{\eta}_{\mathrm{v}\tilde{\theta}}(\tilde{c})\sigma(\tilde{c})\gamma(\tilde{c}) + \lambda(\tilde{c})(2\epsilon(\tilde{c})\lambda(\tilde{c}) + \zeta(\tilde{c})(\lambda(\tilde{c}) + \xi(\tilde{c}))) - \gamma(\tilde{c})\lambda(\tilde{c})^2$
$+4\hat{\rho}(\mathbf{G})\hat{\eta}_{\mathrm{v}\tilde{\theta}}(\tilde{c})\lambda(\tilde{c})(\lambda(\tilde{c}) + \xi(\tilde{c})) + 2\sigma(\tilde{c})\epsilon(\tilde{c})\zeta(\tilde{c})$.
Then $2\hat{\eta}_{\mathrm{v}\tilde{\theta}}(\tilde{c})\sigma(\tilde{c}) \geq \delta(\tilde{c})\iota(\tilde{c})^2$, $\tau(\tilde{c}) \geq 0$, $\phi(\tilde{c}) \geq 0$ and $4\tau(\tilde{c})\phi(\tilde{c}) \geq \chi(\tilde{c})^2$.

Remark 30.2 The notation $\mu := \hat{\mathrm{s}}_{\mathrm{vG1}}$, $\lambda := 2\hat{\mathrm{s}}_{\mathrm{vG2}}$ and $\kappa := -\hat{\mathrm{h}}_{\mathrm{e}}$ is often used and μ is called the **shear viscosity**, $\frac{2}{3}\mu + \lambda$ is called the **bulk viscosity** and κ is called the **heat conductivity**. The condition $\mu = \hat{\mathrm{s}}_{\mathrm{vG1}} \geq 0$ says the shear viscosity is positive. The conditions $\beta \geq 0$ and $\gamma \geq 0$ together imply that

$$\frac{1}{3}\beta(\tilde{c}) + \frac{2}{3}\gamma(\tilde{c}) = \frac{2}{3}\mu(\tilde{c}) + \lambda(\tilde{c}) + \frac{1}{3}\boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma})\hat{\mathrm{s}}_{\mathrm{vG3}}(\tilde{c}) \geq 0$$

where $\tilde{c} = (\hat{\rho}(\mathbf{G}), \theta, \boldsymbol{\gamma}(\mathbf{G}^{-1}\boldsymbol{\gamma}))$, for all $(\mathbf{G}, \theta, \boldsymbol{\gamma}) \in \mathcal{C}$. When $\boldsymbol{\gamma} = \mathbf{0}$ this says the bulk viscosity is positive; however, in general, this may not be the case. The condition $-\kappa = \hat{\mathrm{h}}_{\mathrm{e}} \leq 0$ says that the heat conductivity is positive. It would be interesting to find out if the other inequalities listed above have physical interpretations. ∎

31 Description in a Frame

Let $\mathcal{B}$ be a continuous body system. As discussed in Chapter I, for every point $X \in \mathcal{B}$ there is a tangent space $\mathcal{T}_X$ of $\mathcal{B}$ at X which is a three dimensional linear space. Assume that every tangent space of $\mathcal{B}$ is given the structure of a thermomechanical element, see Definition 16.1.

Let a frame of reference $\mathcal{F}$, with translation space $\mathcal{V}$, be given. Let a time interval I, as defined in Section 6, and a motion $\bar{\mu} : \mathcal{B} \times I \longrightarrow \mathcal{F}$, as defined in Section 7, a temperature process $\bar{\theta} : \mathcal{B} \times I \longrightarrow \mathbb{P}^{\times}$ and a reference placement κ, as defined at the end of Section 7, be given.

Recall, see the notation introduced in Section 7, that $\bar{\theta}_{\mathrm{s}} : \mathcal{M} \longrightarrow \mathbb{P}^{\times}$ denotes the spatial description of the temperature. Let $X \in \mathcal{B}$ be given. Define

$$\bar{\mathbf{g}}(t) := \nabla\bar{\theta}_{\mathrm{s}}(\mu_t(X), t) \quad \text{and} \quad \bar{\mathbf{g}}'(t) := \nabla\bar{\theta}_{\mathrm{s}}^{\bullet}(\mu_t(X), t) \quad \text{for all} \quad t \in I. \tag{31.1}$$

Using the chain rule and (7.12), one can obtain the relations

$$\bar{\boldsymbol{\gamma}} = \bar{\mathbf{M}}^{\top}\bar{\mathbf{g}} \quad \text{and} \quad \bar{\boldsymbol{\gamma}}^{\bullet} = \bar{\mathbf{M}}^{\top}\left(\bar{\mathbf{L}}_{\mathrm{m}}(X, \cdot)^{\top}\bar{\mathbf{g}} + \bar{\mathbf{g}}'\right). \tag{31.2}$$

Let $t \in I$ be given. Let $\mathbf{T} \in \mathrm{Sym}\mathcal{V}$ and $\mathbf{q} \in \mathcal{V}$ denote the Cauchy stress and heat flux at $x = \mu_t(X)$, respectively, and let $\mathbf{S} \in \mathrm{Sym}(\mathcal{T}_X^*, \mathcal{T}_X)$ and $\mathbf{h} \in \mathcal{T}_X$ denote the intrinsic stress and intrinsic heat flux at $X \in \mathcal{B}$ and $t \in I$, respectively. Using $\mathbf{M} := \nabla_X\mu_t$ which is an isomorphism between $\mathcal{T}_X$ and $\mathcal{V}$, see (7.12), we have the relations (see (9.18) and (11.5))

$$\mathbf{T} = \mathbf{M}\mathbf{S}\mathbf{M}^{\top}, \qquad \mathbf{q} = \mathbf{M}\mathbf{h}. \tag{31.3}$$

Now assume that $\mathcal{T}_X$ is a thermoelasto-viscous element, as defined in Definition 29.1. By using the relations in (31.1)–(31.3) one can construct constitutive laws that depend on the frame $\mathcal{F}$ from the frame-free laws given in Section 29. Recalling that $\mathbf{K} = \nabla_X\kappa$, put

$$\mathcal{L} := \{(\mathbf{F}, \theta, \mathbf{g}) \in \mathrm{Lis}\mathcal{V} \times \mathbb{P}^{\times} \times \mathcal{V} \mid (\mathbf{K}^{\top}\mathbf{F}^{\top}\mathbf{F}\mathbf{K}, \theta, \mathbf{K}^{\top}\mathbf{F}^{\top}\mathbf{g}) \in \mathcal{C}\}.$$

We can define the constitutive laws for the stress and heat flux in the frame $\mathcal{F}$, using the abbreviation $\mathbf{M} = \mathbf{F}\mathbf{K}$, by

$$\begin{aligned} \hat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{g}, \mathbf{L}, \theta', \mathbf{g}') &:= \mathbf{M}\hat{\mathbf{S}}(\mathbf{M}^{\top}\mathbf{M}, \theta, \mathbf{M}^{\top}\mathbf{g}, \mathbf{M}^{\top}(\mathbf{L}^{\top} + \mathbf{L})\mathbf{M}, \theta', \mathbf{M}^{\top}(\mathbf{L}^{\top}\mathbf{g} + \mathbf{g}'))\mathbf{M}^{\top}, \\ \hat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{g}, \mathbf{L}, \theta', \mathbf{g}') &:= \mathbf{M}\hat{\mathbf{h}}(\mathbf{M}^{\top}\mathbf{M}, \theta, \mathbf{M}^{\top}\mathbf{g}, \mathbf{M}^{\top}(\mathbf{L}^{\top} + \mathbf{L})\mathbf{M}, \theta', \mathbf{M}^{\top}(\mathbf{L}^{\top}\mathbf{g} + \mathbf{g}')), \end{aligned} \tag{31.4}$$

for all $(\mathbf{F}, \theta, \mathbf{g}) \in \mathcal{L}$ and $(\mathbf{L}, \theta', \mathbf{g}') \in \mathrm{Lin}\mathcal{V} \times \mathbb{R} \times \mathcal{V}$. Similar formulas hold for the constitutive laws for the entropy and free energy. For these two we will use the same symbol to denote the constitutive law in the frame as we did for the frame-free constitutive law. It is a simple exercise to show that the constitutive laws defined in (31.4) satisfy the principle of material frame-indifference.

32 Governing Equations for Fluids

None of the mappings listed in the next two sections are in the material description. To avoid clutter all subscripts "s" are dropped.

Let a continuous body system $\mathcal{B}$ and a frame of reference $\mathcal{F}$ be given, as in the last section. Assume that $\mathcal{B}$ is a fluid homogeneous thermoelasto-viscous material, so that each $\mathcal{T}_X$ has the structure of a fluid thermoelastic-viscous element and that all of these elements are materially isomorphic. Then (31.4) can be used to place all of the terms of the intrinsic stress, see (30.9), (30.13) and (30.14), and the intrinsic heat flux, see (30.15), into the frame $\mathcal{F}$. One finds that the stress and heat flux only depend on the velocity gradient through its symmetric part, $\mathbf{D}$, and that the heat flux only depends on the rate of change of the temperature gradient $\mathbf{g}'$ through[3] $\tilde{\mathbf{g}}' := \mathbf{L}^\top\mathbf{g} + \mathbf{g}'$ (see $(31.2)_2$):

$$\begin{aligned}
&\hat{\mathbf{T}}_{\mathrm{e}}(\mathbf{F},\theta,\mathbf{g}) = \hat{\mathrm{s}}_{e1}(\tilde{c})\mathbf{1}_{\mathcal{V}} + \hat{\mathrm{s}}_{e2}(\tilde{c})\mathbf{g}\otimes\mathbf{g},\\
&\hat{\mathbf{T}}_{\mathrm{vG}}(\mathbf{F},\theta,\mathbf{g})\mathbf{D} = 2\hat{\mathrm{s}}_{\mathrm{vG}1}(\tilde{c})\mathbf{D} + 2\hat{\mathrm{s}}_{\mathrm{vG}2}(\tilde{c})\mathrm{tr}(\mathbf{D})\mathbf{1}_{\mathcal{V}} + 2\hat{\mathrm{s}}_{\mathrm{vG}3}(\tilde{c})\mathrm{tr}(\mathbf{D})\mathbf{g}\otimes\mathbf{g}\\
&\qquad\qquad +2\hat{\mathrm{s}}_{\mathrm{vG}4}(\tilde{c})\left[\mathbf{D}(\mathbf{g}\otimes\mathbf{g}) + (\mathbf{g}\otimes\mathbf{g})\mathbf{D}\right],\\
&\hat{\mathbf{T}}_{\mathrm{v}\theta}(\mathbf{F},\theta,\mathbf{g})\theta' = \hat{\mathrm{s}}_{\mathrm{v}\theta1}(\tilde{c})\theta' 1_{\mathcal{V}} + \hat{\mathrm{s}}_{\mathrm{v}\theta2}(\tilde{c})\theta'(\mathbf{g}\otimes\mathbf{g}),
\end{aligned} \tag{32.1}$$

$$\begin{aligned}
&\hat{\mathbf{q}}_{\mathrm{e}}(\mathbf{F},\theta,\mathbf{g}) = \hat{\mathrm{h}}_{\mathrm{e}}(\tilde{c})\mathbf{g},\\
&\hat{\mathbf{q}}_{\mathrm{vG}}(\mathbf{F},\theta,\mathbf{g})\mathbf{D} = 2\hat{\mathrm{h}}_{\mathrm{vG}1}(\tilde{c})\mathrm{tr}(\mathbf{D})\mathbf{g} + 2\hat{\mathrm{h}}_{\mathrm{vG}2}(\tilde{c})\mathbf{D}\mathbf{g},\\
&\hat{\mathbf{q}}_{\mathrm{v}\theta}(\mathbf{F},\theta,\mathbf{g})\theta' = \hat{\mathrm{h}}_{\mathrm{v}\theta}(\tilde{c})\theta'\mathbf{g},\\
&\hat{\mathbf{q}}_{\mathrm{v}\gamma}(\mathbf{F},\theta,\mathbf{g})\tilde{\mathbf{g}}' = \hat{\mathrm{h}}_{\mathrm{vg}1}(\tilde{c})\tilde{\mathbf{g}}' + \hat{\mathrm{h}}_{\mathrm{vg}2}(\tilde{c})\left(\tilde{\mathbf{g}}'\cdot\mathbf{g}\right)\mathbf{g}.
\end{aligned} \tag{32.2}$$

where $\tilde{c} = (\hat{\rho}(\mathbf{F}),\theta,|\mathbf{g}|)$, for all $(\mathbf{F},\theta,\mathbf{g}) \in \mathcal{L}$ and $(\mathbf{D},\theta',\tilde{\mathbf{g}}') \in \mathrm{Sym}\mathcal{V}\times\mathbb{R}\times\mathcal{V}$.

The different parts of the specific entropy in (30.16) and the specific free energy in (30.17) are given by:

$$\begin{aligned}
&\hat{\eta}_{\mathrm{e}}(\mathbf{F},\theta,\mathbf{g}) = \hat{e}_{\tilde{e}}(\tilde{c})\\
&\hat{\eta}_{\mathrm{vG}}(\mathbf{F},\theta,\mathbf{g})\mathbf{D} = 2\hat{e}_{\mathrm{vG}1}(\tilde{c})\mathbf{D}\mathbf{g}\cdot\mathbf{g} + 2\hat{e}_{\mathrm{vG}2}(\tilde{c})\mathrm{tr}(\mathbf{D})\\
&\hat{\eta}_{\mathrm{v}\theta}(\mathbf{F},\theta,\mathbf{g})\theta' = \hat{e}_{\tilde{\mathrm{v}\theta}}(\tilde{c})\theta'
\end{aligned} \tag{32.3}$$

$$\begin{aligned}
&\hat{\psi}_{\mathrm{e}}(\mathbf{F},\theta,\mathbf{g}) = \hat{\psi}_{\tilde{e}}(\tilde{c})\\
&\nabla_{(1)}\hat{\psi}_{\mathrm{e}}(\mathbf{F},\theta,\mathbf{g}) = \hat{\psi}_{\mathrm{e}1}(\tilde{c})\mathbf{1}_{\mathcal{V}} + \hat{\psi}_{\mathrm{e}2}(\tilde{c})(\mathbf{g}\otimes\mathbf{g})\\
&\nabla_{(2)}\hat{\psi}_{\mathrm{e}}(\mathbf{F},\theta,\mathbf{g}) = \hat{\psi}_{\tilde{2e}}(\tilde{c})
\end{aligned} \tag{32.4}$$

where $\tilde{c} = (\hat{\rho}(\mathbf{F}),\theta,|\mathbf{g}|)$, for all $(\mathbf{F},\theta,\mathbf{g}) \in \mathcal{L}$ and $(\mathbf{D},\theta') \in \mathrm{Sym}\mathcal{V}\times\mathbb{R}$.

Let the mappings

$$\begin{aligned}
&\bar{\mathbf{T}} : \mathcal{M} \longrightarrow \mathrm{Sym}\mathcal{V}, \quad \bar{\eta} : \mathcal{M} \longrightarrow \mathbb{R}, \quad \bar{\mathbf{q}} : \mathcal{M} \longrightarrow \mathcal{V}, \quad \bar{\psi} : \mathcal{M} \longrightarrow \mathbb{R},\\
&\bar{r} : \mathcal{M} \longrightarrow \mathbb{R}, \quad \bar{\mathbf{b}} : \mathcal{M} \longrightarrow \mathcal{V}, \quad \bar{\rho} : \mathcal{M} \longrightarrow \mathbb{R},
\end{aligned} \tag{32.5}$$

give the Cauchy stress, specific entropy, heat flux, specific free energy, external heat absorption, external body force per unit mass and the mass density on $\mathcal{M}$, respectively. These are

[3] This is the covariant rate of $\mathbf{g}$. It is also known as the Cotter–Rivlin rate or the lower-convected rate.

related through three fundamental laws: *balance of forces* (9.3), *balance of energy* (11.8) and *balance of mass* (10.8)

$$\mathrm{div}\ \bar{\mathbf{T}} + \bar{\mathbf{b}} = \mathbf{0}, \tag{32.6}$$

$$\mathrm{tr}(\bar{\mathbf{T}}\bar{\mathbf{D}}) + \bar{r}\bar{\rho} = \mathrm{div}\ \bar{\mathbf{q}} + \bar{\rho}(\bar{\psi} + \bar{\eta}\bar{\theta})^\bullet \tag{32.7}$$

$$\bar{\rho}^\bullet + \mathrm{div}(\bar{\mathbf{v}}\bar{\rho}) = 0. \tag{32.8}$$

The mappings $\bar{\mathbf{T}}, \bar{\eta}, \bar{\mathbf{q}}$ and $\bar{\psi}$ are determined by $\bar{\rho}$ and $\bar{\theta}$ using the constitutive relations given in (32.1)-(32.4). The mappings $\bar{r}$ and $\bar{\mathbf{b}}$ are specified by external constitutive laws. For example, a common set of constitutive laws for $\bar{r}$ and $\bar{\mathbf{b}}$ are

$$\bar{r} = 0 \quad \text{and} \quad \bar{\mathbf{b}} = \bar{\rho}(\mathbf{f}_{\mathrm{g}} - \bar{\mathbf{v}}^\bullet - (\nabla\bar{\mathbf{v}})\bar{\mathbf{v}})$$

where $\mathbf{f}_{\mathrm{g}}$ is the specific force of gravity on the surface of the Earth. However, inside a microwave oven or in the presence of intense sunshine $\bar{r} \neq 0$. When the frame of reference isn't inertial, for example if the frame of reference is determined by the walls of a centrifuge, then $\bar{\mathbf{b}}$ must include centrifugal and Coriolis forces.

Once all of the constitutive laws have been specified (32.6)-(32.8) is a system of partial differential equations that involve $\bar{\theta}, \bar{\rho}$ and $\bar{\mathbf{v}}$. Explicitly, (32.6) and (32.7) become

$$\begin{aligned}
&(\nabla\bar{\theta} \otimes \nabla\bar{\theta})\left(\nabla\bar{\mathrm{T}}_{\mathrm{e2}} + 2\mathrm{tr}(\bar{\mathbf{D}})\nabla\bar{\mathrm{T}}_{\mathrm{vG3}} + 2\bar{\mathrm{T}}_{\mathrm{vG3}}\nabla(\mathrm{tr}\nabla\bar{\mathbf{v}}) + \bar{\theta}^\bullet\nabla\bar{\mathrm{T}}_{\mathrm{v}\theta 2} + \bar{\mathrm{T}}_{\mathrm{v}\theta 2}\nabla\bar{\theta}^\bullet + 4\bar{\mathrm{T}}_{\mathrm{vG4}}\Delta\bar{\mathbf{v}}\right)\\
&+ \left(4\bar{\mathrm{T}}_{\mathrm{vG4}}\bar{\mathbf{D}} + \bar{\mathrm{T}}_{\mathrm{e2}}\mathbf{1}_{\mathcal{V}} + \bar{\mathrm{T}}_{\mathrm{v}\theta 2}\bar{\theta}^\bullet\mathbf{1}_{\mathcal{V}} + 2\bar{\mathrm{T}}_{\mathrm{vG3}}\mathrm{tr}(\bar{\mathbf{D}})\mathbf{1}_{\mathcal{V}}\right)\left(\nabla\bar{\theta}\Delta\bar{\theta} + \nabla^{(2)}\bar{\theta}\nabla\bar{\theta}\right)\\
&+\nabla\bar{\mathrm{T}}_{\mathrm{v}\theta 1}\bar{\theta}^\bullet + \bar{\mathrm{T}}_{\mathrm{v}\theta 1}\nabla\bar{\theta}^\bullet + 2\left((\nabla\bar{\theta}\cdot\nabla\bar{\mathrm{T}}_{\mathrm{vG4}})\bar{\mathbf{D}} + (\bar{\mathbf{D}}\nabla\bar{\theta})\cdot\nabla\bar{\mathrm{T}}_{\mathrm{vG4}}\mathbf{1}_{\mathcal{V}}\right)\nabla\bar{\theta}\\
&+2\left(\bar{\mathrm{T}}_{\mathrm{vG1}}\Delta\bar{\mathbf{v}} + \bar{\mathbf{D}}\nabla\bar{\mathrm{T}}_{\mathrm{vG1}} + \nabla\bar{\mathrm{T}}_{\mathrm{vG2}}\mathrm{tr}(\bar{\mathbf{D}}) + \mathrm{tr}(\nabla^{(2)}\bar{\mathbf{v}}(\nabla\bar{\mathrm{T}}_{\mathrm{vG2}}))\right) + \nabla\bar{\mathrm{T}}_{\mathrm{e1}} + \bar{\mathbf{b}} = \mathbf{0}
\end{aligned} \tag{32.9}$$

and

$$\begin{aligned}
&\left(\bar{\mathrm{T}}_{\mathrm{e1}} + \bar{\mathrm{T}}_{\mathrm{v}\theta 1}\bar{\theta}^\bullet + 2\bar{\mathrm{T}}_{\mathrm{vG3}}\nabla\bar{\theta}\cdot\bar{\mathbf{D}}\nabla\bar{\theta}\right)\mathrm{tr}(\bar{\mathbf{D}}) + 2\left(\bar{\mathrm{T}}_{\mathrm{vG1}}\mathrm{tr}(\bar{\mathbf{D}}^2) + \bar{\mathrm{T}}_{\mathrm{vG2}}(\mathrm{tr}\bar{\mathbf{D}})^2\right)\\
&+ \left(\bar{\mathrm{T}}_{\mathrm{e2}} + \bar{\mathrm{T}}_{\mathrm{v}\theta 2}\bar{\theta}^\bullet\right)(\nabla\bar{\theta}\cdot\bar{\mathbf{D}}\nabla\bar{\theta}) + 4\bar{\mathrm{T}}_{\mathrm{vG4}}|\bar{\mathbf{D}}\nabla\bar{\theta}|^2 + \bar{r}\bar{\rho}\\
&= \left[\bar{\mathbf{h}}_{\mathrm{e}} + 2\bar{\mathbf{h}}_{\mathrm{vG1}}\mathrm{tr}\bar{\mathbf{D}} + \bar{\mathbf{h}}_{\mathrm{vG2}}(\bar{\mathbf{L}}\nabla\bar{\theta} + \nabla\bar{\theta}^\bullet)\cdot\nabla\bar{\theta}\right]\Delta\bar{\theta}\\
&\qquad + \Big[\nabla\bar{\mathbf{h}}_{\mathrm{e}} + 2\mathrm{tr}(\bar{\mathbf{D}})\nabla\bar{\mathbf{h}}_{\mathrm{vG1}} + 2(\bar{\mathbf{h}}_{\mathrm{vG1}} + \bar{\mathbf{h}}_{\mathrm{vG2}})\Delta\mathbf{v} + 2\bar{\mathbf{D}}\nabla\bar{\mathbf{h}}_{\mathrm{vG2}} + \bar{\theta}^\bullet\nabla\bar{\mathbf{h}}_{\mathrm{v}\theta}\\
&\qquad +\bar{\mathbf{h}}_{\mathrm{v}\theta}\nabla\bar{\theta}^\bullet + \bar{\mathbf{h}}_{\mathrm{vG1}}\mathrm{div}\ \bar{\mathbf{L}} + \bar{\mathbf{L}}\nabla\bar{\mathbf{h}}_{\mathrm{vG1}} + (\nabla\bar{\theta}\cdot\nabla\bar{\theta}^\bullet)\nabla\bar{\mathbf{h}}_{\mathrm{vG2}}\Big]\cdot\nabla\bar{\theta}\\
&\qquad +(2\bar{\mathbf{h}}_{\mathrm{vG2}}\bar{\mathbf{D}} + \bar{\mathbf{h}}_{\mathrm{vG1}}\bar{\mathbf{L}})\cdot\nabla^{(2)}\bar{\theta} + \bar{\mathbf{h}}_{\mathrm{vG1}}\Delta\bar{\theta}^\bullet + \nabla\bar{\mathbf{h}}_{\mathrm{vG1}}\cdot\nabla\bar{\theta}^\bullet\\
&\qquad +\bar{\mathbf{h}}_{\mathrm{vG2}}\left[(\nabla\bar{\mathbf{L}}\nabla\bar{\theta} + \nabla^{(2)}\bar{\theta}^\bullet)\nabla\bar{\theta} + \nabla^{(2)}\bar{\theta}(\bar{\mathbf{L}}\nabla\bar{\theta} + \nabla\bar{\theta}^\bullet)\right]\cdot\nabla\bar{\theta}\\
&\qquad +\bar{\rho}\hat{\psi}^\bullet_{\tilde{e}} - \bar{\rho}\bar{\theta}^\bullet\left(\bar{\eta}_{\tilde{e}} + 2\bar{\eta}_{\mathrm{vG1}}|\nabla\bar{\theta}|^2 + 2\bar{\eta}_{\mathrm{vG2}}\mathrm{tr}\bar{\mathbf{D}} + \bar{\eta}_{\mathrm{v}\theta}\bar{\theta}^\bullet\right)\\
&\qquad -\bar{\rho}\bar{\theta}\left[\bar{\eta}^\bullet_{\tilde{e}} + 2\bar{\eta}^\bullet_{\mathrm{vG1}}|\nabla\bar{\theta}|^2 + 4\bar{\eta}_{\mathrm{vG1}}\nabla\bar{\theta}\cdot\nabla\bar{\theta}^\bullet + 2\bar{\eta}^\bullet_{\mathrm{vG2}}\mathrm{tr}\bar{\mathbf{D}} + 2\bar{\eta}_{\mathrm{vG2}}\mathrm{tr}\bar{\mathbf{D}}^\bullet + \bar{\eta}^\bullet_{\tilde{\mathrm{v}\theta}}\bar{\theta}^\bullet + \bar{\eta}_{\tilde{\mathrm{v}\theta}}\bar{\theta}^{\bullet\bullet}\right]
\end{aligned} \tag{32.10}$$

where $\bar{\mathrm{T}}_{\mathrm{e1}} : \mathcal{M} \longrightarrow \mathbb{R}$ is defined by

$$\bar{\mathrm{T}}_{\mathrm{e1}} := \hat{\mathrm{s}}_{\mathrm{e1}} \circ (\bar{\rho}, \bar{\theta}, \nabla\bar{\theta})$$

and similarly for the other coefficient functions. (32.8) and (32.10) are scalar equations while (32.9) is a vector equation. These equations generalize other governing equations that have been proposed, for example, the Navier–Stokes equations.

Before one can attempt to solve the above equations for $\bar{\theta}, \bar{\rho}$ and $\bar{\mathbf{v}}$ initial and boundary conditions must be specified. Frequently this is done by specifying the sets $\mathcal{B}_t$, $t \in I$, and hence the set $\mathcal{M}$. If the fluid is confined to a container then the sets $\mathcal{B}_{\bar{\mu}_t}$, $t \in I$, will all be the same and be equal to the interior of the container. Of course, specifying these initial and boundary conditions does not guarantee that the above equations have a solution.

33 Needed Result

Proposition 33.1 *Let $a, b, c, d, e, f, g, h, i \in \mathbb{R}$ be given. Then the following are equivalent:*

(i)

$$ax^2 + by^2 + cxy + dxz + eyz + fx + gy + hz + i \geq 0 \quad \text{for all} \quad x, y, z \in \mathbb{R}. \tag{33.1}$$

(ii) The numbers a, b, c, d, e, f, g, h and i satisfy the following restrictions:

$$h = d = e = 0, \tag{33.2}$$

$$a \geq 0,\ b \geq 0,\ i \geq 0, \tag{33.3}$$

$$4ab \geq c^2,\ 4ai \geq f^2,\ 4bi \geq g^2, \tag{33.4}$$

$$(4ab - c^2)i \geq ag^2 + bf^2 - cfg. \tag{33.5}$$

Chapter V

Outlook

This thesis is by no means the final word on frame-free continuum thermomechanics. In fact, it should only be considered as a starting point. In Section 1, there were listed six items that were not addressed in this thesis. Each of these issues should be addressed within the framework presented here. By doing this one would obtain a unified and frame-free framework for all of continuum physics.

In Chapter II the theory of simple materials was discussed in detail. However, not all materials are "simple" in that the constitutive laws at a material point may not involve only the body element (tangent space) at that point. One example of such materials are materials of grade two. The study of such materials goes all the way back to Cauchy [EMS], who considered second grade fluids. These kinds of materials were given a rigorous mathematical foundation and studied in [FMCA]. Unfortunately, this formulation is neither frame-free nor is it based on the concept of a state space.

In [FC], Noll briefly outlines what kind of mathematical structures would be necessary to develop such a theory. For materials of grade two, one must consider second order tangent structures in differentiable manifolds. For a detailed discussion of these structures see [FC]. Another crucial concept needed for materials of grade two is that of a couple stress. While such stresses are ruled out for simple materials [PSI], Noll conjectured, in [FC], that materials of grade two cannot be described with stresses alone but must include couple stresses.

The treatment of simple materials with memory given in Chapter III has two issues that should be looked into. In Section 23 a topology on the set of condition processes is proposed. In this topology, there is no notion of "closeness" for processes of different duration. Many materials have responses that are similar even when subjected to processes of different duration. A suitable topology on the the set of processes should capture this effect.

The other issue deals with the approximation theorem, Theorem 26.8. This theorem only holds for processes that are of class C^2 while all of the other material in Chapters II and III only deal with piece-wise C^1 processes. It would be much more satisfying if the C^2 condition of the theorem could be weakened to only C^1 or possibly only piece-wise C^1. This may be possible by using a weighted point-wise bounded variation of the process and its derivative instead of its speed and acceleration. This should be investigated.

There is also more work that could be done regarding consequences of the second law of thermodynamics. In earlier work on materials with fading memory, see [TMM] and [TMM2] for example, it has been shown that in addition to the restrictions found in Section 25 one finds that the free energy of a relaxed state with zero temperature gradient is minimal over all states that relax to that state and that the equilibrium response functions obey the familiar relations of thermostatics. My conjecture is that results of this type should also hold in the framework presented in this thesis. This conjecture should be verified in the future.

Given a fluid thermoelasto-viscous material, as defined in Chapter IV, one should design experiments that will enable one to determine the coefficient functions involved in (32.9) and (32.10). Due to the large number of coefficient functions this will no doubt require a large number of experiments to determine all of them.

The equations (32.8)-(32.10) are so complex that it is difficult to solve them in general. As in the case of the Navier–Stokes equations, it may be possible to analyze various special cases of the above equations. An assumption that is appropriate for many fluids is *incompressibility*. In this case one assumes that the density is constant and the terms in (32.1) give the *extra stress*, i.e., the traceless part of the stress. It follows from (32.8) that we must also have div $\bar{\mathbf{v}} = 0$. It has also been suggested, see [OBA], to consider the case where the material is mechanically incompressible but thermally compressible. In this case div $\bar{\mathbf{v}}$ is not zero, but assumed to be a function of temperature. One could also consider a *no motion* case in which $\bar{\mathbf{v}} = \mathbf{0}$ and the sets $\mathcal{B}_{\bar{\mu}_t}$, $t \in I$, are all the same. This case would just study the effect of heat transfer through the material. Yet another situation is that of a *steady-state* solution. Here one assumes that the spatial time derivatives of $\bar{\mathbf{v}}$ and $\bar{\theta}$ are zero, i.e., $\bar{\mathbf{v}}^{\bullet} = \mathbf{0}$ and $\bar{\theta}^{\bullet} = 0$, and the sets $\mathcal{B}_{\bar{\mu}_t}$, $t \in I$, are all the same.

It is the hope that these results can serve to develop new and better mathematical models to deal with phenomena such as heat transfer (both by conduction and convection), heat pipes, heat exchanges, thermosiphons, Bénard cells, heat sinks and thermophoresis.

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