1967

On compactness and projections

Isidore Fleischer
Carnegie Mellon University

S. P. Franklin

Follow this and additional works at: http://repository.cmu.edu/math
On Compactness and Projections

by

Isidore Fleischer

and

S. P. Franklin

Report 67-20

May, 1967
0. It is a well known and frequently useful fact that whenever a
topological space $X$ is compact, the projection $p$ of $X \times Y$ on
any space $Y$ is a closed mapping ([Du], p.227). (The converse,
although not so well known is also true [M].) Analogues of this
theorem concerning countable compactness, sequential compactness,
$m$-compactness, etc., fairly abound in the literature ([PL], [H],
[I], [N], etc.) In this note, using convergence as our basic con-
cept, we propose to derive improvements and extensions of many of
these results, as well as a number of related facts, in a somewhat
uniform manner.

Whenever $X$ is countably compact and $Y$ is a sequential
space, $p$ is closed. Following Isiwata [I], take $(S)$ to be
the class of space $Y$ such that $v$ is closed for each countably
compact $X$. As Isiwata points out, each subspace and each closed
continuous image of a space in $(p$ is again in $(S$. Hence every sub-
space of a sequential space is in $(S$. (We are unable to decide
whether every space in $(S$ is a subspace of a sequential space.) As
an example of a space not in $(S$, take a point $p\in$N$, where N
is the Stone-Cech compactification of the natural numbers, and con-
sider $N \cup \{p\}$ as a subspace of $\#N$. Then $\#N \setminus \{p\}$ is countably
compact, but the projection of $\#N \setminus \{p\} \times (N \cup \{p\})$ on $N \cup \{p\}$ is
not closed. Thus $N \cup \{p\}$ does not belong to $(\#N)$. It follows
immediately that no superset of $N \cup \{p\}$ belongs to $(S$ and, in
particular $1%//(S$ (see [I]2.3).*

Letting $S_{1}$ be a space consisting of a convergent sequence
and its limit, i.e., homeomorphic to the subset \{0\}\cup\{1/n\mid n\in N\}
of the real line, the following is a strong converse to the first
gassertion of the previous paragraph: if the projection of $X \times S_{1}$
on $p$ is closed, then $X$ is countably compact. Let us denote

*
by (f) the class of spaces \( Y \) such that \( X \) is countably compact whenever the projection of \( X \times y \) on \( Y \) is closed. Hanai has shown that every space which is not a P-space belongs to \( (T) \) ([H, ] Theorem 4). It follows immediately that \( S, dj') \). Conversely, no P-space belongs to \( \emptyset \), so that \( (T) \) is precisely the complement of the class of P-spaces. Also, no non-discrete P-space belongs to \( \emptyset \), from which it follows that the non-discrete spaces in \( (N) \) also belong to \( \emptyset \). However, \( N \cup \{p\} \), since it is not a P-space belongs to \( (T) \) but not to \( \emptyset \). Every non-discrete P-space belongs to neither \( \emptyset \) nor \( \emptyset \).

Since sequential compactness implies countable compactness and every Prechet space is sequential, it follows that whenever \( X \) is sequentially compact and \( Y \) is a Prechet space, \( X \) is closed. Conversely, if the projection of \( X \times S = \text{closed, then} \) X is sequentially compact. These results suggest classes \( (g) \) and \( (f) \) be defined bearing the same relation to sequential compactness as \( (N) \) and \( (j) \) do to countable compactness. Since there are compact spaces which are not sequentially compact (for example \( jSn \), \( jS \)) is empty. Since sequential compactness implies countable compactness \( (S) \) contains \( (S) \). We conjecture that the inclusion is proper.

It is easily seen that the corresponding classes, \( (S) \) and \( (2?c) \), for compactness are respectively all spaces and empty. Given \( Y \), let \( X \) be the set of ordinals less than the initial ordinal of \( X' \).

The unproved assertions of the preceding paragraphs follow readily from the more general considerations which are sketched out in the next section.
1. X and Y are topological spaces; \( n \) is the projection of \( X \times Y \) on Y; \( D \) and \( E \) are classes of nets into \( X, Y \) or \( X \times Y \) closed under composition with and cancellation of maps between these spaces.

The following are equivalent: (i) For every \( S \subseteq X \times Y \), \( Y \) is a limit of an \( E \)-net in \( \text{TT}(S) \) only if \( \text{TT}^{-1}(y) \) contains a limit of a \( D \)-net in \( S \). (ii) For every (not necessarily continuous) function from \( Y \) to \( X \), the range of every convergent \( E \)-net in \( Y \) contains a \( D \)-net converging to the same limit whose image under \( f \) converges in \( X \).

Remarks: (i) implies (iii): If \( S \) contains all limits of its \( D \)-nets, \( \text{TT}(S) \) will contain all limits of its \( E \)-nets. Conversely, if the adjunction of all \( D \)-limits to sets is an idempotent closure operator in \( X \times Y \) (for example, if \( D \) is closed under iteration in the sense of the construction of \( [K] \) p.69), then (iii) implies (i).

(ii) follows for every \( Y \) if every \( E \)-net in \( X \) has a convergent \( D \)-subnet. Conversely, this holds for \( X \) if it satisfies (ii) with any class of \( Y \) containing for each member of \( E \) a point whose neighborhood filter induces on the range of some one-to-one \( E \)-net with the same domain exactly the image of the filter of final subsets. If the ranges of these one-to-one nets in the discrete topology (or/ more generally, in any for which only the eventually constant \( D \)-nets converge) along with their limits actually appear among the \( Y \), then (ii) may be weakened to (iii). (For \( X \) T., take for \( S \) the range of the product \( E \)-net; in general, take the union of its point-closures.)

We have use for two special cases: \( D \) the class of nets and
E the class of sequences; and $D = E$ the class of sequences. We obtain respectively:

For every function from $Y$ to $X$, every convergent sequence in $Y$ has a subsequence whose image clusters in $X$ iff $ir$ sends sequential cluster points onto sequential limits iff $ir$ sends sets containing their sequential cluster points onto sets containing their sequential limits. If $X$ is countably compact this holds for every $Y$; conversely if it holds for some $Y$ containing $S'$, $X$ is countably compact.

For every function from $Y$ to $X$, every convergent sequence in $Y$ has a subsequence whose image converges in $X$ iff $ir$ sends sequential limits onto sequential limits. If $X$ is sequentially compact this holds for every $Y$; conversely if it holds for some $Y$ containing $S'$, $X$ is sequentially compact; if, moreover, it holds for $Y = S'$, then $X$ is sequentially compact if $ir$ only sends sets containing their sequential limits onto sets containing their sequential limits.\(^{[8]}\)

For the remainder of the paper we deal only with $D = E = all$ directed sets with maps. (i) is equivalent to (iii) which says that $TT$ is closed; they hold for all $Y$ iff $X$ is compact. We shall, however, want to take $X$ countably compact and inquire about the corresponding $Y$.

One result follows from what we have shown so far: If $ir$ is closed, it preserves closures of countable sets, hence sends sets containing sequential cluster points on sets containing sequential limits. Thus if $Y$ contains $S'$, $ir$ closed implies $X$ countably compact. The implication already holds if $Y$ only contains a non-closed $F_0$ (Hanai [H\(^n\)]); and for no other $Y$. We shall show this in a somewhat more general setting: If $Y$ contains a non-closed
union of \( \bigcup \) closed sets, then if closed implies that \( X \) contains no locally finite collection of \( \bigcup \) closed sets (else the union of the products of paired closed sets in \( X \) and \( Y \) under any one-to-one pairing would be a closed set with a non-closed projection); conversely, if every \( \bigcup \) closed subsets of \( Y \) have a closed union, then with \( X \) any \( T_{\omega} \) space of cardinality at most \( \omega^\omega \), \( IT \) is closed. (9) Similarly, if \( Y \) contains an ascending well-ordered chain of closed subsets, with each set indexed by a limit ordinal the union of those preceding, which is not closed, and if \( X \) contains a corresponding descending chain of closed sets with empty intersection, then the obvious union in the product is a closed set whose projection is not closed; in particular, if \( Y \) is a non-discrete P-space and \( X \) a suitable transfinite ordinal, \( ir \) is not closed.
FOOTNOTES

(1) Call a set **sequentially** open if no sequence outside the set converges to a point of the set. A space is **sequential** if every sequentially open subset is open \([F,\] \).

(2) This result improves Lemma 1.1 of [I] and one direction of each of Corollary 1.7 of [KL], Corollary 2 of \([H_2]\), and Theorem 1 of [N].

(3) Let us note here that our \([N]\) is apparently slightly different from Isiwata's since he deals only with completely regular spaces. The difference is real, since countably compact subsets of spaces in his \([S]\) are closed \([I]\) Lemma 1.2) while countably compact subsets of a sequential space are closed iff sequential limits are unique \([F_2J]\) Prop. 5.4).

(4) Lynn Imler points out that this improves an unpublished result of E. Michael which asserts that \(\mathbb{N} \cup \{p\}\) cannot be embedded in any Hausdorff sequential space. Since it is not in \([N]\) our result implies it is not a subspace of any sequential space.

(5) A **P-space** is one in which every \(F_\sigma\) is closed \([G-J]\) p. 62).

(6) A space is a **Fréchet** space if the closure of each of its subsets may be obtained by taking the limits of the sequences in the subset (c.f. \([F,\]) \). This is precisely the condition imposed by Isiwata in Lemma 1.1 of \([I]\).

(7) This strengthens the other direction of Theorem 1 of [N] (see (2)).

(8) One might in the interests of symmetry, wish to consider also \(ir\) sending sequential cluster points onto sequential cluster points. This leads to a new kind of compactness implied by ordinary and implying countable; and properly as shown by \(ox\) and \#\(N\)\{p\}; not implying sequential (else ordinary compactness would) and, we suppose, not implied by it, although we have no counterexample.

(9) The authors wish to express their thanks to Renababy for the delectable Japanese meal which inspired this observation.
REFERENCES


