Algebraic models of sets and classes in categories of ideals

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Abstract

We introduce a new sheaf-theoretic construction called the *ideal completion* of a category and investigate its logical properties. We show that it satisfies the axioms for a category of classes in the sense of Joyal and Moerdijk [17], so that the tools of algebraic set theory can be applied to produce models of various elementary set theories. These results are then used to prove the conservativity of different set theories over various classical and constructive type theories.

1 Introduction

It is well known that various type theories may be modelled in certain kinds of categories (cf. [15]). For instance, cartesian closed categories are models of the typed lambda calculus and toposes are models of intuitionistic higher order logic (IHOL). Similarly, Joyal and Moerdijk [17] showed that one can axiomatize a notion of *small map* in a category in such a way that the resulting category will contain an algebraic model of elementary set theory. In this paper we employ a new sheaf theoretic construction called the *ideal completion* in order to relate algebraic models of type theories with category theoretic models of set theories for several type theories and set theories. We also investigate such sheaf models of class theories. Using these sheaf theoretic methods we obtain conservativity results for set theories over type theories and class theories over set theories.

The ideal completion is a category theoretic analogue of the familiar ideal completion of a partially ordered set and has its roots in the related *Ind-completion* studied by the Grothendieck school of algebraic geometry [6]. In particular, the ideal completion $\mathbf{Idl}(\mathcal{C})$ of a category $\mathcal{C}$ is obtained as the subcategory of sheaves $\mathbf{Sh}(\mathcal{C})$ on $\mathcal{C}$ consisting of certain colimits of representable functors. Here the representables are generalizing principal ideals in a partially ordered set and these colimits, directed joins. Now, if $\mathcal{C}$ is a category with sufficient structure to model the type theory in question, the category $\mathbf{Idl}(\mathcal{C})$ will then model an untyped set theory. The three specific type theories under consideration are (typed) first-order intuitionistic logic, a form of extensional

Type Theory	Category	Set Theory	Category
FOL	Heyting pretopos	**BCST**	Basic category of classes
DTT	Π-pretopos	**CST**	Category of classes
IHOL	Topos	**BIST**	Powered category of classes

Table 1: Types, sets and categories.

Martin-Löf style dependent type theory (DTT) and the theory IHOL mentioned above. More specifically, we consider the ideal completion of categories of the following three kinds: Heyting pretoposes (suitable models of first-order intuitionistic logic), Π-pretoposes (models of DTT), and toposes. Here a Π-pretopos is a locally cartesian closed pretopos and is not assumed to possess a universe or arbitrary W-types (cf. [27]).

In summary, in Section 2 we introduce the set theories **BCST** (*basic constructive set theory*), **CST** (*constructive set theory*) and **BIST** (*basic intuitionistic set theory*) and prove that the categories of sets obtained from these theories are, respectively, a Heyting pretopos, a Π-pretopos, and a topos. Note though that the theory **CST** studied in this paper is different from Myhill's **CST** [21]. Much of this section either reviews results from [8] or generalizes them. We then introduce the class theories **BICT** (*basic intuitionistic class theory*) and **BIMK** (*basic intuitionistic Morse-Kelley*) and compare them with the familiar class theories of *von Neumann-Gödel-Bernays* **NGB** and *Morse-Kelley*. In Section 3 the axiomatic theory of categories of classes is developed and soundness and completeness results for the set theories with respect to these categories are obtained. In Section 4 the *ideal completion* of a category is defined and its fundamental properties are developed. It is also shown that if $\mathcal{C}$ is a Heyting pretopos (respectively, a Π-pretopos or a topos), then $\mathbf{Idl}(\mathcal{C})$ contains a category of classes (of the corresponding form). In particular, by the results in Section 3 it then has suitable structure to model **BCST** (respectively, **CST** or **BIST**). Table 1 summarizes the relationships between the type theories, their category theoretic models, the corresponding set theories and the category theoretic models of the set theory. This table is included here as a point of reference for the reader to return to throughout the paper. Finally, in Section 5 we indicate how the inclusion of the ideal completion $\mathbf{Idl}(\mathcal{C})$ in sheaves $\mathbf{Sh}(\mathcal{C})$ on $\mathcal{C}$ may be used to model theories of sets and classes. As an example, it is shown that the theory **BIMK** (in which the sets are classical and the classes are intuitionistic) is conservative over ordinary Zermelo Fraenkel set theory **ZF**.

Related research

We mention some of the research which is most closely related to this paper. The work of Joyal and Moerdijk [17] served to provide, via the notion of small map and category of classes, a general template from which a variety of different problems and set theories could be investigated. One initial application of this approach was the study of predicative set theories by Moerdijk and Palmgren [19, 20] who considered the set theory **CZF** (*Constructive Zermelo-Fraenkel set theory*) and related axioms. The set theory **CZF** was introduced by Aczel (cf. [2, 3, 4]) who showed that it can be interpreted in the dependent type theory $\mathbf{ML_1V}$. The theory $\mathbf{ML_1V}$ is obtained by augmenting basic extensional DTT with a universe V which is also W-type. $\mathbf{ML_1V}$ is, accordingly, stronger than the predicative type theories considered in the present paper. The work of Moerdijk and Palmgren may be understood as extending Aczel's type theoretic interpretation using the tools of category theory. This undertaking yielded, among other things, both a category theoretic account of W-types (cf. [19]) and an algebraic model of **CZF** (cf. [20]).

Subsequent research by Simpson [26] and Butz [11] helped, among other things, to develop both a modified collection of axioms for small maps and a syntactic category approach to proving completeness theorems for set theories. This paper, however, is most closely related to the joint work of Awodey, Butz, Simpson and Streicher [8]. In [8] the set theory **BIST** is introduced and it is shown that **BIST** may be modelled in what we have called *powered categories of classes*.[1] Additionally, it is there shown that any topos $\mathcal{E}$ occurs as the 'sets' in a powered category of classes. Following an idea of Simpson, this is proved by means of a construction which is also called the ideal completion and proceeds in two steps. First, a topos $\mathcal{E}$ is endowed with a system $\mathcal{I}$ of inclusions. These inclusions then permit the definition of (order) ideals. The resulting category $\mathbf{Idl}_{\mathcal{I}}(\mathcal{E})$ consisting of ideals depends essentially on the system of inclusions $\mathcal{I}$. Finally, it is shown that **BIST** is sound and complete with respect to models in categories of the form $\mathbf{Idl}_{\mathcal{I}}(\mathcal{E})$. Although this original version of the ideal completion from [8] had some advantages, a more canonical construction is clearly desired. The Ind-completion and the ideal completion studied in this paper represent such an improvement. It was developed jointly by Rummelhoff [22], Awodey, Forssell and Warren.

The present paper is a summary of the results obtained in the two theses [12] and [27]. Conference versions of these theses appeared in the proceedings of the CT2004 conference as [9] and [10]. Additional attributions of particular results or ideas are provided in the text.

We also mention recent work by Gambino [14] studying presheaf models of constructive set theories. One interesting aspect of this work is that it serves to relate the approach of Joyal and Moerdijk with that of Scott [24]. Finally, the interested reader is referred to [12] or [27] for a more thorough accounting of related research. The reader should also consult [25] or [7] for an accessible overview of the field. The resource [1] provides access to current developments in the field.

[1] The terminology in this area is not yet standardized and we here prefer 'powered categories of classes' to 'categories of classes with powersets' so as to avoid the unpleasant locution 'categories with class structure and powerobjects' when dealing with weaker forms of such categories.

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2 Theories of sets and classes

The theories of sets and classes which we consider are formulated in two languages. First, by a *set theory* we mean a single sorted first-order theory in the *language of set theory* $\mathcal{L}_s := \{\mathsf{S}, \in\}$ where S ('sethood') and $\in$ ('membership') are, respectively, unary and binary predicates. By a *class theory* we mean a two-sorted theory in the *language of class theory* $\mathcal{L}_c := \{\mathsf{S}, \in, \eta\}$. To say that class theory is *two-sorted* means that there are two sorts of variable: *set variables* which are written as lowercase Roman letters, and *class variables* which are written as uppercase Roman letters. There are quantifiers for each sort of variable. The atomic formulae are then stipulated to be those of the form:

$$\mathsf{S}(x),\ x \in y,\ \text{and } x\eta Y,$$

where x and y are any set variables and Y is any class variable. In both cases S is included in the language because we intend to allow urelements or non-sets. The background logic of all theories is taken to be intuitionistic first-order logic unless otherwise mentioned. Where φ is a formula, $\mathrm{FV}(\varphi)$ denotes the set of free variables of φ. We will freely employ the class notation $\{x|\varphi\}$ as in common set theoretical practice. Frequently it will be efficacious to employ *bounded quantification* which is defined as usual; namely, $\forall x \in a.\varphi$ means:

$$\mathsf{S}(a) \wedge \forall x.x \in a \Rightarrow \varphi$$

and $\exists x \in a.\varphi$ means:

$$\mathsf{S}(a) \wedge \exists x.x \in a \wedge \varphi.$$

A formula φ is called Δ_0 if all of its quantifiers are bounded.

Another notational convenience is the introduction of the *set-many quantifier* Ƨ defined as:

$$\text{Ƨ}x.\varphi \quad := \quad \exists y.(\mathsf{S}(y) \wedge \forall x.(x \in y \Leftrightarrow \varphi)), \tag{1}$$

where $y \notin \mathrm{FV}(\varphi)$. We also write:

$$x \subseteq y \quad := \quad \mathsf{S}(x) \wedge \mathsf{S}(y) \wedge \forall z \in x.z \in y.$$

We employ the abbreviation $\mathrm{func}(f, a, b)$ to indicate that f is a functional relation on $a \times b$:

$$\mathrm{func}(f, a, b) \quad := \quad f \subseteq a \times b \wedge \forall x \in a.\exists! y \in b.(x, y) \in f$$

Membership $x \in a \Rightarrow \mathsf{S}(a)$.

Extensionality $(a \subseteq b \wedge b \subseteq a) \Rightarrow a = b$.

Emptyset $\text{Ƨ}z.\bot$.

Pairing $\text{Ƨ}z.z = x \vee z = y$.

Binary Intersection $\mathsf{S}(a) \wedge \mathsf{S}(b) \Rightarrow \text{Ƨ}z.z \in a \wedge z \in b$.

Union $(\forall x \in a.\mathsf{S}(x)) \Rightarrow \text{Ƨ}z.\exists x \in a.z \in x$.

Replacement $\forall x \in a.\exists! y.\varphi \Rightarrow \text{Ƨ}y.\exists x \in a.\varphi$, for any formula φ.

Exponentiation $\mathsf{S}(a) \wedge \mathsf{S}(b) \Rightarrow \text{Ƨ}z.\,\mathrm{func}(z, a, b)$.

Powerset $\mathsf{S}(a) \Rightarrow \text{Ƨ}y.y \subseteq a$.

Δ_0-Separation $\mathsf{S}(a) \Rightarrow \text{Ƨ}z.z \in a \wedge \varphi$, if φ is a Δ_0 formula.

Table 2: Axioms for Set Theories

2.1 Set theories

The particular set theories with which we will be primarily concerned are presented in Table 3. In Table 3 we employ a solid bullet $\bullet$ to indicate that the axiom in question is one of the axioms of the theory and a hollow bullet $\circ$ to indicate a consequence of the axioms. For the sake of brevity the obvious universal quantifiers in the axioms have been omitted. I.e., the axioms should be understood as the universal closures of the formulae enumerated in Table 2.

There are several points worth mentioning in connection with the axiomatizations given in Table 3. First, the form of Δ_0-Separation which holds in **BCST**, **CST** and **BIST** is subject to the stipulation that φ is also well-typed in a sense made precise below (cf. Corollary 2.6). The axiom scheme of Replacement occurring in all three theories is unbounded, and involves no such restrictions (in intuitionistic logic, Replacement does not imply Separation). We also note that the addition of the axiom of infinity to any of these set theories corresponds to the addition of a natural number object to the corresponding category. The addition of the axiom of infinity to any of the theories therefore presents no difficulty and the reader may refer to [8], [27] or [12] for further details. Finally, most of the results contained in this subsection are present, implicitly or explicitly, in [5] or [8].

To begin with, notice that the following schema of *Indexed Union* holds in **BCST**:

$$(\forall x \in a.\text{Ƨ}y.\varphi) \Rightarrow \text{Ƨ}y.\exists x \in a.\varphi.$$

Lemma 2.1. **BCST** $\vdash$ Indexed Union.

Proof. Straightforward using Union and Replacement. $\square$

Axioms	**BCST**	**CST**	**BIST**
Membership	●	●	●
Extensionality, Pairing, Union	●	●	●
Emptyset	●	●	●
Binary Intersection	●	●	●
Replacement	●	●	●
Δ_0-Separation	○	○	○
Exponentiation		●	○
Powerset			●

Table 3: Set Theories

Although **BCST** lacks a separation axiom, it is possible to recover some degree of separation. To this end we define:

$$\varphi[a,x]\text{-Sep} := \mathsf{S}(a) \Rightarrow \text{Ƨ}x.(x \in a \wedge \varphi).$$

Here the free variables a and x need not occur in φ. Additionally we say that a formula φ is *simple* when the following, written $!\varphi$, is provable:

$$\text{Ƨ}z.(z = \emptyset \wedge \varphi) \tag{2}$$

and $z \notin \mathrm{FV}(\varphi)$. The intuition behind simplicity is that certain formulas are sufficiently lacking in logical complexity that their truth values are indeed sets. In particular, we will write t_φ for the subsingleton $\{z|z = \emptyset \wedge \varphi\}$ which we call the *truth value* of φ. Separation holds for such simple formulae:

Lemma 2.2 (Simple Separation)**.** *For any formula φ,* **BCST** $\vdash (\forall x \in a.!\varphi(x)) \Rightarrow \varphi[a,x]$-Sep.

Proof. By assumption $\mathsf{S}(a)$ and for every $x \in a$ the truth value:

$$t_{\varphi(x)} := \{z|z = \emptyset \wedge \varphi(x)\}$$

of $\varphi(x)$ is a set. Suppose $y \in t_{\varphi(x)}$, then $y = \emptyset \wedge \varphi(x)$. But then $\exists! z.z = x \wedge y = \emptyset \wedge \varphi(x)$. By Replacement:

$$q := \{z|\exists y \in t_{\varphi(x)}.z = x \wedge y = \emptyset \wedge \varphi(x)\}$$

is a set. But $\exists y \in t_{\varphi(x)}.z = x \wedge y = \emptyset \wedge \varphi(x)$ is equivalent to $z = x \wedge \varphi(x)$ so that $\{z|z = x \wedge \varphi(x)\}$ is a set for each $x \in a$. The result now follows by Indexed Union. □

Lemma 2.3 (The Equality Axiom)**.** **BCST** *proves the* Equality Axiom *(cf. [25]):*

$$\forall x, y.(\text{Ƨ}z.z = x \wedge z = y).$$

Proof. Let x and y be given. Then $\{x\}$ and $\{y\}$ are sets and, by Binary Intersection, their intersection $\{x\} \cap \{y\}$ is also a set which has the required property. □

Henceforth, given x and y, we write δ_{xy} for the set $\{z|z = x \wedge z = y\}$.

Proposition 2.4. *In* **BCST***:*

1. $!(a = b)$.
2. *If* $\mathsf{S}(a)$ *and* $\forall x \in a.!\varphi(x)$*, then* $!(\exists x \in a.\varphi(x))$ *and* $!(\forall x \in a.\varphi(x))$.
3. $!(x \in a)$*, when* $\mathsf{S}(a)$.
4. *If* $!\varphi$ *and* $!\psi$*, then* $!(\varphi \wedge \psi)$, $!(\varphi \vee \psi)$, $!(\varphi \Rightarrow \psi)$*, and* $!(\neg\varphi)$.
5. *If* $\varphi \vee \neg\varphi$*, then* $!\varphi$.

Proof. See [8] or [27]. □

Corollary 2.5. *Given the other axioms of* **BCST** *the following are equivalent:*

1. *Binary Intersection,*
2. *Equality, and*
3. *Intersection.*

Proof. See [8] or [27]. □

We will now show to what extent Δ_0-separation holds in the set theories under consideration. In a set theory without urelements Δ_0-*Separation* is the following schema:

$$Ƨz.z \in y.\varphi,$$

where φ is a Δ_0-formula. The presence of the sethood predicate $\mathsf{S}(-)$ in the language $\mathcal{L}_s$ makes the statement of this separation principle more complicated since every variable in φ which occurs on the right hand side of the membership relation $\in$ must be a set. This complication is reconciled by the following definitions and the form which Δ_0-Separation takes in $\mathcal{L}_s$ is stated in Corollary 2.6.

Definition 2.1. Let a Δ_0 formula φ and a variable x occurring in φ be fixed. We say that x is an *orphan* if $x \in \mathrm{FV}(\varphi)$. If $x \notin \mathrm{FV}(\varphi)$, then we define the *parent of* x *in* φ to be the variable y such that x occurs as a bound variable of one of the following forms in φ: $\forall x \in y$ or $\exists x \in y$ (note that, possibly after α-renaming, every x which is not an orphan has a unique parent in φ). The *family tree of* x *in* φ, denoted by $\Phi(\varphi, x)$, is the singleton $\{x\}$ if x is an orphan and otherwise it is the tuple $\langle x, y_1, y_2, \ldots, y_n\rangle$ such that the following conditions are satisfied: (i) y_1 is the parent of x in φ, (ii) each y_{m+1} is the parent of y_m for $1 \leq m \leq n-1$, and (iii) y_n is an orphan. The reader may easily verify that, for each variable x occurring in φ, $\Phi(\varphi, x)$ is unique.

Definition 2.2. Given a Δ_0 formula φ and a variable x occurring in φ we adopt the following abbreviation:

$$\begin{aligned}\mathsf{S}(\Phi(\varphi,x)) \quad := \quad & \mathsf{S}(y_n) \wedge \forall y_{n-1} \in y_n. \\ & \mathsf{S}(y_{n-1}) \wedge \forall y_{n-2} \in y_{n-1}.\mathsf{S}(y_{n-2}) \wedge \ldots . \forall x \in y_1.\mathsf{S}(x),\end{aligned}$$

where $\Phi(\varphi, x) = \langle x, y_1, \ldots, y_{n-1}, y_n \rangle$.

Definition 2.3. If φ is a Δ_0 formula of **BCST** such that there are no occurrences of the S predicate in φ and $x_1, \ldots, x_n$ are all of those variables of φ either bound or free which occur on the right hand side of the $\in$ predicate in φ, then we define a formula $\tau(\varphi, m)$ for each $1 \leq m \leq n$ inductively by:

$$\begin{aligned}\tau(\varphi, 0) \quad &:= \quad \top. \\ \tau(\varphi, m+1) \quad &:= \quad \tau(\varphi, m) \wedge \mathsf{S}(\Phi(\varphi, x_{m+1})).\end{aligned}$$

Then $\tau(\varphi) := \tau(\varphi, n)$.

Corollary 2.6 (Δ_0-Separation)**.** *If φ is a Δ_0 formula in which there are no occurrences of S and $x_1, \ldots, x_n$ are all of those free variables of φ that occur on the right hand side of occurrences of $\in$, then:*

$$\mathbf{BCST} \vdash \tau(\varphi) \wedge \mathsf{S}(y) \Rightarrow \text{Ƨ} z \in y.\varphi.$$

Remark. In a theory of $\mathcal{L}_s$ satisfying the *Simple Sethood* axiom, $!\mathsf{S}(x)$, which states that the sethood predicate S is simple the conventional unrestricted version of Δ_0-separation holds.

We will now show that the category of sets of **BCST** form a Heyting pretopos and that the sets of **CST** form a Π-pretopos (what we mean by 'the category of sets' will be made precise shortly). First, we consider quotients of equivalence relations.

Lemma 2.7. *If $\mathsf{S}(a)$ and $r \subseteq a \times a$ is an equivalence relation, then for each $x \in a$ the equivalence class:*

$$[x]_r := \{z | z \in a \wedge (x, z) \in r\}$$

is a set.

Proof. By Simple Separation and Lemma 2.4. □

Lemma 2.8. *If $\mathsf{S}(a)$ and $r \subseteq a \times a$ is an equivalence relation, then the quotient*

$$a/r := \{[x]_r | x \in a\}$$

of the set a modulo r is a set.

Proof. This is an easy application of Replacement. □

Let "**Sets**" be the category consisting of sets and functions between them in **BCST**. More precisely, working in the set theory **BCST**, the objects are those x of **BCST** such that $\mathsf{S}(x)$ and arrows $f : x \longrightarrow y$ are those f of **BCST** such that $\mathrm{func}(f, x, y)$. By the foregoing lemmas and some obvious facts that we omit, we have the following:

Theorem 2.9. **BCST** *proves that* "**Sets**" *is a Heyting pretopos.*

Now consider the category "**Sets**" in **CST**:

Lemma 2.10. *For any object I of "**Sets**", the category* "**Sets**"$/I$ *is equivalent to* "**Sets**"^{I} *where I is regarded as a discrete category.*

Proof. The proof is the same as the usual proof for **Sets** (cf. [18]). □

Given $f : X \longrightarrow Y$ the pullback functor Δ_f : "**Sets**"$/Y \longrightarrow$ "**Sets**"$/X$ serves to reindex a family of sets $(C_y)_{y\in Y}$ as $(C_{f(x)})_{x\in X}$. Note also that given a set I and a family of sets X_i for each $i \in I$, the class $\{X_i | i \in I\}$ is a set by Replacement.

Lemma 2.11. *For any map $f : X \longrightarrow Y$ in* "**Sets**"*, the pullback functor Δ_f :* "**Sets**"$/Y \longrightarrow$ "**Sets**"$/X$ *has both a left adjoint Σ_f and a right adjoint Π_f.*

Proof. We may employ the usual definitions of the adjoints:

$$\begin{array}{ccc} \text{“}\mathbf{Sets}\text{”}^X & \xrightarrow{\Sigma_f} & \text{“}\mathbf{Sets}\text{”}^Y \\ (C_x)_{x\in X} & \longmapsto & (S_y)_{y\in Y}, \end{array}$$

where $S_y := \coprod_{f(x)=y} C_x$, and Π_f:

$$(C_x)_{x\in X} \longmapsto (P_y)_{y\in Y},$$

where $P_y := \prod_{f(x)=y} C_x$. Here the arbitrary product:

$$\prod_{i\in I} X_i := \{f : I \longrightarrow \bigcup_{i\in I} X_i | \forall i \in I. f(i) \in X_i\}$$

is a set. In particular, $\bigcup X_i$ is a set by Union and $(\bigcup X_i)^I$ is a set by Exponentiation. The result follows directly from Lemma 2.4 and Simple Separation. □

By the foregoing lemmas we have proved:

Theorem 2.12. **CST** *proves that* "**Sets**" *is a Π-pretopos.*

Finally, we consider the case where the set theory is the *impredicative* set theory **BIST**. Notice though that the following theorem is the only result of this subsection which involves the Powerset axiom.

Theorem 2.13. **BIST** *proves that* "**Sets**" *is a topos.*

Proof. Since **CST** is a subtheory of **BIST** it follows that "**Sets**" is a Π-pretopos. As such, it remains only to show that there exists a subobject classifier in "**Sets**". It consists of the powerset $\wp(1)$ of the terminal object 1, together with the element $\{\emptyset\}$. The classifying map $\chi_A : B \longrightarrow \wp(1)$ of a subobject $A \subseteq B$ is, then, given by the assignment:

$$\chi_A(b) \quad := \quad t_{b \in A},$$

for $b \in B$. $\square$

2.2 Class theories

The class theories with which we are concerned are called *basic intuitionistic class theory* **BICT** and *basic intuitionistic Morse-Kelley* **BIMK**. In Section 5 we will see that these theories are conservative extensions of the set theories **BIST** and **ZF**, respectively. The axioms of **BICT** and **BIMK** are enumerated in Table 4. See below, definition 2.4, for the symbol $\dagger(X)$. Table 5 indicates, in the same manner employed in Table 3 above, which axioms and axiom schemata define the theories in question and which axioms are derivable therein. Moreover, the familiar theories of *von Neumann-Gödel-Bernays class theory* **NGB** and *Morse-Kelley class theory* **MK** are included alongside for the sake of comparison. Recall that **NGB** is a conservative extension of **ZF**; whereas **MK** is not.

Several remarks about the axiomatizations presented in Table 5 are now in order. First, since the Law of Excluded Middle (LEM) is considered an axiom scheme of **ZF**, LEM is an axiom scheme in **BIMK** ranging over all formulae of the language $\mathcal{L}_s$. That is to say, for each formula $\varphi \in \mathcal{L}_s$, $\varphi \vee \neg\varphi$ is an axiom of **BIMK**. On the other hand, LEM does not hold in **BIMK** for arbitrary formulae of $\mathcal{L}_c$. As will be seen later, this leads to the interesting situation in which the 'sets' of a category are classical whereas the 'classes' in that category are intuitionistic. On the other hand, **NGB** and **MK** are strictly classical in the sense that LEM holds for all formulae of $\mathcal{L}_c$ and **BICT** is strictly intuitionistic. Secondly, note that the addition of the Universal Sethood axiom to **BIMK**, **NGB** and **MK** serves to eliminate urelements. As such, these three theories could instead be formulated in the language $\{\in, \eta\}$, thereby yielding the familiar formulations of **NGB** and **MK**. Thirdly, in all four class theories, the axiom scheme of Class Replacement, whether weak or strong, can be replaced by a single axiom stating that for any class of ordered pairs that forms a functional relation, if the domain is a set, then so is the image. Finally, the axiomatization of **NGB** presented here is not the usual one. In particular, the reader should observe that, using classical logic, **NGB** is finitely axiomatizable (cf. [13]).

Definition 2.4. The *Simple Class Comprehension* axiom of **BIMK** makes use of the notation $\dagger(X)$ where X is a class variable. This is pronounced X *is simple* and holds whenever membership in X is simple, in the sense of (2), that is to say, $\dagger(X)$ is defined by:

$$\dagger(X) \Leftrightarrow \forall x.!(x\eta X),$$

Axioms of $\mathbf{BIST}_0$ Let $\mathbf{BIST}_0$ be **BIST** without the axiom schema of Replacement.

Axioms of $\mathbf{ZF}_0$ Let $\mathbf{ZF}_0$ be **ZF** (regarded as a theory of $\mathcal{L}_s$) without the axiom schema of Replacement.

Law of Excluded Middle (LEM) For all formulae φ of $\mathcal{L}_c$, $\varphi \vee \neg\varphi$.

Universal Sethood $\mathsf{S}(x)$.

Class Extensionality $(\forall z.z\eta X \Leftrightarrow z\eta Y) \Rightarrow X = Y$.

Weak Class Comprehension $\exists X.\forall z.z\eta X \Leftrightarrow \varphi$, for any formula φ such that all class variables of φ are free and $X \notin \mathrm{FV}(\varphi)$.

Strong Class Comprehension $\exists X.\forall z.z\eta X \Leftrightarrow \varphi$, for any formula φ such that $X \notin \mathrm{FV}(\varphi)$.

Weak Class Replacement $\mathsf{S}(a) \wedge \forall x \in a.\exists! y.\varphi \Rightarrow \text{Ƨ} y.\exists x \in a.\varphi$, for any formula φ of $\mathcal{L}_c$ such that all class variables of φ are free.

Strong Class Replacement $\mathsf{S}(a) \wedge \forall x \in a.\exists! y.\varphi \Rightarrow \text{Ƨ} y.\exists x \in a.\varphi$, for any formula φ of $\mathcal{L}_c$.

Class Separation $\exists y.\forall z.z \in y \Leftrightarrow z \in x \wedge z\eta X$.

Simple Class Comprehension For any formula φ with no bound class variables and with all (free) class variables in the list $X_1, \ldots, X_n$:

$$\big(\bigwedge_{i=1}^{n} \dagger(X_i)\big) \Rightarrow (\exists X. \dagger (X) \wedge \forall x.x\eta X \Leftrightarrow \varphi).$$

Table 4: Axioms of class theories.

i.e.,

$$\dagger(X) \Leftrightarrow \forall x.\text{Ƨ} z.(z = \emptyset \wedge x\eta X).$$

Lemma 2.14. *In all four class theories the simple classes are exactly those classes X such that for any set x, the intersection $x \cap X$ is again a set:*

$$\mathbf{BICT} \vdash \dagger(X) \Leftrightarrow \big(\forall x.\mathsf{S}(x) \Rightarrow (\exists z.\mathsf{S}(z) \wedge \forall y.y \in z \Leftrightarrow y \in x \wedge y\eta X)\big)$$

and similarly for **BIMK**, **NGB** *and* **MK**.

Proof. We reason in **BICT**. Assume the right hand side. Let x be given. Form the singleton $s = \{x\}$. Then there is a set $t = s \cap X$. Now, we have that $\forall y \in t.\, \exists! u.\, u = \emptyset$. So by Weak Class Replacement the required set z exists.

Axioms	**BICT**	**BIMK**	**NGB**	**MK**
$\mathbf{BIST}_0$	•	○	○	○
$\mathbf{ZF}_0$		•	•	•
LEM			•	•
Universal Sethood		•	•	•
Class Extensionality	•	•	•	•
Weak Class Replacement	○	○	•	○
Strong Class Replacement	•	•		•
Weak Class Comprehension	○	○	•	○
Strong Class Comprehension	•	•		•
Class Separation			•	•
Simple Class Comprehension		•	○	○

Table 5: Class Theories

Now suppose X is a simple class. Let v be a set. For any x in v, there exists a set w_x such that $\forall y. y \in w_x \Leftrightarrow y = \emptyset \wedge x \eta X$, and so by Weak Class Replacement there exists a set v_x such that $\forall y. y \in v_x \Leftrightarrow y = x \wedge x \eta X$. But then $v \cap X = \bigcup_{x \in v} v_x$. □

Finally, simple classes in **BIMK** are easily recognizable as just the decidable classes:

Lemma 2.15. **BIMK** $\vdash \dagger(X) \Leftrightarrow \forall x. x \eta X \vee \neg x \eta X$.

Proof. By Lemma 2.14, if $\dagger(X)$ then for given x, $\{x\} \cap X$ is a set. Therefore either $\{x\} \cap X = \{x\}$, in which case $x \eta X$, or $\{x\} \cap X = \emptyset$, in which case $\neg x \eta X$. In the other direction, since the intersection of X with a singleton is a set (either the empty set or the singleton itself), for given x, $x \cap X = \bigcup_{y \in x}(X \cap \{y\})$. □

Note that, therefore, if we add the Law of Excluded Middle to **BIMK** we obtain **MK**. That is, **BIMK** + **LEM** = **MK**. We will return to the discussion of these theories in Section 5 once the appropriate semantic material has been developed.

3 Models in categories of classes

In this section we introduce the axiomatic theory of categories of classes (as well as several variants of this notion) and establish soundness and completeness results for **BCST**, **CST** and **BIST**. Our approach is that of algebraic set theory, as developed in [17], [26], [11], [8], and [22].

3.1 Axioms for categories with basic class structure

A category $\mathcal{C}$ is a *positive Heyting category* if it is a regular category with disjoint and stable coproducts such that, for any map $f : A \longrightarrow B$, the pullback functor

$f^* : \mathrm{Sub}_{\mathcal{C}}(B) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(A)$ has a right adjoint $f_* : \mathrm{Sub}_{\mathcal{C}}(A) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(B)$. A *system of small maps* in a positive Heyting category $\mathcal{C}$ is a collection $\mathcal{S}$ of maps of $\mathcal{C}$ satisfying the following axioms:

(S1) $\mathcal{S}$ is closed under composition and all identity arrows are in $\mathcal{S}$.

(S2) If the following is a pullback diagram:

$$\begin{array}{ccc} C' & \longrightarrow & C \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ D' & \longrightarrow & D \end{array}$$

and f is in $\mathcal{S}$, then f' is in $\mathcal{S}$.

(S3) All diagonals $\Delta : C \longrightarrow C \times C$ are in $\mathcal{S}$.

(S4) If e is a cover (i.e., a regular epimorphism) and $g = f \circ e$ is in $\mathcal{S}$,

$$\begin{array}{ccc} C & \overset{e}{\twoheadrightarrow} & D \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & A & \end{array}$$

then f is in $\mathcal{S}$.

(S5) If $f : C \longrightarrow A$ and $g : D \longrightarrow A$ are in $\mathcal{S}$, then so is the copair $[f, g] : C + D \longrightarrow A$.

A map f is called *small* if it is a member of $\mathcal{S}$, and an object C is called *small* if the canonical map $!_C : C \longrightarrow 1$ is small. Similarly, a relation $R \rightarrowtail C \times D$ is said to be a *small relation* if the composite:

$$R \rightarrowtail C \times D \longrightarrow D$$

with the second projection is a small map. Finally, a subobject $A \rightarrowtail C$ is a *small subobject* if $A \rightarrowtail C \times 1$ is a small relation; i.e., exactly when A is a small object.

Definition 3.1. A *category with basic class structure* is a positive Heyting category $\mathcal{C}$ with a system of small maps satisfying:

(P1) For each object C of $\mathcal{C}$ there exists a *power object* $\mathcal{P}_s\,(C)$ and a small *membership relation* $\epsilon_C \rightarrowtail C \times \mathcal{P}_s\,(C)$ such that, for any D and small relation $R \rightarrowtail C \times D$, there exists a unique map $\rho : D \longrightarrow \mathcal{P}_s\, C$ such that the square:

$$\begin{array}{ccc} R & \longrightarrow & \epsilon_C \\ \downarrow & & \downarrow \\ C \times D & \underset{1_C \times \rho}{\longrightarrow} & C \times \mathcal{P}_s\, C \end{array}$$

is a pullback.

As in topos theory we call the unique map ρ in **(P1)** the *classifying map* of R and R the *relation classified by* ρ.

Categories with basic class structure model a typed version of **BCST**, developed in the next subsection. The additional structure required to model typed versions of the Exponentiation and Powerset axioms will be discussed below once some properties of categories with basic class structure have been established.

3.2 The internal language of categories with basic class structure

The approach contained herein is influenced by the work of Rummelhoff [22] and is presented in [27] with more detail than in the present exposition. Developing the internal language in this fashion will expedite the soundness proofs below, as well as aiding in the discussion of categories of ideals in Section 4. Furthermore, we will make some use of the internal language to show that the subcategories of small things have certain category theoretic properties. E.g., if $\mathcal{C}$ is a category with basic class structure, then the subcategory $\mathcal{S}_{\mathcal{C}}$ of small objects is a Heyting pretopos.

More generally, the development of the theory via the internal language allows us to emphasize the contribution of the categorical structure already present in categories with basic class structure and to compare it with the additional structure provided by adding a universal object (cf. subsection 3.6 below). Those readers unfamiliar with the use of the internal language of a category should consult [18] and [15].

Henceforth we will assume that the ambient category $\mathcal{C}$ is a category with basic class structure. The canonical interpretation of a formula φ of the internal language is denoted using Scott brackets $[\![\varphi]\!]$. We denote by π_A the composite

$$\pi_A : \epsilon_A \rightarrowtail A \times \mathcal{P}_s \longrightarrow \mathcal{P}_s\, A.$$

which can be considered as the indexed family of all small subobjects of A. Throughout we employ infix notation for certain distinguished relations and maps as in the use of $x\ \epsilon_C\ y$ for the more cumbersome $\epsilon_C\ (x, y)$. We abbreviate $\forall x_1 : X_1.\forall x_2 : X_2.\forall \ldots.\forall x_n.X_n.\varphi$ by $\forall x_1 : X_1, x_2 : X_2, \ldots, x_n : X_n.\varphi$ and similarly for existential quantifiers. Finally, we write $\forall x\ \epsilon_C\ y$ in place of $\forall x : C.x\ \epsilon_C\ y$.

Lemma 3.1. *1. A relation $R \rightarrowtail C \times D$ is small iff, for some $\rho : D \longrightarrow \mathcal{P}_s\, C$:*

$$\mathcal{C} \models \forall x : C, y : D.R(x, y) \Leftrightarrow x\ \epsilon_C\ \rho(y).$$

2. A map $f : C \longrightarrow D$ is small iff, for some $f^{-1} : D \longrightarrow \mathcal{P}_s\, C$:

$$\mathcal{C} \models \forall x : C, y : D.f(x) = y \Leftrightarrow x\ \epsilon_C\ f^{-1}(y).$$

The following proposition will be one of the most useful tools at our disposal in the study of categories with basic class structure. Indeed, this proposition serves to establish the importance of axiom **(S3)** (which will become all the more obvious with the introduction of the category of ideals below.

Proposition 3.2. *The following are equivalent given* **(S1)**, **(S2)**, **(S4)** *and* **(P1)** *(cf. [8] and [22]):*

1. **(S3)**, *i.e. all diagonals* $\Delta : C \longrightarrow C \times C$ *are in* $\mathcal{S}$.
2. *Regular monomorphisms are small.*
3. *If* $g \circ f$ *is small then* f *is small.*
4. $\in_C$: $\epsilon_C \rightarrowtail C \times \mathcal{P}_s\, C$ *is a small map.*
5. $[\![x : C, u : \mathcal{P}_s\, C, v : \mathcal{P}_s\, C | x \,\epsilon_C\, u \wedge x \,\epsilon_C\, v]\!] \rightarrowtail C \times (\mathcal{P}_s\, C \times \mathcal{P}_s\, C)$ *is a small relation*
6. *Sections are small.*

Proof. For (1)⇒(2) notice that Δ is a regular mono and suppose that $m : A \rightarrowtail B$ is the equalizer of $h, k : B \rightrightarrows C$. Then:

$$\begin{array}{ccc} A & \xrightarrow{h \circ m = k \circ m} & C \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle \Delta} \\ B & \xrightarrow[\langle h,k \rangle]{} & C \times C \end{array}$$

is a pullback and m is small by **(S2)**.

To show that (2)⇒(3) suppose regular monos are small and $g \circ f$ is small where:

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

and consider the pullback:

$$\begin{array}{ccc} P & \xrightarrow{p_2} & B \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ A & \xrightarrow[g \circ f]{} & C. \end{array}$$

There is a canonical map $\zeta : A \longrightarrow P$ such that $p_1 \circ \zeta = 1_A$. By **(S1)** f is a small map.

(3)⇒(1) is trivial. Also (3)⇒(4) is trivial. (4)⇒(1) is by **(S2)**. Both (3)⇒(6) and (6)⇒(1) are trivial.

For (4)⇒(5) notice that if $R \rightarrowtail C \times D$ is a small relation and the map $S \rightarrowtail C \times D$ is small, then $R \wedge S$ is a small relation. (5)⇒(1) is by **(S4)** and the fact that:

$$\mathcal{C} \vDash \forall x : C, y : C. x = y \Leftrightarrow \exists z : C. z \,\epsilon_C\, \{x\}_C \wedge z \,\epsilon_C\, \{y\}_C,$$

where $\{-\}_C : C \longrightarrow \mathcal{P}_s\, C$ exists by **(S1)**. □

Corollary 3.3. *All of the canonical maps* $!_A : 0 \longrightarrow A$ *are small and if* $f : A \longrightarrow B$ *and* $g : C \longrightarrow D$ *are small, then* $f + g : A + C \longrightarrow B + D$ *is also small.*

The reader should be alerted at this point that use of proposition 3.2 and its corollary will often be made without explicit mention.

Proposition 3.4 (Typed Axioms)**.** *The following are true in any category* $\mathcal{C}$ *with basic class structure:*

Extensionality *For each object* C*:*

$$\mathcal{C} \models \forall a, b : \mathcal{P}_s\, C.(\forall x : C.x \;\epsilon_C\; a \Leftrightarrow x \;\epsilon_C\; b) \Rightarrow a = b.$$

Emptyset *For each object* C *there exists a map* $\emptyset_C : 1 \longrightarrow \mathcal{P}_s\, C$ *such that:*

$$\mathcal{C} \models \forall x : C.x \;\epsilon_C\; \emptyset_C \Leftrightarrow \bot.$$

Singleton *For each object* C *the singleton map* $\{-\}_C : C \longrightarrow \mathcal{P}_s\, C$*, which is the classifying map for the diagonal* $\Delta : C \rightarrowtail C \times C$*, is a small monomorphism.*

Binary Union *For each* C *there exists a map* $\cup_C : \mathcal{P}_s\, C \times \mathcal{P}_s\, C \longrightarrow \mathcal{P}_s\, C$ *such that:*

$$\mathcal{C} \models \forall x : C, a, b : \mathcal{P}_s\, C.x \;\epsilon_C\; (a \cup_C b) \Leftrightarrow x \;\epsilon_C\; a \vee x \;\epsilon_C\; b.$$

Product *For each* C *and* D *there exists a map* $\times_{C,D} : \mathcal{P}_s C \times \mathcal{P}_s D \longrightarrow \mathcal{P}_s (C \times D)$ *such that:*

$$\mathcal{C} \models \forall x : C, y : D, a : \mathcal{P}_s C, b : \mathcal{P}_s D.(x, y) \;\epsilon_{C\times D}\; (a \times_{C,D} b) \Leftrightarrow x \;\epsilon_C\; a \wedge y \;\epsilon_D\; b.$$

Pairing *For each* C *there exists a map* $\{-,-\}_C : C \times C \longrightarrow \mathcal{P}_s\, C$ *such that:*

$$\mathcal{C} \models \forall x, y, z : C.x \;\epsilon_C\; \{y, z\}_C \Leftrightarrow x = y \vee x = z.$$

Proof. For Extensionality, let the subobject r be given by the following:

$$[\![a, b : \mathcal{P}_s\, C | (\forall x : C)(x \;\epsilon_C\; a \Leftrightarrow x \;\epsilon_C\; b)]\!] \overset{r}{\rightarrowtail} \mathcal{P}_s\, C \times \mathcal{P}_s\, C.$$

By **(P1)** there exist subobjects S, S' of $C \times R$ classified by $\pi_1 \circ r$ and $\pi_2 \circ r$, respectively. But by assumption $S = S'$. Notice that r factors through the diagonal Δ iff $\pi_1 \circ r = \pi_2 \circ r$ (recall that Δ is the equalizer of π_2 and π_2). Thus, by **(P1)**, R factors through Δ, as required.

For Emptyset it suffices to notice that $[\![x : C | \bot]\!]$ is small.

For Singleton note that by Lemma 3.1 we have that:

$$[\![x, y : C | x \;\epsilon_C\; \{y\}]\!] = \Delta,$$

so that if $\mathcal{C} \models \{x\}_C = \{y\}_C$, then $\mathcal{C} \models x = y$. To see that $\{-\}_C$ is small notice that where:

$$\begin{array}{ccc} C & \xrightarrow{\ p\ } & \epsilon_C \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \in_C} \\ C \times C & \xrightarrow[1_C \times \{-\}_C]{} & C \times \mathcal{P}_s\, C \end{array}$$

we have $\{-\}_C = \pi_C \circ p$. But p is small since it has a retraction.

Binary Union follows from the fact that, by **(S4)** and **(S5)**, the join of two small subobjects is a small subobject. Product is by **(S2)**. Finally, for Pairing, the map $\{-,-\}_C : C \times C \longrightarrow \mathcal{P}_s\, C$ is the composite $\cup_C \circ (\{-\}_C \times \{-\}_C)$. $\square$

The foregoing is a good start, but before we can verify the more sophisticated principles (e.g., Replacement) we must first develop several additional properties of the categories in question.

Proposition 3.5. *$\mathcal{P}_s\,(-)$ is the object part of a covariant endofunctor $\mathcal{P}_s : \mathcal{C} \longrightarrow \mathcal{C}$.*

Proof. As in [17] or [8]. $\square$

Henceforth we write $f_! : \mathcal{P}_s\, C \longrightarrow \mathcal{P}_s\, D$ instead of $\mathcal{P}_s\,(f)$, where $f : C \longrightarrow D$.

Corollary 3.6. *Where $f : C \longrightarrow D$:*

$$\mathcal{C} \models \forall x : D, a : \mathcal{P}_s\, C . x \;\epsilon_D\; f_!(a) \Leftrightarrow \exists y \;\epsilon_C\; a . f(y) = x.$$

Proof. Easy. $\square$

Corollary 3.7. *If $m : C \rightarrowtail D$ is monic, then so is $m_! : \mathcal{P}_s\, C \longrightarrow \mathcal{P}_s\, D$. I.e.,*

$$\mathcal{C} \models \forall x, x' : \mathcal{P}_s\, C . m_!(x) = m_!(x') \Rightarrow x = x'.$$

Proof. By Typed Extensionality and the internal language. $\square$

Corollary 3.8. *If $m : C \rightarrowtail D$ is monic, then:*

$$\begin{array}{ccc} \epsilon_C & \rightarrowtail & \epsilon_D \\ \downarrow & & \downarrow \\ C \times \mathcal{P}_s\, C & \underset{m \times m_!}{\rightarrowtail} & D \times \mathcal{P}_s\, D \end{array}$$

is a pullback.

Proof. Easy. $\square$

Lemma 3.9. *Every small map $f : C \longrightarrow D$ gives rise to an (internal) inverse image map $f^* : \mathcal{P}_s\, D \longrightarrow \mathcal{P}_s\, C$.*

Proof. As in [17] or [8]. $\square$

Lemma 3.10. *If $f : C \longrightarrow D$ is a small map, then:*

$$\mathcal{C} \models \forall x : C, a : \mathcal{P}_s\, D. x \,\epsilon_C\, f^*(a) \Leftrightarrow f(x) \,\epsilon_D\, a,$$

where f^* is inverse image.

Proof. Easy. □

In the following we write $\subseteq_C$ for the subobject of $\mathcal{P}_s\, C \times \mathcal{P}_s\, C$ given by:

$$\subseteq_C \quad := \quad [\![x : \mathcal{P}_s\, C, y : \mathcal{P}_s\, C | \forall z \,\epsilon_C\, x. z \,\epsilon_C\, y]\!].$$

From this description of $\subseteq_C$ it easily follows that $\subseteq_C \rightarrowtail \mathcal{P}_s\, C \times \mathcal{P}_s\, C$ is the equalizer of $\pi_1, \cap_C : \mathcal{P}_s\, C \times \mathcal{P}_s\, C \rightrightarrows \mathcal{P}_s\, C$ and that:

$$\mathcal{C} \models \forall x, y : \mathcal{P}_s\, C. x \subseteq_C y \Leftrightarrow x \cap_C y = x.$$

Lemma 3.11. *If $f : C \longrightarrow D$ is a small map, then $f_! \dashv f^*$ internally. That is:*

$$\mathcal{C} \models \forall x : \mathcal{P}_s\, C, y : \mathcal{P}_s\, D. f_!(x) \subseteq_D y \Leftrightarrow x \subseteq_C f^*(y).$$

Proof. Easy using the internal language. □

Lemma 3.12 (Internal Beck-Chevalley Condition)**.** *If $f : C \longrightarrow D$ is a small map and the following diagram is a pullback:*

$$\begin{array}{ccc} C' & \xrightarrow{g'} & C \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ D' & \xrightarrow[g]{} & D \end{array}$$

then $f^* \circ g_! = g'_! \circ (f')^*$.

Proof. By the external Beck-Chevalley condition. □

3.3 Slicing

In this subsection we first show that the structure of categories with basic class structure is preserved under slicing and prove that small objects are exponentiable.

Theorem 3.13. *If $\mathcal{C}$ is a category with basic class structure and D is an object of $\mathcal{C}$, then $\mathcal{C}/D$ is also a category with basic class structure.*

Proof. The Heyting category structure of $\mathcal{C}$ is easily seen to be preserved under slicing. Also, the collection $\mathcal{S}_D$ of all maps in $\mathcal{C}/D$ that are small in $\mathcal{C}$ is plainly a system of small maps in $\mathcal{C}/D$.

Where $f : C \longrightarrow D$ is an object in $\mathcal{C}/D$ we define the powerobject $\mathcal{P}_s\,(f : C \longrightarrow D)$ as the composite $p_f : V_f \rightarrowtail \mathcal{P}_s\, C \times D \longrightarrow D$ where V_f is defined as follows:

$$V_f \quad := \quad [\![x : \mathcal{P}_s\, C, y : D | f_!(x) \subseteq_D \{y\}_D]\!].$$

Notice that by previous results $V_f = [\![x, y | \forall z\ \epsilon_C\ x.f(z) = y]\!]$. Similarly, we define the membership relation ϵ_f as the composite $M_f \rightarrowtail D \times C \times \mathcal{P}_s\, C \longrightarrow D$ where:

$$M_f \quad := \quad [\![x : D, y : C, z : \mathcal{P}_s\, C | y\ \epsilon_C\ z \wedge \forall x'\ \epsilon_C\ z.f(x') = x]\!].$$

For further details see [8] or [27]. □

Lemma 3.14. *Given $f : B \longrightarrow A$ in $\mathcal{C}$ the pullback functor $\Delta_f : \mathcal{C}/A \longrightarrow \mathcal{C}/B$ preserves all basic class structure.*

Proof. See [8] or [27]. □

We will now show that exponentials D^C exist when C is a small object. We define the exponential in question as a subobject of $\mathcal{P}_s\,(C \times D)$ as follows:

$$D^C \quad := \quad [\![R : \mathcal{P}_s\,(C \times D) | \forall x : C.\exists ! y : D.(x, y)\ \epsilon_{C \times D}\ R]\!].$$

Lemma 3.15. *If C is small, then the following special case of the adjunction $- \times C \dashv -^C$ holds:*

$$\mathrm{Hom}(C, D) \cong \mathrm{Hom}(1, D^C)$$

That is to say, there exists a natural isomorphism $\mathrm{Hom}(C, D) \cong \mathrm{Hom}(1, D^C)$.

Proof. By the internal language (cf. [27]). □

Now, using the fact that $\mathcal{C}/E$ has basic class structure and the pullback functor $\Delta_{!_E} : \mathcal{C} \longrightarrow \mathcal{C}/E$ preserves this structure we arrive at the more general lemma:

Lemma 3.16. *Where C is a small object we have the following natural isomorphism:*

$$\mathrm{Hom}(E \times C, D) \cong \mathrm{Hom}(E, D^C)$$

Corollary 3.17. *Small objects are exponentiable.*

Proposition 3.18. *If $f : C \longrightarrow D$ is a small map, then the pullback functor $\Delta_f : \mathcal{C}/D \longrightarrow \mathcal{C}/C$ has a right adjoint Π_f.*

Proof. Clearly $(f : C \longrightarrow D)$ is a small object in $\mathcal{C}/D$ and, hence, exponentiable there. The existence of the adjoint Π_f then follows as usual (cf. [8]). □

3.4 Additional axioms and the subcategory of small objects

We can now discuss the categorical forms of the Exponentiation and Powerset axioms, which will allow us to validate typed forms of the corresponding set theoretical axioms. We will then discuss the category theoretic structure of the subcategory $\mathcal{S}_{\mathcal{C}}$ consisting of small objects for categories $\mathcal{C}$ with various forms of class structure.

Definition 3.2. A *category with (predicative) class structure* is a category $\mathcal{C}$ with basic class structure which also satisfies the following *exponentiation axiom*:

(E) If $f : C \longrightarrow D$ is a small map, then the functor $\Pi_f : \mathcal{C}/C \longrightarrow \mathcal{C}/D$ (which exists by Proposition 3.18) preserves small maps.

Proposition 3.19. *In a category with class structure if C and D are both small, then so is D^C.*

Proof. Notice that D^C is $\Pi_C \circ \Delta_C(D)$. Moreover, since D is small so is $\Delta_C(D)$. By **(E)** it follows that $D^C \longrightarrow 1$ is also small. □

Proposition 3.20. *If $\mathcal{C}$ is a category with class structure and D is an object of $\mathcal{C}$, then $\mathcal{C}/D$ also has class structure.*

Proof. Use the fact that $(\mathcal{C}/D)/f \cong \mathcal{C}/\operatorname{dom}(f)$. □

Up to this point all of the categories considered in this section will allow us only to model *predicative* set theories. The following axiom will ultimately allow us to model the *impredicative* set theory **BIST**.

Definition 3.3. A *category with powered class structure* is a category $\mathcal{C}$ with basic class structure which also satisfies the following *powerset axiom*:

(P2) The subset relation $\subseteq_C \rightarrowtail \mathcal{P}_s\, C \times \mathcal{P}_s\, C$ is a small relation.

Proposition 3.21. *If $\mathcal{C}$ is a category with powered class structure and D is an object of $\mathcal{C}$, then $\mathcal{C}/D$ also has powered class structure.*

Proof. **(P2)** is equivalent to the claim that, for any object D of $\mathcal{C}$, if C is small in $\mathcal{C}/D$, then $\mathcal{P}_s\, C$ is small in $\mathcal{C}/D$. But this condition is clearly preserved by slicing. □

In the following proposition and theorem we will be concerned with the properties of the full subcategory $\mathcal{S}_{\mathcal{C}} := \mathcal{S}/1$ of $\mathcal{C}$ consisting of small objects and small maps between them.

Proposition 3.22. *Let $\mathcal{C}$ be a category with basic class structure. If $\partial_0, \partial_1 : R \rightrightarrows C \times C$ is an equivalence relation in $\mathcal{S}_{\mathcal{C}}$, then the coequalizer of ∂_0 and ∂_1 exists in $\mathcal{S}_{\mathcal{C}}$ and ∂_0, ∂_1 is its kernel pair.*

Proof. We define the quotient C/R by:

$$C/R \quad := \quad [\![z : \mathcal{P}_s\, C | \exists x : C. \forall y : C. y \;\epsilon_C\; z \Leftrightarrow R(x,y)]\!].$$

Notice that since ∂_0 and ∂_1 are small maps so is $\langle \partial_0, \partial_1 \rangle : R \rightarrowtail C \times C$. As such, $\langle \partial_0, \partial_1 \rangle$ is also a small relation and there exists a unique $\alpha : C \longrightarrow \mathcal{P}_s\, C$ such that:

$$\begin{array}{ccc} R & \xrightarrow{p} & \epsilon_C \\ \downarrow & & \downarrow \\ C \times C & \xrightarrow[1 \times \alpha]{} & C \times \mathcal{P}_s\, C \end{array}$$

is a pullback. That is:

$$\mathcal{C} \vDash \forall x, y : C. R(x,y) \Leftrightarrow x \;\epsilon_C\; \alpha(y).$$

By Typed Extensionality it follows that C/R is the image of α:

$$\mathrm{im}(\alpha) \quad = \quad [\![z : \mathcal{P}_s\, C | \exists x : C. \alpha(x) = z]\!],$$

and, as such, that α factors through $i : C/R \rightarrowtail \mathcal{P}_s\, C$ via a cover $\bar{\alpha}$. Moreover, by **(P1)**, $\bar{\alpha} \circ \partial_0 = \bar{\alpha} \circ \partial_1$ since $\langle \partial_0, \partial_1 \rangle$ is an equivalence relation. Notice that since C is small it follows that $\bar{\alpha}$ is a small map and, by **(S4)**, that C/R is a small object.

Finally, we will show that ∂_0, ∂_1 is the kernel pair of $\bar{\alpha}$; i.e., that:

$$\begin{array}{ccc} R & \xrightarrow{\partial_1} & C \\ {\scriptstyle \partial_0}\downarrow & & \downarrow{\scriptstyle \bar{\alpha}} \\ C & \xrightarrow[\bar{\alpha}]{} & C/R \end{array}$$

is a pullback. Let an object Z and maps $z_0, z_1 : Z \rightrightarrows C$ be given such that $\bar{\alpha} \circ z_0 = \bar{\alpha} \circ z_1$. Then we also have that $\alpha \circ z_0 = \alpha \circ z_1$. Define a map $\eta : Z \longrightarrow \epsilon_C$ by $\eta := p \circ r \circ z_0$, where r is the 'reflexivity' map. Then we have:

$$\begin{aligned} \in \circ \eta \quad &= \quad \langle \partial_0, \alpha \circ \partial_1 \rangle \circ r \circ z_0 \\ &= \quad \langle z_0, \alpha \circ z_0 \rangle \\ &= \quad (1_C \times \alpha) \circ \langle \partial_0, \partial_1 \rangle. \end{aligned}$$

By the universal property of pullbacks there exists a unique map $\bar{\eta} : Z \longrightarrow R$ with $p \circ \bar{\eta} = \eta$ and $\langle \partial_0, \partial_1 \rangle \circ \bar{\eta} = \langle z_0, z_1 \rangle$. Moreover $\bar{\eta}$ is the unique map from Z to R such that $\partial_0 \circ \bar{\eta} = z_0$ and $\partial_1 \circ \bar{\eta} = z_1$. It follows from the fact that covers coequalize their kernel pairs that $\bar{\alpha}$ is a coequalizer of ∂_0 and ∂_1. It is easily seen that if Z together with z_0 and z_1 are in $\mathcal{S}_{\mathcal{C}}$, then so is $\hat{\eta}$. □

Theorem 3.23. *If $\mathcal{C}$ has basic class structure, then $\mathcal{S}_{\mathcal{C}}$ is a Heyting pretopos. If $\mathcal{C}$ has class structure, then $\mathcal{S}_{\mathcal{C}}$ is a Π-pretopos. If $\mathcal{C}$ has full class structure, then $\mathcal{S}_{\mathcal{C}}$ is a topos.*

Proof. By Proposition 3.22 $\mathcal{S}_{\mathcal{C}}$ has coequalizers of equivalence relations. It suffices to show that $\mathcal{S}_{\mathcal{C}}$ is a positive Heyting category. But, this structure is easily seen exist since $\mathcal{C}$ is a positive Heyting category. For instance, to show that $\mathcal{S}_{\mathcal{C}}$ has disjoint finite coproducts note that if C and D are small objects then so is $C + D$ together with the maps $C \longrightarrow C + D$ and $D \longrightarrow C + D$ by **(S5)**. Disjointness and stability are consequences of **(S3)**. Similarly, by the description of $C \times D$ as the pullback of $!_C$ along $!_D$, it follows that $C \times D$ is a small object when C and D are. $\mathcal{S}_{\mathcal{C}}$ is seen to be regular by **(S3)**. Finally, for dual images, let a map $f : C \longrightarrow D$ and a subobject $m : S \rightarrowtail C$ be given in $\mathcal{S}_{\mathcal{C}}$. Consider the subobject $i : \forall_f(m) \rightarrowtail D$. Notice that, in general, if a monomorphism $C \rightarrowtail D$ in a category $\mathcal{C}$ with basic class structure is small, then it is also regular since it is a pullback of the section $\top : 1 \longrightarrow \mathcal{P}_s\, 1$. Moreover since, by Proposition 3.18, Π_f exists and is a right adjoint, it follows that i is a small map.

Proposition 3.19 implies that $\mathcal{S}_{\mathcal{C}}$ is a Π-pretopos when $\mathcal{C}$ has class structure. Finally, the fact that 1 is a small object in $\mathcal{C}$ implies that $\mathcal{S}_{\mathcal{C}}$ is a topos when $\mathcal{C}$ has full class structure. □

3.5 Typed union and replacement

We now show that typed versions of Union and Replacement are valid in categories with basic class structure. To this end, we introduce a typed version of the '$\text{ʔ}z.\varphi$' notation from (1) above as follows:

$$\text{ʔ}x : C.\varphi \quad := \quad \exists y : \mathcal{P}_s\, C.\forall x : C.(x \,\epsilon_C\, y \Leftrightarrow \varphi),$$

where $y \notin \mathrm{FV}(\varphi)$.

Lemma 3.24. *A relation $R \rightarrowtail C \times D$ is small if and only if $\mathcal{C} \models \forall y : D.\text{ʔ}x : C.R(x, y)$.*

Proof. Suppose $R \rightarrowtail C \times D$ is a small relation and $\rho : D \longrightarrow \mathcal{P}_s C$ is the classifying map. Then by Lemma 3.1 we have $\mathcal{C} \models \forall y : D.\forall x : C.R(x, y) \Leftrightarrow x \,\epsilon_C\, \rho(y)$. The conclusion may be seen to follow from this (use ρ to witness the existential).

For the other direction suppose $\mathcal{C} \models \forall y : D.\text{ʔ}x : C.R(x, y)$. Then, by Typed Extensionality:

$$\mathcal{C} \models \forall y : D.\exists! z : \mathcal{P}_s\, C.\forall x : C(x \,\epsilon_C\, z \Leftrightarrow R(x, y)),$$

and there is a map $\rho : D \longrightarrow \mathcal{P}_s\, C$ with the requisite property. □

Proposition 3.25 (Typed Union)**.** *For all C:*

$$\mathcal{C} \models \forall a : \mathcal{P}_s\, (\mathcal{P}_s\, C).\text{ʔ}z : C.\exists x \,\epsilon_{\mathcal{P}_s\, C}\, a.z \,\epsilon_C\, x.$$

Proof. Let H be defined as:

$$H \quad := \quad [\![x : C, y : \mathcal{P}_s\, C, z : \mathcal{P}_s\, (\mathcal{P}_s\, C) | y \,\epsilon_{\mathcal{P}_s\, C}\, z \wedge x \,\epsilon_C\, y]\!],$$

and note that the projection:

$$H \rightarrowtail C \times \mathcal{P}_s\, C \times \mathcal{P}_s\, (\mathcal{P}_s\, C) \longrightarrow \mathcal{P}_s\, (\mathcal{P}_s\, C)$$

is small. By **(S4)** it follows that $[\![x : C, z : \mathcal{P}_s\, (\mathcal{P}_s\, C) | \exists y\ \epsilon_{\mathcal{P}_s\, C}\ z \wedge x\ \epsilon_C\ y]\!]$ is a small relation. We write $\bigcup_C : \mathcal{P}_s\, (\mathcal{P}_s\, C) \longrightarrow \mathcal{P}_s\, C$ for the classifying map. □

Proposition 3.26 (Typed Replacement)**.** *For all C and D:*

$$\mathcal{C} \vDash \forall a : \mathcal{P}_s\, C.(\forall x\ \epsilon_C\ a.\exists! y : D.\varphi) \Rightarrow (\text{Ƨ}y : D.\exists x\ \epsilon_C\ a.\varphi).$$

Proof. Let $a : 1 \longrightarrow \mathcal{P}_s\, C$ be given with $1 \Vdash \forall x\ \epsilon_C\ a.\exists! y : D.\varphi$. Let $\alpha \rightarrowtail C$ be the small subobject classified by a. Then the assumption yields a map $f : \alpha \longrightarrow C \longrightarrow D$ such that:

$$\Gamma(f) \quad = \quad [\![x : \alpha, y : D | \varphi(x, y)]\!].$$

Moreover, the image of f is the subobject:

$$I \quad := \quad [\![y : D | \exists x\ \epsilon_C\ a.\varphi(x, y)]\!].$$

Since α is a small subobject it follows by **(S4)** that I is also a small subobject. We may now pull the general problem back as usual. □

3.6 Universal objects and categories of classes

The set theories introduced in Section 2 are untyped (or, as we prefer to think of things, mono-typed) theories; yet the internal languages of the categories we have been considering are typed languages. As such, we will introduce a technical device which will allow us to model untyped theories. The use of universal objects for this purpose originated in [26] and has its roots in Scott's earlier work on modelling the untyped lambda calculus (cf. [23]) in the type calculus.

Definition 3.4. A *universal object* in a category $\mathcal{C}$ is an object U of $\mathcal{C}$ such that for any object C there exists a monomorphism $m : C \rightarrowtail U$. Similarly, in a category $\mathcal{C}$ with basic class structure, a *universe* is an object U together with a monomorphism $\iota : \mathcal{P}_s\, (U) \rightarrowtail U$.

Notice that the monomorphisms m and ι in the definition need not be unique. Here we consider universal objects in categories with various forms of class structure. Note that such an object is always a universe. In Section 4.5 we will see how to turn a universe into a universal object.

Definition 3.5. A *basic category of classes* is a category $\mathcal{C}$ with basic class structure satisfying the additional *universal object* axiom:

(U) There exists a universal object U.

Similarly, a *category of classes* is a category with class structure satisfying **(U)** and a *powered category of classes* is a category with powered class structure satisfying **(U)**.

We will now turn to proving that **BCST** is sound and complete with respect to models in basic categories of classes, that **CST** is sound and complete with respect to models in categories of classes, and that **BIST** is sound and complete with respect to models in powered categories of classes.

3.7 Soundness and completeness

In order to interpret the theories in question in basic categories of classes (respectively, categories of classes or powered categories of classes) we must choose a monomorphism $\iota : \mathcal{P}_s U \rightarrowtail U$ (this is because **(U)** is consistent with the existence of multiple monos $\mathcal{P}_s U \rightarrowtail U$). This presents no difficulty constructively since, in the examples considered below, the models are constructed with a distinguished ι already in mind. An *interpretation* of **BCST** in a basic category of classes $\mathcal{C}$ is a conventional interpretation $[\![-]\!]$ of the first-order structure $(\in, \mathsf{S})$ with respect to the object U, determined by the following conditions:

- $[\![\mathsf{S}(x)]\!]$ is defined to be:
$$\mathcal{P}_s U \overset{\iota}{\rightarrowtail} U.$$
- $[\![x \in y]\!]$ is interpreted as the subobject:
$$\epsilon_U \overset{\in}{\rightarrowtail} U \times \mathcal{P}_s U \overset{1\times\iota}{\rightarrowtail} U \times U.$$

Remark. We write $(\mathcal{C}, U) \vDash \varphi$ to indicate that φ is satisfied by the interpretation. As above $\mathcal{C} \vDash \varphi$ indicates that φ is true in the internal language and $Z \Vdash \varphi$ means that Z forces φ.

Several technical lemmas are needed in order to transfer results about the typed internal language to the untyped set theories in question.

Lemma 3.27. *If $a : 1 \longrightarrow U$ and $1 \Vdash \mathsf{S}(a)$ via some map $\bar{a} : 1 \longrightarrow \mathcal{P}_s U$ (i.e., $\iota \circ \bar{a} = a$), then:*

$$[\![x|\mathsf{S}(x) \wedge (\forall y)(y \in x \Rightarrow y \in a)]\!] \quad = \quad \mathcal{P}_s\, \alpha,$$

where $i : \alpha \rightarrowtail U$ is the small subobject classified by $\bar{a}$ and $\mathcal{P}_s\, \alpha$ is regarded as a subobject of U via $\iota \circ i_!$.

Proof. See [27]. □

Theorem 3.28 (Soundness of **BCST**)**. BCST** *is sound with respect to models in basic categories of classes.*

Proof. The Membership axiom is trivial, and all of the other axioms follow from the fact that their typed analogues are valid in the internal languages of categories with basic class structure (see Propositions 3.4, 3.25 and 3.26). □

In order to prove the soundness of **CST** one needs the following lemma:

Lemma 3.29. *If $a, b : 1 \rightrightarrows U$ factor through ι via $\bar{a}$ and $\bar{b}$, respectively, and $i : \alpha \rightarrowtail U$ and $j : \beta \rightarrowtail U$ are the subobjects classified by $\bar{a}$ and $\bar{b}$, respectively, then:*

$$\beta^\alpha \quad = \quad [\![z | z \subseteq a \times b \wedge \forall x \in a . \exists ! y \in b . \langle x, y \rangle \in z]\!],$$

where the exponential β^α is regarded as a subobject of U.

Proof. Straightforward, see [27]. □

Theorem 3.30 (Soundness of **CST**)**. CST** *is sound with respect to models in predicative categories of classes.*

Proof. All that remains to be checked is that $(\mathcal{C}, U) \vDash$ Exponentiation where $\mathcal{C}$ is a predicative category of classes.

First, observe that for any $a, b : 1 \longrightarrow U$ factoring through $\iota : \mathcal{P}_s\, U \rightarrowtail U$ via maps $\bar{a}$ and $\bar{b}$, respectively, the subobject $[\![z |\, \mathrm{func}(z, a, b)]\!]$ is small. For there exist small subobjects α and β of U corresponding to $\bar{a}$ and $\bar{b}$. And since these subobjects are small, so is the exponential β^α by Proposition 3.19. By the foregoing lemma and Proposition 3.27, it follows that:

$$\begin{aligned} \beta^\alpha \quad &= \quad [\![z | z \subseteq a \times b \wedge \forall x \in a . \exists ! y \in b . \langle x, y \rangle \in z]\!] \qquad (3) \\ &= \quad [\![z |\, \mathrm{func}(z, a, b)]\!]. \end{aligned}$$

The result now follows from the fact that, given $a, b : Z \rightrightarrows U$ such that $Z \Vdash \mathsf{S}(a) \wedge \mathsf{S}(b)$, we may pull the problem back to $\mathcal{C}/Z$ along $\Delta_{!_Z}$. □

Theorem 3.31 (Soundness of **BIST**)**. BIST** *is sound with respect to models in powered categories of classes.*

Proof. By **(P2)**, the relation $\subseteq_U \rightarrowtail \mathcal{P}_s\, U \times \mathcal{P}_s\, U$ is small and has a classifying map $\rho : \mathcal{P}_s\, U \longrightarrow \mathcal{P}_s\, \mathcal{P}_s\, U$. Therefore:

$$(\mathcal{C}, U) \models \forall x : \mathcal{P}_s\, U . \exists z : \mathcal{P}_s\, \mathcal{P}_s\, U . \forall y : \mathcal{P}_s\, U . y \in_{\mathcal{P}_s\, U} z \Leftrightarrow y \subseteq_U x.$$

whence

$$(\mathcal{C}, U) \models \forall x : U . \mathsf{S}(x) \Rightarrow \exists z : U . \mathsf{S}(z) \wedge \forall y : U . y \in z \Leftrightarrow y \subseteq x,$$

as required. □

In order to prove completeness theorems for **BCST**, **CST** and **BIST** we employ the familiar *syntactic category construction* discussed e.g. in [15]. It provides an illuminating perspective on the theory of small maps and its relation to logical definability. This approach to completeness theorems for algebraic set theory is to be found in [8] and was originally used in [26]. For proofs of the relevant facts the reader is referred therefore to [8], [12] and [27].

Theorem 3.32 (Completeness)**.** *For any formula φ of $\mathcal{L}$, if $(\mathcal{C}, U) \models \varphi$ for all models $(\mathcal{C}, U)$ with $\mathcal{C}$ a category of classes, then* **BCST** $\vdash \varphi$*. Similarly, if $(\mathcal{C}, U) \models \varphi$ for all models with $\mathcal{C}$ a predicative category of classes, then* **CST** $\vdash \varphi$*. Finally, if $(\mathcal{C}, U) \models \varphi$ for all models with $\mathcal{C}$ a full category of classes, then* **BIST** $\vdash \varphi$*.*

Proof. As indicated above the proof proceeds by defining, for the theory T (whichever of the set theories of which we are proving the completeness), the *syntactic category $\mathcal{C}_T$ of T* and showing that it is a category of classes of the appropriate kind. Specifically, the objects of $\mathcal{C}_T$ are α-equivalence classes of formulae in context, which are written $\{\vec{x}|\varphi\}$. An arrow $\{\vec{x}|\varphi\} \longrightarrow \{\vec{y}|\psi\}$ is then a provable equivalence class of formulae θ in context with $\mathrm{FV}(\theta) \subseteq \{\vec{x}, \vec{y}\}$ which are provably functional. Maps are written as $[\theta]$. As in [8] small maps are defined to be those maps $[\theta] : \{\vec{x}|\varphi\} \longrightarrow \{\vec{y}|\psi\}$ such that:

$$\psi \quad \vdash_{\vec{y}} \quad ☡\vec{x}.\theta(\vec{x}, \vec{y})$$

is provable in T. Given an object $\{\vec{x}|\varphi\}$ of **BCST** the powerobject is given by the following definition:

$$\mathcal{P}_s(\{\varphi\}) \quad := \quad \{\vec{y}|\mathsf{S}(\vec{y}) \wedge \forall \vec{x} \in \vec{y}.\varphi\}.$$

Finally, the universal object U is defined as follows:

$$U \quad := \quad \{u|u = u\}.$$

Completeness then follows from the completeness of logic with respect to the universal model in the syntactic category (cf. [15]). □

We obtain analogous theorems for the theories $\mathbf{BCST}^+$, $\mathbf{CST}^+$ and $\mathbf{BIST}^+$, obtained by augmenting the theories in question with the Axiom of Infinity, if we restrict attention only to those basic categories of classes (respectively, categories of classes or powered categories of classes) $\mathcal{C}$ such that there exists a natural number object in the subcategory $\mathcal{S}_\mathcal{C}$ of small objects and maps.

4 The category of ideals

In this section we introduce and study the *category of ideals* $\mathbf{Idl}(\mathcal{C})$ over a category $\mathcal{C}$. The category of ideals arises, as it turns out, as the full subcategory of sheaves $\mathbf{Sh}(\mathcal{C})$ on $\mathcal{C}$ consisting of exactly those sheaves with small diagonals, for a suitable notion of small map in the category of sheaves. Basic properties of $\mathbf{Idl}(\mathcal{C})$ are developed in Section 4.1. In Section 4.2 we prove that if $\mathcal{C}$ is a topos, then $\mathbf{Idl}(\mathcal{C})$ is a category with powered class structure. The predicative analogue of this result is proved in Section 4.3 where it is shown that $\mathbf{Idl}(\mathcal{C})$ has basic class structure when $\mathcal{C}$ is a Heyting pretopos. In Section 4.4 we show that $\mathbf{Idl}(\mathcal{C})$ has class structure when $\mathcal{C}$ is a Π-pretopos. Finally, in Section 4.5 we show how to add a universal object, and we state our main new results. We conclude the section by considering a special property of ideals related to the set-theoretic axiom scheme of Collection.

4.1 Small maps in sheaves

Let a small pretopos $\mathcal{E}$ be given. Consider the category $\mathbf{Sh}(\mathcal{E})$ of sheaves on $\mathcal{E}$, for the coherent covering [16, A2.1.11(b)], consisting of finite epimorphic families. Recall that the Yoneda embedding $y : \mathcal{E} \hookrightarrow \mathbf{Sh}(\mathcal{E})$ is a full and faithful Heyting functor [15, D3.1.17].

We intend to build a class category in $\mathbf{Sh}(\mathcal{E})$ in which the representables are the small objects. First, we define a system $\mathcal{S}$ of small maps on $\mathbf{Sh}(\mathcal{E})$ by including in $\mathcal{S}$ just the morphisms of $\mathbf{Sh}(\mathcal{E})$ with 'representable fibers' in the following sense:

Definition 4.1 (Small Map)**.** A morphism $f : A \longrightarrow B$ in $\mathbf{Sh}(\mathcal{E})$ is *small* if for any morphism with representable domain $g : yD \longrightarrow B$, there exists an object C in $\mathcal{E}$, and morphisms f', g' in $\mathbf{Sh}(\mathcal{E})$ fitting into a pullback as follows:

$$\begin{array}{ccc} yC & \xrightarrow{f'} & yD \\ {\scriptstyle g'}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & B. \end{array}$$

Thus, in this sense, *small maps pull representables back to representables.*

Proposition 4.1. *$\mathcal{S}$ satisfies axioms* ***(S1)****,* ***(S2)****, and* ***(S5)****.*

Proof. **(S1)** and **(S2)** follow easily from the familiar two pullbacks lemma.

For **(S5)**, the pullback of, say, $yD \xrightarrow{h} C$ along $(f, g) : A + B \longrightarrow C$ is the coproduct of the pullback of h along f and of h along g. But this is representable, since representables are closed under finite coproducts in $\mathbf{Sh}(\mathcal{E})$. □

We move to consider **(S3)**. A *directed diagram* (in any category $\mathcal{C}$) is a functor $I \longrightarrow \mathcal{C}$ where I is a directed preorder. A small directed diagram in $\mathcal{C}$ in which (the image of) every morphism is a monomorphism in $\mathcal{C}$ we shall call an *ideal diagram.* An ideal diagram has no non-trivial parallel pairs, and is therefore also a filtered diagram (and every small filtered diagram in which the image of every morphism is a monomorphism can be reindexed as an ideal diagram).

Definition 4.2 (Ideal over $\mathcal{E}$)**.** An object A in the category $\widehat{\mathcal{E}} = \mathbf{Sets}^{\mathcal{E}^{op}}$ of presheaves is an *ideal over* $\mathcal{E}$ if it can be written as a colimit of an ideal diagram $I \longrightarrow \mathcal{E}$ of representables,

$$A \;\cong\; \varinjlim_{I}(yC_i).$$

We denote the full *subcategory of ideals* in $\widehat{\mathcal{E}}$ by $\mathbf{Idl}(\mathcal{E})$.

Lemma 4.2. *Every ideal is a sheaf.*

Proof. Since an ideal diagram is a filtered diagram, filtered colimits commute with finite limits, being a sheaf is a finite limit condition, and all representables are sheaves, all such presheaves are also sheaves. □

In accordance with a conjecture by André Joyal, it now turns out that the ideals over the pretopos $\mathcal{E}$ are exactly the sheaves for which **(S3)** holds, i.e. for which the diagonal $A \rightarrowtail A \times A$ is a small map.

Lemma 4.3. *Any sheaf F can be written as a colimit (in $\widehat{\mathcal{E}}$) of representables $F \cong \varinjlim_I (yC_i)$ where I has the property that for any two objects i, j in I, there is an object k in I and morphisms $i \longrightarrow k$ and $j \longrightarrow k$.*

Proof. We may write a sheaf F as the colimit of the composite functor:

$$\int F \xrightarrow{\pi} \mathcal{E} \xrightarrow{y} \widehat{\mathcal{E}},$$

where $\int F$ is the category of elements of F, and π is the forgetful functor. The objects in $\mathbf{Sh}(\mathcal{E})$ can be characterized as the functors $\mathcal{E}^{\mathrm{op}} \longrightarrow \mathbf{Sets}$ which preserve monomorphisms and finite products. It follows that $\int F$ has the required property, since for any two objects (A, a), (B, b) in $\int F$ (with $a \in FA$, $b \in FB$),

$$(A, a) \longrightarrow (A + B, \langle a, b\rangle) \longleftarrow (B, b).$$

(By the coproduct $A+B$, we mean the coproduct in $\mathcal{E}$, hence the product $A \times B$ in $\mathcal{E}^{\mathrm{op}}$, which is sent to the product $FA \times FB$ in **Sets**.) □

Theorem 4.4. *For any sheaf F, the following are equivalent:*

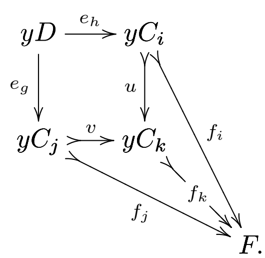
1. *F is an ideal.*
2. *The diagonal $F \rightarrowtail F \times F$ is a small map.*
3. *For all arrows with representable domain $f : yC \longrightarrow F$, the image of f in sheaves is representable, $f : yC \twoheadrightarrow yD \rightarrowtail F$, for some D in $\mathcal{E}$.*

Proof. (1)⇒(2):

We write F as an ideal diagram of representables, $F = \varinjlim_I (yC_i)$. Note that the pullback of any arrow $f : A \longrightarrow F \times F$ along $\Delta : F \longrightarrow F \times F$ is the equalizer of the pair $\pi_1 f, \pi_2 f : A \rightrightarrows F$. Thus let $g, h : yD \rightrightarrows F$ be given, and we must verify that their equalizer $e : E \rightarrowtail yD$ is representable. Recall that, in $\widehat{\mathcal{E}}$, if we are given a colimit $\varinjlim_I (yC_i)$ and an arrow $f : yX \longrightarrow \varinjlim_I (yC_i)$, then f factors through the base of the colimiting cocone, i.e.

$$\begin{array}{ccc} yX & \xrightarrow{e} & yC_i \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f_i} & \\ & \varinjlim_I (yC_i) & \end{array}$$

for some e and some i (where f_i is an arrow of the colimiting cocone). Hence we may factor h as $f_i \circ e_h : yX \longrightarrow yC_i \longrightarrow \varinjlim_I (yC_i)$ and g as $f_j \circ e_g : yX \longrightarrow C_j \longrightarrow \varinjlim_I (yC_i)$. Since the diagram is directed, there is a C_k and arrows u, v such that the two triangles in the following commute:



Since f_k is monic, the equalizer $e : E \rightarrowtail yD$ of $h = f_k u e_h$ and $g = f_k v e_g$ is precisely the equalizer of ue_h and ve_g. But Yoneda preserves and reflects equalizers, so we may conclude that the equalizer of h and g is representable, $E \cong yC$ for some C.

(2)⇒(3):

Let $yD \xrightarrow{f} F$ be given. The kernel pair k_1', k_2' of f can be described as the pullback:

$$\begin{array}{ccc} K' & \longrightarrow & F \\ {\scriptstyle (k_1', k_2')}\downarrow & \lrcorner & \downarrow{\scriptstyle \Delta} \\ yD \times yD & \xrightarrow[f \times f]{} & F \times F. \end{array}$$

Since $yD \times yD \cong y(D \times D)$ is representable and the diagonal of F is small, K' is representable, $K' \cong yK$. Hence, for suitable $k_1, k_2 : K \rightrightarrows D$, we may rewrite the kernel pair as

$$yK \underset{yk_2}{\overset{yk_1}{\rightrightarrows}} yD \xrightarrow{f} F.$$

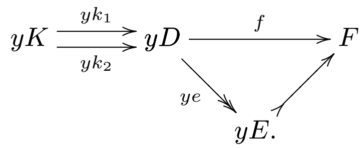
The kernel pair is an equivalence relation in $\widehat{\mathcal{E}}$. Since Yoneda is full and faithful and cartesian, $k_1, k_2 : K \rightrightarrows D$ is an equivalence relation in $\mathcal{E}$. Since $\mathcal{E}$ is effective, there is a coequalizer

$$K \underset{k_2}{\overset{k_1}{\rightrightarrows}} D \overset{e}{\twoheadrightarrow} E,$$

such that k_1 and k_2 is the kernel pair of e. Since the Yoneda embedding preserves pullbacks and covers into $\mathbf{Sh}(\mathcal{E})$,

$$yK \underset{yk_2}{\overset{yk_1}{\rightrightarrows}} yD \overset{ye}{\twoheadrightarrow} yE$$

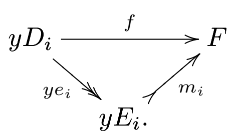
is a coequalizer diagram in $\mathbf{Sh}(\mathcal{C})$. This gives us, then, the required epi-mono factorization:



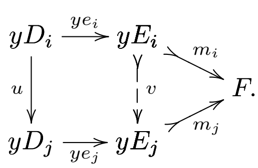
(3)⇒(1):

Step 1: To construct an ideal diagram of representables.

We write F as a colimit $F = \varinjlim_I (yD_i)$, in accordance with Lemma 4.3 (so that I is the category of elements of F). Now, for each $i \in I$, factor in sheaves the cocone arrow $f_i : yD_i \longrightarrow F$:



For $u : yD_i \longrightarrow yD_j$ in the diagram I, consider the following:

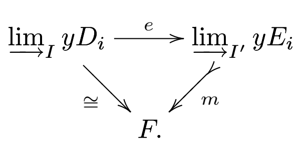


Since $f_i = f_j u$, it follows that f_i factors through yE_j, which gives us the mono v, making the triangle in the diagram commute (to see this, the diagram must be considered in $\mathbf{Sh}(\mathcal{E})$, where e_i is a cover). Since m_j is monic, the square commutes.

The new diagram I' of the yE_i and v thus obtained is an ideal diagram, since I has the directedness property described in Lemma 4.3, and any parallel pair of arrows in I collapse in I' by the construction.

Step 2: To show $F \cong \varinjlim_{I'} (yE_i)$

Observe that the maps ye_i, for i in I, in the diagram above induce a morphism $e : \varinjlim_I yD_i \longrightarrow \varinjlim_{I'} yE_i$, and the maps m_i induce a monomorphism $\varinjlim_{I'} yE_i \rightarrowtail F$, such that the following commutes:



Thus m is also an isomorphism. □

In order to ensure that **(S3)** is satisfied, we therefore narrow our attention from $\mathbf{Sh}(\mathcal{E})$ to the full subcategory of ideals, denoted $\mathbf{Idl}(\mathcal{E})$. We shall see that no further restriction is needed. First, we verify that $\mathbf{Idl}(\mathcal{E})$ is a positive Heyting category:

Lemma 4.5. $\mathbf{Idl}(\mathcal{E})$ *is closed under subobjects and finite limits.*

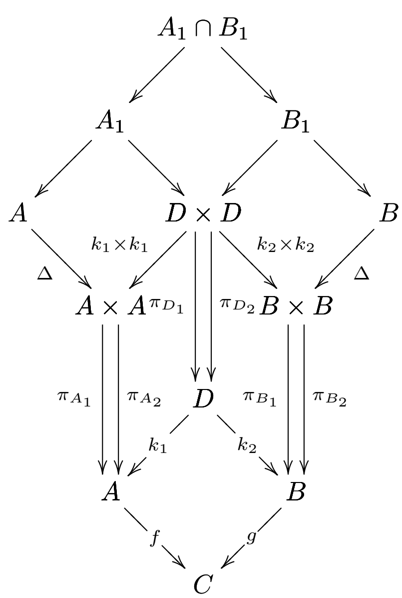
Proof. We use the description of ideals as sheaves with small diagonal. That $\mathbf{Idl}(\mathcal{E})$ is closed under subobjects follows from **(S2)**. Notice that, since it is an isomorphism, $\Delta : 1 \rightarrowtail 1 \times 1$ is small. If A, B are ideals and C is any sheaf, we consider the pullback:

$$\begin{array}{ccc} D & \xrightarrow{k_2} & B \\ {\scriptstyle k_1}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$$

Now, if we pull the diagonals back:

$$\begin{array}{ccc} A_1 & \longrightarrow & A \\ {\scriptstyle \alpha}\downarrow & \lrcorner & \downarrow{\scriptstyle \Delta} \\ D \times D & \xrightarrow[k_1 \times k_1]{} & A \times A \end{array} \qquad \begin{array}{ccc} B_1 & \longrightarrow & B \\ {\scriptstyle \beta}\downarrow & \lrcorner & \downarrow{\scriptstyle \Delta} \\ D \times D & \xrightarrow[k_2 \times k_2]{} & B \times B \end{array}$$

By a diagram chase, the diagonal of D is $A_1 \cap B_1$, which is small since smallness is preserved by pullback and composition. We draw the diagram in which to chase:



Where all squares not involving projections are pullback squares. $\square$

Lemma 4.6. $\mathbf{Idl}(\mathcal{E})$ *is closed under finite coproducts, and inclusion maps are small.*

Proof. $0 \longrightarrow 0 \times 0$ is iso, so small.

Now, the terminal object 1 in $\mathbf{Sh}(\mathcal{E})$ is representable, and so is $1+1$, since Yoneda preserves finite coproducts. The inclusion $i_1 : 1 \longrightarrow 1+1$ is therefore small. But coproducts in $\mathbf{Sh}(\mathcal{E})$ being disjoint, the following is a pullback:

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_1} \\ A+B & \xrightarrow[!_A+!_B]{} & 1+1 \end{array}$$

So by **(S2)**, the inclusion map i_A is small.

The diagonal of $A+B$ can be regarded as the disjoint union of the diagonal of A and of B:

$$\begin{array}{ccccc} A & \xrightarrow{p_A} & A+B & \xleftarrow{p_B} & B \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\ A \times A & \dashrightarrow & (A+B)\times(A+B) & \dashleftarrow & B \times B \\ & {\scriptstyle p_{A\times A}}\searrow & \uparrow{\scriptstyle \cong} & \swarrow{\scriptstyle p_{B\times B}} & \\ & & (A\times A)+(A\times B)+(B\times A)+(B\times B) & & \end{array}$$

By smallness of coproduct inclusions and isos, and applying **(S5)**, if A, B are ideals then so is $A+B$. □

Proposition 4.7. $\mathbf{Idl}(\mathcal{E})$ *is positive Heyting, with the structure inherited from* $\mathbf{Sh}(\mathcal{E})$.

Proof. We have done finite limits and finite coproducts. For a morphism $f : A \longrightarrow B$ of ideals, $\mathrm{im}(f)$ is an ideal, since there is a monomorphism $\mathrm{im}(f) \rightarrowtail B$. The cover $e : A \twoheadrightarrow \mathrm{im}(f)$ is the coequalizer of its kernel pair in $\mathbf{Sh}(\mathcal{E})$, the kernel pair is the same in $\mathbf{Idl}(\mathcal{E})$, so e is also a regular epimorphism in $\mathbf{Idl}(\mathcal{E})$.

For dual images, since $\mathbf{Idl}(\mathcal{E})$ is closed under subobjects and finite limits can be taken in sheaves, dual images can also be taken in sheaves. □

Lemma 4.8. *(S4) is satisfied in* $\mathbf{Idl}(\mathcal{E})$.

Proof. Let $a : A \twoheadrightarrow B$ and $b : B \longrightarrow C$ be given, and assume $b \circ a$ is small. Let $yG \longrightarrow C$ be given, and consider the following two pullbacks diagram:

$$\begin{array}{ccccc} yD & \twoheadrightarrow & E & \longrightarrow & yG \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ A & \xrightarrow[a]{} & B & \xrightarrow[b]{} & C \end{array}$$

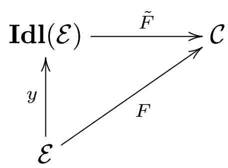
By Theorem 4.4, the image of a representable is a representable in $\mathbf{Idl}(\mathcal{E})$. Hence E in the diagram above is representable. □

We summarize the results of this subsection:

Theorem 4.9. *For any pretopos $\mathcal{E}$, the full subcategory $\mathbf{Idl}(\mathcal{E}) \hookrightarrow \mathbf{Sh}(\mathcal{E})$ of ideals is a positive Heyting category with a system of small maps satisfying axioms **(S1)**-**(S5)**.*

We conclude by noting a characterizing feature of $\mathbf{Idl}(\mathcal{E})$ of which we make extensive use in the following sections. Namely, $\mathbf{Idl}(\mathcal{E})$ is the *ideal completion* of $\mathcal{E}$, in the following sense:

Proposition 4.10. $\mathbf{Idl}(\mathcal{E})$ *has colimits of ideal diagrams and if $\mathcal{C}$ is a category with ideal colimits, and $F : \mathcal{E} \longrightarrow \mathcal{C}$ is a functor which preserves monomorphisms, then there is a unique (up to natural isomorphism) extension $\tilde{F} : \mathbf{Idl}(\mathcal{E}) \longrightarrow \mathcal{C}$ of F such that $\tilde{F}$ is* ideal continuous*, in the sense of preserving ideal colimits, and such that the following commutes up to isomorphism:*



Proof. $\mathbf{Idl}(\mathcal{E})$ has colimits of ideal diagrams since any such diagram is an ideal diagram of representables (cf. [15, C2]). Given F, write $E = \varinjlim_I (yC_i)$ and set $\tilde{F}(E) = \varinjlim_I (FC_i)$. □

4.2 Powerobjects in Idl($\mathcal{E}$) for a Topos $\mathcal{E}$

In this section, we will show that when $\mathcal{E}$ is a topos $\mathbf{Idl}(\mathcal{E})$ is a category with powered class structure. To this end all that remains is to verify is that axioms **(P1)** and **(P2)** are satisfied in $\mathbf{Idl}(\mathcal{E})$. For this we will use the topos powerobjects from $\mathcal{E}$ to construct powerobjects in $\mathbf{Idl}(\mathcal{E})$. We rely heavily on the characterization of $\mathbf{Idl}(\mathcal{E})$ as the colimits of ideal diagrams of representables.

In a topos $\mathcal{E}$, the covariant powerobject functor $P : \mathcal{E} \to \mathcal{E}$, which sends an object A to its powerobject PA and a morphism $f : A \longrightarrow B$ to the (topos) direct image morphism $Pf : PA \longrightarrow PB$, preserves monomorphisms. Therefore, if we have an ideal $A = \varinjlim_{i \in I}(yA_i)$ in $\mathbf{Sh}(\mathcal{E})$, we may apply $P(-)$ to obtain another ideal $\mathcal{P}_s\, A = \varinjlim_{i \in I}(yPA_i)$.

Lemma 4.11. *Let $A = \varinjlim_{i \in I}(yA_i)$ where I is an ideal diagram. Then the ideal $\mathcal{P}_s\, A := \varinjlim_{i \in I}(yPA_i)$ together with the relation $\epsilon_A := \varinjlim_{i \in I}(y\epsilon_{A_i}) \rightarrowtail A \times \mathcal{P}_s\, A$ satisfies axiom **(P1)**.*

Proof. First, we should complete the definition of the subobject ϵ_A. In any class category, and in any topos, any monomorphism $u : A \rightarrowtail B$ leads to the following pullback:

$$\begin{array}{ccc} \epsilon_A & \longrightarrow & \epsilon_B \\ \downarrow & \lrcorner & \downarrow \\ A \times PA & \xrightarrow[u \times Pu]{} & B \times PB \end{array}$$

Now, take $A = \varinjlim_I (yA_i)$. Let $u : yA_i \rightarrowtail yA_j$ be an arrow of that diagram. Since I is filtered, $\varinjlim_I (yA_i) \times \varinjlim_I (yPA_i) \cong \varinjlim_I (y(A_i \times PA_i))$, and the pullback

$$\begin{array}{ccc} y\epsilon_{A_i} & \xrightarrow{\epsilon_u} & y\epsilon_{A_j} \\ \downarrow & \lrcorner & \downarrow \\ y(A_i \times PA_i) & \xrightarrow[u \times Pu]{} & y(A_j \times PA_j) \end{array}$$

serves to illustrate what the arrows are in the diagram $\epsilon_A := \varinjlim_{i \in I} (y\epsilon_{A_i})$, and what the monomorphism $\epsilon_A \rightarrowtail A \times \mathcal{P}_s\, A$ is. It follows from the construction that ϵ_A is a small relation.

Now let $yC \rightarrowtail A$ be a small subobject of the ideal $A = \varinjlim_I (yA_i)$. The inclusion arrow $yC \rightarrowtail \varinjlim_I (yA_i)$ factors through some colimiting cocone morphism $yA_i \rightarrowtail A$, and we get the following diagram, in which $\gamma : 1 \longrightarrow PA_i$ classifies $C \rightarrowtail A_i$:

$$\begin{array}{ccccc} yC & \longrightarrow & y\epsilon_{A_i} & \rightarrowtail & \varinjlim_I (\epsilon_{yA_i}) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ yA_i \times 1 & \xrightarrow{Id \times \gamma} & yA_i \times yPA_i & \rightarrowtail & \varinjlim_I (yA_i \times yPA_i) \\ \downarrow & \lrcorner & \downarrow & \swarrow\!\nearrow\ = & \\ \varinjlim_I (yA_i) \times 1 & \longrightarrow & \varinjlim_I (yA_i \times yPA_i) & & \end{array}$$

from which we can conclude that the global point $1 \xrightarrow{\gamma} yPA_i \rightarrowtail \varinjlim_I (yPA_i)$ classifies $yC \rightarrowtail A$. We now observe that nothing prevents this argument from going through in the slightly more general case when yC is a small relation $yC \rightarrowtail A \times yD$, for some fixed $D \in \mathcal{E}$, so that we instead get a classifying map $\rho : yD \longrightarrow \mathcal{P}_s\, A$ such that:

$$\begin{array}{ccc} yC & \longrightarrow & \epsilon_A \\ \downarrow & \lrcorner & \downarrow \\ A \times yD & \xrightarrow[Id \times \rho]{} & A \times \mathcal{P}_s\, A \end{array}$$

For the general situation with an ideal $X \cong \varinjlim_J (yC_j)$, consider a small relation $R \rightarrowtail A \times X$. Since $\pi_2 : R \longrightarrow X$ pulls representables back to representables, and since pullbacks commute with filtered colimits, we obtain a reindexing of R as a colimit of a diagram over J of representables $\pi_2^*(yD_j) =: yC_j$ by considering the pullback:

$$\begin{array}{ccc} \varinjlim_J(\pi_2^*(yD_j)) & \longrightarrow & \varinjlim_J(yD_j) \\ \cong\downarrow \ \lrcorner & & \downarrow\cong \\ R & \xrightarrow[\pi_2]{} & X \end{array}$$

This allows us to consider each index $j \in J$ separately. Applying the foregoing case to each $yC_j \rightarrowtail A \times yD_j$ gives a cocone of classifying arrows $\rho_j : yD_j \longrightarrow \mathcal{P}_s\, A$ making pullbacks:

$$\begin{array}{ccc} yC_j & \longrightarrow & \epsilon_A \\ \downarrow \ \lrcorner & & \downarrow \\ A \times yD_j & \longrightarrow & A \times \mathcal{P}_s\, A \end{array}$$

We thus have the classifying map $X \longrightarrow \mathcal{P}_s\, A$, as the unique one determined by the cocone $(\rho_j)_j$. □

It follows from Lemma 4.11 and Theorem 4.9 that for ideal diagrams I and J, if $\varinjlim_I(yC_i) \cong \varinjlim_J(yD_j)$, then $\varinjlim_I(yPC_i) \cong \varinjlim_J(yPD_j)$, as the system of small maps determine the power objects up to isomorphism in a class category (**(P2)** not needed). Hence power objects for ideals may be defined in the manner of Lemma 4.11.

Lemma 4.12. *If $\mathcal{E}$ is a topos, then* $\mathbf{Idl}(\mathcal{E})$ *satisfies axiom* ***(P2)***.

Proof. We need to construct an internal power set map $\mathcal{P} : \mathcal{P}_s\, A \longrightarrow \mathcal{P}_s\, \mathcal{P}_s\, A$, that is, a classifying map for $\subseteq_A \rightarrowtail \mathcal{P}_s\, A \times \mathcal{P}_s\, A$. If $A = \varinjlim_I(yA_i)$, then $\mathcal{P}_s\, A = \varinjlim_I(yPA_i)$ and $\mathcal{P}_s\, \mathcal{P}_s\, A = \varinjlim_I(yPPA_i)$. In any category with powered class structure, $\mathcal{E}$ in particular, the following square commutes for any $f : A \longrightarrow B$:

$$\begin{array}{ccc} PA & \xrightarrow{\mathcal{P}_A} & PPA \\ {\scriptstyle P(f)}\downarrow & & \downarrow{\scriptstyle PP(f)} \\ PB & \xrightarrow[\mathcal{P}_B]{} & PPB \end{array}$$

and if f is a monomorphism, then the square is a pullback. This allows us to construct the power set map $\mathcal{P} : \mathcal{P}_s\, A \longrightarrow \mathcal{P}_s\, \mathcal{P}_s\, A$ directly out of the maps $\mathcal{P}_{A_i} : PA_i \longrightarrow PPA_i$ for $i \in I$. Correspondingly, in a category with powered class structure, $\mathcal{E}$ in particular, if $f : A_i \rightarrowtail A_j$ is a monomorphism, then

$$\begin{array}{ccc} \subseteq_{A_i} & \rightarrowtail & \subseteq_{A_j} \\ \downarrow \ \lrcorner & & \downarrow \\ PA_i \times PA_i & \xrightarrow[Pf \times Pf]{} & PA_j \times PA_j \end{array}$$

is a pullback, and we can define the subobject $\varinjlim_I(y \subseteq_{A_i}) \rightarrowtail \varinjlim_I(yA_i) \times \varinjlim_I(yA_i) = \mathcal{P}_s\, A \times \mathcal{P}_s\, A$. It is now straightforward to verify that

$$\begin{array}{ccc}
\varinjlim_I(\subseteq_{A_i}) & \longrightarrow & \varinjlim_I(\epsilon_{PA_i}) \\
\downarrow & & \downarrow \\
\varinjlim_I(PA_i) \times \varinjlim_I(PA_i) & \xrightarrow[Id \times \mathcal{P}]{} & \varinjlim_I(PA_i) \times \varinjlim_I(PPA_i)
\end{array}$$

is a pullback, and the verification that $\varinjlim_I(\subseteq_{A_i}) \cong \subseteq_A$ is a similar diagram chase. $\square$

To summarize, then:

Theorem 4.13. *If $\mathcal{E}$ is a topos, then $\mathbf{Idl}(\mathcal{E})$ is a category with powered class structure with respect to the small maps given in Lemma 4.11.*

4.3 Predicative Powerobjects in $\mathbf{Idl}(\mathcal{C})$

We will strengthen the result of the foregoing section by showing that even if $\mathcal{C}$ is only a Heyting pretopos the category $\mathbf{Idl}(\mathcal{C})$ is a category with basic class structure. In light of Theorem 4.9 it remains to identify the powerobject structure on $\mathbf{Idl}(\mathcal{C})$. This is made more complicated than in the topos case due to the fact that the category $\mathcal{C}$ does not possess a powerobject structure of its own.

In order to motivate the definition of the powerobjects $\mathcal{P}_s\,(X)$ in $\mathbf{Idl}(\mathcal{C})$ notice that when $\mathcal{E}$ is a topos and C is an object of $\mathcal{E}$ we have $\mathcal{P}_s\,(yC) \cong y(\Omega^C)$ and at any object E in $\mathcal{E}$:

$$\begin{array}{rcl}
\mathcal{P}_s\,(yC)(E) & \cong & y(\Omega^C)(E) \\
& = & \mathrm{Hom}_{\mathcal{E}}(E, \Omega^C) \\
& \cong & \mathrm{Hom}_{\mathcal{E}}(E \times C, \Omega) \\
& \cong & \mathrm{Sub}_{\mathcal{E}}(E \times C).
\end{array}$$

Dropping both the assumption that the indexing category is a topos and that we are working in presheaves, we therefore adopt the following provisional definition of the powerobject of yC in $\mathbf{Idl}(\mathcal{C})$:

$$\mathcal{P}_s\,(yC) \;:=\; \mathrm{Sub}_{\mathcal{C}}(- \times C),$$

where the contravariant action is by pullback. We then extend $\mathcal{P}_s\,(-)$ continuously to ideals $X = \varinjlim_i yC_i$ by:

$$\mathcal{P}_s\,(X) \;:=\; \varinjlim_i \mathcal{P}_s\,(yC_i).$$

We will show that this definition of $\mathcal{P}_s\,(X)$ is justified by first showing that $\mathrm{Sub}_{\mathcal{C}}(- \times C)$ is indeed an ideal and then that the functor:

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\mathrm{Sub}^r_{\mathcal{C}}} & \mathbf{Idl}(\mathcal{C}) \\
C & \longmapsto & \mathrm{Sub}_{\mathcal{C}}(- \times C),
\end{array}$$

preserves monomorphisms. We then apply Proposition 4.10 to obtain an extension $\mathcal{P}_s : \mathbf{Idl}(\mathcal{C}) \longrightarrow \mathbf{Idl}(\mathcal{C})$ which will be seen to be a powerobject functor in the sense of satisfying **(P1)**.

Similarly, the membership relation $\epsilon_X \rightarrowtail X \times \mathcal{P}_s\, X$ is defined as the restriction of the sheaf (hence also of the presheaf) membership relation to $\mathcal{P}_s\, X$. Explicitly, for a representable yC, an ideal $X := \varinjlim_i yC_i$ and an object D of the base category $\mathcal{C}$:

$$\begin{array}{rcl} \epsilon_{yC}\,(D) & := & \{\langle f, S\rangle \in yC(D) \times \mathcal{P}_s\,(yC)(D) | \Gamma(f) \le S\}, \text{ and} \\ \epsilon_X & := & \varinjlim_i \epsilon_{yC_i}\ , \end{array}$$

where $\Gamma(f) \rightarrowtail D \times C$ is the graph of f and $\le$ is the partial ordering of subobjects in $\mathcal{C}$. I.e., $\Gamma(f) \le S$ indicates that in $\mathrm{Sub}_{\mathcal{C}}(D \times C)$ the subobject $\Gamma(f)$ factors through S. We first prove that the foregoing definition of powerobjects in $\mathbf{Idl}(\mathcal{C})$ does actually determine an ideal.

Lemma 4.14. *If $\mathcal{C}$ is a pretopos and C is an object of $\mathcal{C}$, then the purported powerobject presheaf $\mathcal{P}_s\,(yC) := \mathrm{Sub}_{\mathcal{C}}(- \times C)$ is a sheaf.*

Proof. Notice that $\mathcal{P}_s\,(yC)(0) \cong \{*\}$ and, since coproducts in $\mathcal{C}$ are stable, $\mathcal{P}_s\,(yC)(A+B) \cong \mathcal{P}_s\,(yC)(A) \times \mathcal{P}_s\,(yC)(B)$. Suppose $f : A \longrightarrow\!\!\triangleright B$ is a cover and let $h, k : Z \rightrightarrows \mathrm{Sub}_{\mathcal{C}}(B \times C)$ be given such that $\mathrm{Sub}_{\mathcal{C}}(f \times C) \circ h = \mathrm{Sub}_{\mathcal{C}}(f \times C) \circ k$. Then, for any $z \in Z$, $h(z), k(z) \in \mathrm{Sub}_{\mathcal{C}}(B \times C)$ and the pullback P of $h(z)$ along $f \times 1_C$ is also the pullback of $k(z)$ along $f \times 1_C$. But covers are preserved under pullback in $\mathcal{C}$ so that $h(z) = k(z)$ by the uniqueness of image factorizations. $\square$

Proposition 4.15. *If $\mathcal{C}$ is a Heyting pretopos and C is an object of $\mathcal{C}$, then the purported powerobject $\mathcal{P}_s\,(yC)$ is an ideal.*

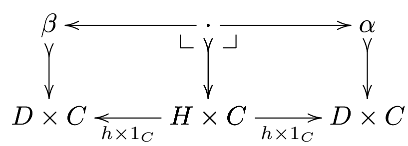
Proof. Since $\mathcal{C}$ is effective it suffices by Proposition 4.4 to show that $\mathcal{P}_s\,(yC)$ has a small diagonal. To that end let $yD \longrightarrow \mathcal{P}_s\,(yC) \times \mathcal{P}_s\,(yC)$ be given and consider the following diagram:

$$\begin{array}{ccccc} & & yD & & \\ & & \Big\downarrow{\scriptstyle i} & & \\ \mathcal{P}_s\,(yC) & \overset{\Delta}{\rightarrowtail} & \mathcal{P}_s\,(yC) \times \mathcal{P}_s\,(yC) & \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} & \mathcal{P}_s\,(yC) \end{array}$$

We will show that the equalizer of $i_1 := \pi_1 \circ i$ and $i_2 := \pi_2 \circ i$ is representable, which clearly suffices.

By the Yoneda lemma there are subobjects α and β of $D \times C$ classified by i_1 and i_2, respectively. We want to find some H and $h : H \longrightarrow D$ in $\mathcal{C}$ such that the result of pulling α back along $h \times 1_C$ is the same as the result of pulling β

back along $h \times 1_C$:

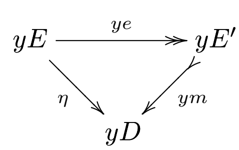


Define the subobject H of D as follows:

$$H \quad := \quad [\![x : D|\forall z : C.\alpha(x,z) \Leftrightarrow \beta(x,z)]\!],$$

and write $h : H \rightarrowtail D$ for the canonical inclusion. It is then a trivial application of the internal language to see that, by definition of H, α and β both pull back to the same thing along $h \times 1_C$. Therefore, $i_1 \circ yh = i_2 \circ yh$.

To see that $yh : yH \rightarrowtail yD$ is the equalizer suppose given some $\eta : X \longrightarrow yD$ with $i_1 \circ \eta = i_2 \circ \eta$. It suffices to assume that X is representable, so suppose $X \cong yE$. Consider the image factorization yE' of η:



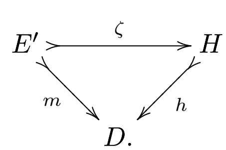
Notice that $i_1 \circ ym = i_2 \circ ym$ since ye' is a cover. In particular, the Yoneda lemma implies that:

$$\mathcal{C} \models \forall x : E', y : C.\alpha(m(x), y) \Leftrightarrow \beta(m(x), y). \tag{4}$$

As a special case of (4) we obtain that $m : E' \rightarrowtail D$ factors through $h : H \rightarrowtail D$; i.e.:

$$E' \Vdash \forall z : C.\alpha(m,z) \Leftrightarrow \beta(m,z).$$

That is, there exists a map $\zeta : E' \longrightarrow H$ such that the following triangle commutes:



Clearly, $\zeta \circ e$ is the unique map $E \longrightarrow H$ such that $h \circ \zeta \circ e = \eta$ since h is a momomorphism. Therefore we have shown that $yh : yH \longrightarrow yD$ is the equalizer of i_1 and i_2, as required. □

Lemma 4.16. *The functor* $\mathrm{Sub}^r_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbf{Idl}(\mathcal{C})$ *defined by*

$$\mathrm{Sub}^r_{\mathcal{C}}(C) := \mathrm{Sub}_{\mathcal{C}}(- \times C),$$

preserves monomorphisms.

Proof. A map $f : D \longrightarrow C$ induces a natural transformation $\varphi : \mathrm{Sub}_{\mathcal{C}}(- \times D) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(- \times C)$ given at an object E of $\mathcal{C}$ by:

$$\begin{array}{ccc} S \in \mathrm{Sub}_{\mathcal{C}}(E \times D) & \overset{\varphi_E}{\longmapsto} & S' \in \mathrm{Sub}_{\mathcal{C}}(E \times C), where \\ S' & := & (1_E \times f)_!(S). \end{array}$$

As such, we define $\mathrm{Sub}^r_{\mathcal{C}}(f) := \varphi$. Notice that φ is natural because $\mathcal{C}$ satisfies the Beck-Chevalley condition.

If f is monic, then each component φ_E is monic and, by the Yoneda lemma, φ is monic (since the monomorphisms, like other limits, in $\mathbf{Idl}(\mathcal{C})$ agree with those in $\widehat{\mathcal{C}}$). $\square$

Definition 4.3. For any object $X = \varinjlim_i yC_i$ of $\mathbf{Idl}(\mathcal{C})$, where $\mathcal{C}$ is a Heyting pretopos, we have by Proposition 4.10 and the foregoing lemma that there is a unique functor $\mathcal{P}_s : \mathbf{Idl}(\mathcal{C}) \longrightarrow \mathbf{Idl}(\mathcal{C})$ with:

$$\begin{array}{rcl} \mathcal{P}_s\,(X) & \cong & \mathcal{P}_s\,(\varinjlim_i yC_i) \\ & \cong & \varinjlim_i \mathrm{Sub}^r_{\mathcal{C}}(C_i) \\ & = & \varinjlim_i \mathrm{Sub}_{\mathcal{C}}(- \times C_i). \end{array}$$

We will now show that the axiom **(P1)** holds in $\mathbf{Idl}(\mathcal{C})$ where $\mathcal{C}$ is a Heyting pretopos. It will be more efficient to break the proof into several steps. Also, notice that we write $\in_X$ for the membership relation in $\widehat{\mathcal{C}}$ and ϵ_X for the membership relation in $\mathbf{Idl}(\mathcal{C})$. Similarly, we write $\mathcal{P}X$ for the power object in $\widehat{\mathcal{C}}$ and $\mathcal{P}_s\,X$ for the small power object in $\mathbf{Idl}(\mathcal{C})$.

Lemma 4.17. *Given any small relation $R \overset{r}{\rightarrowtail} X \times Y$ in $\mathbf{Idl}(\mathcal{C})$ there exists a unique classifying map $\hat{r} : Y \longrightarrow \mathcal{P}_s\,X$.*

Proof. First consider the case where $R \rightarrowtail yC \times yD$. Then in $\widehat{\mathcal{C}}$ both of the following squares (and the outer rectangle):

$$\begin{array}{ccc} \epsilon_{yC} & \rightarrowtail & \in_{yC} \\ \downarrow & & \downarrow \\ yC \times \mathcal{P}_s\,yC & \underset{1 \times i}{\rightarrowtail} & yC \times \mathcal{P}yC \\ \downarrow & & \downarrow \\ \mathcal{P}_s\,yC & \underset{i}{\rightarrowtail} & \mathcal{P}yC \end{array}$$

are pullbacks where $\in_{yC}$ and $\mathcal{P}yC$ are the presheaf membership and powerobject relations and i is the inclusion of $\mathcal{P}_s\,yC$ into $\mathcal{P}yC$ ($\mathcal{P}_s\,yC$ is, by definition, a subfunctor of $\mathcal{P}yC$). Notice that R is representable since r is a small relation. In particular, $R = yE$ for some object E of $\mathcal{C}$ and $r = ye$. So, using the 'twist' isomorphism $\tilde{} : C \times D \cong D \times C$, we have a relation $\tilde{e} : E \rightarrowtail D \times C$. By the Yoneda lemma such an element corresponds to a map $\hat{r} : yD \longrightarrow \mathcal{P}_s\,yC$.

We will now show that the canonical classifying map $\rho : yD \longrightarrow \mathcal{P}yC$ in $\widehat{\mathcal{C}}$ factors through $\hat{r}$. I.e., we show that:

$$\begin{array}{ccc} yD & \xrightarrow{\hat{r}} & \mathcal{P}_s\, yC \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle i} & \\ & \mathcal{P}yC & \end{array}$$

commutes. Notice that, by the two pullbacks lemma, this will suffice to show that $\hat{r}$ is a classifying map for R in $\mathbf{Idl}(\mathcal{C})$. By the proof of the Yoneda lemma the action of $\hat{r}$ on a given member f of $yD(F)$ is:

$$f \quad \longmapsto \quad \mathcal{P}_s\,(yC)(f)(\tilde{e}).$$

But, $\rho_F(f) = (yf \times 1_{yC})^*(y\tilde{e}) = i(\mathcal{P}_s\,(yC)(f)(\tilde{e}))$.

For uniqueness suppose that $q : yD \longrightarrow \mathcal{P}_s\, yC$ such that:

$$\begin{array}{ccc} yE & \longrightarrow & \epsilon_{yC} \\ \downarrow & & \downarrow \\ yC \times yD & \xrightarrow[1\times q]{} & yC \times \mathcal{P}_s\, yC \end{array}$$

is a pullback. Then, in $\widehat{\mathcal{C}}$, ye is the pullback of $\in_{yC}$ along $i \circ q$ and along $i \circ \hat{r} = \rho$. Since ρ is unique with this property it follows that $i \circ \hat{r} = i \circ q$ and, since i is monic, $q = \hat{r}$.

Now, for any ideal $X \cong \varinjlim_i yC_i$ and small relation $r : R \rightarrowtail X \times yD$, R must be representable since the projection:

$$R \rightarrowtail X \times yD \longrightarrow yD$$

is small. I.e., $R \cong yE$ for some E. Therefore there exists a factorization of r:

$$R \rightarrowtail yC_i \times yD \rightarrowtail X \times yD$$

for some i. Thus indeed $\mathrm{Hom}(-, \mathcal{P}_s\, X) \cong \varinjlim_i \mathrm{Hom}(-, \mathcal{P}_s\,(yC_i))$. □

Lemma 4.18. *For any ideal X, $\epsilon_X \rightarrowtail X \times \mathcal{P}_s\, X$ is a small relation.*

Proof. It clearly suffices to verify this for the case where X is a representable yC. Let $yD \rightarrowtail \mathcal{P}_s\, yC$ be given. Then there is a $r : R \rightarrowtail C \times D$ in $\mathcal{C}$ such that:

$$\begin{array}{ccc} yR & \xrightarrow{\pi \circ yr} & yD \\ \downarrow & & \downarrow \\ \epsilon_{yC} & \xrightarrow[\pi_{yC}]{} & \mathcal{P}_s\, yC \end{array}$$

is a pullback, as required. □

Corollary 4.19. *Any relation $R \rightarrowtail X \times Y$ such that there exists a unique classifying map $\rho : Y \longrightarrow \mathcal{P}_s\, X$ is a small relation.*

Proof. By **(S2)** and the fact that ϵ_X is a small relation. □

Putting the foregoing together we have the following proposition:

Proposition 4.20. *If $\mathcal{C}$ is a Heyting pretopos and $X \cong \varinjlim_i yC_i$ is an object of* $\mathbf{Idl}(\mathcal{C})$*, then $\mathcal{P}_s(X) = \varinjlim_i \mathrm{Sub}_{\mathcal{R}}(- \times C_i)$ is a powerobject.*

Moreover, when combined with the fact that axioms **(S1)**-**(S5)** are satisfied in pretoposes we have shown the following:

Theorem 4.21. *If $\mathcal{C}$ is a Heyting pretopos, then* $\mathbf{Idl}(\mathcal{C})$ *is a category with basic class structure.*

Remark. Alex Simpson was the first to prove Proposition 4.15. His proof differed from that given here; it did not use Joyal's small diagonal condition from Theorem 4.4.

4.4 Exponentiation

We now extend the results of the preceding subsection by showing that if $\mathcal{C}$ is a Π-pretopos, then $\mathbf{Idl}(\mathcal{C})$ satisfies condition **(E)** from Section 3.4. First we need the following beautiful and useful fact:

Proposition 4.22. *If $\mathcal{C}$ is a small category and P is an object of* $\mathbf{Idl}(\mathcal{C})$*, then:*

$$\mathbf{Idl}(\mathcal{C})/P \quad \simeq \quad \mathbf{Idl}(\int_{\mathcal{C}} \mathcal{P}),$$

where $\int_{\mathcal{C}} P$ is the category of elements of P *as in [18].*

Proof. It is well known that

$$\widehat{\mathcal{C}}/P \simeq \widehat{\int_{\mathcal{C}} \mathcal{P}}.$$

In particular, there are two functors $R : \widehat{\mathcal{C}}/P \longrightarrow \widehat{\int_{\mathcal{C}} \mathcal{P}}$ and $L : \widehat{\int_{\mathcal{C}} \mathcal{P}} \longrightarrow \widehat{\mathcal{C}}/P$ such that $L \dashv R$ and the two maps are pseudo-inverse to one another. These functors are defined as follows:

- $R(\eta : F \longrightarrow P)$ is a functor given by:

  $$(c, C) \quad \longmapsto \quad \mathrm{Hom}_{\widehat{\mathcal{C}}/P}(\tilde{c} : yC \longrightarrow P, \eta : F \longrightarrow P),$$

  where $\tilde{c}$ is the map in $\widehat{\mathcal{C}}$ corresponding to the element $c \in P(C)$ by the Yoneda lemma.

- $L(F) := \varinjlim_{\mathcal{J}} \pi \circ i$ where $\mathcal{J} := \int_{\int_{\mathcal{C}} P}$, $i : \int_{\mathcal{C}} P \longrightarrow \widehat{\mathcal{C}}/P$ is the map taking an object (c, C) to the corresponding $\tilde{c} : yC \longrightarrow P$ as above and π is the projection from the category of elements.

We begin by showing that if $(\eta : F \longrightarrow P)$ is an object of $\mathbf{Idl}(\mathcal{C})/P$, then $R(P)$ is isomorphic to an object of $\mathbf{Idl}(\int_{\mathcal{C}} \mathcal{P})$. Let η be given as mentioned. Then, since F is an ideal we have $F \cong \varinjlim_{\mathcal{I}} yD_i$ with maps $\mu_i : yD_i \longrightarrow F$ making up the cocone.

We define a functor $G : \mathcal{I} \longrightarrow \int_{\mathcal{C}} P$ such that $\varinjlim_{\mathcal{I}} yG_i \cong R(\eta)$ and $\varinjlim_{\mathcal{I}} yG_i$ is an object of $\mathbf{Idl}(\int_{\mathcal{C}} \mathcal{P})$. Let $G(i) := \widetilde{\eta \circ \mu_i}$ be the object corresponding via the Yoneda lemma to $\eta \circ \mu_i$. Given $f : i \longrightarrow j$ in $\mathcal{I}$, let $G(f) := D(f)$. G is easily seen to be functorial.

Next, let $T := \varinjlim_{\mathcal{I}} yG$. We now define an isomorphism $\varphi : R(\eta) \longrightarrow T$. If $f \in R(\eta)(c, C)$ then we have $f : yC \longrightarrow F$. By familiar properties of representables, there exists an i together with a map $yl : yC \longrightarrow yD_i$ such that $\mu_i \circ yl = f$. Now, an element of $T(c, C)$ is an equivalence class $[g : C \longrightarrow D_i]_\sim$ where $g : C \longrightarrow D_i \sim g' : C \longrightarrow D_{i'}$ if and only if there exists an object i'' of $\mathcal{I}$ together with maps $h : i \longrightarrow i''$ and $h' : i' \longrightarrow i''$ such that $D(h) \circ g = D(h') \circ g'$. So we define $\varphi_{(c,C)}(f) := [l]_\sim$. The naturality of φ follows from the fact that $\mathcal{I}$ is filtered and the maps $\mu_k : yD_k \longrightarrow F$ are monic.

Now we need an inverse map $\psi : T \longrightarrow R(\eta)$. If $[g : C \longrightarrow D_i]_\sim \in T(c, C)$, then let $\psi_{(c,C)}([g]_\sim) := \mu_i \circ yg$. This definition is independent of choice of representative by the fact that $\mathcal{I}$ is filtered and naturality is straightforward.

Finally, it is straightforward to verify, using the fact that $\mathcal{I}$ is filtered, that $\varphi \circ \psi = 1_T$. Moreover, $\psi \circ \varphi = 1_{R(\eta)}$ is trivial. Furthermore, G is easily seen to preserve monomorphisms. As such, we have shown that $R(\eta)$ is an ideal in $\mathbf{Idl}(\int_{\mathcal{C}} \mathcal{P})$.

Similarly, given an object F of $\mathbf{Idl}(\int_{\mathcal{C}} \mathcal{P})$ it follows from the fact that $\pi : \int_{\int_{\mathcal{C}} P} F \longrightarrow \int_{\mathcal{C}} P$ and $i : \int_{\mathcal{C}} P \longrightarrow \widehat{\mathcal{C}}/P$ both preserve monomorphisms that $L(F)$ is an object of $\mathbf{Idl}(\mathcal{C})/P$. $\square$

Theorem 4.23. *If $\mathcal{C}$ is a Π-pretopos, then $\mathbf{Idl}(\mathcal{C})$ is a category with class structure.*

Proof. All that remains is to verify that $\mathbf{Idl}(\mathcal{C})$ satisfies **(E)**.

First, we show that given $!_{yC} : yC \longrightarrow 1$ and $f : X \longrightarrow yC$ the map $\Pi_{!_{yC}}(f) \longrightarrow 1$ is small. By definition we have the following pullback square:

$$\begin{array}{ccc} \Pi_{!_{yC}}(f) & \longrightarrow & X^{yC} \\ \downarrow & & \downarrow{\scriptstyle f^{yC}} \\ 1 & \xrightarrow[\widetilde{\pi_{yC}}]{} & yC^{yC} \end{array}$$

where $\widetilde{\pi_{yC}}$ is the transpose of 1_{yC}. However, since f is small it follows that X is representable. I.e., $X \cong yE$ for some E. But since $\mathcal{R}$ is a Π-pretopos it follows that:

$$\begin{aligned} yC^{yC} &\cong y(C^C), \text{ and} \\ yE^{yC} &\cong y(E^C). \end{aligned}$$

Therefore f^{yC} is a small map and by **(S2)** so is the map $\pi_{!_{yC}}(f) \longrightarrow 1$.

The general case now follows from the foregoing proposition. □

4.5 Universes in $\mathbf{Idl}(\mathcal{E})$

We move to find a universe in $\mathbf{Idl}(\mathcal{C})$ where $\mathcal{C}$ is a Heyting pretopos. We are particularly interested in universes U which include $\mathcal{C}$, in the sense that for every representable yC there is a monomorphism $yC \rightarrowtail U$. This will allow us to conclude that every Heyting pretopos occurs, up to equivalence, as the small objects of a basic category of classes with a universal object, and analogously for Π-pretoposes and toposes.

Since the powerobject functor $\mathcal{P}_s(-)$ is ideal continuous (cf. Proposition 4.10), we may find fixed points for it by standard means (further details and examples can be found in both [8] and [22]). For present purposes we compose $\mathcal{P}_s(-)$ with the ideal continuous functor $C \longmapsto A + C$ for a fixed A in $\mathbf{Idl}(\mathcal{C})$, to obtain the functor G_A defined by $C \longmapsto A + \mathcal{P}_s(C)$. To construct a universal object, we wish for every representable to have a monomorphism into our universe, so we take as our starting point $A := \coprod_{C \in \mathcal{C}} yC$ (where the coproduct is taken in sheaves). This is an ideal, for it is the colimit of the ideal diagram of finite coproducts of representables, which themselves are representable since $yA + yB \cong y(A + B)$, with arrows the coproduct inclusions.

Now consider the ideal diagram of ideals:

$$A \overset{i_A}{\rightarrowtail} A + \mathcal{P}_s A \overset{1_A + (i_A)_!}{\rightarrowtail} A + \mathcal{P}_s(A + \mathcal{P}_s A) \rightarrowtail \dots$$

Where i_A is the coproduct inclusion. Call the colimit U. Then, since the functor G_A is ideal continuous:

$$A + \mathcal{P}_s U \cong U.$$

Therefore, there exists a universe U consisting of the 'class' A of atoms and the 'class' $\mathcal{P}_s U$ of 'sets'. We note that with respect to the powerobject endofunctor $\mathcal{P}_s(-) : \mathbf{Idl}(\mathcal{C}) \longrightarrow \mathbf{Idl}(\mathcal{C})$ this construction makes U the free $\mathcal{P}_s$-algebra over A.

The so-constructed object U is not yet a universal object, however. We obtain, finally, our basic category of classes (respectively, category of classes or powered category of classes) containing $\mathcal{C}$ as the small objects by 'cutting out' the part of $\mathbf{Idl}(\mathcal{C})$ that we need.

Lemma 4.24. *If $\mathcal{C}$ is a category with basic class structure (respectively, class structure or powered class structure) and U is a universe in $\mathcal{C}$, then the full subcategory $\downarrow(U) \rightarrowtail \mathcal{C}$ of objects A in $\mathcal{C}$ such that there exists a monomorphism $A \rightarrowtail U$ is a basic category of classes (respectively, a category of classes or a powered category of classes) with the structure it inherits from $\mathcal{C}$ and with U as its universal object.*

Proof. Using the results of Section 3.7 it is straightforward to verify that $\downarrow(U)$ is closed under the Heyting and basic class (respectively, class or powered class) structure. □

We now summarize our chief results as follows:

Theorem 4.25. *Every Heyting pretopos (respectively, Π-pretopos or topos) $\mathcal{C}$ occurs, up to equivalence, as the small objects in a basic category of classes (respectively, category of classes or powered category of classes) with a universal object.*

Corollary 4.26. *Every Heyting pretopos $\mathcal{C}$ is a model of the set theory* **BCST** *in a canonical way, namely in its own ideal completion* $\mathbf{Idl}(\mathcal{C})$*,*

$$\big(\downarrow(U), U\big) \models \mathbf{BCST}.$$

Moreover, the same holds for Π-pretoposes and **CST***, and toposes and* **BIST***.*

Remark. The particular universe U constructed above validates the *Decidable Sethood* condition:

$$\forall x.\mathsf{S}(x) \vee \neg\mathsf{S}(x)$$

since $\mathcal{P}_s\, U \rightarrowtail U$ is a coproduct inclusion. As a further consequence, Separation thus holds for *all* bounded formulae. I.e., if φ is a Δ_0 formula, then the (universally quantified) statement:

$$\mathsf{S}(x) \rightarrow \text{Ƨ}y \in x.\phi$$

is validated by this universe (cf. the discussion in Section 2.1 and the conditions in Corollary 2.6). It also satisfies some further conditions such as $\in$-induction, but we will not pursue this here (cf. [22]).

Moreover, if $\mathcal{C}$ is a Boolean topos, then this U validates the principle of excluded middle for all simple formulae (and, *a fortiori* all Δ_0 formulae). The reader is directed to [22] for a more detailed discussion of this approach to the construction of universes.

Remark. Recall that, as a set theoretic axiom in the language $\mathcal{L}_s$ from Section 2, *Strong Collection* is the following axiom:

$$\mathsf{S}(a) \wedge (\forall x \in a.\exists y.\varphi) \Rightarrow \exists b.(\mathsf{S}(b) \wedge \mathrm{coll}(x \in a, y \in b, \varphi),$$

where:

$$\mathrm{coll}(x \in a, y \in b, \varphi) \quad := \quad (\forall x \in a.\exists y \in b.\varphi) \wedge (\forall y \in b.\exists x \in a.\varphi).$$

The category theoretic formulation of this axiom is given by the following additional condition on small maps (cf. [17]):

(S6) For any cover $p : D \longrightarrow\!\!\triangleright C$ and $f : C \longrightarrow A$ in $\mathcal{S}$ there exists a quasi-pullback square:

$$\begin{array}{ccccc} C' & \longrightarrow & D & \overset{p}{\twoheadrightarrow} & C \\ {\scriptstyle f'}\downarrow & & & & \downarrow{\scriptstyle f} \\ A' & & \underset{h}{\xrightarrow{\qquad\qquad}\!\!\!\!\rightarrow} & & A \end{array}$$

such that h is a cover and f' is in $\mathcal{S}$.

Whenever $\mathcal{C}$ is a Heyting pretopos its ideal completion $\mathbf{Idl}(\mathcal{C})$ will satisfy a principle (called *small covers*) which in turn implies **(S6)** (cf. [8] or [27]). Thus, in particular, for any Heyting pretopos $\mathcal{C}$ and universe U in $\mathbf{Idl}(\mathcal{C})$ constructed as in Section 4.5:

$$(\downarrow(U), U) \models \text{Strong Collection}.$$

Awodey et al [8] have obtained, for the (impredicative) theory $\mathbf{BIST}_C$, obtained by adding Strong Collection to the axioms of **BIST**, a strengthening of the completeness theorem from Section 3. This so-called 'topos-completeness' result states that $\mathbf{BIST}_C$ is complete with respect to models of the form $(\downarrow(U), U)$ where $\mathcal{E}$ is a topos and U is a universe in $\mathbf{Idl}(\mathcal{E})$.

5 Sheaf models of class theories

Any small category $\mathcal{C}$ with powered class structure can be embedded into sheaves $\mathbf{Sh}(\mathcal{C})$ (coherent coverage) on $\mathcal{C}$ by the Yoneda embedding, which is a full and faithful—and therefore conservative—Heyting functor (into $\mathbf{Sh}(\mathcal{C})$). Similarly, for a topos $\mathcal{E}$, the category $\mathbf{Idl}(\mathcal{E})$ has powered class structure and the inclusion functor $\mathbf{Idl}(\mathcal{E}) \hookrightarrow \mathbf{Sh}(\mathcal{E})$ is a full and faithful Heyting functor, and so our construction below can be carried out for this case as well. The leading idea is to use the higher-order structure present in sheaf categories to model class theory extensions of the set theories of the embedded categories, $\mathcal{C} \hookrightarrow \mathbf{Sh}(\mathcal{C})$ or $\mathbf{Idl}(\mathcal{E}) \hookrightarrow \mathbf{Sh}(\mathcal{E})$. Thus, in particular, to consider categories $\mathcal{C}$ that model **BIST** or **ZF** and obtain models in $\mathbf{Sh}(\mathcal{C})$ of the class theories **BICT** or **BIMK** introduced in Section 2.2. Here we provide details of the **BIMK** case and direct the reader to [12] for details of the, similar, **BICT** case. Because the Yoneda embedding is a conservative functor, these class theories are conservative extensions of the set theories we begin with, and this fact manifests itself insofar as we have to choose between full separation or full comprehension. This is a question of which sheaf to choose to interpret the sort of classes: Here we use the full sheaf powerobject of the universe to interpret the sort of classes, and define a particular subobject of this powerobject to be the 'collection of separable classes'. This results in a full comprehension scheme for classes, but separation only holds for classes in the defined subobject. One could, of course, choose to interpret the sort of classes as this subobject instead, but then one would not have full comprehension.

For the purpose of brevity, we shall restrict ourselves, here, to the special case where $(\mathcal{C}, \iota : \mathcal{P}_s U \cong U)$ is a small class category modelling **ZF**, e.g. $\mathcal{C}$ could be the syntactic category of **ZF**. Notice that in $\mathbf{Sh}(\mathcal{C})$ there is both the 'small' powerobject $y(\mathcal{P}_s U)$ and the 'full' sheaf powerobject $P(yU)$. For notational convenience we will write $\in \rightarrowtail yU \times yU$ for the relation $y(\epsilon_U \rightarrowtail U \times U)$ and $\eta \rightarrowtail yU \times P(yU)$ for the 'full' sheaf membership relation. The presence of both the 'small' membership relation inherited from $\mathcal{C}$ and the 'full' sheaf membership relation allows us to interpret the two sorted language $\mathcal{L}_c$ of sets and classes in $\mathbf{Sh}(\mathcal{C})$. In the following we describe this structure and some of the formulas which are validated therein.

5.1 A sheaf model of BIMK

As in Section 2 we use lower case variables for set variables, and upper case variables for classes. The sort of sets will be yU, and the sort of classes PyU. We now need to interpret both the set membership relation $\in$ and the class membership relation η of $\mathcal{L}_c$. In $\mathbf{Sh}(\mathcal{C})$ we interpret these relations as follows:

- $[\![x \in y]\!]$ is defined to be the subobject:

$$y\big(\,\epsilon_U \rightarrowtail U \times \mathcal{P}_s\, U \rightarrowtail U \times U\big).$$

  (recall that $y(U \times U) \cong yU \times yU$)

- $[\![x\eta X]\!]$ is defined to be:

$$\eta \rightarrowtail yU \times P(yU).$$

We refer to this $\mathcal{L}_c$ structure in $\mathbf{Sh}(\mathcal{C})$ as $\langle \mathbf{Sh}(\mathcal{C}), \epsilon_U, \eta\rangle$ and its restriction to the language $\mathcal{L}_s$ as $\langle \mathbf{Sh}(\mathcal{C}), \epsilon_U\rangle$. Since the structure $\langle \mathbf{Sh}(\mathcal{C}), \epsilon_U\rangle$ is just the image under Yoneda of the structure that interprets $\mathcal{L}_s$ in $\mathcal{C}$ (and we have assumed that $\mathcal{C}$ is a model of **ZF**), and Yoneda is logical, we immediately get the following:

Lemma 5.1. $\langle \mathbf{Sh}(\mathcal{C}), \epsilon_U\rangle \models \mathbf{ZF}$*, where* **ZF** *includes the Law of Excluded Middle for every formula of* $\mathcal{L}_s$*.*

Equally immediate, using familiar properties of toposes, is the next lemma:

Lemma 5.2. $\langle \mathbf{Sh}(\mathcal{C}), \epsilon_U, \eta\rangle$ *models Class Extensionality and Class Comprehension. I.e.,*

1. $\langle \mathbf{Sh}(\mathcal{C}), \epsilon_U, \eta\rangle \models \forall X, Y.(\forall x.x\eta X \Leftrightarrow x\eta Y) \Rightarrow X = Y.$
2. $\langle \mathbf{Sh}(\mathcal{C}), \epsilon_U, \eta\rangle \models \exists X.\forall z.z\eta X \Leftrightarrow \varphi$*, for any formula* φ *such that* $X \notin \mathrm{FV}(\varphi)$*.*

Replacement, however, is less obvious.

Lemma 5.3. $\langle \mathbf{Sh}(\mathcal{C}), \epsilon_U, \eta\rangle$ *models Strong Class Replacement. I.e., the image of a set under a functional relation is a set. Formally, for any formula* φ*:*

$$\langle \mathbf{Sh}(\mathcal{C}), \epsilon_U, \eta\rangle \models (\forall y \in x.\exists! z.\varphi) \Rightarrow (\exists u.\forall z.z \in u \Leftrightarrow \exists y \in x.\varphi)$$

with $u \notin \mathrm{FV}(\varphi)$*.*

Proof. It makes no essential difference to our proof whether φ has additional parameters or not, so we will assume not. We must show, then, that a generalized element $\alpha : X \longrightarrow y(\mathcal{P}_s\, U)$ that factors through $[\![x : y(\mathcal{P}_s\, U) | \forall y \in x.\exists! z.\varphi]\!] \rightarrowtail y(\mathcal{P}_s\, U)$ also factors through $[\![x : y(\mathcal{P}_s\, U) | \exists u.\forall z.z \in u \Leftrightarrow \exists y \in$

$x.\varphi]\!] \rightarrowtail y(\mathcal{P}_s\, U)$. It suffices, since the representables generate the topos of sheaves, to test the claim for a representable, so that $X = yC$. Now, we can slice over a representable without affecting any properties we need for the sake of this proof. Specifically, slicing preserves first-order logic; $\mathcal{C}/C$ is still a category with powered class structure; $\mathbf{Sh}(\mathcal{C})/yC$ is a topos; the embedding $\mathcal{C}/C \longrightarrow \mathbf{Sh}(\mathcal{C})/yC$, defined by $f : D \longrightarrow C \mapsto yf : yD \longrightarrow yC$, is full, faithful, and Heyting; and $y(\mathcal{C})$ generates $\mathbf{Sh}(\mathcal{C})$. Therefore, we can assume without loss of generality that our generalized element is a point $\alpha : 1 \longrightarrow y(\mathcal{P}_s\, U)$, such that

$$\mathbf{Sh}(\mathcal{C}) \models \forall y \in \alpha.\exists!z.\varphi$$

We reason internally in $\mathbf{Sh}(\mathcal{C})$ with the necessary translations from $\mathcal{L}_c$ usually left implicit. It more than suffices to find another point $\beta : 1 \longrightarrow y\mathcal{P}_s\, U$ such that:

$$\mathbf{Sh}(\mathcal{C}) \models \forall z.z \in \beta \Leftrightarrow \exists y \in \alpha.\varphi$$

We now use the fact that Yoneda is full and faithful and preserves and reflects finite limits and regular epimorphisms to jump back and forth between $\mathcal{C}$ and $\mathbf{Sh}(\mathcal{C})$, thus allowing us to use the class structure on $\mathcal{C}$. As a point of $y\mathcal{P}_s\, U$, the point α represents a small subobject yA,

$$\begin{array}{ccc} yA := [\![x : U | x\epsilon\alpha]\!] & \longrightarrow & y\epsilon \\ \downarrow & \lrcorner & \downarrow \\ yU \times 1 & \xrightarrow[1_{yU} \times \alpha]{} & yU \times y\mathcal{P}_s\, U \end{array}$$

such that φ defines a functional relation on $y(A \times U) \cong yA \times yU$. This functional relation is the graph of a unique morphism $yf : yA \longrightarrow yU$, which must be in the image of y since Yoneda is full. The image factorization of f must again be small, by axiom S4 for categories with basic class structure, thereby giving us another small subobject B of U classified by a point β:

$$\begin{array}{ccc} yB & \longrightarrow & y\epsilon \\ \downarrow & \lrcorner & \downarrow \\ yU \times 1 & \xrightarrow[1_{yU} \times \beta]{} & yU \times y\mathcal{P}_s\, U \end{array}$$

Since we obtained yB as the image of $yA := [\![x : U | x\epsilon\alpha]\!]$ under the functional relation φ, we must have that

$$\mathbf{Sh}(\mathcal{C}) \models \forall z.z \in \beta \Leftrightarrow \exists y \in \alpha.\varphi$$

as hoped for. □

Consulting table 5 on page 12 which lists the axioms of **BIMK**, we see that only the axiom of Simple Class Comprehension (SCC) remains to verify. This axiom is formulated in terms of a defined predicate $\dagger(-)$, which in our structure $\langle \mathbf{Sh}(\mathcal{C}), \epsilon_U, \eta \rangle$ is interpreted by a subobject of $P(yU)$:

$$[\![X \mid \dagger(X)]\!] \quad := \quad [\![X \mid \forall x.!(x\eta X)]\!] \rightarrowtail P(yU)$$

But, since the axiom of Simple Class Comprehension itself plays no role in the proofs of lemmas 2.14 and 2.15, soundness allows us to conclude that,

$$[\![X \mid \dagger(X)]\!] \quad = \quad [\![X \mid \forall x.(x\eta X) \vee \neg(x\eta X)]\!] \tag{5}$$

in $\mathrm{Sub}_{\mathbf{Sh}(\mathcal{C})}(P(yU))$.

This subobject has a more simple characterization as follows. Since Yoneda preserves the terminal object and finite coproducts, we have that:

$$y(1+1) \cong y1 + y1 \cong 1+1$$

This object, $1+1$, is a decidable subobject classifier in both $\mathcal{C}$ and $\mathbf{Sh}(\mathcal{C})$. In $\mathcal{C}$ it is in fact a subobject classifier tout court, since $\mathcal{C}$ is Boolean, and it can also be shown to be the small power object of 1, such that $P_s 1 \cong 1+1$. In $\mathbf{Sh}(\mathcal{C})$, however, it is a subobject of the sheaf subobject classifier:

$$1+1 \xrightarrow{[\top,\bot]} \Omega \cong P1$$

where $\top : 1 \longrightarrow \Omega$ as usual denotes the point 'true' of Ω, and $\bot$ is its complement. Now, $P(yU) \cong P1^{yU}$, and the monomorphism $[\top,\bot] : 1+1 \rightarrowtail \Omega \cong P1$ yields a monomorphism:

$$[\top,\bot]^{yU} : (1+1)^{yU} \rightarrowtail P1^{yU} \cong P(yU)$$

which is easily seen to represent the desired subobject:

$$(1+1)^{yU} \quad \cong \quad [\![X \mid \dagger(X)]\!] \rightarrowtail P(yU) \tag{6}$$

Now that we have a grip on the interpretation of the $\dagger(-)$ predicate in $\mathbf{Sh}(\mathcal{C})$, let us recall what the axiom of Simple Class Comprehension looks like. We want to show that for any formula φ in $\mathcal{L}_c$ such that all class variables $X_1, \ldots, X_n$ are free in φ,

$$\langle \mathbf{Sh}(\mathcal{C}), \epsilon_U, \eta \rangle \models \bigwedge_{j=1}^{n} \dagger(X_j) \Rightarrow (\exists X. \dagger(X) \wedge \forall x. x\eta X \Leftrightarrow \varphi).$$

For simplicity, we may restrict ourselves to the case with only one free variable Y in φ, as the argument we are about to go through readily generalizes. Diagrammatically, then, we have to establish the following inclusion of subobjects:

$$\begin{array}{ccc} [\![Y \mid \dagger(Y)]\!] & \rightarrowtail \dashrightarrow & [\![Y \mid \exists X. \dagger(X) \wedge \forall x. x\eta X \Leftrightarrow \varphi]\!] \\ & \searrow \quad \swarrow & \\ & P(yU) & \end{array}$$

We do this by showing that

$$[\![Y|\dagger(Y)]\!] \cap [\![Y|\exists X.\dagger(X) \wedge \forall x.x\eta X \Leftrightarrow \varphi]\!] = [\![Y|\dagger(Y)]\!]$$

in $\mathrm{Sub}_{\mathbf{Sh}(\mathcal{C})}(P(yU))$. By 6, $[\![Y|\dagger(Y)]\!]$ is the monomorphism

$$[\top, \bot]^{yU} : (1+1)^{yU} \rightarrowtail P1^{yU}.$$

Consider the following pullback:

$$\begin{array}{ccc}
[\![Y : (1+1)^{yU}|\exists X.\dagger(X) \wedge \forall x.x\eta X \Leftrightarrow \varphi']\!] & \rightarrowtail & (1+1)^{yU} \\
\downarrow & \lrcorner & \downarrow \\
[\![Y : P1^{yU}|\exists X.\dagger(X) \wedge \forall x.x\eta X \Leftrightarrow \varphi]\!] & \rightarrowtail & P1^{yU}
\end{array} \tag{7}$$

We claim that the top arrow is an isomorphism, i.e. that the following holds in $\mathbf{Sh}(\mathcal{C})$:

$$\forall Y : (1+1)^{yU}.\exists X : (1+1)^{yU}.\forall x : P1^{yU}.x\eta X \Leftrightarrow \varphi'$$

As already used, pulling back respects first-order logic, so that if we consider a formula φ' fitting into a pullback

$$\begin{array}{ccc}
[\![Y : (1+1)^{yU}|\varphi']\!] & \rightarrowtail & (1+1)^{yU} \\
\downarrow & & \downarrow \\
[\![Y : P1^{yU}|\varphi]\!] & \rightarrowtail & P1^{yU}
\end{array} \tag{8}$$

then up to equivalence this formula can be obtained from φ by replacing the $P1^{yU}$-sorted free variable (Y) with a $(1+1)^{yU}$-sorted variable (Y again, say), and replacing the η membership predicate with its restriction (by pullback) to $yU \times (1+1)^{yU}$, which we call η'. The following is easily checked:

Lemma 5.4.

$$\eta' := [\![x : yU, X : (1+1)^{yU}|x\eta' X]\!] \rightarrowtail yU \times (1+1)^{yU}$$

is a complemented subobject. Moreover, it is therefore classified by the evaluation morphism of $(1+1)^{yU}$*:*

$$\begin{array}{ccc}
\eta' & \longrightarrow & 1 \\
\downarrow & \lrcorner & \downarrow \top \\
yU \times (1+1)^{yU} & \xrightarrow[eval]{} & 1+1
\end{array}$$

Lemma 5.5. *For the formula φ' of (8),*

$$[\![x : yU, Y : (1+1)^{yU} | \varphi']\!] \rightarrowtail yU \times (1+1)^{yU}$$

is a complemented subobject.

Proof. Since any sheaf is a colimit of representables, the claim will follow if we can show that for any arrow with representable source, $\langle f_1, f_2\rangle : yC \longrightarrow yU \times (1+1)^{yU}$, the pullback along $\langle f_1, f_2\rangle$:

$$\begin{array}{ccc} [\![c : yC | \varphi'(f_1(c), f_2(c))]\!] & \longrightarrow & [\![x : yU, Y : (1+1)^{yU} | \varphi'(x, Y)]\!] \\ \big\downarrow \lrcorner & & \big\downarrow \\ yC & \xrightarrow[\langle f_1, f_2\rangle]{} & yU \times (1+1)^{yU} \end{array} \tag{9}$$

is a complemented subobject. Consider first the following two pullbacks diagram:

$$\begin{array}{ccccc} [\![x : yU, c : yC | x\eta' f_2(c)]\!] & \longrightarrow & \eta' & \longrightarrow & 1 \\ \big\downarrow \lrcorner & & \big\downarrow \lrcorner & & \big\downarrow \top \\ yU \times yC & \xrightarrow[Id \times f_2]{} & yU \times (1+1)^{yU} & \xrightarrow[eval]{} & 1+1 \end{array}$$

Since 1 and $1+1$ are representable and Yoneda preserves products, the outer pullback square can be taken in $\mathcal{C}$. Therefore, the subobject $[\![x : yU, c : yC | x\eta' f_2(c)]\!]$ is representable. Now, consider (9). Apart from the variable $Y : (1 = 1)^{yU}$ and the predicate η', the formula φ' is constructed only with variables of sort yU and the $y\epsilon_U$ predicate. Therefore, the fact that the subobject $[\![x : yU, c : yC | x\eta' f_2(c)]\!]$ is representable allows us to construct the subobject $[\![c : yC | \varphi'(f_1(c), f_2(c))]\!]$ entirely within the subcategory $y : \mathcal{C} \hookrightarrow \mathbf{Sh}(\mathcal{C})$ (keeping in mind that Yoneda is Heyting). So the subobject $[\![c : yC | \varphi'(f_1(c), f_2(c))]\!]$ is itself representable, and therefore complemented ($\mathcal{C}$ being Boolean). □

With lemma 5.5, we get a classifying arrow $\varrho : yU \times (1+1)^{yU} \longrightarrow 1+1$, and by considering the transpose $\widetilde{\varrho}$, we obtain the following two pullbacks diagram:

$$\begin{array}{ccccc} [\![x : yU, Y : (1+1)^{yU} | \varphi']\!] & \longrightarrow & \eta' & \longrightarrow & 1 \\ \big\downarrow \lrcorner & & \big\downarrow \lrcorner & & \big\downarrow \top \\ yU \times (1+1)^{yU} & \xrightarrow[Id \times \widetilde{\varrho}]{} & yU \times (1+1)^{yU} & \xrightarrow[eval]{} & 1+1 \end{array}$$

From the left pullback square we conclude that the following statement in the internal language of $\mathbf{Sh}(\mathcal{C})$ holds in $\mathbf{Sh}(\mathcal{C})$:

$$\mathbf{Sh}(\mathcal{C}) \models \forall Y : (1+1)^{yU}.\exists X : (1+1)^{yU}.\forall x.x\eta' X \Leftrightarrow \varphi'$$

And hence the left vertical arrow in diagram (7) is an isomorphism, as claimed. Thus we conclude:

Lemma 5.6.

$$\langle \mathbf{Sh}(\mathcal{C}), \epsilon_U, \eta \rangle \models \text{Simple Class Comprehension}.$$

Finally, by lemma 5.1, lemma 5.2, lemma 5.3, and lemma 5.6 put together:

Proposition 5.7. $\langle \mathbf{Sh}(\mathcal{C}), \epsilon_U, \eta \rangle$ *is a model of* **BIMK***:*

$$\langle \mathbf{Sh}(\mathcal{C}), \epsilon_U, \eta \rangle \models \mathbf{BIMK}.$$

Recall now that we assumed of the powered category of classes $\mathcal{C}$ only that it be small, Boolean and a model of **ZF**. In particular, $\mathcal{C}$ could be the syntactic category of **ZF**, so that the model of **ZF** in it is the generic one. This model is conservative, in the sense that only statements provable in **ZF** are true in it. Since Yoneda is a conservative functor, the same holds for the model we have constructed in $\mathbf{Sh}(\mathcal{C})$. That is to say, for any sentence φ in $\mathcal{L}_c$ that is also a sentence in $\mathcal{L}_s$:

$$\langle \mathbf{Sh}(\mathcal{C}), \epsilon_U, \eta \rangle \models \varphi \;\text{ implies }\; \mathbf{ZF} \vdash \varphi$$

Therefore,

Proposition 5.8. **BIMK** *is a conservative extension of* **ZF***.*

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