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Characterization of Sobolev and BV Spaces

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CHARACTERIZATION OF SOBOLEV AND BV SPACES

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This work presents some new characterizations of Sobolev spaces and the space of functions of Bounded Variation. Additionally it gives new proofs of continuity and lower semicontinuity theorems due to Reshetnyak.

TABLE OF CONTENTS

1.0 INTRODUCTION 1

1.1 Characterizing Sobolev and BV spaces 1

1.1.1 Imaging Applications 8

1.2 Reshetnyak's Theorems 10

2.0 PRELIMINARIES 13

2.1 Banach Spaces and Duality 13

2.1.1 Locally Convex Topological Vector Spaces 13

2.1.2 Duality and Weak, Weak-Star Topologies 17

2.2 Measures 19

2.2.1 Measures and Integration 19

2.2.2 Measures and Differentiation 32

2.2.3 Measures as a Dual Space 38

2.3 Spaces of Integrable Functions 41

2.3.1 Lebesgue spaces 41

2.3.2 Lebesgue spaces on Euclidean space 46

2.3.3 Sobolev Spaces 48

2.3.4 Functions of Bounded Variation 53

2.4 Convexity 56

3.0 CHARACTERIZATION OF SOBOLEV AND BV SPACES 58

3.1 Some Preliminaries 58

3.2 A Few Useful Estimates 62

3.3 New Characterizations 68

3.4 Counterexamples 77
4.0 MEASURE SEMICONTINUITY THEOREMS 82
4.1 Proof of Theorem 16 82
4.2 Proof of Theorem 17 85
BIBLIOGRAPHY 91

1.0 INTRODUCTION

1.1 CHARACTERIZING SOBOLEV AND BV SPACES

In the recent paper [12], Bourgain, Brezis, and Mironescu studied the limiting behavior of the semi-norm

$$|f|_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}}\, dydx\right)^{\frac{1}{p}},$$

of the fractional Sobolev spaces $W^{s,p}$, $0 < s < 1$, $1 < p < \infty$. This semi-norm was introduced by Gagliardo in [44], to characterize the space of traces of functions in $W^{1,p}$, $p > 1$. It is well known that $|f|_{W^{s,p}(\Omega)}$ does not converge to

$$|f|_{W^{1,p}(\Omega)} := \left(\int_\Omega |\nabla f|^p\, dx\right)^{\frac{1}{p}}$$

when $s \to 1^-$. Bourgain, Brezis, and Mironescu [12] recognized that this difficulty is a question of scaling. Indeed, they were able to show that when Ω is a smooth, bounded domain,

$$\lim_{s\to 1^-}(1-s)|f|^p_{W^{s,p}(\Omega)} = \frac{K_{p,N}}{p}|f|^p_{W^{1,p}(\Omega)}, \tag{1.1}$$

for all $f \in L^p(\Omega)$, $1 < p < \infty$, where $|f|_{W^{1,p}} := \infty$ if $f \notin W^{1,p}(\Omega)$. Here, $K_{p,N} > 0$ only depends on p and N. This important result has been extended in several directions. Maz'ya and Shaposhnikova [63] proved that for $f \in \bigcup_{0<s<1} W^{s,p}_0(\mathbb{R}^N)$,

$$\lim_{s\to 0^+} s|f|^p_{W^{s,p}(\mathbb{R}^N)} = C_{p,N}||f||^p_{L^p(\mathbb{R}^N)}. \tag{1.2}$$

Kolyada and Lerner [54] extended these results to general Besov spaces $B^s_{p,\theta}$, while Milman [65] generalized (1.1) and (1.2) to the setting of interpolation spaces, by establishing continuity of the real and complex interpolation spaces at the endpoints.

Another important consequence of (1.1) is that the analysis led to a new characterization of the Sobolev spaces $W^{1,p}(\Omega)$, $1 < p < \infty$.

Consider the family of mollifiers

$$\rho_\epsilon \geq 0, \quad \int_{\mathbb{R}^N} \rho_\epsilon(x)\, dx = 1, \tag{1.3}$$

$$\lim_{\epsilon \to 0} \int_{|x|>\delta} \rho_\epsilon(x)\, dx = 0 \quad \text{for all } \delta > 0, \tag{1.4}$$

$$\rho_\epsilon \text{ is radial, that is, } \rho_\epsilon(x) = \hat{\rho}_\epsilon(|x|), \quad x \in \mathbb{R}^N. \tag{1.5}$$

In [12], Bourgain, Brezis, and Mironescu proved the following result.

Theorem 1 ([12], Theorem 2)**.** *Suppose $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain, $1 < p < \infty$, and ρ_ϵ satisfy (1.3), (1.4), and (1.5). Then for $f \in L^p(\Omega)$,*

$$\lim_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \hat{\rho}_\epsilon(|x-y|)\, dydx = K_{p,N} |f|^p_{W^{1,p}(\Omega)}, \tag{1.6}$$

where $|f|_{W^{1,p}(\Omega)} := \infty$ if $f \notin W^{1,p}(\Omega)$.

Note that (1.1) follows from Theorem 1 by taking

$$\rho_\epsilon(x) := \frac{\chi_{[0,R]}(|x|)}{R^{\epsilon p}} \frac{p\epsilon}{|x|^{N-p\epsilon}},$$

where $\epsilon = (1-s)$, $R > 0$ is chosen bigger than the diameter of Ω, and by writing

$$(1-s) \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{N+sp}}\, dydx = \frac{R^{\epsilon p}}{p} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \hat{\rho}_\epsilon(|x-y|)\, dydx,$$

since $R^{\epsilon p} \to 1$ as $\epsilon \to 0$.

The case $p = 1$ is a little delicate, and if $f \in W^{1,1}(\Omega)$, then the equality (1.6) holds. However, assuming the left-hand-side of (1.6) is finite is not enough to conclude $f \in W^{1,1}(\Omega)$. The following theorem is the appropriate extension to $p = 1$.

Theorem 2 ([12], Theorem 3′)**.** *Suppose $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain and ρ_ϵ satisfy (1.3), (1.4), and (1.5). Then there exist constants $C_1, C_2 > 0$ such that for every $f \in L^1(\Omega)$,*

$$C_1|Df|(\Omega) \leq \liminf_{\epsilon\to 0} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\hat{\rho}_\epsilon(|x-y|)\,dydx$$
$$\leq \limsup_{\epsilon\to 0} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\hat{\rho}_\epsilon(|x-y|)\,dydx \leq C_2|Df|(\Omega),$$

where $|Df|(\Omega)$ is the total variation of the measure Df, the distributional derivative of f, and $|Df|(\Omega) = +\infty$ if $f \notin BV(\Omega)$.

In one dimension, Bourgain, Brezis, and Mironescu were able to obtain $C_1 = C_2 = 1$, so that the BV semi-norm is actually the limit as in the $W^{1,p}$ case. This limit characterization was completed for $N \geq 2$ independently by Ambrosio [3] and Dávila [34], who proved the following result.

Theorem 3 ([34], Theorem 1)**.** *Suppose $\Omega \subset \mathbb{R}^N$ be open, bounded domain with Lipschitz boundary and ρ_ϵ satisfy (1.3), (1.4), and (1.5). Then for $f \in L^1(\Omega)$,*

$$\lim_{\epsilon\to 0} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|}\hat{\rho}_\epsilon(|x-y|)\,dydx = K_{1,N}|Df|(\Omega), \tag{1.7}$$

where $|Df|(\Omega) = +\infty$ if $f \notin BV(\Omega)$.

Note that for *smooth* domains, Theorems 1 and 3 give new characterizations of the spaces $W^{1,p}(\Omega)$, $1 < p < \infty$, and $BV(\Omega)$. However, these characterizations fail for arbitrary open, bounded sets, as Brezis [15, Remark 5] gives a construction of a bounded open set Ω and a function $f \in W^{1,\infty}(\Omega)$ such that

$$\lim_{\epsilon\to 0} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p}\hat{\rho}_\epsilon(|x-y|)\,dydx = +\infty,$$

where ρ_ϵ satisfy (1.3), (1.4), and (1.5) (see a related construction in Theorem 13). Thus $f \in W^{1,p}(\Omega)$ for every p, and yet the iterated integral is infinite. In this construction, Ω is specifically chosen such that points close with respect to the Euclidean distance are far with respect to the geodesic distance d_Ω in Ω. This leads to the following questions of Brezis [15] and Ponce [70].

Open Question 1 (Brezis, [15])

For $\Omega \subset \mathbb{R}^N$ open and ρ_ϵ satisfying (1.3), (1.4), and (1.5), does $f \in L^p(\Omega)$ and

$$\limsup_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{d_\Omega(x,y)^p} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dydx < +\infty \tag{1.8}$$

imply that $f \in W^{1,p}(\Omega)$?

Open Question 2 (Ponce, [70])

For $\Omega \subset \mathbb{R}^N$ open and ρ_ϵ satisfy (1.3), (1.4), and (1.5), does $f \in L^p(\Omega)$ and

$$\limsup_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dydx < +\infty \tag{1.9}$$

imply that $f \in W^{1,p}(\Omega)$?

The main purpose of this thesis is to provide answers to these questions and to give a characterization of the spaces $W^{1,p}(\Omega)$, $1 < p < \infty$ and $BV(\Omega)$ for arbitrary domains.

Remark 4. *We remark that if we can prove the conjecture of Brezis, then we obtain the conjecture of Ponce, since the inequality $|x - y| \leq d_\Omega(x,y)$ implies that the functional conjectured by Brezis is less than or equal to the functional of Ponce. Thus, we will only focus on proving the conjecture of Brezis. However, we will also later give a counterexample that demonstrates they are in fact different conditions.*

Following the work of Ponce [70], we replace the hypothesis that ρ_ϵ are radial with a weaker condition. Precisely, we assume there exist $\{v_i\}_{i=1}^N \subset \mathbb{R}^N$ and a $\delta > 0$ such that for all $\sigma_i \in C_\delta(v_i)$ the set $\{\sigma_i\}_{i=1}^N$ is linearly independent, where

$$C_\delta(v) := \left\{ w \in \mathbb{R}^N \backslash \{0\} : \frac{v}{|v|} \cdot \frac{w}{|w|} > 1 - \delta \right\},$$

and

$$\liminf_{\epsilon \to 0} \int_{C_\delta(v_i)} \rho_\epsilon(x)\, dx > 0 \text{ for all } i = 1, \dots, N. \tag{1.10}$$

Remark 5. *Given a linearly independent set $\{v_i\}_{i=1}^N$, by using the continuity of the determinant it is always possible to find a $\delta > 0$ small enough such that for all $\sigma_i \in C_\delta(v_i)$ the set $\{\sigma_i\}_{i=1}^N$ is linearly independent. However, we additionally require that condition (1.10) holds for these cones to ensure the coercivity of the limiting measure, so that it is, in a sense, equivalent to the Hausdorff surface measure and we can draw conclusions similar to the ones in the radial case.*

Our main result is the following characterization of $W^{1,p}(\Omega)$, $1 < p < \infty$, for *arbitrary* open sets Ω.

Theorem 6. *Let $\Omega \subset \mathbb{R}^N$ be open, let ρ_ϵ satisfy (1.3), (1.4), and (1.10), let $1 < p < \infty$ and $1 \le q < \infty$, with $1 \le q \le \frac{N}{N-p}$ if $p < N$, and let $f \in L^1_{\mathrm{loc}}(\Omega)$. Then $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and $\nabla f \in L^p(\Omega; \mathbb{R}^N)$ if and only if*

$$\lim_{r\to 0} \limsup_{\epsilon\to 0} \int_{\Omega_r} \left(\int_{\Omega_r} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx < +\infty. \tag{1.11}$$

Moreover, if ρ_ϵ satisfy (1.5), then there exists

$$\lim_{r\to 0} \lim_{\epsilon\to 0} \int_{\Omega_r} \left(\int_{\Omega_r} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx = K_{p,q,N} \int_\Omega |\nabla f|^p \, dx, \tag{1.12}$$

where

$$K_{p,q,N} := \left(\int_{S^{N-1}} |e_1 \cdot \sigma|^{pq} \, d\mathcal{H}^{N-1}(\sigma) \right)^{\frac{1}{q}}.$$

Here, for $r > 0$,

$$\Omega_r := \left\{ x \in \Omega \ : \ |x| < \frac{1}{r}, \ \ \mathrm{dist}(x, \partial\Omega) > r \right\}. \tag{1.13}$$

Remark 7. *Without the hypothesis (1.5), we cannot in general expect convergence of the whole sequence. However, we can still prove that there exist a subsequence $\{\epsilon_j\}$ and a probability measure $\mu \in M(S^{N-1})$ such that*

$$\begin{aligned} \lim_{r\to 0} \lim_{j\to\infty} \int_{\Omega_r} \left(\int_{\Omega_r} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_{\epsilon_j}(x-y)\, dy \right)^{\frac{1}{q}} dx \\ = \int_\Omega \left(\int_{S^{N-1}} (|\nabla f(x) \cdot \sigma|^p)^q \, d\mu(\sigma) \right)^{\frac{1}{q}} dx. \end{aligned}$$

Beyond the interest of the characterization, we will demonstrate that (1.8) implies the same estimates as condition (1.11) in the case $q = 1$, so that the proof of this theorem will imply the sufficiency of condition (1.8). In this way we obtain proofs of the conjectures of [15], the substance of which is contained in the following corollary.

Corollary 8. *Let $\Omega \subset \mathbb{R}^N$ open, $1 < p < \infty$, ρ_ϵ satisfy (1.3), (1.4), and (1.5), $f \in L^p(\Omega)$ and*

$$\limsup_{\epsilon\to 0} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{d_\Omega(x,y)^p} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dy dx < +\infty.$$

Then $f \in W^{1,p}(\Omega)$.

Analogous to the smooth boundary case, when $p = 1$ our result gives the following characterization of $BV(\Omega)$ for Ω an arbitrary open set.

Theorem 9. *Let $\Omega \subset \mathbb{R}^N$ be open, let ρ_ϵ satisfy (1.3), (1.4), and (1.10), let $1 \le q < \infty$ with $1 \le q \le \frac{N}{N-1}$ if $N > 1$, and let $f \in L^1_{\mathrm{loc}}(\Omega)$. Then $f \in BV_{\mathrm{loc}}(\Omega)$ and $Df \in M_b(\Omega; \mathbb{R}^N)$ if and only if*

$$\lim_{r \to 0} \limsup_{\epsilon \to 0} \int_{\Omega_r} \left(\int_{\Omega_r} \left(\frac{|f(x) - f(y)|}{|x - y|} \right)^q \rho_\epsilon(x - y)\, dy \right)^{\frac{1}{q}} dx < +\infty. \tag{1.14}$$

Moreover, if ρ_ϵ satisfy (1.5), then there exists

$$\lim_{r \to 0} \lim_{\epsilon \to 0} \int_{\Omega_r} \left(\int_{\Omega_r} \left(\frac{|f(x) - f(y)|^p}{|x - y|^p} \right)^q \rho_\epsilon(x - y)\, dy \right)^{\frac{1}{q}} dx = K_{1,q,N} |Df|(\Omega).$$

Remark 10. *Again, without the hypothesis (1.5) we are able to show that there exist a subsequence $\{\epsilon_j\}$ and a probability measure $\mu \in M(S^{N-1})$ such that*

$$\begin{aligned} &\lim_{r \to 0} \lim_{j \to \infty} \int_{\Omega_r} \left(\int_{\Omega_r} \left(\frac{|f(x) - f(y)|}{|x - y|} \right)^q \rho_{\epsilon_j}(x - y)\, dy \right)^{\frac{1}{q}} dx \\ &= \int_\Omega \left(\int_{S^{N-1}} \left(\left| \frac{dDf}{d|Df|}(x) \cdot \sigma \right| \right)^q d\mu(\sigma) \right)^{\frac{1}{q}} d|Df|(x), \end{aligned}$$

where $\frac{dDf}{d|Df|}$ is the Radon–Nikodym of Df with respect to $|Df|$.

As before, we are able to argue that the estimates in Theorem 9 (with $q = 1$) imply the proof of the corresponding conjecture in BV.

Corollary 11. *Let $\Omega \subset \mathbb{R}^N$ open, ρ_ϵ satisfy (1.3), (1.4), and (1.5), $f \in L^1(\Omega)$ and*

$$\limsup_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{d_\Omega(x, y)} \hat{\rho}_\epsilon(d_\Omega(x, y))\, dydx < +\infty.$$

Then $f \in BV(\Omega)$.

These corollaries answer the question of the sufficiency of (1.8) for arbitrary domains, however, it is still of interest to consider the necessity. It turns out that it is not necessary, as we are able to give a counterexample that demonstrates the class of functions for which (1.8) is finite can be strictly contained in $W^{1,p}(\Omega)$ (or $BV(\Omega)$). As they have been proven equivalent for extension domains in $\mathbb{R}^N$, the key ingredient here is to examine issues of boundary regularity. Extension domains are precisely those for which the standard Sobolev embeddings can be expected, and so we examine a construction of Fraenkel [43] that shows for general domains the Sobolev embedding theorem fails. Extending his analysis to our problem, we are able to prove the following theorem.

Theorem 12. *There exists an open set $\Omega \subset \mathbb{R}^2$, an $f \in W^{1,2}(\Omega)$, and ρ_ϵ satisfying (1.3), (1.4), and (1.5) such that*

$$\lim_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^2}{d_\Omega(x,y)^2} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dydx = +\infty.$$

We mention that it is not difficult to modify Theorem 12 to extend the proof to other values of p, including the case $p = 1$, such that the iterated integral is infinite. This is accomplished simply by changing the parameters in the construction, demonstrating that there is nothing special about the case $p = 2$.

Finally, we have the follow theorem constructing a domain Ω and a function $f \in W^{1,\infty}(\Omega)$ such that the functional (1.9) is infinite, which in particular demonstrates that (1.8) and (1.9) are genuinely different conditions, since any $f \in W^{1,\infty}(\Omega)$ is necessarily Lipschitz with respect to the geodesic distance (see, for example, [18]), and therefore finite on the functional (1.8).

Theorem 13. *There exists an open set $\Omega \subset \mathbb{R}^2$, $\hat{\rho}_\epsilon$ satisfying (1.3), (1.4), and (1.5), and an $f \in W^{1,\infty}(\Omega)$ such that for every $p > 1$*

$$\lim_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dydx = +\infty.$$

1.1.1 Imaging Applications

As Theorems 6 and 9 have indicated, we will be concerned with the functional

$$J^{p,q}_{\epsilon,r}(f) := \int_{\Omega_r} \left(\int_{\Omega_r} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx, \tag{1.15}$$

whose limit in ϵ and r characterizes $W^{1,p}(\Omega)$ for $p > 1$ and $BV(\Omega)$ for $p = 1$. The functional (1.15) is the same as the one introduced in [12], aside from two specific modifications. The first of which is the approach of Ω by subsets with compact closure and positive distance to the boundary ($\Omega_r \subset\subset \Omega$), which along with a measure support truncation lemma is the key to allowing our proofs to go through for arbitrary open Ω. The second of these modifications is the addition of the variable q, which enables us to prove a localization result that has applications in image processing. This is because the non-local functionals we are concerned with are one class of examples of recently introduced non-local functionals in image processing by Gilboa and Osher [47], whose aim is to improve effectiveness in image denoising and reconstruction. Since for the purpose of the applications in imaging the domain can be assumed to be sufficiently regular (usually a rectangle), it is better to consider the functional

$$J^{p,q}_{\epsilon}(f) := \int_{\Omega} \left(\int_{\Omega} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx, \tag{1.16}$$

as it relates to the non-local imaging functionals.

Although the total variation model of Rudin, Osher, and Fatemi [72] has been highly successful in such problems, it has had notable difficulties in preserving fine structures, details, and textures (since blurring is common), as well as the highly undesirable staircase effect (where smooth affine regions are replaced by piecewise constant regions). This model is mathematically represented via the minimization problem

$$\min \left\{ |Df|\,(\Omega) + \alpha \int_{\Omega} |f\,(x) - f_0\,(x)|^2 \, dx : \; f \in BV(\Omega) \right\}, \tag{1.17}$$

where $f_0 \in L^2(\Omega)$ is given. Seeking to overcome the above difficulties, in some recent work, Gilboa and Osher [46], [47] (see also [11]) propose a systematic and coherent framework for non-local image and signal processing. They specifically address the problem of image reconstruction and segmentation for images with repetitive structures and fine textures, and introduce a non-local version of (1.17) to correct the blurring and staircasing problems mentioned. The idea is that any point in the image domain is (ideally) allowed to interact directly with any other point. The use of information beyond the local function value gives them some freedom in the reconstruction of an image. The gradient-based regularizing functional introduced by Gilboa and Osher in [47] takes the form

$$J(f) := \int_\Omega \phi\left(\int_\Omega |f(x) - f(y)|^2 w(x,y)\, dy\right) dx, \tag{1.18}$$

where $\Omega \subset \mathbb{R}^N$ is an open set (in imaging $N = 2$), $f : \Omega \to \mathbb{R}$, $\phi : [0,\infty) \to [0,\infty)$ is a function convex in $\sqrt{s}$ with $\phi(0) = 0$, and w is a positive and symmetric weight function that measures the interaction between different values of x, y. The prototype model for ϕ is the function $\phi(s) = \sqrt{s}$, which leads to the non-local functional

$$J_{NL-TV}(f) := \int_\Omega \sqrt{\int_\Omega |f(x) - f(y)|^2\, w(x,y)\, dydx}. \tag{1.19}$$

This corresponds to the functional (1.16) when $p = 1$, $q = 2$, and $w = w_\varepsilon(x,y) = \frac{\rho_\varepsilon(|x-y|)}{|x-y|^2}$. Thus, our result shows that the non-local functional (1.19) converges to a constant times the total variation, when the mass of $\{\rho_\varepsilon\}_\varepsilon$ concentrates at origin. This shows that the non-local minimization problem, in some sense, localizes to the Rudin, Osher, Fatemi model (see [9] for more relationships between non-local minimization problems and their corresponding local forms).

We finally remark that there is a large body of work on related non-local functionals, including papers addressing compactness (see [12], [69]), applications to problems and further questions (see [17], [15]), extended looks at non-radial mollifiers (see [70], [69]), Γ-convergence of non-local functionals ([70]), and other characterizations of Sobolev spaces ([67], [68]). Our work is related to these papers, and all of them relate to the localization of non-local functionals. It is then natural that our techniques follow closely the work in [12], [15], [34], and [70], with the mentioned modifications specific to our aim and technical requirements for the proofs to work.

1.2 RESHETNYAK'S THEOREMS

One of the primary tools used to extend the results we have mentioned previously in $BV(\Omega)$ are two semicontinuity theorems for functions of measures. These two results are originally due to Reshetnyak [71], who in 1968 proved the continuity and lower semicontinuity of functionals of measures with respect to weak convergence of measures. More precisely, in [71], the following theorems are given.

Theorem 14. *Let X be a locally compact, separable metric space and λ_n, $\lambda \in M_b(X;\mathbb{R}^m)$. Assume that $\lambda_n \overset{*}{\rightharpoonup} \lambda$ in $(C_b(X;\mathbb{R}^m))'$ and that*

$$\lim_{n\to\infty} \int_X g\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right)\, d|\lambda_n| = \int_X g\left(x, \frac{d\lambda}{d|\lambda|}(x)\right)\, d|\lambda| \tag{1.20}$$

for some[1] *continuous function $g : X \times \mathbb{R}^m \to \mathbb{R}$, positively 1-homogeneous and strictly convex in the second variable, satisfying the growth condition $|g(x,z)| \le C|z|$ for each $(x,z) \in X \times \mathbb{R}^m$ and for some $C > 0$. Then*

$$\lim_{n\to\infty} \int_X f\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right)\, d|\lambda_n| = \int_X f\left(x, \frac{d\lambda}{d|\lambda|}(x)\right)\, d|\lambda| \tag{1.21}$$

for every continuous function $f : X \times \mathbb{R}^m \to \mathbb{R}$ satisfying the growth condition $|f(x,z)| \le C_1|z|$ for each $(x,z) \in X \times \mathbb{R}^m$ and for some $C_1 > 0$.

Theorem 15. *Let X be a locally compact, separable metric space and λ_n, $\lambda \in M_b(X;\mathbb{R}^m)$; if $\lambda_n \overset{*}{\rightharpoonup} \lambda$ in $(C_b(X;\mathbb{R}^m))'$, then*

$$\liminf_{n\to\infty} \int_X f\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right)\, d|\lambda_n| \ge \int_X f\left(x, \frac{d\lambda}{d|\lambda|}(x)\right)\, d|\lambda|$$

for every continuous function $f : X \times \mathbb{R}^m \to \mathbb{R}$, positively 1-homogeneous and convex in the second variable, satisfying the growth condition $|f(x,z)| \le C|z|$ for each $(x,z) \in X \times \mathbb{R}^m$ and for some $C > 0$.

[1] The English translation of this quantifier say 'for each', when in fact the original Russian says 'for some'.

These theorems are used in a variety of areas in the calculus of variations ranging from problems in relaxation ([1],[4],[5],[10]), estimates in Γ-convergence ([53],[57],[60]), anisotropic surface energies studied in continuum mechanics ([37],[38],[39],[51]) and various other applications ([2],[21],[41]). Proofs to variants of Theorems 14 and 15 have been given in [6], [60], and [71], and although the statement of the hypotheses differs, the technique is essentially the same. The idea has been to construct sequences of measures in the product space $X \times S^{m-1}$, extract a limit via compactness, and use a disintegration theorem (see [6], Theorem 2.28) to project the limiting object for analysis. There has been some work involving arguments specific to particular problems, for example, time-dependent problems [53], as well as the desire to consider f that are not necessarily 1-homogeneous ([55], [56]). However, these arguments either use the original theorem or are applicable only in a more specific context.

We are able to show that in the Euclidean setting it is possible to give simple proofs of Theorems 14 and 15 which do not make use of the disintegration theorem. Note that the assumption $X \subset \mathbb{R}^N$ is not as restrictive as it looks, since locally compact topological vector spaces are finite dimensional (see Section 1.9 in [73])[2]. Moreover, the applications of Theorems 14 and 15 are generally to problems involving functions of bounded variation $BV(\Omega;\mathbb{R}^m)$ (the space of functions in $L^1(\Omega;\mathbb{R}^m)$ whose distributional derivative is an element of $M_b(\Omega;\mathbb{R}^{mN})$). Precisely, we prove the following continuity theorem (see [6]).

Theorem 16. *Let $\Omega \subset \mathbb{R}^N$ be open, $\lambda_n, \lambda \in M_b(\Omega;\mathbb{R}^m)$ such that*

$$\lambda_n \overset{*}{\rightharpoonup} \lambda \text{ in } (C_0(\Omega;\mathbb{R}^m))' \text{ and } |\lambda_n|(\Omega) \to |\lambda|(\Omega). \tag{1.22}$$

Then

$$\lim_{n\to\infty} \int_\Omega f\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right)\, d|\lambda_n| = \int_\Omega f\left(x, \frac{d\lambda}{d|\lambda|}(x)\right)\, d|\lambda|$$

for every continuous and bounded function $f : \Omega \times S^{m-1} \to \mathbb{R}$.

Note that although hypotheses (1.22) of Theorem 16 seem to differ from those in Theorem 14 (namely, $\lambda_n \overset{*}{\rightharpoonup} \lambda$ in $(C_b(\Omega;\mathbb{R}^m))'$ and (1.20)), they are in fact equivalent, as we will demonstrate (see Remark 157).

[2]Thus, if the metric on X comes from a norm or is compatible with the topology of a topological vector space, then X is automatically finite dimensional.

Moreover, as the projection techniques are typically used for both the continuity and lower semicontinuity theorems, we also give an alternative proof of Theorem 15 in the Euclidean setting. In view of the applications (see [4], [5], [10]) we additionally study lower semicontinuity with respect to the weak-star convergence in $(C_0(\Omega;\mathbb{R}^m))'$, which requires f to be non-negative but allows f to take the value $+\infty$. The appropriate hypothesis in this setting are as follows.

Theorem 17. *Let $\Omega \subset \mathbb{R}^N$ be open and λ_n, $\lambda \in M_b(\Omega;\mathbb{R}^m)$; if $\lambda_n \overset{*}{\rightharpoonup} \lambda$ in $(C_0(\Omega;\mathbb{R}^m))'$, then*

$$\liminf_{n\to\infty} \int_\Omega f\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right)\, d|\lambda_n| \geq \int_\Omega f\left(x, \frac{d\lambda}{d|\lambda|}(x)\right)\, d|\lambda| \tag{1.23}$$

for every lower semicontinuous function $f : \Omega \times \mathbb{R}^m \to [0,\infty]$, positively 1-homogeneous and convex in the second variable.

Moreover, if we assume that $\lambda_n \overset{*}{\rightharpoonup} \lambda$ in $(C_b(\Omega;\mathbb{R}^m))'$, then (1.23) holds for every lower semicontinuous function $f : \Omega \times \mathbb{R}^m \to (-\infty,\infty]$, positively 1-homogeneous and convex in the second variable such that

$$f(x,z) \geq b(x)\cdot z \tag{1.24}$$

for some $b \in C_b(\Omega;\mathbb{R}^m)$.

The organization of the thesis will be as follows. We will first develop some preliminaries of duality, measures, and L^p spaces. We will then introduce the Sobolev and Bounded Variation spaces and give some basic results of functions in these spaces. After recalling some notions of convexity, we will then be in a position to discuss and prove our main results. We will first address the new characterization of these spaces, and then proceed to the proofs of the Reshetnyak semicontinuity theorems.

2.0 PRELIMINARIES

We begin by developing all the requisite preliminaries in the sections that follow, which should allow the reader to familiarize themselves with the notation, as well as to give a complete basis for all necessary results we will use in the sequel.

2.1 BANACH SPACES AND DUALITY

2.1.1 Locally Convex Topological Vector Spaces

In this section we develop the notion of locally convex topological vector spaces, whose topological dual spaces have nice compactness properties as a result of the Banach-Alaoglu theorem. We begin with the definition of a vector space.

Definition 18. *A* vector space*, or* linear space*, over* $\mathbb{R}$ *is a nonempty set* X*, whose elements are called* vectors*, together with two operations,* addition *and* multiplication by scalars*,*

$$\begin{array}{ccc} X \times X \to X & & \mathbb{R} \times X \to X \\ & \textit{and} & \\ (x, y) \mapsto x + y & & (t, x) \mapsto tx \end{array}$$

with the properties that

(i) $(X, +)$ *is a commutative group, that is,*

- *a.* $x + y = y + x$ *for all* $x, y \in X$ *(commutative property),*
- *b.* $x + (y + z) = (x + y) + z$ *for all* $x, y, z \in X$ *(associative property),*
- *c. there is a vector* $0 \in X$*, called* zero*, such that* $x + 0 = 0 + x$ *for all* $x \in X$*,*

d. *for every* $x \in X$ *there exists a vector in* X*, called the* opposite *of* x *and denoted* $-x$*, such that* $x + (-x) = 0$*,*

(ii) *for all* $x, y \in X$ *and* $s, t \in \mathbb{R}$*,*

a. $s\,(tx) = (st)\,x$*,*

b. $1x = x$*,*

c. $s\,(x + y) = (sx) + (sy)$*,*

d. $(s + t)x = (sx) + (tx)$*.*

Normed spaces are a primary example of vector spaces, equipped with a topological structure induced by the norm.

Definition 19. *A* normed space *is a pair* $(X, \|\cdot\|)$*, where* X *is a vector space and* $\|\cdot\| : X \to [0, \infty)$ *is a* norm*, that is,*

(i) $\|x\| = 0$ *if and only if* $x = 0$*.*

(ii) $\|tx\| = |t|\,\|x\|$ *for all* $t \in \mathbb{R}$ *and* $x \in X$*.*

(iii) ***(Triangle inequality)*** $\|x + y\| \leq \|x\| + \|y\|$ *for all* x*,* $y \in X$*.*

In normed spaces, since the norm defines a metric we have a notion of completeness in terms of the norm.

Definition 20. *Given a normed space* $(X, \|\cdot\|)$*, we can define the distance function* $d : X \times X \to [0, \infty)$*, defined by*

$$d\,(x, y) := \|x - y\|\,, \quad v, w \in X.$$

We say that X *is a* Banach *space if it is a complete metric space, that is, if every Cauchy sequence* $\{x_n\} \subset X$ *converges to some element in* X*.*

Unfortunately, infinite dimensional Banach spaces have poor compactness properties with respect to the norm topology. However, there is some compensation in that we are able to define a weaker topology on such spaces that maintains some of the space's structure while regaining this lack of compactness. To this end we need the notion of a locally convex topological vector space.

Definition 21. *Given a vector space X over $\mathbb{R}$ endowed with a topology τ, the pair (X, τ) is called a* topological vector space *if the functions*

$$\begin{array}{ccc} X \times X \to X, & \multirow{2}{*}{\textit{and}} & \mathbb{R} \times X \to X, \\ (x, y) \mapsto x + y, & & (t, x) \mapsto tx, \end{array}$$

are continuous.

Definition 22. *A topological vector space (X, τ) is* locally convex *if it has a local base at* 0 *consisting of convex sets.*

In particular, we will be interested in topological vector spaces whose local bases are balanced.

Definition 23. *Let X be a vector space over $\mathbb{R}$ and let $E \subset X$. The set E is said to be* balanced*, or* circled*, if $tx \in E$ for all $x \in E$ and $t \in [-1, 1]$.*

The following proposition establishes the connection between a locally convex topological vector space and its base.

Proposition 24. *A locally convex topological vector space admits a local base at the origin consisting of balanced convex neighborhoods of zero.*

Finally, we define a semi-norm, which will be used to construct the topology of a locally convex topological vector space in the theorem that follows.

Definition 25. *Let X be a vector space. A map $p : X \to \mathbb{R}$ is called*

(i) positively homogeneous of degree $\alpha \geq 0$ *if*

$$p(tx) = t^{\alpha} p(x)$$

for all $x \in X$ and $t > 0$,

(ii) subadditive *if*

$$p(x + y) \leq p(x) + p(y)$$

for all $x, y \in X$,

(iii) sublinear *if it is positively homogeneous of degree one and subadditive*

(iv) a seminorm *if it is subadditive and*

$$p(tx) = |t| \, p(x)$$

for all $x \in X$ and $t \in \mathbb{R}$.

We also will need the Minkowski functional of a set, defined in what follows.

Let X be a vector space and let $E \subset X$. The function $p_E : X \to [0, \infty]$, defined by

$$p_E(x) := \inf \{s > 0 : \, x \in sE\}, \quad x \in X,$$

is called the *gauge*, or *Minkowski functional,* of E. Note that if $x \in E$, then $p_E(x) \leq 1$. Hence,

$$E \subset \{x \in X : \, p_E(x) \leq 1\}. \tag{2.1}$$

The definitions and theorems thus far are motivated by the following result, which gives necessary and sufficient conditions for a family of semi-norms to construct a locally convex topological vector space.

Theorem 26. *If $\mathcal{F}$ is a balanced, convex local base of 0 for a locally convex topological vector space (X, τ), then the family $\{p_U : U \in \mathcal{F}\}$ is a family of continuous seminorms. Conversely, given a family $\mathcal{P}$ of seminorms on a vector space X, let $\mathcal{B}$ be the collection of all finite intersections of sets of the form*

$$B_p(0, r) := \{x \in X : \, p(x) < r\}, \quad p \in \mathcal{P}, \; r > 0.$$

Then $\mathcal{B}$ is a balanced, convex local base of 0 for a topology τ that turns X into a locally convex topological vector space such that each p is continuous with respect to τ.

2.1.2 Duality and Weak, Weak-Star Topologies

As a result of Theorem 26, we have the following constructions of the *weak topology* and *weak-star topology.*

If X and Y are topological vector spaces, then the vector space of all continuous linear operators from X to Y is denoted by $\mathcal{L}(X;Y)$. In the special case $Y=\mathbb{R}$, the space $\mathcal{L}(X;\mathbb{R})$ is called the *dual space* of X and it is denoted by X'. The elements of X' are also called continuous linear *functionals.*

The bilinear (i.e., linear in each variable) mapping

$$\begin{aligned}\langle\cdot,\cdot\rangle_{X',X} : X'\times X &\to \mathbb{R} \\ (L,x) &\mapsto L(x)\end{aligned} \tag{2.2}$$

is called the *duality pairing.*

Given a locally convex topological vector space X, for each $L\in X'$ the function $p_L : X\to[0,\infty)$ defined by

$$p_L(x) := |L(x)|\,,\quad x\in X, \tag{2.3}$$

is a seminorm. In view of Theorem 26, the family of seminorms $\{p_L\}_{L\in X'}$ generates a locally convex topology $\sigma(X,X')$ on the space X, called the *weak topology*, such that each p_L is continuous with respect to $\sigma(X,X')$. In turn, this implies that every $L\in X'$ is $\sigma(X,X')$ continuous.

Given a topological vector space (X,τ), for each $x\in X$ the function $p_x : X'\to[0,\infty)$ defined by

$$p_x(L) := |L(x)|\,,\quad L\in X', \tag{2.4}$$

is a seminorm. In view of Theorem 26, the family of seminorms $\{p_x\}_{x\in X}$ generates a locally convex topology $\sigma(X',X)$ on the space X', called the *weak-star topology*, such that each p_x is continuous with respect to $\sigma(X',X)$.

The usefulness of the above constructions is in the following compactness theorem.

Theorem 27 (Banach–Alaoglu)**.** *If V is a neighborhood of 0 in a locally convex topological vector space (X,τ), then*

$$K := \{L\in X' : \ |L(x)|\le 1 \text{ for every } x\in V\}$$

is weak-star compact.

A consequence of the above theorem is the following corollary of sequential weak-star compactness (when X is separable, which will be the case for our applications of the result).

Corollary 28 (Bolzano-Weierstrass)**.** *Let V be a neighborhood of 0 in a separable locally convex topological vector space (X, τ) and let $\{L_n\} \subset X'$ be such that*

$$|L_n(x)| \leq 1 \text{ for every } x \in V \text{ and for all } n \in \mathbb{N}.$$

Then there exists a subsequence $\{L_{n_k}\}$ that is weakly star convergent. In particular, if X is a separable normed space and $\{L_n\} \subset X'$ is any bounded sequence in X', then there exists a subsequence that is weakly star convergent.

In certain spaces the weak and weak-star topologies coincide, and so the above compactness result is then stronger.

Definition 29. *A normed space $(X, \|\cdot\|)$ is* reflexive *if $J(X) = X''$.*

Here, J is the injection of X into X'', where we identify elements of X as linear functionals on X', that is, for $x \in X$ and $L \in X'$, we define

$$J(x)(L) := L(x),$$

so that $J(x)$ is a continuous linear functional on X'. In the case a space is reflexive, it is possible to identify X with its bidual X'', and we have the following corollary of the above compactness theorem.

Corollary 30. *Let $(X, \|\cdot\|)$ be a reflexive Banach space and let $\{x_n\} \subset X$ be a bounded sequence. Then there exists a subsequence that is weakly convergent.*

Thus it is useful to know when a space is reflexive, to determine whether this stronger compactness property holds. This is always the case for a uniformly convex Banach space, the substance of the following definition and theorem.

Definition 31. *A normed space $(X, \|\cdot\|)$ is* uniformly convex *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in X$, with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| > \varepsilon$,*

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

Although it is not a necessary condition, since it is possible to place different equivalent norms on the same Banach space, the following theorem asserts that it is a sufficient condition in determining reflexivity.

Theorem 32 (Milman-Pettis)**.** *Let* $(X, \|\cdot\|)$ *be a uniformly convex Banach space. Then* X *is reflexive.*

2.2 MEASURES

2.2.1 Measures and Integration

We now recall some of the basic results of measure and Lebesgue integration theory.

We first define outer measures, an important tool in the theory of measure and integration. Beyond their usefulness in constructing measures, they have the added advantage of alleviating anxiety over measurability of sets and functions, the specter of many a beginning analyst's dreams.

Definition 33. *Let* X *be a nonempty set. A map* $\mu^* : \mathcal{P}(X) \to [0, \infty]$ *is an* outer measure *if*

(i) $\mu^*(\emptyset) = 0$,

(ii) $\mu^*(E) \leq \mu^*(F)$ *for all* $E \subset F \subset X$,

(iii) $\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$ *for every countable collection* $\{E_n\} \subset \mathcal{P}(X)$.

Definition 34. *Let* X *be a nonempty set and let* $\mu^* : \mathcal{P}(X) \to [0, \infty]$ *be an outer measure. A set* $E \subset X$ *has* σ-finite μ^* outer measure *if it can be written as a countable union of sets of finite outer measure;* μ^* *is said to be* σ-finite *if* X *has* σ*-finite* μ^* *outer measure;* μ^* *is said to be* finite *if* $\mu^*(X) < \infty$.

Our next definition is the notion of measurability, and since this is a prerequisite to Lebesgue integration theory, we will therefore be careful to give all the necessary definitions, theorems, and corollaries concerning measurability.

Definition 35. *Let X be a nonempty set and let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure. A set $E \subset X$ is said to be* μ^*-measurable *if*

$$\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \setminus E)$$

for all sets $F \subset X$.

Definition 36. *Let X be a nonempty set. A collection $\mathfrak{M} \subset \mathcal{P}(X)$ is a* σ-algebra *if*

(i) $\emptyset \in \mathfrak{M}$,

(ii) if $E \in \mathfrak{M}$ then $X \setminus E \in \mathfrak{M}$,

(iii) if $\{E_n\} \subset \mathfrak{M}$ then $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{M}$.

To highlight the dependence of the σ-algebra $\mathfrak{M}$ on X we will sometimes use the notation $\mathfrak{M}(X)$. If $\mathfrak{M}$ is a σ-algebra then the pair $(X, \mathfrak{M})$ is called a *measurable space*. For simplicity we will often apply the term measurable space only to X.

Using De Morgan's laws and (ii) and (iii), it follows that a σ-algebra is closed under countable intersection.

Let X be a nonempty set. Given any subset $\mathcal{F} \subset \mathcal{P}(X)$ the smallest (in the sense of inclusion) σ-algebra that contains $\mathcal{F}$ is given by the intersection of all σ-algebras on X that contain $\mathcal{F}$.

If X is a topological space, then the *Borel σ-algebra* $\mathcal{B}(X)$ is the smallest σ-algebra containing all open subsets of X.

Definition 37. *Let X be a nonempty set, let $\mathfrak{M} \subset \mathcal{P}(X)$ be a σ-algebra. A map $\mu : \mathfrak{M} \to [0, \infty]$ is called a* (positive) measure *if*

$$\mu(\emptyset) = 0, \quad \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

for every countable collection $\{E_n\} \subset \mathfrak{M}$ of pairwise disjoint sets. The triple $(X, \mathfrak{M}, \mu)$ is said to be a measure space.

By restricting ourselves to a σ-algebra of measurable sets we make a trade-off in not being able to measure all sets while gaining the countable additivity property listed above. This gain is, in fact, substantial, as the consequences of this will be seen later in Proposition 40, and then again in integration in the Lebesgue monotone and dominated convergence theorems.

Definition 38. *Given a measure space $(X, \mathfrak{M}, \mu)$, the measure μ is said to be* complete *if for every $E \in \mathfrak{M}$ with $\mu(E) = 0$ it follows that every $F \subset E$ belongs to $\mathfrak{M}$.*

The following theorem is due to Carathéodory, and is one of the primary reasons we began with the notion of outer measure.

Theorem 39 (Carathéodory)**.** *Let X be a nonempty set and let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure. Then*

$$\mathfrak{M}^* := \{E \subset X : \ E \text{ is } \mu^*\text{-measurable}\} \tag{2.5}$$

is a σ-algebra and $\mu^* : \mathfrak{M}^* \to [0, \infty]$ is a complete measure.

And now we are able to more precisely state the gain of restricting ourselves to the consideration of measures alluded to before.

Proposition 40. *Let $(X, \mathfrak{M}, \mu)$ be a measure space.*

(i) If $\{E_n\}$ is an increasing sequence of subsets of $\mathfrak{M}$ then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

(ii) If $\{E_n\}$ is a decreasing sequence of subsets of $\mathfrak{M}$ and $\mu(E_1) < \infty$ then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

Finally, we recall the following important theorem on the measure of disjoint sets with respect to a finite measure.

Theorem 41. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and assume $\mu : \mathfrak{M} \to [0, \infty)$. Then if $\{E_i\}_{i\in I}$ are pairwise disjoint sets, we have that $\mu(E_i) > 0$ for at most countably many i.*

We now introduce the notions of measurable and integrable functions. As previously mentioned, we will spend some time to mention a number of results about measurable functions, as this is the primary technical point we must be concerned with before introducing integration, which is essential in the definition of many of the functional spaces we will be concerned with later.

Definition 42. *Let X and Y be nonempty sets, and let $\mathfrak{M}$ and $\mathfrak{N}$ be algebras on X and Y, respectively. A function $f : X \to Y$ is said to be* measurable *if $f^{-1}(F) \in \mathfrak{M}$ for every set $F \in \mathfrak{N}$.*

If X and Y are topological spaces, $\mathfrak{M} := \mathcal{B}(X)$ and $\mathfrak{N} := \mathcal{B}(Y)$, then a measurable function $f : X \to Y$ will be called a *Borel function.*

The structure of a σ-algebra and inverse function operations imply that it is sufficient to test measurability only on a subset of the σ-algebra, like a particular family which generates it, as the following proposition demonstrates.

Proposition 43. *If $\mathfrak{M}$ is a σ-algebra on a set X and $\mathfrak{N}$ is the smallest σ-algebra that contains a given family $\mathcal{G}$ of subsets of a set Y, then $f : X \to Y$ is measurable if and only if $f^{-1}(F) \in \mathfrak{M}$ for every set $F \in \mathcal{G}$.*

We have that the composition of measurable functions is again a measurable function.

Proposition 44. *Let $(X, \mathfrak{M})$, $(Y, \mathfrak{N})$, $(Z, \mathfrak{O})$ be measurable spaces and let $f : X \to Y$ and $g : Y \to Z$ be two measurable functions. Then $g \circ f = X \to Z$ is measurable.*

Therefore, we are able to perform some commonly used operations on measurable functions and conclude that the result is measurable.

Corollary 45. *Let $(X, \mathfrak{M})$ be a measurable space and let $f : X \to \mathbb{R}$ (respectively $f : X \to [-\infty, \infty]$) be a measurable function. Then f^2, $|f|$, , f^+, f^-, cf, where $c \in \mathbb{R}$, are measurable.*

Remark 46. *If $c = 0$ and $f : X \to [-\infty, \infty]$ the function cf is defined to be identically equal to zero.*

Further, our considerations will include scalar-valued functions, and their derivatives, which can be identified with vector-valued functions. We therefore need to connect measurability of vector-valued functions with the previously introduced notion of measurability.

Given two measurable spaces $(X, \mathfrak{M})$ and $(Y, \mathfrak{N})$ we denote by $\mathfrak{M} \otimes \mathfrak{N} \subset \mathcal{P}(X \times Y)$ the smallest σ-algebra that contains all sets of the form $E \times F$, where $E \in \mathfrak{M}$, $F \in \mathfrak{N}$. Then $\mathfrak{M} \otimes \mathfrak{N}$ is called the *product σ-algebra* of $\mathfrak{M}$ and $\mathfrak{N}$.

Proposition 47. *Let $(X, \mathfrak{M})$, $(Y_1, \mathfrak{N}_1), \ldots, (Y_n, \mathfrak{N}_n)$ be measurable spaces and consider*

$$(Y_1 \times \ldots \times Y_n, \mathfrak{N}_1 \otimes \ldots \otimes \mathfrak{N}_n).$$

Then the vector-valued function $f : X \to Y_1 \times \ldots \times Y_n$ is measurable if and only if its components $f_i : X \to Y_i$ are measurable functions for all $i = 1, \ldots, n$.

Theorem 48. *Let X be a complete, separable metric space and let $f : X \to \mathbb{R}$ be a Borel function (or a continuous function). Then for every Borel set $B \subset X$, the set $f(B)$ is Lebesgue measurable (but not necessarily a Borel set).*

The following corollary gives us some more information on the operations that can be performed on measurable functions so that the result is measurable.

Corollary 49. *Let $(X, \mathfrak{M})$ be a measurable space and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be two measurable functions. Then $f + g$, fg, $\min\{f, g\}$, $\max\{f, g\}$ are measurable.*

Remark 50. *The previous corollary continues to hold if $\mathbb{R}$ is replaced by $[-\infty, \infty]$, provided $f + g$ are well-defined, i.e., $(f(x), g(x)) \notin \{\pm(\infty, -\infty)\}$ for all $x \in X$. Concerning fg, we define $(fg)(x) := 0$ whenever $f(x)$ or $g(x)$ is zero.*

Moreover, given a sequence of measurable functions, the limit (inferior or superior) is also a measurable function.

Proposition 51. *Let $(X, \mathfrak{M})$ be a measurable space and let $f_n : X \to [-\infty, \infty]$, $n \in \mathbb{N}$, be measurable functions. Then $\sup_n f_n$, $\inf_n f_n$, $\liminf_{n\to\infty} f_n$, and $\limsup_{n\to\infty} f_n$ are measurable.*

Remark 52. *The previous proposition uses in a crucial way the fact that $\mathfrak{M}$ is a σ-algebra.*

The delicate issues of sets of measure zero can generally be avoided by redefining a function on such a set. Again, we must be concerned with measurability, as the following definitions and theorems make clear.

Definition 53. *Let* $(X, \mathfrak{M})$ *and* $(Y, \mathfrak{N})$ *be two measurable spaces, and let* $\mu : \mathfrak{M} \to [0, \infty]$ *be a measure. Given a function* $f : X \setminus E \to Y$ *where* $\mu(E) = 0$, f *is said to be* measurable over X *if* $f^{-1}(F) \in \mathfrak{M}$ *for every set* $F \in \mathfrak{N}$.

Proposition 54. *Let* $(X, \mathfrak{M})$ *and* $(Y, \mathfrak{N})$ *be two measurable spaces and let* $f : X \to Y$ *be a measurable function. Let* $\mu : \mathfrak{M} \to [0, \infty]$ *be a complete measure. If* $g : X \to Y$ *is a function such that* $f(x) = g(x)$ *for* μ *a.e.* $x \in X$, *then* g *is measurable.*

Corollary 55. *Let* $(X, \mathfrak{M})$ *be a measurable space and let* $f_n : X \to [-\infty, \infty]$, $n \in \mathbb{N}$, *be measurable functions. Let* $\mu : \mathfrak{M} \to [0, \infty]$ *be a complete measure. If there exists* $\lim_{n\to\infty} f_n(x)$ *for* μ *a.e.* $x \in X$, *then* $\lim_{n\to\infty} f_n$ *is measurable.*

We are now in a position to introduce the notion of integral. We begin be integration of the simple functions, and then define integration for an arbitrary measurable function.

Definition 56. *Let* X *be a nonempty set and let* $\mathfrak{M}$ *be a* σ*-algebra on* X. *A* simple function *is a measurable function* $s : X \to \mathbb{R}$ *whose range consists of finitely many points.*

If $c_1, \ldots, c_\ell$ are the distinct values of s, then we write

$$s = \sum_{n=1}^{\ell} c_n \chi_{E_n},$$

where χ_{E_n} is the *characteristic function* of the set $E_n := \{x \in X : s(x) = c_n\}$, i.e.,

$$\chi_{E_n}(x) := \begin{cases} 1 & \text{if } x \in E_n, \\ 0 & \text{otherwise.} \end{cases}$$

If μ is a (positive) measure on X and $s \geq 0$, then for every measurable set $E \in \mathfrak{M}$ we define the *Lebesgue integral* of s over E as

$$\int_E s \, d\mu := \sum_{n=1}^{\ell} c_n \mu(E_n \cap E), \tag{2.6}$$

where if $c_n = 0$ and $\mu(E_n \cap E) = \infty$, then we use the convention

$$c_n \mu(E_n \cap E) := 0.$$

Theorem 57. *Let X be a nonempty set, let $\mathfrak{M}$ be a σ-algebra on X, and let $f : X \to [0, \infty]$ be a measurable function. Then there exists a sequence $\{s_n\}$ of simple functions such that*

$$0 \leq s_1(x) \leq s_2(x) \leq \ldots \leq s_n(x) \to f(x)$$

for every $x \in X$. The convergence is uniform on any set on which f is bounded from above.

In view of the previous theorem, if $f : X \to [0, \infty]$ is a measurable function, then we define its *(Lebesgue) integral* over a measurable set E as

$$\int_E f \, d\mu := \sup \left\{ \int_E s \, d\mu : \, s \text{ simple}, \, 0 \leq s \leq f \right\}.$$

We list below some basic properties of Lebesgue integration for nonnegative functions.

Proposition 58. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f, g : X \to [-\infty, \infty]$ be two measurable functions.*

(i) If $0 \leq f \leq g$, then $\int_E f \, d\mu \leq \int_E g \, d\mu$ for any measurable set E.
(ii) If $c \in [0, \infty]$, then $\int_E cf \, d\mu = c \int_E f \, d\mu$ (here we set $0\infty := 0$).
(iii) If $E \in \mathfrak{M}$ and $f(x) = 0$ for μ a.e. $x \in E$, then $\int_E f \, d\mu = 0$, even if $\mu(E) = \infty$.
(iv) If $E \in \mathfrak{M}$ and $\mu(E) = 0$, then $\int_E f \, d\mu = 0$, even if $f \equiv \infty$ in E.
(v) $\int_E f \, d\mu = \int_X \chi_E f \, d\mu$ for any measurable set E.

The next results are central in the theory of integration of nonnegative functions, a result of the countable additivity of the measure and measurability of the functions being integrated.

Theorem 59 (Lebesgue monotone convergence theorem)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f_n : X \to [0, \infty]$ be a sequence of measurable functions such that*

$$0 \leq f_1(x) \leq f_2(x) \leq \ldots \leq f_n(x) \to f(x)$$

for every $x \in X$. Then f is measurable and

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Remark 60. *The previous theorem continues to hold if we assume that* $f_n(x) \to f(x)$ *for* μ *a.e.* $x \in X$. *Indeed, in view of Proposition 58(iv), it suffices to re-define* f_n *and* f *to be zero in the set of measure zero in which there is no pointwise convergence.*

Corollary 61. *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* f, $g : X \to [0, \infty]$ *be two measurable functions. Then*

$$\int_X (f + g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu.$$

Corollary 62. *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* $f_n : X \to [0, \infty]$ *be a sequence of measurable functions. Then*

$$\sum_{n=1}^{\infty} \int_X f_n\, d\mu = \int_X \sum_{n=1}^{\infty} f_n\, d\mu.$$

Lemma 63 (Fatou lemma)**.** *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space.*

(i) If $f_n : X \to [0, \infty]$ *is a sequence of measurable functions, then*

$$f := \liminf_{n\to\infty} f_n$$

is a measurable function and

$$\int_X f\, d\mu \le \liminf_{n\to\infty} \int_X f_n\, d\mu;$$

(ii) if $f_n : X \to [-\infty, \infty]$ *is a sequence of measurable functions such that*

$$f_n \le g$$

for some measurable function $g : X \to [0, \infty]$ *with* $\int_X g\, d\mu < \infty$, *then*

$$f := \limsup_{n\to\infty} f_n$$

is a measurable function and

$$\int_X f\, d\mu \ge \limsup_{n\to\infty} \int_X f_n\, d\mu.$$

Having been careful to deal with measurability concerns, the integral can in turn give information about the function being integrated, as the following corollary suggests.

Corollary 64. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f : X \to [0, \infty]$ be a measurable function. Then*

$$\int_X f \, d\mu = 0$$

if and only if $f(x) = 0$ for μ a.e. $x \in X$.

More generally, we will have the need to integrate functions which are not necessarily positive. Therefore, in order to extend the notion of integral to functions of arbitrary sign, consider $f : X \to [-\infty, \infty]$ and set

$$f^+ := \max\{f, 0\}, \quad f^- := \max\{-f, 0\}.$$

Note that $f = f^+ - f^-$, $|f| = f^+ + f^-$, and f is measurable if and only if f^+ and f^- are measurable. Also, if f is bounded, then so are f^+ and f^-, and in view of Theorem 57, f is then the uniform limit of a sequence of simple functions.

Definition 65. *Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $f : X \to [-\infty, \infty]$ be a measurable function. Given a measurable set $E \in \mathfrak{M}$, if at least one of the two integrals $\int_E f^+ \, d\mu$ and $\int_E f^- \, d\mu$ is finite, then we define the* (Lebesgue) integral of f *over the measurable set E by*

$$\int_E f \, d\mu := \int_E f^+ \, d\mu - \int_E f^- \, d\mu.$$

If both $\int_E f^+ \, d\mu$ and $\int_E f^- \, d\mu$ are finite, then f is said to be (Lebesgue) integrable *over the measurable set E.*

In the special case that μ is the Lebesgue measure, we denote $\int_E f \, d\mathcal{L}^N$ simply by

$$\int_E f \, dx.$$

If $(X, \mathfrak{M}, \mu)$ is a measure space, with X a topological space, and if $\mathfrak{M}$ contains $\mathcal{B}(X)$, then $f : X \to [-\infty, \infty]$ is said to be *locally integrable* if it is Lebesgue integrable over every compact set.

A measurable function $f : X \to [-\infty, \infty]$ is Lebesgue integrable over the measurable set E if and only if

$$\int_E |f| \, d\mu < \infty.$$

Remark 66. *If* $(X, \mathfrak{M}, \mu)$ *is a measure space and* $f : X \to [-\infty, \infty]$ *is Lebesgue integrable, then the set* $\{x \in X : |f(x)| = \infty\}$ *has measure zero, while the set* $\{x \in X : |f(x)| > 0\}$ *is* σ*-finite.*

As a result of the structure of measurable functions and construction of the integral, we have the following properties for integrable functions.

Proposition 67. *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* $f,\ g : X \to [-\infty, \infty]$ *be two integrable functions.*

(i) If $\alpha,\ \beta \in \mathbb{R}$*, then* $\alpha f + \beta g$ *is integrable and*

$$\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu.$$

(ii) $\left|\int_X f\, d\mu\right| \leq \int_X |f|\, d\mu$.

If $(X, \mathfrak{M}, \mu)$ is a measure space and $f,\ g : X \to [-\infty, \infty]$ are two measurable functions such that $f(x) = g(x)$ for μ a.e. $x \in X$, then, defining

$$E := \{x \in X : f(x) \neq g(x)\},$$

we have that $E \in \mathfrak{M}$, with $\mu(E) = 0$. Hence

$$\begin{aligned}\int_X f^{\pm}\, d\mu &= \int_X \left(f^{\pm}\chi_E + f^{\pm}\chi_{X\setminus E}\right)\, d\mu \\ &= \int_E f^{\pm}\, d\mu + \int_{X\setminus E} f^{\pm}\, d\mu \\ &= \int_{X\setminus E} g^{\pm}\, d\mu = \int_E g^{\pm}\, d\mu + \int_{X\setminus E} g^{\pm}\, d\mu \\ &= \int_X g^{\pm}\, d\mu.\end{aligned}$$

Thus $\int_X f\, d\mu$ is well-defined if and only if $\int_X g\, d\mu$ is well-defined, and in this case we have

$$\int_X f\, d\mu = \int_X g\, d\mu. \tag{2.7}$$

This shows that the Lebesgue integral does not distinguish functions that coincide μ a.e. in X. This motivates our later considerations of measurable functions as equivalence classes, a concept we will define more precisely later.

Finally, if $F \in \mathfrak{M}$ is such that $\mu(F) = 0$ and $f : X \setminus F \to [-\infty, \infty]$ is a measurable function in the sense of Definition 53, then we define the *(Lebesgue) integral of* f over the measurable set E as the Lebesgue integral of the function

$$g(x) := \begin{cases} f(x) & \text{if } x \in X \setminus F, \\ 0 & \text{otherwise,} \end{cases}$$

provided $\int_E g \, d\mu$ is well-defined. Note that in this case

$$\int_E g \, d\mu = \int_E \tilde{v} \, d\mu,$$

where

$$\tilde{v}(x) := \begin{cases} f(x) & \text{if } x \in X \setminus F, \\ w(x) & \text{otherwise,} \end{cases}$$

and w is an arbitrary measurable function defined on F. If the measure μ is complete, then $\int_E g \, d\mu$ is well-defined if and only if $\int_{E \setminus F} f \, d\mu$ is well-defined.

For functions of arbitrary sign we have the following convergence result.

Theorem 68 (Lebesgue dominated convergence theorem)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $f_n : X \to [-\infty, \infty]$ be a sequence of measurable functions such that*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for μ a.e. $x \in X$. If there exists a Lebesgue integrable function g such that

$$|f_n(x)| \leq g(x)$$

for μ a.e. $x \in X$ and all $n \in \mathbb{N}$, then f is Lebesgue integrable and

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0.$$

In particular,

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

Corollary 69. *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and let* $f_n : X \to [-\infty, \infty]$ *be a sequence of measurable functions. If*

$$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty,$$

then the series $\sum_{n=1}^{\infty} f_n(x)$ *converges for* μ *a.e.* $x \in X$*, the function*

$$f(x) := \sum_{n=1}^{\infty} f_n(x),$$

defined for μ *a.e.* $x \in X$*, is integrable, and*

$$\sum_{n=1}^{\infty} \int_X f_n \, d\mu = \int_X \sum_{n=1}^{\infty} f_n \, d\mu.$$

Beyond the previously stated results on integration, we will make use of integration in product spaces, where we again find use for outer measures. Given two measures, their product can be defined easily on rectangles, and with some work, can be used to construct an outer measure on the product space. This motivates our use of outer measures, as we can then apply Carathéodory's theorem to construct a *product measure*. The following theorem shows that rectangles are measurable in the product and gives a formula for computing their measure.

Definition 70. *Let* $(X, \mathfrak{M}, \mu)$ *and* $(Y, \mathfrak{N}, \nu)$ *be two measure spaces. For* $E \in \mathcal{P}(X)$ *and* $F \in \mathcal{P}(Y)$*, we define the outer measure* $(\mu \times \nu)^* : \mathcal{P}(X) \times \mathcal{P}(Y) \to [0, \infty]$ *by*

$$(\mu \times \nu)^* (E \times F) := \mu(E)\nu(F)$$

Then by Carthéodory's Theorem we have the restriction of $(\mu \times \nu)^*$ to the σ-algebra of $(\mu \times \nu)^*$ measurable sets is a measure. We denote this σ-algebra as $\mathfrak{M} \times \mathfrak{N}$. We have the following theorem relating the product of the σ-algebras $\mathfrak{M}$ and $\mathfrak{N}$ with this new σ-algebra.

Theorem 71. *Let* $(X, \mathfrak{M}, \mu)$ *and* $(Y, \mathfrak{N}, \nu)$ *be two measure spaces. If* $F \in \mathfrak{M}$ *and* $G \in \mathfrak{N}$*, then* $F \times G$ *is* $(\mu \times \nu)^*$*-measurable and*

$$(\mu \times \nu)(F \times G) = \mu(F)\,\nu(G). \tag{2.8}$$

In particular, $\mathfrak{M} \otimes \mathfrak{N} \subset \mathfrak{M} \times \mathfrak{N}$.

More generally, any set in the cartesian product can be covered by a rectangle with the same measure.

Corollary 72. *Let* $(X, \mathfrak{M}, \mu)$ *and* $(Y, \mathfrak{N}, \nu)$ *be two measure spaces. If* $E \subset X \times Y$*, then there exists a set* $R \in \mathfrak{M} \otimes \mathfrak{N}$ *containing* E *such that*

$$(\mu \times \nu)^* (E) = (\mu \times \nu) (R) .$$

It is important to understand measurability of the sections of a set in the product space, which we demonstrate in the following theorem.

Theorem 73. *Let* $(X, \mathfrak{M}, \mu)$ *and* $(Y, \mathfrak{N}, \nu)$ *be two measure spaces. Assume that* μ *and* ν *are complete and* $E \in \mathfrak{M} \times \mathfrak{N}$ *has* σ*-finite* $\mu \times \nu$ *measure. Then for* μ *a.e.* $x \in X$ *the* section

$$E_x := \{y \in Y : (x, y) \in E\}$$

belongs to the σ*-algebra* $\mathfrak{N}$ *and for* ν *a.e.* $y \in Y$ *the section*

$$E_y := \{x \in X : (x, y) \in E\}$$

belongs to the σ*-algebra* $\mathfrak{M}$*. Moreover, the functions* $y \mapsto \mu (E_y)$ *and* $x \mapsto \nu (E_x)$ *are measurable and*

$$(\mu \times \nu) (E) = \int_Y \mu (E_y) \, d\nu (y) = \int_X \nu (E_x) \, d\mu (x) .$$

When integrating non-negative functions, we have the following very powerful theorem for exchanging the order of integration.

Theorem 74 (Tonelli)**.** *Let* $(X, \mathfrak{M}, \mu)$ *and* $(Y, \mathfrak{N}, \nu)$ *be two measure spaces. Assume that* μ *and* ν *are complete and* σ*-finite, and let* $f : X \times Y \to [0, \infty]$ *be an* $\mathfrak{M} \times \mathfrak{N}$ *measurable function. Then for* μ *a.e.* $x \in X$ *the function* $f (x, \cdot)$ *is measurable and the function* $\int_Y f (\cdot, y) \, d\nu (y)$ *is measurable. Similarly, for* ν *a.e.* $y \in Y$ *the function* $f (\cdot, y)$ *is measurable and the function* $\int_X f (x, \cdot) \, d\mu (x)$ *is measurable. Moreover,*

$$\begin{aligned} \int_{X \times Y} f (x, y) \, d (\mu \times \nu) (x, y) &= \int_X \left(\int_Y f (x, y) \, d\nu (y) \right) d\mu (x) \\ &= \int_Y \left(\int_X f (x, y) \, d\mu (x) \right) d\nu (y) . \end{aligned}$$

Assuming integrability, we have the following equally powerful theorem for functions of an arbitrary sign.

Theorem 75 (Fubini)**.** *Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be two measure spaces. Assume that μ and ν are complete, and let $f : X \times Y \to [-\infty, \infty]$ be $\mu \times \nu$-integrable. Then for μ a.e. $x \in X$ the function $f(x, \cdot)$ is ν-integrable, and the function $\int_Y f(\cdot, y)\, d\nu(y)$ is μ-integrable.*

Similarly, for ν a.e. $y \in Y$ the function $f(\cdot, y)$ is μ-integrable, and the function $\int_X f(x, \cdot)\, d\mu(x)$ is ν-integrable. Moreover,

$$\begin{aligned}\int_{X\times Y} f(x,y)\, d(\mu\times\nu)(x,y) &= \int_X \left(\int_Y f(x,y)\, d\nu(y)\right) d\mu(x)\\ &= \int_Y \left(\int_X f(x,y)\, d\mu(x)\right) d\nu(y).\end{aligned}$$

Finally, we recall a corollary to Fubini's Theorem, Minkowski's inequality for integrals.

Theorem 76. *Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be two measure spaces. Assume that μ and ν are complete and σ-finite, and $f : X \times Y \to [0, \infty]$ be an $\mathfrak{M} \times \mathfrak{N}$ measurable function, and let $1 \le p \le \infty$. Then*

$$\left(\int_Y \left(\int_X |f(x,\cdot)|\, d\mu(x)\right)^p d\nu(y)\right)^{\frac{1}{p}} \le \int_X \left(\int_Y |f(x,\cdot)|^p\, d\nu(y)\right)^{\frac{1}{p}} d\mu(x)$$

2.2.2 Measures and Differentiation

Beyond the necessity of measures in the theory of Lebesgue integration, our adaptation of the blow-up argument of Fonseca and Müller [40] requires that we study their differentiability properties, in particular when the underlying space is Euclidean. The results that follow make up a version of the fundamental theorem of calculus, which in the full generality of measures is called the Radon-Nikodym theorem.

Definition 77. *Let $(X, \mathfrak{M})$ be a measurable space and let μ, $\nu : \mathfrak{M} \to [0, \infty]$ be two measures. The measure ν is said to be* absolutely continuous with respect to μ, *and we write $\nu \ll \mu$, if for every $E \in \mathfrak{M}$ with $\mu(E) = 0$ we have $\nu(E) = 0$.*

Theorem 78. *[Radon–Nikodym]Let* $(X, \mathfrak{M})$ *be a measurable space and let* μ*,* $\nu : \mathfrak{M} \to [0, \infty]$ *be two measures, with* μ σ*-finite and* ν *absolutely continuous with respect to* μ*. Then there exists a unique (up to sets of measure* μ *zero) measurable function* $f : X \to [0, \infty]$ *such that*

$$\nu(E) = \int_E f \, d\mu$$

for every $E \in \mathfrak{M}$*.*

The assumption of absolute continuity along with an assumption of a finite or σ-finite measure ensures that the Radon-Nikodym derivative is well-defined. More generally, given two measures, one of which is positive and finite or σ-finite, the Radon-Nikodym theorem continues to hold for the *absolutely continuous part* of the measure, a result we make more precise with several lemmata detailing decomposition properties of measures. One construction for this part is as follows.

Lemma 79. *Let* $(X, \mathfrak{M})$ *be a measurable space and let* μ*,* $\nu : \mathfrak{M} \to [0, \infty]$ *be two measures. For every* $E \in \mathfrak{M}$ *define*

$$\nu_{ac}(E) := \sup \left\{ \int_E f \, d\mu : \ f : X \to [0, \infty] \ \text{measurable}, \right. \\ \left. \int_{E'} f \, d\mu \leq \nu(E') \ \text{for all } E' \subset E, E' \in \mathfrak{M} \right\}. \tag{2.9}$$

Then ν_{ac} *is a measure, with* $\nu_{ac} \ll \mu$*, and for each* $E \in \mathfrak{M}$ *the supremum in the definition of* ν_{ac} *is actually attained by a function* f *admissible for* $\nu_{ac}(E)$*. Moreover, if* ν_{ac} *is* σ*-finite, then* f *may be chosen independently of the set* E*.*

Besides the absolutely continuous part, there can be a remaining piece of the measure, the *singular part*.

Definition 80. *Let* $(X, \mathfrak{M})$ *be a measurable space and let* μ*,* $\nu : \mathfrak{M} \to [0, \infty]$ *be two measures.* μ*,* ν *are said to be* mutually singular*, and we write* $\nu \perp \mu$*, if there exist two disjoint sets* X_μ*,* $X_\nu \in \mathfrak{M}$ *such that* $X = X_\mu \cup X_\nu$ *and for every* $E \in \mathfrak{M}$ *we have*

$$\mu(E) = \mu(E \cap X_\mu), \quad \nu(E) = \nu(E \cap X_\nu).$$

In a similar way to the construction of the absolutely continuous part, we are able to define the singular part of a measure as follows.

Lemma 81. *Let* $(X, \mathfrak{M})$ *be a measurable space and let* μ, $\nu : \mathfrak{M} \to [0, \infty]$ *be two measures. For every* $E \in \mathfrak{M}$ *define*

$$\nu_s(E) := \sup\{\nu(F) : F \subset E, F \in \mathfrak{M}, \mu(F) = 0\}. \tag{2.10}$$

Then ν_s *is a measure and for each* $E \in \mathfrak{M}$ *the supremum in the definition of* ν_s *is actually attained by a measurable set.*

Moreover, if ν_s *is* σ*-finite, then* $\nu_s \perp \mu$.

The preceding results are important in the Lebesgue decomposition theorem.

Theorem 82 (Lebesgue decomposition theorem)**.** *Let* $(X, \mathfrak{M})$ *be a measurable space and let* μ, $\nu : \mathfrak{M} \to [0, \infty]$ *be two measures, with* μ σ*-finite. Then*

$$\nu = \nu_{ac} + \nu_s \tag{2.11}$$

with $\nu_{ac} \ll \mu$. *Moreover, if* ν *is* σ*-finite, then* $\nu_s \perp \mu$ *and the decomposition* (2.11) *is unique, that is, if*

$$\nu = \overline{\nu}_{ac} + \overline{\nu}_s,$$

for some measures $\overline{\nu}_{ac}$, $\overline{\nu}_s$, *with* $\overline{\nu}_{ac} \ll \mu$ *and* $\overline{\nu}_s \perp \mu$, *then*

$$\nu_{ac} = \overline{\nu}_{ac} \quad \text{and} \quad \nu_s = \overline{\nu}_s.$$

We will also be concerned with the Radon-Nikodym theorem as applied to one positive measure and one measure which is *signed*, and we therefore must introduce the notion of signed measures. This will be further developed as we consider a subset of these signed measures as a functional space.

Definition 83. *Let* $(X, \mathfrak{M})$ *be a measurable space. A* signed measure *is a function* $\lambda : \mathfrak{M} \to [-\infty, \infty]$ *such that*

(i) $\lambda(\emptyset) = 0$;

(ii) λ *takes at most one of the two values* ∞ *and* $-\infty$, *that is, either* $\lambda : \mathfrak{M} \to (-\infty, \infty]$ *or* $\lambda : \mathfrak{M} \to [-\infty, \infty)$;

(iii) for every countable collection $\{E_i\} \subset \mathfrak{M}$ of pairwise disjoint sets we have

$$\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \lambda(E_n).$$

The following lemma enables us to decompose a signed measure into two positive measures, thus regaining the structure of positive measures previously introduced.

Lemma 84. *Let $(X, \mathfrak{M})$ be a measurable space and let $\lambda : \mathfrak{M} \to [-\infty, \infty]$ be a signed measure. For every $E \in \mathfrak{M}$ define*

$$\lambda^+(E) := \sup\{\lambda(F) : \ F \subset E, \ F \in \mathfrak{M}\}, \tag{2.12}$$

$$\begin{aligned} \lambda^-(E) &:= -\inf\{\lambda(F) : \ F \subset E, \ F \in \mathfrak{M}\} \\ &= \sup\{-\lambda(F) : \ F \subset E, \ F \in \mathfrak{M}\}. \end{aligned} \tag{2.13}$$

Then λ^+ and λ^- are measures. Moreover, if $\lambda : \mathfrak{M} \to [-\infty, \infty)$, then for every $E \in \mathfrak{M}$ we have

$$\lambda^+(E) = \sup\left\{\lambda(F) : \ F \subset E, \ F \in \mathfrak{M}, \ \lambda^-(F) = 0\right\}, \tag{2.14}$$

λ^+ is finite, and $\lambda = \lambda^+ - \lambda^-$.

More generally, the Lebesgue decomposition theorem is true for signed measures, if we allow modify our definitions of absolutely continuous and singular to include a potentially signed measure.

Definition 85. *Let $(X, \mathfrak{M})$ be a measurable space, let $\mu : \mathfrak{M} \to [0, \infty]$ a measure and $\lambda : \mathfrak{M} \to [-\infty, \infty]$ be a signed measure.*

(i) λ is said to be absolutely continuous with respect to μ*, and we write $\lambda \ll \mu$, if $\lambda(E) = 0$ whenever $E \in \mathfrak{M}$ and $\mu(E) = 0$.*

(ii) λ and μ are said to be mutually singular*, and we write $\lambda \perp \mu$, if there exist two disjoint sets X_μ, $X_\lambda \in \mathfrak{M}$ such that $X = X_\mu \cup X_\lambda$ and for every $E \in \mathfrak{M}$ we have*

$$\mu(E) = \mu(E \cap X_\mu), \quad \lambda(E) = \lambda(E \cap X_\lambda).$$

Note that if $\lambda \ll \mu$, then $\lambda^+ \ll \mu$ and $\lambda^- \ll \mu$.

If $(X, \mathfrak{M})$ is a measurable space, $\lambda : \mathfrak{M} \to [-\infty, \infty]$ is a signed measure, and $\mu : \mathfrak{M} \to [0, \infty]$ is a σ-finite (positive) measure, then by applying the Lebesgue decomposition theorem to λ^+ and μ (respectively to λ^- and μ) we can write

$$\lambda^+ = \left(\lambda^+\right)_{ac} + \left(\lambda^+\right)_s, \quad \lambda^- = \left(\lambda^-\right)_{ac} + \left(\lambda^-\right)_s,$$

where the measures $(\lambda^+)_{ac}$ and $(\lambda^+)_s$ are defined in (2.9) and (2.10), and $(\lambda^+)_{ac}$, $(\lambda^-)_{ac} \ll \mu$. Hence we can apply the Radon–Nikodym theorem to find two measurable functions f^+, $f^- : X \to [0, \infty]$ such that

$$\left(\lambda^+\right)_{ac}(E) = \int_E f^+ \, d\mu, \quad \left(\lambda^-\right)_{ac}(E) = \int_E f^- \, d\mu$$

for every $E \in \mathfrak{M}$. The functions f^+ and f^- are unique up to a set of μ measure zero

Since either λ^+ or λ^- is finite we may define

$$\lambda_{ac} := \left(\lambda^+\right)_{ac} - \left(\lambda^-\right)_{ac}, \quad \lambda_s := \left(\lambda^+\right)_s - \left(\lambda^-\right)_s, \quad f := f^+ - f^-.$$

Then λ_{ac} is a signed measure with $\lambda_{ac} \ll \mu$. Note that if λ is positive then so are λ_{ac} and λ_s.

Theorem 86 (Lebesgue decomposition theorem)**.** *Let $(X, \mathfrak{M})$ be a measurable space, let $\lambda : \mathfrak{M} \to [-\infty, \infty]$ be a signed measure, and let $\mu : \mathfrak{M} \to [0, \infty]$ be a σ-finite (positive) measure. Then*

$$\lambda = \lambda_{ac} + \lambda_s$$

with $\lambda_{ac} \ll \mu$, and

$$\lambda_{ac}(E) = \int_E f \, d\mu$$

for all $E \in \mathfrak{M}$. Moreover, if λ is σ-finite then $\lambda_s \perp \mu$ and the decomposition is unique, that is, if

$$\lambda = \overline{\lambda}_{ac} + \overline{\lambda}_s,$$

for some signed measures $\overline{\lambda}_{ac}$, $\overline{\lambda}_s$, with $\overline{\lambda}_{ac} \ll \mu$ and $\overline{\lambda}_s \perp \mu$, then

$$\lambda_{ac} = \overline{\lambda}_{ac} \quad \textit{and} \quad \lambda_s = \overline{\lambda}_s.$$

We call λ_{ac} and λ_s, respectively, the *absolutely continuous part* and the *singular part of* λ with respect to μ, and often we write

$$f = \frac{d\lambda}{d\mu}.$$

When $X \subset \mathbb{R}^N$ we use the notation $Q(x,r)$ to denote the cube in $\mathbb{R}^N$ centered at x with radius r. The Besicovitch derivation theorem is ubiquitous in papers in the Calculus of Variations, and the main aim of our introduction of differentiation of measures.

Theorem 87. *[Besicovitch derivation theorem]Let* $\mu, \nu : \mathcal{B}\left(\mathbb{R}^N\right) \to [0,\infty]$ *be measures finite on compact sets. Then there exists a Borel set* $M \subset \mathbb{R}^N$*, with* $\mu(M) = 0$*, such that for any* $x \in \mathbb{R}^N \setminus M$*,*

$$\frac{d\nu}{d\mu}(x) = \lim_{r \to 0^+} \frac{\nu\left(\overline{Q(x,r)}\right)}{\mu\left(\overline{Q(x,r)}\right)} \in \mathbb{R} \tag{2.15}$$

and

$$\lim_{r \to 0^+} \frac{\nu_s\left(\overline{Q(x,r)}\right)}{\mu\left(\overline{Q(x,r)}\right)} = 0, \tag{2.16}$$

where

$$\nu = \nu_{ac} + \nu_s, \quad \nu_{ac} \ll \mu, \quad \nu_s \perp \mu. \tag{2.17}$$

In particular, in the case of a locally integrable function, we call the set of $x \in \mathbb{R}^N$ for which the above holds *Lebesgue points*, a result we make more precise in the following theorem.

Theorem 88. *[Lebesgue differentiation theorem]Let* $\mu : \mathcal{B}\left(\mathbb{R}^N\right) \to [0,\infty]$ *be a measure finite on compact sets and let* $f : \mathbb{R}^N \to \mathbb{R}$ *be locally integrable. Then there exists a Borel set* $E \subset \mathbb{R}^N$*, with* $\mu(E) = 0$*, such that for every* $x \in \mathbb{R}^N \setminus E$*,*

$$\lim_{r \to 0^+} \frac{1}{\mu\left(\overline{B(x,r)}\right)} \int_{\overline{B(x,r)}} f \, d\mu = f(x). \tag{2.18}$$

We say that $x \in \mathbb{R}^N$ *is a* Lebesgue point *if (2.18) holds at* x*.*

Moreover, by enlarging the bad set, we have the following stronger result.

Corollary 89. *Let $\mu : \mathcal{B}\left(\mathbb{R}^N\right) \to [0,\infty]$ be a measure finite on compact sets and let $f : \mathbb{R}^N \to [-\infty,\infty]$ be a locally integrable function. Then there exists a Borel set $F \subset \mathbb{R}^N$, with $\mu(F) = 0$, such that for every $x \in \mathbb{R}^N \setminus F$,*

$$\lim_{r\to 0^+} \frac{1}{\mu\left(\overline{B(x,r)}\right)} \int_{\overline{B(x,r)}} |f(y) - f(x)| \, d\mu(y) = 0. \tag{2.19}$$

Here, we have that $F \supset E$, but in practice since $\mu(F) = \mu(E) = 0$, we are not particularly concerned with the difference in these two sets.

2.2.3 Measures as a Dual Space

In this section, we consider X a locally compact, separable metric space.

Definition 90. *A topological space (X, τ) is* locally compact *if every point has a neighborhood whose closure is compact.*

This nice topological structure of the underlying space X implies that the continuous linear functionals on the space of continuous functions vanishing at infinity are precisely the *Radon measures*. As we saw in the subsection Duality and Weak, Weak-Star Topologies, this implies weak-star compactness of norm-bounded sequences of Radon measures, once we are able to establish this duality via the Riesz representation theorem.

As we saw previously, duality is something we established between two vector spaces with topological structure, and measures as we have considered them are not a vector space. However, if we consider the functional space of finite signed measures which are regular (that is, which satisfy (2.29)), we are then in a position to apply the previous results. We therefore must first extend our notion of integration to include integration with respect to signed measures, which we can define thanks to the Lebesgue decomposition theorem. Given a signed measure λ, we say that a bounded function $f : X \to \mathbb{R}$ is *integrable* with respect to λ if it is integrable with respect to λ^+ and λ^-, and we set

$$\int_X f \, d\lambda = \int_X f \, d\lambda^+ - \int_X f \, d\lambda^-.$$

Definition 91. *Let X be a locally compact, separable metric space and $\mathfrak{M}$ be the Borel σ-algebra. The space $M_b(X, \mathfrak{M})$ of all* finite signed Radon measures *consists of all finite signed measures λ such that for every $E \in \mathfrak{M}$,*

$$\lambda^+(E) = \inf\left\{\lambda^+(A) : A \text{ open, } A \supset E\right\} \tag{2.20}$$
$$= \sup\left\{\lambda^+(C) : C \text{ closed, } C \subset E\right\} \tag{2.21}$$

and

$$\lambda^-(E) = \inf\left\{\lambda^-(A) : A \text{ open } A \supset E\right\}$$
$$= \sup\left\{\lambda^-(C) : C \text{ closed } C \subset E\right\}.$$

We will use $M_b(X)$ in the sequel, suppressing the dependence on the Borel σ-algebra. We use the notation $M_b(X; \mathbb{R}^m)$ to denote the space of $\mathbb{R}^m$-valued finite signed Radon measures.

Theorem 92. *The space $M_b(X)$ is a Banach space with the norm*

$$|\lambda|(X) := \sup\left\{\sum_{i=1}^{\infty} |\lambda(E_i)| : X = \bigcup_{i=1}^{\infty} E_i\right\}$$

Proposition 93. *When $\lambda \in M_b(X; \mathbb{R}^m)$, every $\phi \in C_0(X; \mathbb{R}^m)$ is integrable with respect to λ.*

Here, we use ϕ in place of f to denote a continuous function (as opposed to the integration theory previously developed with respect to an arbitrary measurable function). The notation $\phi \in C_0(X; \mathbb{R}^m)$ is used to denote the space of continuous functions who vanish at infinity (which can be seen as the completion of the space of ($\mathbb{R}^m$-valued) continuous functions with compact support in the sup norm).

The following result is known as the Riesz representation theorem in $C_0(X; \mathbb{R}^m)$, characterizing the dual as the space of ($\mathbb{R}^m$-valued) finite signed Radon measures.

Theorem 94. *Let X be a locally compact, separable metric space and $\mathfrak{M}$ be the Borel σ-algebra. Then every bounded linear functional $L : C_0(X;\mathbb{R}^m) \to \mathbb{R}$ is represented by a unique $\lambda \in M_b(X;\mathbb{R}^m)$ in the sense that*

$$L(\phi) = \int_X \phi \cdot d\lambda \quad \text{for every } \phi \in C_0(X;\mathbb{R}^m). \tag{2.22}$$

Moreover, the norm of L coincides with the total variation norm $|\lambda|(X)$. Conversely, every functional of the form (2.22), where $\lambda \in M_b(X;\mathbb{R}^m)$, is a bounded linear functional on $C_0(X;\mathbb{R}^m)$.

This theorem allow us to identify $M_b(X;\mathbb{R}^m)$ and the dual of $C_0(X;\mathbb{R}^m)$, and in what follows we write $(C_0(X;\mathbb{R}^m))'$ for $M_b(X;\mathbb{R}^m)$ to emphasize this relationship. In particular, this implies (via the Corollary of Bolzano-Weierstrass in Duality and Weak, Weak-Star Topologies) that bounded sets of $M_b(X;\mathbb{R}^m)$ are weak-star compact. Moreover, from the definition of the weak-star topology, given $\lambda_n, \lambda \in M_b(X;\mathbb{R}^m)$, we have that λ_n converges weakly-star to λ in $(C_0(X;\mathbb{R}^m))'$ (for which we will use the notation $\lambda_n \overset{*}{\rightharpoonup} \lambda$ in $(C_0(X;\mathbb{R}^m))'$) when

$$\lim_{n\to\infty} \int_X \phi \cdot d\lambda_n = \int_X \phi \cdot d\lambda \tag{2.23}$$

for every $\phi \in C_0(X;\mathbb{R}^m)$.

There are many topologies we can put on the finite signed Radon measures, and in particular we will be concerned with a frequently used one from probability. By Theorem 26, the linear functionals defined by integration against the set of $\mathbb{R}^m$-valued continuous and bounded functions on X equipped with the sup norm makes $M_b(X;\mathbb{R}^m)$ into a locally convex topological vector space. We no longer have the same compactness, since these spaces are not dual to each other, but it justifies defining another weak convergence for $\lambda_n, \lambda \in M_b(X;\mathbb{R}^m)$. We write that $\lambda_n \overset{*}{\rightharpoonup} \lambda$ in $(C_b(X;\mathbb{R}^m))'$ if (2.23) holds for all $\phi \in C_b(X;\mathbb{R}^m)$. We remark that, in general, weak convergence in $(C_b(X;\mathbb{R}^m))'$ is stronger than the weak-star convergence in $(C_0(X;\mathbb{R}^m))'$; however, under additional assumptions that we will later make, they are equivalent (see Remark 156).

2.3 SPACES OF INTEGRABLE FUNCTIONS

2.3.1 Lebesgue spaces

We give a basic introduction to the L^p spaces, since the many of the commonly used theorems in L^p spaces also find use in techniques in the Sobolev spaces.

Let $(X, \mathfrak{M}, \mu)$ be a measure space. For $1 \leq p < \infty$, we define the space

$$M^p(X) := \left\{ f : f : X \to [-\infty, \infty] \text{ measurable and } \|f\|_{M^p(X)} < \infty \right\},$$

where

$$\|f\|_{M^p(X)} := \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

For $p = \infty$, we define

$$M^\infty(X) := \{ f : f : X \to \mathbb{R} \text{ measurable and bounded} \},$$

where

$$\|f\|_{M^\infty(X)} := \sup_{x \in X} |f(x)|.$$

Next we study the triangle inequality to determine if the above integral and supremum define norms on the spaces M^p and M^∞.

Let q be the *Hölder conjugate exponent* of p, i.e.,

$$q := \begin{cases} \frac{p}{p-1} & \text{if } 1 < p < \infty, \\ \infty & \text{if } p = 1, \\ 1 & \text{if } p = \infty. \end{cases}$$

Note that, with an abuse of notation, we have

$$\frac{1}{p} + \frac{1}{q} = 1.$$

In the sequel, the Hölder conjugate exponent of p will often be denoted by p'.

Hölder's inequality is a very important one for our purposes.

Theorem 95 (Hölder's inequality)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space, let $1 \leq p \leq \infty$, and let q be its Hölder conjugate exponent. If f, $g : X \to [-\infty, \infty]$ are measurable functions then*

$$\int_X |fg| \, d\mu \leq \left(\int_X |f|^p \, d\mu \right)^{1/p} \left(\int_X |g|^q \, d\mu \right)^{1/q} \tag{2.24}$$

if $1 < p < \infty$,

$$\int_X |fg| \, d\mu \leq \sup_{x \in X} |g(x)| \int_X |f| \, d\mu \tag{2.25}$$

if $p = 1$, and

$$\int_X |fg| \, d\mu \leq \sup_{x \in X} |f(x)| \int_X |g| \, d\mu \tag{2.26}$$

if $p = \infty$. In particular, if $f \in M^p(X)$ and $g \in M^p(X)$ then $fg \in M^1(X)$.

Minkowski's inequality implies the triangle inequality when we consider the previously introduced norm.

Theorem 96 (Minkowski's inequality)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space, let $1 \leq p \leq \infty$, and let f, $g : X \to [-\infty, \infty]$ be measurable functions. Then,*

$$\|f + g\|_{M^p(X)} \leq \|f\|_{M^p(X)} + \|g\|_{M^p(X)} \tag{2.27}$$

whenever $\|f + g\|_{M^p(X)}$ is well-defined. In particular, if $f, g \in M^p(X)$, then $f + g \in M^p(X)$ and (2.27) holds.

In view of the previous theorem we now have that for $1 \leq p < \infty$, properties (ii) and (iii) of Definition 19 are satisfied. The problem is property (i). Indeed, if

$$\|f\|_{L^p} = \left(\int_X |f|^p \, d\mu \right)^{1/p} = 0,$$

then by Corollary 64 there exists a set $E \in \mathfrak{M}$ with $\mu(E) = 0$ such that $f(x) = 0$ for all $x \in X \setminus E$. This does not imply that the function f is zero. For example, the Dirichlet function

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise,} \end{cases}$$

has exactly this property (using the Lebesgue measure).

To circumvent this problem, given two measurable functions f, $g : X \to [-\infty, \infty]$, we say that f is *equivalent* to g, and we write

$$f \sim g \text{ if } f(x) = g(x) \text{ for } \mu \text{ a.e. } x \in X. \tag{2.28}$$

Note that $\sim$ is an equivalence relation in the class of measurable functions. Moreover, if $f(x) = 0$ for μ a.e. $x \in X$, then $f \sim 0$, or, equivalently, f belongs to equivalence class $[0]$.

Definition 97. *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $1 \le p < \infty$. We define*

$$L^p(X) := M^p(X) \diagup \sim = \left\{ [f] : \ f : X \to [-\infty, \infty] \text{ measurable and } \|f\|_{M^p(X)} < \infty \right\}.$$

In the space $L^p(X)$ we define the norm

$$\|[f]\|_{L^p(X)} := \|f\|_{M^p(X)}.$$

Note that $\|[f]\|_{L^p}$ does not depend on the choice of the representative. We now have that $(L^p(X), \|\cdot\|_{L^p})$ is a normed space, since properties (i)-(ii) of Definition 19 are satisfied.

Let's now consider the case $p = \infty$. Unlike the case $1 \le p < \infty$, the supremum of a function changes if we change the function even at one point. Thus, we cannot take as a norm $\|[f]\|_{L^\infty(X)} := \sup_{x \in X} |f(x)|$. What we need is a notion of supremum that does not change if we modify a function on a set of measure zero.

Let $(X, \mathfrak{M}, \mu)$ be a measure space. Given a measurable function $f : X \to [-\infty, \infty]$ we define the *essential supremum* $\operatorname{esssup} f$ of the function f as

$$\operatorname{esssup} f := \inf \{ t \in \mathbb{R} : \ f(x) \le t \text{ for } \mu \text{ a.e. } x \in X \}.$$

Note that if $M := \operatorname{esssup} f < \infty$, then by taking $t_n := M + \frac{1}{n}$ we can find $E_n \in \mathfrak{M}$ with $\mu(E_n) = 0$ such that

$$f(x) \le M + \frac{1}{n} \text{ for all } x \in X \setminus E_n.$$

Take

$$E_\infty := \bigcup_{n=1}^{\infty} E_n.$$

Then $\mu(E_\infty) \le \sum_{n=1}^{\infty} \mu(E_n) = 0$, and if $x \in X \setminus E_\infty$, then

$$f(x) \le M + \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$, we get that $f(x) \le M$ for all $x \in X \setminus E_\infty$. Conversely, if there are $t \in \mathbb{R}$ and $E \in \mathfrak{M}$ with $\mu(E) = 0$ such that $f(x) \le t$ for all $x \in X \setminus E$, then by definition of $\operatorname{esssup} f$, we have that $\operatorname{esssup} f \le t < \infty$. This shows that $\operatorname{esssup} f < \infty$ if and only if the function f is bounded from above except on a set of measure zero.

Moreover, if $f \sim g$ then $\operatorname{esssup} f = \operatorname{esssup} g$. This leads us to the following definition.

Definition 98. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. We define*

$$L^\infty(X) := \{[f] : f : X \to [-\infty, \infty] \textit{ measurable and } \operatorname{esssup} |f| < \infty\}.$$

In the space $L^\infty(X)$ we define the norm

$$\|[f]\|_{L^\infty} := \operatorname{esssup} |f|.$$

Indeed, properties (i) and (ii) of Definition 19 are satisfied. To prove property (iii), note that if $[f]$ and $[g]$ belong to $L^\infty(X)$, then there exist $E, F \in \mathfrak{M}$ with $\mu(E) = \mu(F) = 0$ such that $|f(x)| \le \operatorname{esssup} |f|$ for all $x \in X \setminus E$ and $|g(x)| \le \operatorname{esssup} |g|$ for all $x \in X \setminus F$. Hence,

$$|f(x) + g(x)| \le |f(x)| + |g(x)| \le \operatorname{esssup} |f| + \operatorname{esssup} |g|$$

for all $x \in X \setminus (E \cup F)$, which implies that $\operatorname{esssup} |f + g| \le \operatorname{esssup} |f| + \operatorname{esssup} |g|$. Thus, the triangle inequality holds.

Remark 99. *Note that in Hölder's inequality one can replace (2.25) and (2.26) with*

$$\int_X |fg| \, d\mu \le \operatorname{esssup} |g| \int_X |f| \, d\mu$$

and

$$\int_X |fg| \, d\mu \le \operatorname{esssup} |f| \int_X |g| \, d\mu,$$

respectively. Indeed, in the first case, since $|g(x)| \le \operatorname{esssup} |g|$ for all $x \in X \setminus E$, where $E \in \mathfrak{M}$ with $\mu(E) = 0$, we have that

$$\begin{aligned} \int_X |fg| \, d\mu &= \int_{X \setminus E} |f| \, |g| \, d\mu \le \int_{X \setminus E} |f| \operatorname{esssup} |g| \, d\mu \\ &= \operatorname{esssup} |g| \int_{X \setminus E} |f| \, d\mu \le \operatorname{esssup} |g| \int_X |f| \, d\mu. \end{aligned}$$

With an abuse of notation, from now on we identify a measurable function $f : X \to [-\infty, \infty]$ with its equivalence class $[f]$. Note that this is very dangerous.

By identifying functions with their equivalence classes $[f]$, it follows from Minkowski's inequality that $\|\cdot\|_{L^p}$ is a norm on $L^p(X)$.

Theorem 100. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Then $L^p(X)$ is a Banach space for $1 \le p \le \infty$.*

Next we study some density results for $L^p(X)$ spaces.

Theorem 101. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Then the family of all simple functions in $L^p(X)$ is dense in $L^p(X)$ for $1 \le p \le \infty$.*

As we have studied the Radon measures, we have seen that there is some nice interplay between the signed finite Radon measures and some continuous functions that vanish at infinity. We further have the following important theorem, asserting the density of continuous functions with compact support when the measure is regular (although not necessarily finite).

Theorem 102. *Let $(X, \mathfrak{M}, \mu)$ be a measure space, with $X \subset \mathbb{R}^N$ open, and $\mathfrak{M} \supset \mathcal{B}(X)$. Assume that*

$$\mu(E) = \sup\{\mu(C) : C \text{ closed}, C \subset E\} = \inf\{\mu(A) : A \text{ open}, A \supset E\} \tag{2.29}$$

for every set $E \in \mathfrak{M}$ with finite measure. Then $L^p(X) \cap C_c(X)$ is dense in $L^p(X)$ for $1 \le p < \infty$.

The following theorem implies that when X is separable, bounded sequences in $L^p(X)$ are weakly compact.

Theorem 103. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. Then $L^p(X)$ is uniformly convex for every $1 < p < \infty$. In particular, Theorem 32 implies that $L^p(X)$ is reflexive for $1 < p < \infty$.*

Moreover, the following theorem characterizing the dual space of $L^p(X)$ enables us to better understand the consequences of weak compactness.

Theorem 104 (Riesz representation theorem in L^p)**.** *Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $1 < p < \infty$. Then every bounded linear functional $L : L^p(X) \to \mathbb{R}$ is represented by a* *unique* $g \in L^{p'}(X)$ *in the sense that*

$$L(f) = \int_X fg\, d\mu \quad \text{for every } f \in L^p(X). \tag{2.30}$$

Moreover, the norm of L *coincides with* $\|g\|_{L^{p'}}$. *Conversely, every functional of the form* (2.30), *where* $g \in L^{p'}(X)$, *is a bounded linear functional on* $L^p(X)$.

2.3.2 Lebesgue spaces on Euclidean space

Here we consider the case where $\Omega \subset \mathbb{R}^N$ is open and study some results of approximation by *mollifiers*. We continue the notation Ω_δ as defined in (1.13).

Given a nonnegative bounded function $\varphi \in L^1(\mathbb{R}^N)$ with

$$\operatorname{supp}\varphi \subset \overline{B(0,1)}, \quad \int_{\mathbb{R}^N} \varphi(x)\, dx = 1, \tag{2.31}$$

for every $\delta > 0$ we define

$$\varphi_\delta(x) := \frac{1}{\delta^N}\varphi\left(\frac{x}{\delta}\right), \quad x \in \mathbb{R}^N.$$

The functions φ_δ are called *mollifiers*.

Note that $\operatorname{supp}\varphi_\delta \subset \overline{B(0,\delta)}$. Hence, given an open set $\Omega \subset \mathbb{R}^N$ and a function $f \in L^1_{\text{loc}}(\Omega)$, we may define

$$f_\delta(x) := (f * \varphi_\delta)(x) = \int_\Omega \varphi_\delta(x-y) f(y)\, dy \tag{2.32}$$

whenever $\operatorname{dist}(x, \partial\Omega) > \delta$

The function f_δ is called a *mollification* of f.

Note that if $x \in \Omega$, then $f_\delta(x)$ is well-defined for all $0 < \delta < \operatorname{dist}(x, \partial\Omega)$. Thus, it makes sense to talk about $\lim_{\delta\to 0^+} f_\delta(x)$. We will use this fact without further mention.

Remark 105. *In the applications we will consider two important types of mollifiers:*

(i) φ *is the (renormalized) characteristic function of the unit ball, that is*

$$\varphi(x) := \frac{1}{\alpha_N}\chi_{B(0,1)}(x), \quad x \in \mathbb{R}^N,$$

where $\alpha_N = \mathcal{L}^N(B(0,1))$;

(ii) φ *is the* C_c^∞ *function*

$$\varphi(x) := \begin{cases} c \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases} \tag{2.33}$$

where we choose $c > 0$ *so that (2.31) is satisfied. In this case, the functions* φ_δ *are called* standard mollifiers.

The first main result of this subsection is the following theorem.

Theorem 106. *Let* $\Omega \subset \mathbb{R}^N$ *be an open set, let* $\varphi \in L^1(\mathbb{R}^N)$ *be a nonnegative bounded function satisfying (2.31), and let* $f \in L^1_{\text{loc}}(\Omega)$.

(i) If $f \in C(\Omega)$*, then* $f_\delta \to f$ *as* $\delta \to 0^+$ *uniformly on compact subsets of* Ω.

(ii) For every Lebesgue point $x \in \Omega$ *(and so for* $\mathcal{L}^N$ *a.e.* $x \in \Omega$*),* $f_\delta(x) \to f(x)$ *as* $\delta \to 0^+$.

(iii) If $1 \leq p \leq \infty$*, then*

$$\|f_\delta\|_{L^p(\Omega_\delta)} \leq \|f\|_{L^p(\Omega)} \tag{2.34}$$

for every $\delta > 0$ *and*

$$\|f_\delta\|_{L^p(\Omega_\delta)} \to \|f\|_{L^p(\Omega)} \text{ as } \delta \to 0^+. \tag{2.35}$$

(iv) If $f \in L^p(\Omega)$*,* $1 \leq p < \infty$*, then for any open set* $\Omega' \subset \Omega$ *with* $\operatorname{dist}(\Omega', \partial\Omega) > 0$,

$$\lim_{\delta \to 0^+} \left(\int_{\Omega'} |f_\delta - f|^p \, dx\right)^{\frac{1}{p}} = 0,$$

so that $f_\delta \to f$ *in* $L^p(\Omega')$.

Moreover, Jensen's Inequality implies that the above inequality between a function and its mollification applies to any for $\omega : [0, \infty) \to [0, \infty)$ convex. More precisely, we have

$$\begin{aligned} \omega(|f_\delta(x)|) &= \omega\left(\left|\int_{\mathbb{R}^N} f(x-y)\varphi_\delta(y)\, dy\right|\right) \\ &\leq \int_{\mathbb{R}^N} \omega(|f(x-y)|)\varphi_\delta(y)\, dy = (\omega \circ |f|)_\delta(x), \end{aligned}$$

for every $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) > \delta$.

2.3.3 Sobolev Spaces

Definition 107. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 \leq p \leq \infty$. The* Sobolev space *$W^{1,p}(\Omega)$ is the space of all functions $f \in L^p(\Omega)$ whose distributional first order partial derivatives belong to $L^p(\Omega)$, that is, for all $i = 1, \ldots, N$ there exists a function $g_i \in L^p(\Omega)$ such that*

$$\int_\Omega f \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega g_i \varphi\, dx$$

for all $\varphi \in C_c^\infty(\Omega)$. The function g_i is called the weak *or* distributional partial derivative *of f with respect to x_i and is denoted $\frac{\partial f}{\partial x_i}$.*

For $f \in W^{1,p}(\Omega)$ we set

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N}\right).$$

As usual, we define $W^{1,p}(\Omega; \mathbb{R}^d)$ as the space of all functions $f = (f_1, \ldots, f_d)$ such that $f_i \in W^{1,p}(\Omega)$ for all $i = 1, \ldots, d$. Also,

$$W^{1,p}_{\text{loc}}(\Omega) := \left\{f \in L^1_{\text{loc}}(\Omega) : \ f \in W^{1,p}(\Omega') \text{ for all open sets } \Omega' \subset\subset \Omega\right\}.$$

The Sobolev spaces are frequently useful for their structure as a Banach space, the substance of the next theorem.

Theorem 108. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 \leq p \leq \infty$. Then*

(i) the space $W^{1,p}(\Omega)$ is a Banach space with the norm

$$\|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega;\mathbb{R}^N)};$$

(ii) the space $H^1(\Omega) := W^{1,2}(\Omega)$ is an Hilbert space with the inner product

$$\langle f, g\rangle_{H^1(\Omega)} := \int_\Omega fg\, dx + \sum_{i=1}^N \int_\Omega \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i}\, dx.$$

We could also have considered other equivalent norms, however the above norm is uniformly convex for $1 < p < \infty$. Thus we can again use Theorem 32 to conclude reflexivity of the spaces.

Theorem 109. *$W^{1,p}(\Omega)$ is uniformly convex for every $1 < p < \infty$. In particular, Theorem 32 implies that $W^{1,p}(\Omega)$ is reflexive for $1 < p < \infty$.*

We also have the following result on separability of $W^{1,p}(\Omega)$.

Theorem 110. *$W^{1,p}(\Omega)$ is separable for every $1 \leq p < \infty$.*

In particular, the previous two theorems, Theorem 104 (Riesz representation theorem in $L^p(\Omega)$), and Corollary 28 (Bolzano-Weierstrass compactness corollary to the Banach-Alaoglu theorem) imply the following important result.

Theorem 111 (Compactness)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 < p < \infty$. Assume that $\{f_n\} \subset W^{1,p}(\Omega)$ is bounded. Then there exist a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f \in W^{1,p}(\Omega)$ such that $f_{n_k} \rightharpoonup f$ in $L^p(\Omega)$ and $\frac{\partial f_{n_k}}{\partial x_i} \rightharpoonup \frac{\partial f}{\partial x_i}$ in $L^p(\Omega)$ for all $i = 1, \ldots, N$. Moreover, we can show that in fact $f_{n_k} \to f$ in $L^p_{\text{loc}}(\Omega)$, and with boundary regularity even in $L^p(\Omega)$.*

Our introduction of mollifiers for the L^p spaces was motivated by their application in Sobolev and Bounded Variation spaces. When we restrict our attention to standard mollifiers, the resulting regularity of f_δ is vastly improves. Moreover, mollifiers have the nice property of approaching functions in the appropriate topology.

Theorem 112. *Let $\Omega \subset \mathbb{R}^N$ be an open set, let $\varphi \in L^1(\mathbb{R}^N)$ be defined as in (2.33), and let $f \in L^1_{\text{loc}}(\Omega)$. Then $f_\delta \in C^\infty(\Omega_\delta)$ for all $0 < \delta < 1$ and for every multi-index α,*

$$\frac{\partial^\alpha f_\delta}{\partial x^\alpha}(x) = \left(f * \frac{\partial^\alpha \varphi_\delta}{\partial x^\alpha}\right)(x) = \int_{\mathbb{R}^N} \frac{\partial^\alpha \varphi_\delta}{\partial x^\alpha}(x - y) f(y)\, dy \tag{2.36}$$

for all $x \in \Omega_\delta$.

An important corollary of the above result is the following.

Corollary 113. *Let $\Omega \subset \mathbb{R}^N$ be an open set, let $\varphi \in L^1(\mathbb{R}^N)$ be defined as in (2.33), and let $f \in W^{1,p}(\Omega)$. Then $f_\delta \to f$ in $W^{1,p}_{\text{loc}}(\Omega)$.*

The above result is an important one, in that it is the foundation for the assertion that spaces of weakly differentiable functions and strongly differentiable functions are one and the same, the celebrated result of Meyers and Serrin.

Theorem 114 (Meyers–Serrin)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 \leq p < \infty$. Then the space $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$.*

This result was very important in connecting Sobolev functions and classically differentiable functions, and led to a rich body of work on understanding the behavior of Sobolev functions. One of the nice properties of such functions is given in the next theorem relating weak partial derivatives (which are derivatives in the sense of distributions) with the classical partial derivatives (derivatives as an linear map). It is also a theorem characterizing $W^{1,p}(\Omega)$.

Theorem 115 (Absolute Continuity on Lines)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 \leq p < \infty$. A function $f \in L^p(\Omega)$ belongs to the space $W^{1,p}(\Omega)$ if and only if it has a representative $\overline{f}$ that is absolutely continuous on $\mathcal{L}^{N-1}$ a.e. line segments of Ω that are parallel to the coordinate axes, and whose first order (classical) partial derivatives belong to $L^p(\Omega)$. Moreover the (classical) partial derivatives of $\overline{f}$ agree $\mathcal{L}^N$ a.e. with the weak derivatives of f.*

As a consequence (and very practical application) of Theorem 115, we can show that the composition, product, or reflection of Sobolev functions is again a Sobolev function (modulo some details on the proper exponents).

The following theorem is very much related to our later characterization of the Sobolev spaces, providing us with some context of some known *non-local* characterizations.

Let $\Omega \subset \mathbb{R}^N$ be an open set and for every $i = 1, \ldots, N$ and $h > 0$, let

$$\Omega_h := \{x \in \Omega : \ x + he_i \in \Omega\}.$$

Theorem 116. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $f \in W^{1,p}(\Omega)$, $1 \leq p < \infty$. Then for every $i = 1, \ldots, N$ and $h > 0$,*

$$\int_{\Omega_h} \frac{|f(x + he_i) - f(x)|^p}{h^p}\, dx \leq \int_\Omega \left|\frac{\partial f}{\partial x_i}(x)\right|^p dx \tag{2.37}$$

and

$$\lim_{h\to 0^+} \left(\int_{\Omega_h} \frac{|f(x + he_i) - f(x)|^p}{h^p}\, dx\right)^{\frac{1}{p}} = \left(\int_\Omega \left|\frac{\partial f}{\partial x_i}(x)\right|^p dx\right)^{\frac{1}{p}}. \tag{2.38}$$

Conversely, if $f \in L^p(\Omega)$, $1 < p < \infty$, is such that

$$\liminf_{h\to 0^+} \left(\int_{\Omega_h} \frac{|f(x + he_i) - f(x)|^p}{h^p}\, dx\right)^{\frac{1}{p}} < \infty \tag{2.39}$$

for every $i = 1, \ldots, N$, then $f \in W^{1,p}(\Omega)$.

Some further results we will need include the embeddings of Sobolev spaces into "better spaces", better meaning with more integrability than might be expected, and in fact into spaces of continuous functions for large exponents p.

If $1 \leq p < N$, define $p^* = \frac{Np}{N-p}$.

Theorem 117 (Sobolev–Gagliardo–Nirenberg Embedding)**.** *Let $1 \leq p < N$. Then there exists a constant $C = C(N,p) > 0$ such that for every function $f \in W^{1,p}(\mathbb{R}^N)$,*

$$\left(\int_{\mathbb{R}^N} |f(x)|^{p^*}\, dx\right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^N} |\nabla f(x)|^p\, dx\right)^{\frac{1}{p}}. \tag{2.40}$$

In particular, $W^{1,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $L^q\left(\mathbb{R}^N\right)$ for all $p \leq q \leq p^*$.

When the boundary of Ω is not sufficiently smooth, there are some pathological examples that show the above embedding fails. There is some compensation, however, as a result of the underlying structure of $\mathbb{R}^N$, and so we can still expect a local version of the above embedding even in this case, as the following theorem demonstrates.

Corollary 118 (Local Sobolev–Gagliardo–Nirenberg Embedding)**.** *Let $1 \leq p < N$, and $\Omega' \subset\subset \Omega$. Then there exists a constant $C = C(N,p,\Omega') > 0$ such that for every function $f \in W^{1,p}_{\text{loc}}(\Omega)$,*

$$\left(\int_{\Omega'} |f(x)|^{p^*}\, dx\right)^{\frac{1}{p^*}} \leq C \left(\int_{\Omega} |\nabla f(x)|^p\, dx\right)^{\frac{1}{p}}. \tag{2.41}$$

We previously asserted that weak convergence in $W^{1,p}(\Omega)$ implies strong convergence of the functions and weak convergence of the derivatives. This is partially a consequence of the following estimates, which are generally useful in compactness questions of Sobolev functions.

Lemma 119. *Let $1 \leq p < \infty$ and let $f \in W^{1,p}\left(\mathbb{R}^N\right)$. Then for all $h \in \mathbb{R}^N \setminus \{0\}$,*

$$\int_{\mathbb{R}^N} |f(x+h) - f(x)|^p\, dx \leq |h|^p \int_{\mathbb{R}^N} |\nabla f(x)|^p\, dx.$$

Lemma 120. *Let $1 \leq p < \infty$ and let $f \in W^{1,p}\left(\mathbb{R}^N\right)$. For $k \in \mathbb{N}$ consider standard mollifiers as defined in (2.33). Then*

$$\int_{\mathbb{R}^N} |(f * \varphi_k)(x) - f(x)|^p\, dx \leq \frac{C(N,p)}{k^p} \int_{\mathbb{R}^N} |\nabla f(x)|^p\, dx.$$

When $p = N$, we actually improve our embedding space to arbitrarily large q.

Theorem 121. *The space $W^{1,N}\left(\mathbb{R}^N\right)$ is continuously embedded in $L^q\left(\mathbb{R}^N\right)$ for all $N \leq q < \infty$.*

We also have a local version of the above result.

Corollary 122. *Let $\Omega \subset \mathbb{R}^N$, and $\Omega' \subset\subset \Omega$. The space $W^{1,N}(\Omega)$ is continuously embedded in $L^q(\Omega')$ for all $N \leq q < \infty$.*

Finally, when $p > N$, we can in fact conclude continuity of Sobolev functions.

We recall that, given an open set $\Omega \subset \mathbb{R}^N$, a function $f : \Omega \to \mathbb{R}$ is Hölder continuous with exponent $\alpha > 0$ if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

for all $x, y \in \Omega$. We define the space $C^{0,\alpha}\left(\overline{\Omega}\right)$ as the space of all bounded functions that are Hölder continuous with exponent α.

The next theorem shows that if $p > N$ a function $u \in W^{1,p}\left(\mathbb{R}^N\right)$ has a representative in the space $C^{0,1-\frac{N}{p}}\left(\mathbb{R}^N\right)$.

Theorem 123 (Morrey)**.** *Let $N < p < \infty$. Then the space $W^{1,p}\left(\mathbb{R}^N\right)$ is continuously embedded in $C^{0,1-\frac{N}{p}}\left(\mathbb{R}^N\right)$. Moreover, if $f \in W^{1,p}\left(\mathbb{R}^N\right)$ and $\bar{f}$ is its representative in $C^{0,1-\frac{N}{p}}\left(\mathbb{R}^N\right)$, then*

$$\lim_{|x|\to\infty} \bar{f}(x) = 0.$$

We also have the following local corollary.

Corollary 124. *Let $\Omega \subset \mathbb{R}^N$ and $N < p < \infty$. Then for all $\Omega' \subset\subset \Omega$ the space $W^{1,p}(\Omega)$ is continuously embedded in $C^{0,1-\frac{N}{p}}(\Omega')$.*

As a consequence of the above theorem we obtain the following result.

Corollary 125. *If $f \in W^{1,p}\left(\mathbb{R}^N\right)$, $N < p < \infty$, and $\bar{f}$ is its representative in $C^{0,1-\frac{N}{p}}\left(\mathbb{R}^N\right)$, then $\bar{f}$ is differentiable at $\mathcal{L}^N$ a.e. $x \in \mathbb{R}^N$ and the weak partial derivatives of f coincide with the (classical) partial derivatives of $\bar{f}$ $\mathcal{L}^N$ a.e. in $\mathbb{R}^N$.*

As mentioned, several embeddings that are valid for the entire space $\mathbb{R}^N$ continue to hold for domains with sufficient regularity, which we will call extension domains. This includes, for example, bounded open sets with C^2 boundary. More precisely, we have the following definition.

Definition 126. *Given* $1 \leq p \leq \infty$, *an open set* $\Omega \subset \mathbb{R}^N$ *is called an* extension domain for the Sobolev space $W^{1,p}(\Omega)$ *if there exists a continuous linear operator*

$$\mathcal{E} : W^{1,p}(\Omega) \to W^{1,p}\left(\mathbb{R}^N\right)$$

with the property that $\mathcal{E}(u)(x) = u(x)$ *for all* $u \in W^{1,p}(\Omega)$ *and for* $\mathcal{L}^N$ *a.e.* $x \in \Omega$.

Note that the *extension operator* $\mathcal{E}$ strongly depends on p. However, as mentioned, smooth domains (or even *nice* Lipschitz domains) are extension domains for all $1 \leq p < \infty$.

2.3.4 Functions of Bounded Variation

Let $\Omega \subset \mathbb{R}^N$ be an open set.

Definition 127. *The space of functions of* Bounded Variation $BV(\Omega)$ *is the space of all functions* $f \in L^1(\Omega)$ *whose distributional first order partial derivatives belong to* $M_b(\Omega)$, *that is, for all* $i = 1, \ldots, N$ *there exists a finite signed Radon measure* $\lambda_i \in M_b(\Omega)$ *such that*

$$\int_\Omega f \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega \varphi \, d\lambda_i$$

for all $\varphi \in C_c^\infty(\Omega)$.

For $f \in BV(\Omega)$ we set

$$Df := (\lambda_1, \ldots, \lambda_N).$$

Again, we define $BV\left(\Omega; \mathbb{R}^d\right)$ as the space of all functions $f = (f_1, \ldots, f_d)$ such that $f_i \in BV(\Omega)$ for all $i = 1, \ldots, d$. Also,

$$BV_{\mathrm{loc}}(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\Omega) : \; f \in BV(\Omega') \text{ for all open sets } \Omega' \subset\subset \Omega \right\}.$$

Many of the results true in $W^{1,p}(\Omega)$ are true for $BV(\Omega)$, modulo exponents. For example, BV is a Banach space.

Theorem 128. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then the space $BV(\Omega)$ is a Banach space with the norm*

$$\|f\|_{BV(\Omega)} := \|f\|_{L^1(\Omega)} + |Df|(\Omega).$$

In practice, however, the norm topology is too strong a topology to work with, and more frequently we use the topology induced by the *strict convergence.*

Definition 129. *The space $BV(\Omega)$ equipped with the metric*

$$d(f,g) := ||f-g||_{L^1(\Omega)} + ||Df|(\Omega) - |Dg|(\Omega)|$$

is a complete, separable metric space.

With the topology induced by this metric, we have the following result on mollification.

Corollary 130. *Let $\Omega \subset \mathbb{R}^N$ be an open set, let $\varphi \in L^1\left(\mathbb{R}^N\right)$ be defined as in (2.33), and let $f \in BV(\Omega)$. Then for any $A \subset \Omega$ such that $Df(\partial A) = 0$ we have that $f_\delta \to f$ strictly in $BV(A)$.*

Here we have used the symbol ∂ to denote the boundary of the set A. The above result and the argument of Meyers and Serrin implies the following density result.

Theorem 131. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 \le p < \infty$. Then the space $C^\infty(\Omega) \cap BV(\Omega)$ is dense in $BV(\Omega)$, with respect to the strict convergence.*

As in the Sobolev case, we have the following embedding theorems.

Theorem 132 (BV Sobolev Embedding)**.** *There exists a constant $C = C(N) > 0$ such that for every function $f \in BV(\mathbb{R}^N)$,*

$$\left(\int_{\mathbb{R}^N} |f(x)|^{1^*}\, dx\right)^{\frac{1}{1^*}} \le C|Df|(\mathbb{R}^N) \tag{2.42}$$

In particular, $BV\left(\mathbb{R}^N\right)$ is continuously embedded in $L^q\left(\mathbb{R}^N\right)$ for all $1 \le q \le 1^*$.

Again, when the boundary of Ω is not smooth, we have a local version of the above embedding.

Corollary 133 (Local BV Sobolev Embedding)**.** *Let $\Omega' \subset\subset \Omega$. Then there exists a constant $C = C(N, \Omega') > 0$ such that for every function $f \in BV_{\text{loc}}(\Omega)$,*

$$\left(\int_{\Omega'} |f(x)|^{1^*} \, dx \right)^{\frac{1}{1^*}} \leq C|Df|(\Omega). \tag{2.43}$$

In particular, the case $N = 1$ is much easier to treat, since one dimensional BV functions are bounded.

Theorem 134. *Let $N = 1$ and $f \in BV(\Omega)$. Then $f \in L^\infty(\Omega)$.*

We also have the following characterization of BV, again pertinent in the context of our thesis.

Theorem 135. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $f \in BV(\Omega)$. Then for every $i = 1, \ldots, N$ and $h > 0$,*

$$\int_{\Omega_h} \frac{|f(x + he_i) - f(x)|}{h} \, dx \leq |\lambda_i|(\Omega) \tag{2.44}$$

and

$$\lim_{h \to 0^+} \left(\int_{\Omega_h} \frac{|f(x + he_i) - f(x)|}{h} \, dx \right) = |\lambda_i|(\Omega) \tag{2.45}$$

Conversely, if $f \in L^1(\Omega)$ is such that

$$\liminf_{h \to 0^+} \int_{\Omega_h} \frac{|f(x + he_i) - f(x)|}{h} \, dx < \infty \tag{2.46}$$

for every $i = 1, \ldots, N$, then $f \in BV(\Omega)$.

One of the primary difference between the space BV and the Sobolev spaces lies in the compactness properties. Since BV is not reflexive, we embed it in a dual space and utilize the weak-star compactness of this space. This weak-star compactness is precisely the weak-star convergence of finite signed Radon measures introduced in the section Measures as a Dual Space, as the distributional derivatives of BV functions are precisely finite signed Radon measures. Thus, we have the following compactness theorem.

Theorem 136 (Compactness in BV)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open set. Assume that $\{f_n\} \subset BV(\Omega)$ is bounded. Then there exist a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f \in BV(\Omega)$ such that $f_{n_k} \rightharpoonup f$ in $L^1(\Omega)$ and $\lambda_{i,n_k} \overset{*}{\rightharpoonup} \lambda_i$ in $(C_0(\Omega))'$ for all $i = 1, \ldots, N$. Moreover, we have that $f_{n_k} \to f$ in $L^1_{\text{loc}}(\Omega)$, and if the boundary has sufficient regularity the convergence is in $L^1(\Omega)$.*

2.4 CONVEXITY

Convexity is an important notion from many applications, and we will in particular need the notion of convexity for the purposes of lower semicontinuity of functions of measures. In our setting, the functions will be positively 1-homogenous, so that convexity can be reduced to different inequalities than are standard.

Definition 137. *We say that* $g : \mathbb{R}^m \to (-\infty, +\infty]$ *is* positively 1-homogeneous *if*

$$g(tz) = tg(z)$$

for all $t > 0$ *and all* $z \in \mathbb{R}^m$*.*

Note that the definition for positive 1-homogeneity is not uniform throughout the literature, particularly when functions can take the value $+\infty$.

Definition 138. *We say that* $g : \mathbb{R}^m \to (-\infty, +\infty]$ *is* convex *if the inequality*

$$g(\theta z + (1-\theta)w) \leq \theta g(z) + (1-\theta)g(w)$$

holds for all $\theta \in [0, 1]$ *and for all* $z, w \in \mathbb{R}^m$*.*

If g is positively 1-homogeneous, this definition is equivalent to the inequality

$$g(z + w) \leq g(z) + g(w) \tag{2.47}$$

holding for all $z, w \in \mathbb{R}^m$. Following Reshetnyak [71], we say that a positively 1-homogeneous function $g : \mathbb{R}^m \to (-\infty, +\infty]$ is *strictly convex* if the inequality (2.47) is strict, except when $w = 0$ or $z = tw$ for some $t > 0$. Note that this definition is not standard.

For any convex function, we have Jensen's Inequality, which states that an integral is increased by bringing a convex function inside it. We will have need of this inequality in the following form.

Theorem 139. *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space, assume* $\mu(X) = 1$*, and let* $g : [0, \infty] \to [0, \infty]$ *be convex. Then*

$$g\left(\int_X |f(x)|\, d\mu(x)\right) \leq \int_X (g \circ |f|)(x)\, d\mu(x), \tag{2.48}$$

for all $f \in L^1(X)$*.*

Finally, we recall an important theorem on representations for convex functions.

Theorem 140. *Let $E \subset \mathbb{R}^N$ be a Lebesgue measurable set and let $g : E \times \mathbb{R}^m \to \mathbb{R}$ be a lower semicontinuous function such that $g(x, \cdot)$ is convex and positively 1-homogenous for all $x \in E$. Then there exists a sequence of continuous functions $b_i : E \to \mathbb{R}^m$ such that*

$$g(x, z) = \sup_i b_i(x) \cdot z$$

for all $x \in E$ and $z \in \mathbb{R}^m$. Moreover, if g is non-negative then b_i may be taken to be bounded.

3.0 CHARACTERIZATION OF SOBOLEV AND BV SPACES

In this chapter, we prove Theorems 6, 9, 12, and their corollaries.

3.1 SOME PRELIMINARIES

For a set $E \subset \mathbb{R}^N$ and $r > 0$, continuing definition (1.13), we define a fattening of E and approach of E by compact subset as

$$E^r := \{x \in \mathbb{R}^N : \ \mathrm{dist}(x, E) < r\}, \tag{3.1}$$

$$E_r := \Big\{x \in E : \ |x| < \frac{1}{r}, \ \mathrm{dist}(x, \partial E) > r\Big\}, \tag{3.2}$$

so that $E_r \subset E \subset E^r$.

As stated in Chapter 1, we assume $\{\rho_\epsilon\} \subset L^1(\mathbb{R}^N)$ satisfy (1.3), (1.4), and (1.10), so that if $E \subset \mathbb{R}^N$ is bounded and measurable, then

$$\lim_{\epsilon \to 0} \int_E |x| \rho_\epsilon(x) \, dx = 0, \tag{3.3}$$

since fixing $\delta > 0$, by (1.3) we have

$$\begin{aligned} \limsup_{\epsilon \to 0} \int_E |x|\rho_\epsilon(x)\, dx &\le \limsup_{\epsilon \to 0} \left(\int_{\{|x|>\delta\}\cap E} |x|\rho_\epsilon(x)\, dx + \int_{|x|\le\delta} |x|\rho_\epsilon(x)\, dx \right) \\ &\le C \lim_{\epsilon \to 0} \int_{|x|>\delta} \rho_\epsilon(x)\, dx + \delta \end{aligned}$$

and (3.3) follows by sending $\delta \to 0$, along with the equality (1.4).

We are interested in utilizing the coercivity condition (1.10) to understand the behavior of a family $\{\rho_\epsilon\}$ which is not necessarily radial. This condition (1.10) implies the following lemma establishing some coercivity with respect to the uniform measure (see [70] for the introduction of this condition).

Lemma 141. *Let ρ_ϵ satisfy (1.3), (1.4), and (1.10), and let $\{\mu_\epsilon\} \subset M(S^{N-1})$ be the measures defined by*

$$\mu_\epsilon(F) := \int_F \int_0^\infty \rho_\epsilon(t\sigma)t^{N-1}\, dt d\mathcal{H}^{N-1}(\sigma) \tag{3.4}$$

for $F \subset S^{N-1}$ Borel. Then there exist a subsequence $\{\epsilon_j\}$, with $\epsilon_j \to 0^+$, and μ in $M(S^{N-1})$ such that $\mu_{\epsilon_j} \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$. Moreover, for every $p > 0$ there exists $\alpha > 0$ such that for every $v \in \mathbb{R}^N$, we have

$$\int_{S^{N-1}} |v \cdot \sigma|^p \, d\mu(\sigma) \geq \alpha |v|^p. \tag{3.5}$$

Proof. Using polar coordinates and (1.3), we have that

$$\mu_\epsilon(S^{N-1}) = \int_{S^{N-1}} \int_0^\infty \rho_\epsilon(t\sigma)t^{N-1}\, dt d\mathcal{H}^{N-1}(\sigma) = 1.$$

Thus, $||\mu_\epsilon||_{M(S^{N-1})} = 1$ and so up to a subsequence, $\mu_{\epsilon_j} \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$ with $||\mu||_{M(S^{N-1})} = 1$ (since $1 \in C(S^{N-1})$). Let $\{v_i\}_{i=1}^N$ be the linearly independent set of vectors given in (1.10). We claim there exists an $\epsilon_0 > 0$ with the property that for all $v \in \mathbb{R}^N$ there exists an i such that

$$|v \cdot \sigma| \geq \epsilon_0 |v| \tag{3.6}$$

for all $\sigma \in C_\delta(v_i) \cap S^{N-1}$. By rescaling we restrict ourselves to the case $v \in S^{N-1}$, and we proceed by contradiction. If not, then there exist a sequence $\{\epsilon_n\}$ tending to zero, $w_n \in S^{N-1}$, and $\sigma_{i,n} \in C_\delta(v_i)$, $i = 1, \ldots, N$, so that up to a subsequence, which we will not relabel, $w_n \to w_0 \in S^{N-1}$ and $\sigma_{i,n} \to \sigma_{i,0} \in C_\delta(v_i)$, with

$$|w_0 \cdot \sigma_{i,0}| = 0$$

for all $i = 1, \ldots, N$. However, since the $\{\sigma_{i,0}\}_{i=1}^N$ form a linearly independent set (see Remark 5), we have a contradiction. Thus, (3.6) holds. Define

$$c := \min_i \liminf_{j\to\infty} \int_{C_\delta(v_i)} \rho_{\epsilon_j}(x)\, dx.$$

By (1.10), we have that $c > 0$. Given $v \in \mathbb{R}^N$, let i be such that (3.6) holds; then by (3.4) and Tonelli's theorem we compute

$$\begin{aligned}\int_{S^{N-1}} |v\cdot\sigma|^p \, d\mu_{\epsilon_j}(\sigma) &\geq \int_{C_\delta(v_i)\cap S^{N-1}} |v\cdot\sigma|^p \, d\mu_{\epsilon_j}(\sigma) \\ &\geq (\epsilon_0|v|)^p \int_{C_\delta(v_i)\cap S^{N-1}} \int_0^\infty \rho_{\epsilon_j}(t\sigma)t^{N-1} \, dt d\mathcal{H}^{N-1}(\sigma) \\ &= (\epsilon_0|v|)^p \int_{C_\delta(v_i)} \rho_{\epsilon_j}(x) \, dx.\end{aligned}$$

Letting $j \to \infty$, using the fact that $\mu_{\epsilon_j} \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$, and the definition of c, we have

$$\int_{S^{N-1}} |v\cdot\sigma|^p \, d\mu(\sigma) \geq c\epsilon_0^p |v|^p.$$

Define $\alpha := c\epsilon_0^p$, and the result is demonstrated. □

Remark 142. *If ρ_ϵ satisfy (1.5), then $\mu_\epsilon = \mu = \mathcal{H}^{N-1}$ and there is no need to pass to a subsequence, since we may rewrite equation (3.4) as*

$$\begin{aligned}\mu_\epsilon(F) &= \int_F \int_0^\infty \rho_\epsilon(t\sigma)t^{N-1} \, dt d\mathcal{H}^{N-1}(\sigma) \\ &= \int_F \int_0^\infty \hat{\rho}_\epsilon(t|\sigma|)t^{N-1} \, dt d\mathcal{H}^{N-1}(\sigma) \\ &= \int_F d\mathcal{H}^{N-1}(\sigma),\end{aligned}$$

where we have used (1.3).

Since boundary regularity is not assumed, we must avoid calculations which might be near the boundary. Thus, the following measure truncation lemma is an essential tool in our proof of Theorems 6 and 9. We demonstrate that restricting the support (truncation in the domain) of ρ_ϵ gives the same measure in the weak-star limit. More precisely, consider, for every fixed $\eta > 0$, ρ_ϵ^η defined by

$$\rho_\epsilon^\eta := \rho_\epsilon \chi_{B(0,\eta)}. \tag{3.7}$$

This gives rise to a measure μ_ϵ^η defined by

$$\mu_\epsilon^\eta(F) := \int_F \int_0^\infty \rho_\epsilon^\eta(t\sigma)t^{N-1} \, dt d\mathcal{H}^{N-1}(\sigma) \tag{3.8}$$

for $F \subset S^{N-1}$ Borel, so that again applying the Radon–Nikodym theorem, for $\mathcal{H}^{N-1}$ a.e. $\sigma \in S^{N-1}$,

$$\frac{d\mu_\epsilon^\eta}{d\mathcal{H}^{N-1}}(\sigma) = \int_0^\infty \rho_\epsilon^\eta(t\sigma)t^{N-1}\,dt = \int_0^\eta \rho_\epsilon(t\sigma)t^{N-1}\,dt.$$

Lemma 143. *Let ρ_ϵ satisfy (1.3) and (1.4), and let $\{\mu_\epsilon\} \subset M(S^{N-1})$ be the corresponding measures defined in (3.4). Let $\epsilon_j \to 0^+$ and assume that $\mu_{\epsilon_j} \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$. Then for every $\eta > 0$, $\mu_{\epsilon_j}^\eta \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$, where $\mu_{\epsilon_j}^\eta$ are the measures defined in (3.8).*

Proof. We begin by proving that $\mu_{\epsilon_j} - \mu_{\epsilon_j}^\eta \to 0$ in $M(S^{N-1})$. For $f \in C(S^{N-1})$, with $\max_{S^{N-1}}|f| = 1$, using spherical coordinates we have

$$\begin{aligned}\left|\int_{S^{N-1}} f\,d\mu_{\epsilon_j}^\eta - \int_{S^{N-1}} f\,d\mu_{\epsilon_j}\right| &= \left|\int_{S^{N-1}}\int_\eta^\infty f(\sigma)\rho_{\epsilon_j}(t\sigma)t^{N-1}\,dtd\mathcal{H}^{N-1}(\sigma)\right| \\ &\le \max_{S^{N-1}}|f|\int_{|x|>\eta}\rho_{\epsilon_j}(x)\,dx = \int_{|x|>\eta}\rho_{\epsilon_j}(x)\,dx.\end{aligned}$$

Taking the supremum over all such f, we get

$$||\mu_{\epsilon_j}^\eta - \mu_{\epsilon_j}||_{M(S^{N-1})} \le \int_{|x|>\eta}\rho_{\epsilon_j}(x)\,dx \to 0$$

as $j \to \infty$ by (1.4). Thus, $\mu_{\epsilon_j} - \mu_{\epsilon_j}^\eta \to 0$ in $M(S^{N-1})$. Since $\mu_{\epsilon_j} \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$, it follows that $\mu_{\epsilon_j}^\eta \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$. □

Note also that by the definition of ρ_ϵ^η, the fact that $\rho_\epsilon^\eta \le \rho_\epsilon$, (1.3), (1.4), and (3.3), we have that the following properties of ρ_ϵ^η hold

$$\rho_\epsilon^\eta \ge 0, \quad \int_{\mathbb{R}^N}\rho_\epsilon^\eta(x)\,dx \le 1, \tag{3.9}$$

$$\lim_{\epsilon\to 0^+}\int_{|x|>\delta}\rho_\epsilon^\eta(x)\,dx = 0 \quad \text{for all } \delta > 0, \tag{3.10}$$

$$\lim_{\epsilon\to 0^+}\int_E |x|\rho_\epsilon^\eta(x)\,dx = 0 \tag{3.11}$$

for every $E \subset \mathbb{R}^N$ bounded and measurable.

3.2 A FEW USEFUL ESTIMATES

In this section we prove some lemmata, which will be used in the sequel. Proofs of variants of these results can be found in [12], [15], and [70]. We adapt these proofs to our setting allowing for truncated mollifiers and for an additional q in the integrand, and present the proofs for the convenience of the reader. We use the notation (3.1) and (3.2).

Lemma 144. *Let $A \subset \mathbb{R}^N$ be open and bounded and let $f \in C^2(\overline{A^\eta})$ for some $\eta > 0$. Then*

$$|f(x) - f(y) - \nabla f(x) \cdot (x-y)| \leq C^f |x-y|^2$$

for all $x \in A$ and $y \in A^\eta$, where C^f depends upon $||f||_{C^2(\overline{A^\eta})}$.

Proof. Fix $x \in A$ and $y \in A^\eta$. If $|x-y| < \eta$, then the segment of endpoints x and y is contained in $\overline{A^\eta}$, and so we may apply Taylor's formula to obtain

$$|f(x) - f(y) - \nabla f(x) \cdot (x-y)| \leq C(N) ||\nabla^2 f||_{L^\infty(\overline{A^\eta})} |x-y|^2.$$

On the other hand, if $|x-y| > \eta$, we may estimate

$$|f(x) - f(y) - \nabla f(x) \cdot (x-y)| \leq \left(\frac{2}{\eta^2} ||f||_{L^\infty(\overline{A^\eta})} + \frac{1}{\eta} ||\nabla f||_{L^\infty(\overline{A^\eta})} \right) |x-y|^2,$$

and defining

$$C^f := C(N) ||\nabla^2 f||_{L^\infty(\overline{A^\eta})} + \frac{2}{\eta^2} ||f||_{L^\infty(\overline{A^\eta})} + \frac{1}{\eta} ||\nabla f||_{L^\infty(\overline{A^\eta})},$$

the result is demonstrated. □

Lemma 145. *Let Ω and ρ_ϵ be as in Theorem 6, let $A \subset \Omega$ be open and bounded with $\operatorname{dist}(A, \partial\Omega) > 0$, let $r \geq 1$ and let $f \in C^2(\overline{A^\eta})$, where $0 < \eta < \operatorname{dist}(A, \partial\Omega)$. Let μ_{ϵ_j} be the subsequence obtained from Lemma 143. Then for every $x \in A$, we have*

$$\lim_{j \to \infty} \int_{A^\eta} \left(\frac{|f(x) - f(y)|}{|x-y|} \right)^r \rho^\eta_{\epsilon_j}(x-y)\, dy = \int_{S^{N-1}} |\nabla f(x) \cdot \sigma|^r \, d\mu(\sigma),$$

where $\rho^\eta_{\epsilon_j}$ is the family of truncated mollifiers introduced in (3.7).

Proof. First, we demonstrate that in the limit, the difference quotient averages over A^η behave like the derivative averages over A^η. We then use Tonelli's theorem and the weak-star convergence of the measures to prove the result.

Step 1: We prove that for $x \in A$,

$$\limsup_{j\to\infty} \int_{A^\eta} \left(\frac{|f(x)-f(y)|}{|x-y|}\right)^r \rho^\eta_{\epsilon_j}(x-y)\,dy$$
$$= \limsup_{j\to\infty} \int_{A^\eta} \left|\nabla f(x)\cdot \frac{x-y}{|x-y|}\right|^r \rho^\eta_{\epsilon_j}(x-y)\,dy.$$

Set $M_f := ||\nabla f||_{L^\infty(\overline{A^\eta})}$. By the mean value theorem, for all $s,t \in [0, M_f]$,

$$|s^r - t^r| \leq r M_f^{r-1}|s-t|.$$

Thus, for $x \in A$ and $y \in A^\eta$ we can estimate the difference

$$\left|\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^r - \left|\nabla f(x)\cdot\frac{x-y}{|x-y|}\right|^r\right|$$
$$\leq r M_f^{r-1}\frac{|f(x)-f(y)-\nabla f(x)\cdot(x-y)|}{|x-y|} \leq r M_f^{r-1} C^f |x-y|,$$

where C^f is the constant given in Lemma 144. Therefore,

$$\int_{A^\eta}\left|\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^r - \left|\nabla f(x)\cdot\frac{x-y}{|x-y|}\right|^r\right| \rho^\eta_{\epsilon_j}(x-y)\,dy$$
$$\leq C^f_r \int_{A^\eta} |x-y|\rho^\eta_{\epsilon_j}(x-y)\,dy,$$

where $C^f_r := r M_f^{r-1} C^f$. Making the change of variables $h = x-y$ and using monotonicity of the integral, we obtain that the right-hand side of the previous inequality is less than or equal to

$$C^f_r \int_{|h|<\eta} |h|\rho^\eta_{\epsilon_j}(h)\,dh.$$

Using (3.11) and sending $j \to \infty$, we conclude

$$\limsup_{j\to\infty} \int_{A^\eta} \left(\frac{|f(x)-f(y)|}{|x-y|}\right)^r \rho^\eta_{\epsilon_j}(x-y)\,dy$$
$$= \limsup_{j\to\infty} \int_{A^\eta} \left|\nabla f(x)\cdot \frac{x-y}{|x-y|}\right|^r \rho^\eta_{\epsilon_j}(x-y)\,dy.$$

Step 2: We will show that for each $x \in A$,

$$\lim_{j\to\infty} \int_{A^\eta} \left|\nabla f(x) \cdot \frac{x-y}{|x-y|}\right|^r \rho^\eta_{\epsilon_j}(x-y)\, dy = \int_{S^{N-1}} |\nabla f(x)\cdot\sigma|^r \, d\mu(\sigma). \tag{3.12}$$

Since $\rho^\eta = 0$ if $|x-y| > \eta$, we may use polar coordinates to write

$$\begin{aligned}\int_{A^\eta} \left|\nabla f(x) \cdot \frac{x-y}{|x-y|}\right|^r \rho^\eta_{\epsilon_j}(x-y)\, dy &= \int_{B(x,\eta)} \left|\nabla f(x) \cdot \frac{x-y}{|x-y|}\right|^r \rho^\eta_{\epsilon_j}(x-y)\, dy \\ &= \int_{S^{N-1}} |\nabla f(x)\cdot\sigma|^r \int_0^\eta \rho^\eta_{\epsilon_j}(t\sigma) t^{N-1}\, dt d\mathcal{H}^{N-1}(\sigma),\end{aligned}$$

and using the definition of $\mu^\eta_{\epsilon_j}$ (see (3.8)), we have

$$\int_{S^{N-1}} |\nabla f(x)\cdot\sigma|^r \int_0^\eta \rho^\eta_{\epsilon_j}(t\sigma) t^{N-1}\, dt d\mathcal{H}^{N-1}(\sigma) = \int_{S^{N-1}} |\nabla f(x)\cdot\sigma|^r \, d\mu^\eta_{\epsilon_j}(\sigma).$$

Now, since the function $\sigma \mapsto |\nabla f(x)\cdot\sigma|^r$ is continuous, we may let $j\to\infty$ and use Lemma 143 to obtain (3.12). □

Lemma 146. *Let Ω and ρ_ϵ be as in Theorem 6, let $A \subset \Omega$ be open and bounded, with $\gamma := \operatorname{dist}(A, \partial\Omega) > 0$, let $1 \le p, q < \infty$, and let $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Then for all $0 < \eta < \frac{\gamma}{3}$ we have*

$$\begin{aligned}&\int_{A^\eta} \left(\int_{A^\eta} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q \rho^\eta_\epsilon(x-y)\, dy\right)^{\frac{1}{q}} dx \\ &\qquad \le \int_{A^{2\eta}} \left(\int_{B(0,\eta)} \left|\nabla f(y)\cdot\frac{h}{|h|}\right|^{pq} \rho^\eta_\epsilon(h)\, dh\right)^{\frac{1}{q}} dy.\end{aligned}$$

Proof. Making the change of variables $y = x + h$, and using the fact that $\rho^\eta_\epsilon = 0$ outside $B(0,\eta)$, we have

$$\begin{aligned}&\int_{A^\eta} \left(\int_{A^\eta} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q \rho^\eta_\epsilon(x-y)\, dy\right)^{\frac{1}{q}} dx \\ &\quad = \int_{A^\eta} \left(\int_{B(0,\eta)} \left(\frac{|f(x+h)-f(x)|^p}{|h|^p}\right)^q \rho^\eta_\epsilon(h)\, dh\right)^{\frac{1}{q}} dx.\end{aligned}$$

For $0 < \delta < \eta < \frac{\gamma}{3}$, we have that f_δ (see (2.33)) is well defined in A^η, and so we may apply the fundamental theorem of calculus to f_δ to write

$$\begin{aligned}
&\int_{A^\eta} \left(\int_{B(0,\eta)} \left(\frac{|f_\delta(x+h) - f_\delta(x)|^p}{|h|^p} \right)^q \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}} dy \\
&\quad = \int_{A^\eta} \left(\int_{B(0,\eta)} \left(\int_0^1 \left| \nabla f_\delta(x+th) \cdot \frac{h}{|h|} \right| dt \right)^{pq} \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}} dx =: I.
\end{aligned}$$

Then Theorem 139 (Jensen's inequality) and Theorem 76 (Minkowski's inequality for integrals) imply

$$I \leq \int_{A^\eta} \int_0^1 \left(\int_{B(0,\eta)} \left(\left| \nabla f_\delta(x+th) \cdot \frac{h}{|h|} \right| \right)^{pq} \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}} \, dt dx,$$

while Tonelli's theorem and the change of variables $y = x + th$ yield

$$I \leq \int_{A^{2\eta}} \left(\int_{B(0,\eta)} \left| \nabla f_\delta(y) \cdot \frac{h}{|h|} \right|^{pq} \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}} \, dy, \tag{3.13}$$

where we have used the fact that $|h| < \eta$ and that the integrand is non-negative. We thus conclude that

$$\begin{aligned}
&\int_{A^\eta} \left(\int_{A^\eta} \left(\frac{|f_\delta(x) - f_\delta(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx \\
&\quad \leq \int_{A^{2\eta}} \left(\int_{B(0,\eta)} \left| \nabla f_\delta(y) \cdot \frac{h}{|h|} \right|^{pq} \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}} \, dy.
\end{aligned} \tag{3.14}$$

Now, define

$$g_\delta(y) := \left(\int_{B(0,\eta)} \left| \nabla f_\delta(y) \cdot \frac{h}{|h|} \right|^{pq} \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}}.$$

Then we have

$$\begin{aligned}
g_\delta(y) &\leq \left(\int_{B(0,\eta)} |\nabla f_\delta(y)|^{pq} \, \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}} \\
&= |\nabla f_\delta(y)|^p \left(\int_{B(0,\eta)} \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}} \\
&\leq |\nabla f_\delta(y)|^p \,.
\end{aligned}$$

Therefore, by Corollary 113, we are justified in letting $\delta \to 0$ in and applying Fatou's lemma and Lebesgue dominated convergence theorem to equation (3.14) obtain the result. $\square$

Remark 147. *The hypothesis* $0 < \eta < \frac{\gamma}{3}$ *is a technical assumption to ensure* f_δ *is well defined in the region being considered. In the case in which* Ω *is an extension domain for* $W^{1,p}$ *(and hence we can extend* f *to all of* $\mathbb{R}^N$*) we have no need for this assumption. We also note that it is here that we have implicitly used the truncation of* ρ_ϵ, ρ_ϵ^η, *to ensure that the change of variables does not leave the domain of definition of the function* f. *This can also be bypassed in the case in which* Ω *is an extension domain.*

Next we extend the previous lemma to the BV case. We remark that the calculations in the next proof are identical to those of the previous one until the final limiting step, where Df is only a measure and not a function, and so we use the Reshetnyak continuity theorem instead of Lebesgue dominated convergence theorem to pass to the limit.

Lemma 148. *Let* Ω *and* ρ_ϵ *be as in Theorem 6 with* $p = 1$, *let* $A \subset \Omega$ *be open and bounded, with* $\gamma := \operatorname{dist}(A, \partial\Omega) > 0$, *let* $1 \leq q < \infty$, *and let* $f \in BV_{\mathrm{loc}}(\Omega)$. *Then for all* $0 < \eta < \frac{\gamma}{3}$ *such that* $|Df|(\partial A^{2\eta}) = 0$, *we have*

$$\int_{A^\eta} \left(\int_{A^\eta} \left(\frac{|f(x) - f(y)|}{|x-y|} \right)^q \rho_\epsilon^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx$$
$$\leq \int_{A^{2\eta}} \left(\int_{B(0,\eta)} \left| \frac{dDf}{d|Df|}(x) \cdot \frac{h}{|h|} \right|^q \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}} d|Df|(x),$$

where $\frac{dDf}{d|Df|}$ *is the Radon-Nikodym derivative of* Df *with respect to* $|Df|$.

Proof. We proceed as in the previous proof with $p = 1$ up to (3.13). Thus, we have

$$I \leq \int_{A^{2\eta}} \left(\int_{B(0,\eta)} \left| \nabla f_\delta(x) \cdot \frac{h}{|h|} \right|^q \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}} dx. \tag{3.15}$$

Consider the Radon measures $\nu_\delta \in M_b(A^\eta; \mathbb{R}^N)$ defined by

$$\nu_\delta\,(F) := \int_F \nabla f_\delta(x)\, dx$$

for $F \subset A^\eta$ Borel, and let $\Psi_{\epsilon,\eta} : \mathbb{R}^N \to [0, \infty)$ defined by

$$\Psi_{\epsilon,\eta}\,(v) := \left(\int_{B(0,\eta)} \left| v \cdot \frac{h}{|h|} \right|^q \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}}, \quad v \in \mathbb{R}^N. \tag{3.16}$$

Then by 1-homogeneity inequality (3.15) can be rewritten as

$$I \leq \int_{A^{2\eta}} \Psi_{\epsilon,\eta}\left(\frac{d\nu_\delta}{d\,|\nu_\delta|}(x)\right)\, d|\nu_\delta|(x).$$

Now, since $\nu_\delta \overset{*}{\rightharpoonup} Df$ in $M_b(\Omega_r^{2\eta};\mathbb{R}^N)$, $|\nu_\delta|(A^{2\eta}) \to |Df|(A^{2\eta})$ (as a result of the assumption $|Df|(\partial A^{2\eta}) = 0$, by applying Corollary 130), it follows by Reshetnyak's continuity theorem (see Theorem 16) that

$$\int_{A^{2\eta}} \Psi_{\epsilon,\eta}\left(\frac{d\nu_\delta}{d\,|\nu_\delta|}(x)\right)\, d|\nu_\delta|(x) \to \int_{A^{2\eta}} \Psi_{\epsilon,\eta}\left(\frac{dDf}{d\,|Df|}(x)\right)\, d|Df|(x).$$

Combining this convergence with Fatou's lemma as in the proof of Lemma 146, we have that

$$\begin{aligned}
&\int_{A^\eta}\left(\int_{A^\eta}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^q \rho_\epsilon^\eta(x-y)\,dy\right)^{\frac{1}{q}} dx\\
&\qquad \leq \int_{A^{2\eta}}\left(\int_{B(0,\eta)}\left|\frac{dDf}{d|Df|}(x)\cdot\frac{h}{|h|}\right|^q \rho_\epsilon^\eta(h)\,dh\right)^{\frac{1}{q}} d|Df|(x),
\end{aligned}$$

which concludes the proof. □

The next two lemmata are due to an observation of Stein (see [15] or [70]), adapted to our setting.

Lemma 149. *Let Ω and ρ_ϵ be as in Theorem 6, let $A \subset \Omega$ be open and bounded, with $\gamma := \operatorname{dist}(A, \partial\Omega) > 0$, $1 \leq p, q < \infty$, and $f \in L^1_{\mathrm{loc}}(\Omega)$. Then for all $0 < \delta < \eta < \frac{\gamma}{3}$ we have*

$$\begin{aligned}
&\int_A\left(\int_A\left(\frac{|f_\delta(x)-f_\delta(y)|^p}{|x-y|^p}\right)^q \rho_\epsilon^\eta(x-y)\,dy\right)^{\frac{1}{q}} dx\\
&\qquad \leq \int_{A^\eta}\left(\int_{A^\eta}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q \rho_\epsilon^\eta(x-y)\,dy\right)^{\frac{1}{q}} dx,
\end{aligned}$$

where f_δ is the mollification of f (see (2.33)).

Proof. We begin by writing

$$\begin{aligned}
&\int_A \left(\int_A \left(\frac{|f_\delta(x) - f_\delta(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx \\
&= \int_A \left(\int_A \left(\frac{|\int_{B(0,\delta)} [f(x-z) - f(y-z)] \psi_\delta(z)\, dz|}{|x-y|} \right)^{pq} \rho_\epsilon^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx \\
&\le \int_A \left(\int_A \left(\int_{B(0,\delta)} \frac{|f(x-z) - f(y-z)|}{|x-y|} \psi_\delta(z)\, dz \right)^{pq} \rho_\epsilon^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx =: I,
\end{aligned}$$

Applying Theorem 139 (Jensen's inequality) and Theorem 76 (Minkowski's inequality for integrals) as in Lemma 146, followed by Tonelli's theorem, we have

$$I \le \int_{B(0,\delta)} \int_A \left(\int_A \left(\frac{|f(x-z) - f(y-z)|}{|x-y|} \right)^{pq} \rho_\epsilon^\eta(x-y)\, dy \right)^{\frac{1}{q}} \psi_\delta(z)\, dx dz.$$

Then making the change of variables $w = x + z$, $v = y + z$, for $z \in B(0, \delta)$, along with non-negativity of the integrand, we have

$$I \le \int_{B(0,\delta)} \int_{A^\eta} \left(\int_{A^\eta} \left(\frac{|f(w) - f(v)|^p}{|w-v|^p} \right)^q \rho_\epsilon^\eta(w-v)\, dv \right)^{\frac{1}{q}} \psi_\delta(z)\, dw dz.$$

Finally, integrating in z and using $\int_{B(0,\delta)} \psi_\delta(z)\, dz = 1$, we obtain the result. $\square$

3.3 NEW CHARACTERIZATIONS

In this section, we prove several results of independent interest that lead up to a characterization of $W^{1,p}(\Omega)$ and $BV(\Omega)$ for Ω an arbitrary open set. We begin by proving the sufficiency of conditions (1.11) and (1.14) in Theorems 6 and 9.

Theorem 150. *Let Ω and ρ_ϵ be as in Theorem 6, let $1 < p < \infty$, $1 \le q < \infty$, and let $f \in L^1_{\mathrm{loc}}(\Omega)$. Assume*

$$\lim_{r\to 0} \limsup_{\epsilon\to 0} \int_{\Omega_r} \left(\int_{\Omega_r} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx < +\infty.$$

Then $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and $\nabla f \in L^p(\Omega;\mathbb{R}^N)$. Moreover, there exist $\epsilon_j \to 0^+$ and a probability measure $\mu \in M(S^{N-1})$ (independent of f) such that for all $0 < \eta < \frac{r}{3}$,

$$\begin{aligned} &\lim_{r\to 0} \liminf_{j\to\infty} \int_{\Omega_r^\eta} \left(\int_{\Omega_r^{2\eta}} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_{\epsilon_j}(x-y)\, dy \right)^{\frac{1}{q}} dx \\ &\ge \int_{\Omega} \left(\int_{S^{N-1}} \left(|\nabla f(x)\cdot\sigma|^p\right)^q \, d\mu(\sigma) \right)^{\frac{1}{q}} dx. \end{aligned}$$

Proof. Define

$$C := \lim_{r\to 0} \limsup_{\epsilon\to 0} \int_{\Omega_r} \left(\int_{\Omega_r} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx < \infty.$$

By the monotonicity of the integrals over Ω_r we have that for any $\eta < \frac{r}{3}$,

$$\limsup_{\epsilon\to 0} \int_{\Omega_r^\eta} \left(\int_{\Omega_r^{2\eta}} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx \le C,$$

where $\Omega_r^\eta := (\Omega_r)^\eta$. But since $\rho_\epsilon^\eta \le \rho_\epsilon$, we have that

$$\limsup_{\epsilon\to 0} \int_{\Omega_r^\eta} \left(\int_{\Omega_r^{2\eta}} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx \le C. \tag{3.17}$$

Fix $0 < \eta < \frac{r}{3}$, and for any $0 < \delta < \eta$ apply Lemma 149 to obtain

$$\begin{aligned} &\int_{\Omega_r} \left(\int_{\Omega_r^\eta} \left(\frac{|f_\delta(x)-f_\delta(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx \\ &\quad\le \int_{\Omega_r^\eta} \left(\int_{\Omega_r^{2\eta}} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx. \end{aligned}$$

Let μ_ϵ be the measures defined in (3.4). By Lemma 141 there exist a subsequence $\{\epsilon_j\}$, with $\epsilon_j \to 0^+$, and a probability measure $\mu \in M(S^{N-1})$ such that $\mu_{\epsilon_j} \overset{*}{\rightharpoonup} \mu$ in $M(S^{N-1})$. Since $f_\delta \in C^2\left(\overline{\Omega_r^{2\eta}}\right)$ with $\Omega_r^{2\eta}$ open and bounded, by Lemma 145 for every $x \in \Omega_r$,

$$\lim_{j\to\infty} \int_{\Omega_r^\eta} \left(\frac{|f_\delta(x)-f_\delta(y)|}{|x-y|} \right)^{pq} \rho_{\epsilon_j}^\eta(x-y)\, dy = \int_{S^{N-1}} |\nabla f_\delta(x)\cdot\sigma|^{pq}\, d\mu(\sigma).$$

Thus, applying Fatou's lemma and the fact that $t^{\frac{1}{q}}$ is continuous, we have that

$$\begin{aligned}
&\int_{\Omega_r} \left(\int_{S^{N-1}} |\nabla f_\delta(x) \cdot \sigma|^{pq} \; d\mu(\sigma) \right)^{\frac{1}{q}} dx \qquad (3.18) \\
&\quad \leq \liminf_{j \to \infty} \int_{\Omega_r} \left(\int_{\Omega_r^\eta} \left(\frac{|f_\delta(x) - f_\delta(y)|^p}{|x-y|^p} \right)^q \rho_{\epsilon_j}^\eta (x-y) \; dy \right)^{\frac{1}{q}} dx \\
&\quad \leq \liminf_{j \to \infty} \int_{\Omega_r^\eta} \left(\int_{\Omega_r^{2\eta}} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p} \right)^q \rho_{\epsilon_j}^\eta (x-y) \; dy \right)^{\frac{1}{q}} dx \leq C
\end{aligned}$$

so that

$$\int_{\Omega_r} \left(\int_{S^{N-1}} |\nabla f_\delta(x) \cdot \sigma|^{pq} \; d\mu(\sigma) \right)^{\frac{1}{q}} dx \leq C.$$

However, Lemma 141 implies

$$\int_{\Omega_r} |\nabla f_\delta(x)|^p \; dx \leq \frac{C}{\alpha} \tag{3.19}$$

for some constant $\alpha > 0$ (independent of r). Then Theorems 111, 112 and Corollary 113 combined with these bounds on ∇f_δ imply $f \in W^{1,p}_{\text{loc}}(\Omega)$ and $\nabla f \in L^p(\Omega_r; \mathbb{R}^N)$. Finally, letting $r \to 0$ we obtain $\nabla f \in L^p(\Omega; \mathbb{R}^N)$.

To prove the last part of the statement, let $\delta \to 0$ in (3.18) (utilizing $\rho_{\epsilon_j}^\eta \leq \rho_{\epsilon_j}$) and use Fatou's lemma to obtain

$$\begin{aligned}
&\int_{\Omega_r} \left(\int_{S^{N-1}} |\nabla f(x) \cdot \sigma|^{pq} \; d\mu(\sigma) \right)^{\frac{1}{q}} dx \\
&\quad \leq \liminf_{j \to \infty} \int_{\Omega_r^\eta} \left(\int_{\Omega_r^{2\eta}} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p} \right)^q \rho_{\epsilon_j} (x-y) \; dy \right)^{\frac{1}{q}} dx. \qquad (3.20)
\end{aligned}$$

It now suffices to let $r \to 0$ and use Lebesgue monotone convergence theorem. $\square$

The analogous result for $p = 1$ is the following theorem.

Theorem 151. *Let Ω and ρ_ϵ be as in Theorem 6, let $1 \leq q < \infty$, and let $f \in L^1_{\text{loc}}(\Omega)$. Assume*

$$\lim_{r \to 0} \limsup_{\epsilon \to 0} \int_{\Omega_r} \left(\int_{\Omega_r} \left(\frac{|f(x) - f(y)|}{|x-y|} \right)^q \rho_\epsilon (x-y) \; dy \right)^{\frac{1}{q}} dx < +\infty.$$

Then $f \in BV_{\mathrm{loc}}(\Omega)$ and $Df \in M_b(\Omega;\mathbb{R}^N)$. Moreover, there exist $\epsilon_j \to 0^+$ and a probability measure $\mu \in M(S^{N-1})$ (independent of f) such that for all $0 < \eta < \frac{r}{3}$,

$$\begin{aligned}
&\lim_{r\to 0}\lim_{j\to\infty}\int_{\Omega_r^\eta}\left(\int_{\Omega_r^{2\eta}}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^q \rho_{\epsilon_j}(x-y)\,dy\right)^{\frac{1}{q}} dx \\
&\geq \int_\Omega \left(\int_{S^{N-1}}\left|\frac{dDf}{d\,|Df|}(x)\cdot\sigma\right|^q\,d\mu(\sigma)\right)^{\frac{1}{q}} d\,|Df|\,(x)\,,
\end{aligned}$$

where $\frac{dDf}{d|Df|}$ is the Radon–Nikodym of Df with respect to $|Df|$.

Proof. We proceed as in the previous theorem up to (3.19), which now becomes

$$\int_{\Omega_r} |\nabla f_\delta(x)|\,dx \leq \frac{C}{\alpha},$$

and as before, Theorem 136 and Corollary 130 combined with the bounds on ∇f_δ imply $f \in BV_{\mathrm{loc}}(\Omega)$ and $Df \in M_b(\Omega_r;\mathbb{R}^N)$. Finally, letting $r \to 0$ we obtain $f \in BV_{\mathrm{loc}}(\Omega)$ with $Df \in M_b(\Omega;\mathbb{R}^N)$.

To prove the last part of the statement, observe that the function $\Psi : \mathbb{R}^N \to [0,\infty)$, defined by

$$\Psi(v) := \left(\int_{S^{N-1}} |v\cdot\sigma|^q\,d\mu(\sigma)\right)^{\frac{1}{q}}, \quad v \in \mathbb{R}^N, \tag{3.21}$$

is convex and positively homogeneous of degree one, and again consider the Radon measures $\nu_\delta \in M_b(\Omega_r;\mathbb{R}^N)$ defined by

$$\nu_\delta(F) := \int_F \nabla f_\delta(x)\,dx$$

for $F \subset \Omega_r$ Borel. We rewrite (3.18) (again utilizing $\rho_{\epsilon_j}^\eta \leq \rho_{\epsilon_j}$) as

$$\begin{aligned}
&\int_{\Omega_r}\Psi\left(\frac{d\nu_\delta}{d\,|\nu_\delta|}(x)\right)\,d\,|\nu_\delta|\,(x) \\
&\qquad \leq \liminf_{j\to\infty}\int_{\Omega_r^\eta}\left(\int_{\Omega_r^{2\eta}}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q \rho_{\epsilon_j}(x-y)\,dy\right)^{\frac{1}{q}} dx.
\end{aligned}$$

Since $\nu_\delta \overset{*}{\rightharpoonup} Df$ in $M_b(\Omega_r;\mathbb{R}^N)$, it follows by Reshetnyak's lower semicontinuity theorem (see Theorem 17) that

$$\begin{aligned}
&\int_{\Omega_r}\Psi\left(\frac{dDf}{d\,|Df|}(x)\right)\,d\,|Df|\,(x) \\
&\qquad \leq \liminf_{j\to\infty}\int_{\Omega_r^\eta}\left(\int_{\Omega_r^{2\eta}}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q \rho_{\epsilon_j}(x-y)\,dy\right)^{\frac{1}{q}} dx.
\end{aligned} \tag{3.22}$$

It now suffices to let $r \to 0$ and use Lebesgue monotone convergence theorem. $\square$

Using Theorems 150 and 151 we can now prove Corollaries 8 and 11.

Proof of Corollary 8. Let $f \in L^p(\Omega)$ satisfy (1.8). Then for every $\eta < \frac{r}{3}$ and every $\Omega_r \subset \Omega$ we have

$$\limsup_{\epsilon \to 0} \int_{\Omega_r^\eta} \int_{\Omega_r^{2\eta}} \frac{|f(x) - f(y)|^p}{d_\Omega(x,y)^p} \hat{\rho}_\epsilon^\eta(d_\Omega(x,y))\, dydx < +\infty.$$

Now, since $\hat{\rho}_\epsilon^\eta = 0$ if $d_\Omega(x,y) > \eta$, for each x we can restrict ourselves to integration over y such that $d_\Omega(x,y) \le \eta$. Then since $|x-y| \le d_\Omega(x,y) \le \eta$, this implies that

$$\limsup_{\epsilon \to 0} \int_{\Omega_r^\eta} \int_{\Omega_r^{2\eta}} \frac{|f(x) - f(y)|^p}{d_\Omega(x,y)^p} \hat{\rho}_\epsilon^\eta(d_\Omega(x,y))\, dydx < +\infty$$

However, since $\eta < \frac{r}{3}$, for $x \in \Omega_r^\eta$ and $y \in \Omega_r^{2\eta}$, we have that the segment containing x and y is contained in Ω, so that $d_\Omega(x,y) = |x-y|$, and thus

$$\begin{aligned} &\int_{\Omega_r^\eta} \int_{\Omega_r^{2\eta}} \frac{|f(x) - f(y)|^p}{d_\Omega(x,y)^p} \hat{\rho}_\epsilon^\eta(d_\Omega(x,y))\, dydx \\ &\qquad = \int_{\Omega_r^\eta} \int_{\Omega_r^{2\eta}} \frac{|f(x) - f(y)|^p}{|x-y|^p} \hat{\rho}_\epsilon^\eta(|x-y|)\, dydx, \end{aligned}$$

so that

$$\limsup_{\epsilon \to 0} \int_{\Omega_r^\eta} \int_{\Omega_r^{2\eta}} \frac{|f(x) - f(y)|^p}{|x-y|^p} \hat{\rho}_\epsilon^\eta(|x-y|)\, dydx < +\infty.$$

We can now proceed as in the proof of Theorem 150 starting from equation (3.17) to conclude that $\nabla f \in L^p(\Omega; \mathbb{R}^N)$, and therefore, $f \in W^{1,p}(\Omega)$. $\square$

Remark 152. *Lemma 143 is essential in the proof of Corollary 8, as it ensures that truncation of the mollifiers does not destroy coercivity of the limiting measure, which was necessary for our comparison of the geodesic and Euclidean distances (our analysis hinged on the equality $d_\Omega(x,y) = |x-y|$ for certain x and y). This analysis implies that the same argument applies to Corollary 11, where we invoke the argument of Theorem 151 instead of Theorem 150.*

Next we prove the necessity of conditions (1.11) and (1.14) in Theorems 6 and 9.

Theorem 153. *Let Ω and ρ_ϵ be as in Theorem 6, let $1 < p < \infty$ and $1 \le q < \infty$, with $1 \le q \le \frac{N}{N-p}$ if $p < N$, let $0 < \eta < \frac{r}{3}$, and let $f \in W^{1,p}_{\text{loc}}(\Omega)$ with $\nabla f \in L^p(\Omega; \mathbb{R}^N)$. Then*

$$\begin{aligned}
\int_{\Omega_r^\eta} \left(\int_{\Omega_r^\eta} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx \\
\le \int_\Omega |\nabla f(x)|^p \, dx + C_{p,q,\eta,r} ||f||^p_{L^{pq}(\Omega_r)} \left(\int_{|x| > \frac{\eta}{2}} \rho_\epsilon(x)\, dx \right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. Fix $r > 0$, and let $0 < \eta < \frac{r}{3}$. Consider

$$\begin{aligned}
&\int_{\Omega_r^\eta} \left(\int_{\Omega_r^\eta} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx \\
&\quad = \int_{\Omega_r^\eta} \left(\int_{|x-y|<\eta} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx \\
&\quad + \int_{\Omega_r^\eta} \left(\int_{|x-y|>\eta} \left(\frac{|f(x) - f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx \\
&\quad =: I \; + \; II.
\end{aligned}$$

Considering II, we have

$$II \le \frac{2^{p-1}}{\eta^p} \int_{\Omega_r^\eta} \left(\int_{|x-y|>\eta} |f(x)|^{pq} + |f(y)|^{pq} \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx.$$

Applying Hölder's inequality, we have

$$\begin{aligned}
II &\le \frac{2^{p-1}}{\eta^p} |\Omega_r^\eta|^{1-\frac{1}{q}} \left(\int_{\Omega_r^\eta} \int_{|x-y|>\eta} |f(x)|^{pq} + |f(y)|^{pq} \rho_\epsilon(x-y)\, dy dx \right)^{\frac{1}{q}} \\
&\le \frac{2^{p-1}}{\eta^p} |\Omega_r^\eta|^{1-\frac{1}{q}} \left(\int_{\Omega_r}^\eta \int_{|x-y|>\eta} |f(x)|^{pq} + |f(y)|^{pq} \rho_\epsilon(x-y)\, dy dx \right)^{\frac{1}{q}}.
\end{aligned}$$

Separating terms and applying Tonelli's theorem, we have that

$$\begin{aligned}
&\int_{\Omega_r^\eta} \int_{|x-y|>\eta} |f(x)|^{pq} + |f(y)|^{pq} \rho_\epsilon(x-y)\, dy dx \\
&\qquad = \int_{\Omega_r^\eta} |f(x)|^{pq} \int_{|x-y|>\eta} \rho_\epsilon(x-y)\, dy dx \\
&\qquad + \int_{\Omega_r^\eta} |f(y)|^{pq} \int_{|x-y|>\eta} \rho_\epsilon(x-y)\, dx dy
\end{aligned}$$

so that we may bound

$$II \leq \frac{2^{p-1}}{\eta^p} |\Omega_r^\eta|^{1-\frac{1}{q}} ||f||^p_{L^{pq}(\Omega_r^\eta)} \left(2 \int_{|h|>\eta} \rho_\epsilon(h)\, dh \right)^{\frac{1}{q}}.$$

Thus, if we define $C_{p,q,\eta,r} := |\Omega_r^\eta|^{1-\frac{1}{q}} \frac{2^p}{\eta^p}$, we obtain the necessary estimate for II. As for I, we may apply Lemma 146 to conclude that

$$\begin{aligned} I &\leq \int_{\Omega_r^\eta} \left(\int_{|x-y|<\eta} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx \\ &\leq \int_{\Omega_r^{2\eta}} \left(\int_{B(0,\eta)} \left| \nabla f(x) \cdot \frac{h}{|h|} \right|^{pq} \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}} dx \qquad (3.23) \\ &\leq \int_\Omega |\nabla f(x)|^p\, dx \end{aligned}$$

and the result is demonstrated. □

Remark 154. *We will later use the fact that if the domain of integration in the inner integral is increased, it does not change the estimate (3.23).*

Theorem 155. *Let Ω and ρ_ϵ be as in Theorem 6, let $1 \leq q < \infty$ with $1 \leq q \leq \frac{N}{N-1}$ if $N > 1$, and let $f \in BV_{\mathrm{loc}}(\Omega)$ with $Df \in M_b(\Omega; \mathbb{R}^N)$. Then for all $0 < \eta < \frac{r}{3}$ such that $|Df|(\partial\Omega_r^\eta) = 0$ we have*

$$\begin{aligned} &\int_{\Omega_r^\eta} \left(\int_{\Omega_r^\eta} \left(\frac{|f(x)-f(y)|}{|x-y|} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx \\ &\qquad \leq |Df|(\Omega) + C_{1,q,\eta,r} \left(||f||_{L^q(\Omega_r)} \int_{|h|>\eta} \rho_\epsilon(h)\, dh \right)^{\frac{1}{q}}. \end{aligned}$$

Proof. We proceed as in Theorem 153, and we must again obtain bounds on I and II. We can use the bounds on I as Theorem 153, while for II we utilize Lemma 148 to conclude that

$$\begin{aligned} II &= \int_{\Omega_r^\eta} \left(\int_{|x-y|<\eta} \left(\frac{|f(x)-f(y)|}{|x-y|} \right)^q \rho_\epsilon^\eta(x-y)\, dy \right)^{\frac{1}{q}} dx \\ &\leq \int_{\Omega_r^{2\eta}} \left(\int_{B(0,\eta)} \left| \frac{dDf}{d|Df|}(x) \cdot \frac{h}{|h|} \right|^q \rho_\epsilon^\eta(h)\, dh \right)^{\frac{1}{q}} d|Df|(x) \leq |Df|(\Omega), \qquad (3.24) \end{aligned}$$

and the result is demonstrated. □

We are now able to prove Theorems 6 and 9.

Proof of Theorem 6. Let $f \in L^1_{\text{loc}}(\Omega)$ be such that

$$\lim_{r\to 0}\limsup_{\epsilon\to 0}\int_{\Omega_r}\left(\int_{\Omega_r}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q \rho_\epsilon(x-y)\,dy\right)^{\frac{1}{q}} dx < +\infty.$$

Then applying Theorem 150 we have that $f \in W^{1,p}_{\text{loc}}(\Omega)$ and $\nabla f \in L^p(\Omega;\mathbb{R}^N)$, with the inequality (3.20),

$$\begin{aligned}&\int_{\Omega_r}\left(\int_{S^{N-1}}(|\nabla f(x)\cdot\sigma|^p)^q\ d\mu(\sigma)\right)^{\frac{1}{q}} dx\\ &\qquad\leq \liminf_{j\to\infty}\int_{\Omega_r^\eta}\left(\int_{\Omega_r^{2\eta}}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q \rho_{\epsilon_j}(x-y)\,dy\right)^{\frac{1}{q}} dx.\end{aligned}$$

Conversely, let $f \in W^{1,p}_{\text{loc}}(\Omega)$ and $\nabla f \in L^p(\Omega;\mathbb{R}^N)$. Then applying Theorem 153 we have that

$$\begin{aligned}&\int_{\Omega_r^\eta}\left(\int_{\Omega_r^\eta}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q \rho_\epsilon(x-y)\,dy\right)^{\frac{1}{q}} dx\\ &\qquad\leq \int_\Omega |\nabla f(x)|^p\ dx + C_{p,q,\eta,r}||f||^p_{L^{pq}(\Omega_r^\eta)}\left(\int_{|h|>\eta}\rho_\epsilon(h)\,dh\right)^{\frac{1}{q}}.\end{aligned}$$

Based upon what interval q lies in, we may apply one of the embedding theorems Theorem 118, Theorem 122, or Theorem 124, which combined with the convergence

$$\int_{|h|>\eta}\rho_\epsilon(h)\,dh \to 0,$$

by (1.4), we have that the second term vanishes as $\epsilon \to 0$. Combining this with the estimate from (3.23) in Theorem 153, we have

$$\begin{aligned}&\liminf_{j\to\infty}\int_{\Omega_r^\eta}\left(\int_{\Omega_r^{2\eta}}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q \rho_{\epsilon_j}(x-y)\,dy\right)^{\frac{1}{q}} dx\\ &\qquad\leq \liminf_{j\to\infty}\int_{\Omega_r^{2\eta}}\left(\int_{B(0,\eta)}\left|\nabla f(x)\cdot\frac{h}{|h|}\right|^{pq}\rho^\eta_{\epsilon_j}(h)\,dh\right)^{\frac{1}{q}} dx\\ &\qquad= \liminf_{j\to\infty}\int_{\Omega_r^{2\eta}}\left(\int_{S^{N-1}}|\nabla f(x)\cdot\sigma|^{pq}\ d\mu^\eta_{\epsilon_j}(\sigma)\right)^{\frac{1}{q}} dx\\ &\qquad= \int_{\Omega_r^{2\eta}}\left(\int_{S^{N-1}}|\nabla f(x)\cdot\sigma|^{pq}\ d\mu(\sigma)\right)^{\frac{1}{q}} dx.\end{aligned}$$

Combining these two estimates we have

$$\int_{\Omega_r}\left(\int_{S^{N-1}}|\nabla f(x)\cdot\sigma|^{pq}\ d\mu(\sigma)\right)^{\frac{1}{q}}dx$$

$$\leq \liminf_{j\to\infty}\int_{\Omega_r^\eta}\left(\int_{\Omega_r^{2\eta}}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q\rho_{\epsilon_j}(x-y)\,dy\right)^{\frac{1}{q}}dx$$

$$\leq \int_{\Omega_r^{2\eta}}\left(\int_{S^{N-1}}|\nabla f(x)\cdot\sigma|^{pq}\ d\mu(\sigma)\right)^{\frac{1}{q}}dx,$$

and thus

$$\int_{\Omega_r}\left(\int_{S^{N-1}}|\nabla f(x)\cdot\sigma|^{pq}\ d\mu(\sigma)\right)^{\frac{1}{q}}dx$$

$$\leq \liminf_{j\to\infty}\int_{\Omega_r^\eta}\left(\int_{\Omega_r^{2\eta}}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q\rho_{\epsilon_j}(x-y)\,dy\right)^{\frac{1}{q}}dx$$

$$\leq \int_{\Omega}\left(\int_{S^{N-1}}|\nabla f(x)\cdot\sigma|^{pq}\ d\mu(\sigma)\right)^{\frac{1}{q}}dx,$$

and finally sending $r\to 0$ the result is demonstrated.

When ρ_ϵ satisfy (1.5), by Remark 142 we have that $\mu=\mathcal{H}^{N-1}$. In this case, utilizing rotational invariance of $\mathcal{H}^{N-1}$ on the sphere S^{N-1} we have

$$\left(\int_{S^{N-1}}|\nabla f(x)\cdot\sigma|^{pq}\ d\mu(\sigma)\right)^{\frac{1}{q}}=|\nabla f(x)|^p\left(\int_{S^{N-1}}|e_1\cdot\sigma|^{pq}\ d\mathcal{H}^{N-1}(\sigma)\right)^{\frac{1}{q}}$$

$$=K_{p,q,N}|\nabla f(x)|^p.$$

Since Remark 142 asserts the convergence of the full sequence in the radial case, we are hence able to conclude the limit (1.12) exists. □

Proof of Theorem 9. Let $f\in L^1_{\mathrm{loc}}(\Omega)$ be such that

$$\lim_{r\to 0}\limsup_{\epsilon\to 0}\int_{\Omega_r}\left(\int_{\Omega_r}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^q\rho_\epsilon(x-y)\,dy\right)^{\frac{1}{q}}dx<+\infty.$$

Then applying Theorem 151 we have that $f\in BV_{\mathrm{loc}}(\Omega)$ and $Df\in M_b(\Omega;\mathbb{R}^N)$, with the inequality (3.22)

$$\int_{\Omega_r}\left(\int_{S^{N-1}}\left|\frac{dDf}{d\,|Df|}(x)\cdot\sigma\right|^q\ d\mu(\sigma)\right)^{\frac{1}{q}}d\,|Df|\,(x)$$

$$\leq \liminf_{j\to\infty}\int_{\Omega_r^\eta}\left(\int_{\Omega_r^{2\eta}}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^q\rho_{\epsilon_j}(x-y)\,dy\right)^{\frac{1}{q}}dx.$$

Conversely, let $f \in BV_{\text{loc}}(\Omega)$ and $Df \in M_b(\Omega;\mathbb{R}^N)$. Then applying Theorem 155 we have that

$$\begin{aligned}\int_{\Omega_r^\eta}&\left(\int_{\Omega_r^\eta}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^q \rho_\epsilon(x-y)\,dy\right)^{\frac{1}{q}} dx\\ &\le |Df|(\Omega) + C_{p,q,\eta,r}\left(||f||_{L^q(\Omega_r^\eta)}\int_{|h|>\eta}\rho_\epsilon(h)\,dh\right)^{\frac{1}{q}}.\end{aligned}$$

Taking the limit as $\epsilon \to 0$, our bounds on q and Theorems 133, 134 implies that the second right-hand-side term vanishes, so that letting $r \to 0$, we see that the left-hand-side is finite. To prove the final part of the statement, we reason as in the proof of Theorem 6. Given r, choose η such that $|Df|(\partial\Omega_r^{2\eta}) = 0$ (which we may do by applying Theorem 41). Combining inequality (3.22) with (3.24) (and using non-negativity of the integrand as in the last inequalities in Theorem 6), we have

$$\begin{aligned}\int_{\Omega_r}&\left(\int_{S^{N-1}}\left|\frac{dDf}{d|Df|}(x)\cdot\sigma\right|^q d\mu(\sigma)\right)^{\frac{1}{q}} d|Df|\\ &\le \liminf_{j\to\infty}\int_{\Omega_r^\eta}\left(\int_{\Omega_r^{2\eta}}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^q \rho_{\epsilon_j}(x-y)\,dy\right)^{\frac{1}{q}} dx\\ &\le \int_{\Omega}\left(\int_{S^{N-1}}\left|\frac{dDf}{d|Df|}(x)\cdot\sigma\right|^q d\mu(\sigma)\right)^{\frac{1}{q}} d|Df|(x).\end{aligned}$$

Finally, sending $r \to 0$ the result is demonstrated.

Under the assumption ρ_ϵ satisfy (1.5), we reason as in the previous proof to conclude the corresponding convergence of the functional to $K_{1,q,N}|Df|(\Omega)$. □

3.4 COUNTEREXAMPLES

In this section, we provide proofs to Theorems 12 and 13 in the form of counterexamples.

Proof of Theorem 12. This is based on a counterexample of Fraenkel [43], constructed from work by Courant and Hilbert. We find a set Ω, and $f \in W^{1,2}(\Omega)$ such that

$$\liminf_{\epsilon\to 0} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^2}{d_\Omega(x,y)^2} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dydx = +\infty,$$

where $\hat{\rho}_\epsilon$ are radial mollifiers satisfying (1.3) and (1.4). Initially posed as an example of when the embedding of $W^{1,2}$ into L^q fails for $q > 2$, due to a lack of regularity of the boundary, the construction from Section 2.2, Example (i) of [43] is as follows. Let $N = 2$, $p = 2$, and construct Ω as follows. Let

$$h_j := j^{-\frac{3}{2}}, \quad \delta_j := j^{-\frac{5}{2}}, \quad c_n := \sum_{i=1}^n h_i, \tag{3.25}$$

and use these sequences to define *rooms* R_j and *passages* P_{j+1},

$$\begin{aligned}
R_j &:= (c_j - h_j, c_j) \times \left(-\frac{1}{2}h_j, \frac{1}{2}h_j\right), \\
P_{j+1} &:= [c_j, c_j + h_{j+1}] \times \left(-\frac{1}{2}\delta_{j+1}, \frac{1}{2}\delta_{j+1}\right), \\
\Omega &:= \bigcup_{i \text{ odd}} R_i \cup P_{i+1},
\end{aligned}$$

so that Ω is open. Given Ω, we define for j odd

$$f(x) := \begin{cases} K_j := \frac{j}{\log 2j} & x \in R_j \\ K_j + (K_{j+2} - K_j)\frac{x-c_j}{h_j+1} & x \in P_{j+1}. \end{cases}$$

As mentioned, in [43] Fraenkel demonstrates that $f \in W^{1,2}(\Omega)$, but $f \notin L^q(\Omega)$ for $q > 2$, so that Ω is not an extension domain. We continue this example, letting $\hat{\rho}(x) = \frac{1}{\alpha_2}\chi_{[0,1)}(|x|)$, and $\hat{\rho}_\epsilon(x) = \frac{1}{\epsilon^2}\hat{\rho}(\frac{x}{\epsilon})$. Consider

$$\begin{aligned}
I &:= \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^2}{d_\Omega(x,y)^2} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dydx \\
&\geq \sum_{4h_i<\epsilon} \int_{R_i} \int_{R_{i+2}} \frac{|f(x)-f(y)|^2}{d_\Omega(x,y)^2} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dydx,
\end{aligned}$$

where we have thrown away the integral for all but neighboring rooms, and have began summing for $4h_i < \epsilon$. Since $d_\Omega(x,y) < 4h_i < \epsilon$ for $x \in R_i$, $y \in R_{i+2}$, we have $\hat{\rho}_\epsilon(d_\Omega(x,y)) = \frac{C}{16\epsilon^2}$, and $\frac{1}{d_\Omega(x,y)^2} \geq \frac{1}{(4h_i)^2}$, so that

$$I \geq \frac{C}{16\epsilon^2} \sum_{4h_i<\epsilon} \int_{R_i} \int_{R_{i+2}} \frac{\left|\frac{i}{\log(2i)} - \frac{i+2}{\log(2i+4)}\right|^2}{(4h_i)^2} \, dydx.$$

Now, $|R_i| = h_i^2$, while $|R_{i+2}| = h_{i+2}^2$, so that

$$I \geq \frac{C}{\epsilon^2} \sum_{4h_i<\epsilon} h_{i+2}^2 \left|\frac{i}{\log(2i)} - \frac{i+2}{\log(2i+4)}\right|^2.$$

By (3.25), and solving the equation $4h_i < \epsilon$ for i in terms of ϵ, we have

$$I \geq \frac{C}{\epsilon^2} \sum_{i>(\frac{4}{\epsilon})^{\frac{2}{3}}} \frac{1}{(i+2)^3} \left|\frac{i}{\log(2i)} - \frac{i+2}{\log(2i+4)}\right|^2.$$

However, considering the square term, we find a common denominator and expand to see that

$$\begin{aligned}
\left|\frac{i}{\log(2i)} - \frac{i+2}{\log(2i+4)}\right|^2 &= \left|\frac{i\left(\log(2i) + \log(1+\frac{2}{i})\right) - (i+2)\log(2i)}{\log(2i)\log(2i+4)}\right|^2 \\
&= \left|\frac{\log(1+\frac{2}{i})^i}{\log(2i)\log(2i+4)} - \frac{2}{\log(2i+4)}\right|^2 \\
&\geq \frac{-4\log(1+\frac{2}{i})^i}{\log(2i)(\log(2i+4))^2} + \frac{4}{(\log(2i+4))^2} \\
&= \frac{2\log(2i) - 4\log(1+\frac{2}{i})^i}{\log(2i)(\log(2i+4))^2} + \frac{2}{(\log(2i+4))^2} \\
&\geq \frac{2}{(\log(2i+4))^2},
\end{aligned}$$

whenever i is large enough. Using this lower bound with the above inequality for I, we have

$$I \geq \frac{C}{16\epsilon^2} \sum_{i>(\frac{4}{\epsilon})^{\frac{2}{3}}} \frac{2}{(i+2)^3(\log(2i+4))^2},$$

for ϵ small. Now, the function $i \mapsto \frac{2}{(i+2)^3(\log(2i+4))^2}$ is decreasing, and so we may use the integral test to determine the convergence of the series. Thus,

$$\begin{aligned} I &\geq \frac{C}{16\epsilon^2} \int_{(\frac{4}{\epsilon})^{\frac{2}{3}}}^{\infty} \frac{2}{(x+2)^3(\log(2x+4))^2}\, dx \\ &\geq \frac{C}{16\epsilon^2} \int_{(\frac{8}{\epsilon})^{\frac{2}{3}}}^{\infty} \frac{2}{x^3(\log(x))^2}\, dx \end{aligned}$$

We utilize L'Hôpital's rule to calculate the limit of the right hand side

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{C}{16\epsilon^2} \int_{(\frac{8}{\epsilon})^{\frac{2}{3}}}^{\infty} \frac{2}{x^3(\log(x))^2}\, dx &= \lim_{\epsilon \to 0} \frac{C}{32\epsilon} \frac{2}{4^3\epsilon^{-2}(\log(4\epsilon^{\frac{-2}{3}}))^2} \frac{8}{3}\epsilon^{\frac{-5}{3}} \\ &= \lim_{\epsilon \to 0} \tilde{C} \frac{1}{\epsilon^{\frac{2}{3}}(\log(4\epsilon^{\frac{-2}{3}}))^2}, \end{aligned}$$

and since $x\log^2(x) \to 0$ as $x \to 0$, we conclude that the limit of the right hand side is $+\infty$, so that

$$\liminf_{\epsilon \to 0} I = +\infty,$$

and the result is demonstrated. $\square$

Proof of Theorem 13. If we consider in $N = 2$ the unit disc without the positive x-axis, and the function

$$f(x,y) = \begin{cases} \operatorname{sign} y & \frac{1}{3} < x < 1, \\ 0 & x < 0, \end{cases}$$

and connected by affine functions between. Then for ρ_ϵ of the form

$$\hat{\rho}_\epsilon(|x|) = \tilde{\rho}_\epsilon(|x|) + \chi_{[1-\epsilon,1]}(|x|),$$

where $\int \tilde{\rho}_\epsilon = 1 - C(\epsilon)$ where $C(\epsilon) \to 0$ as $\epsilon \to 0$, and $\int \chi_{[1-\epsilon,1]} = C(\epsilon)$. Then $\int \hat{\rho}_\epsilon = 1$ and $\hat{\rho}_\epsilon = 1$ in a small tubular neighborhood of points of distance approximately 1. Let $Q_\epsilon(x,y)$ be a cube centered at (x,y) with side length ϵ, and $Q_\epsilon^+, Q_\epsilon^-$ be the upper and lower halves of such a cube. Then we can calculate

$$\begin{aligned} &\liminf_{\epsilon \to 0} \int_{Q_\epsilon^+(\frac{1}{2},0)} \int_{Q_\epsilon^-(\frac{1}{2},0)} \frac{2}{|x-y|^p} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dy dx \\ &\leq \limsup_{\epsilon \to 0} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p} \hat{\rho}_\epsilon(d_\Omega(x,y))\, dy dx \end{aligned}$$

we can see that for every $\epsilon > 0$ we can find a small region inside the first cube where the inner integral is infinite (since $\hat{\rho}_\epsilon$ is 1 in a small region and $\frac{1}{|h|^p}$ is not integrable in two dimensions for any $p > 1$). This family ρ_ϵ is also an example of why we require the specific bounds on q as given in the main results of the paper. If ρ_ϵ are radial scalings of one function, or more generally, we can get control on the support as a function of ϵ, then we can relax the hypothesis on q to be $1 \leq q < \infty$ for any value of p. $\square$

4.0 MEASURE SEMICONTINUITY THEOREMS

In this chapter, we prove Theorems 16 and 17.

Let us first remark the additional information we obtain by assuming $\Omega \subset \mathbb{R}^N$ and (1.22), as alluded to in Measures as a Dual Space.

Remark 156. *The convergence assumptions in (1.22) imply convergence in a topology stronger than the weak-star topology. As a result, if $A \subset \Omega$ is open with $\overline{A} \subset \Omega$ compact and $|\lambda|(\partial A) = 0$, then*

$$\lim_{n\to\infty} \int_A \phi \cdot d\lambda_n = \int_A \phi \cdot d\lambda \tag{4.1}$$

for every $\phi \in C_b(\Omega; \mathbb{R}^m)$, and

$$\lim_{n\to\infty} \int_A \psi \, d|\lambda_n| = \int_A \psi \, d|\lambda| \tag{4.2}$$

for every $\psi \in C_b(\Omega)$.

4.1 PROOF OF THEOREM 16

Proof. We claim it is enough to demonstrate

$$\lim_{n\to\infty} \int_{\Omega'} f\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right) \, d|\lambda_n| = \int_{\Omega'} f\left(x, \frac{d\lambda}{d|\lambda|}(x)\right) \, d|\lambda| \tag{4.3}$$

for every $\Omega' \subset \Omega$ open with $\overline{\Omega'} \subset \Omega$ compact and $|\lambda|(\partial\Omega') = 0$. If this is the case, we may estimate the boundary layer by

$$\int_{\Omega\setminus\overline{\Omega'}} f\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right) \, d|\lambda_n| \leq M \, |\lambda_n| \, (\Omega \setminus \Omega') \, , \tag{4.4}$$

$$\int_{\Omega\setminus\overline{\Omega'}} f\left(x, \frac{d\lambda}{d|\lambda|}(x)\right) \, d|\lambda| \leq M \, |\lambda| \, (\Omega \setminus \Omega') \, , \tag{4.5}$$

where $M := \sup_{(x,z)\in\Omega\times S^{m-1}} |f(x,z)|$. Computing the limit of (4.4), we have

$$\begin{aligned}\lim_{n\to\infty} |\lambda_n|(\Omega\setminus\Omega') &= \lim_{n\to\infty} |\lambda_n|(\Omega) - \lim_{n\to\infty} |\lambda_n|(\Omega') \\ &= |\lambda|(\Omega) - |\lambda|(\Omega') = |\lambda|(\Omega\setminus\Omega'),\end{aligned}$$

where we have used the the fact that $|\lambda|(\partial\Omega') = 0$ to apply the convergence in equation (4.2) with $\psi = 1$. We can then choose Ω' appropriately to make (4.4) and (4.5) arbitrarily small.

We therefore proceed to prove (4.3). Define $\tilde{f} : \Omega \times \overline{B(0,1)} \to \mathbb{R}$ by

$$\tilde{f}(x,z) = \begin{cases} f\left(x, \frac{z}{|z|}\right)|z| & \text{if } 0 < |z| \le 1, \\ 0 & \text{if } z = 0. \end{cases}$$

Then since f is bounded and continuous, we have that $\tilde{f}$ is bounded and continuous. Further, since $\overline{\Omega'}$ is compact, $\tilde{f} : \Omega' \times \overline{B(0,1)} \to \mathbb{R}$ is uniformly continuous. Thus, for every $\delta > 0$, there exists an $C_\delta > 0$ such that

$$\left|\tilde{f}(x,y) - \tilde{f}(x,z)\right| \le C_\delta |y-z|^2 + \delta \tag{4.6}$$

for all $x \in \Omega'$ and for all $y, z \in \overline{B(0,1)}$. To obtain this estimate, let $\delta > 0$ be given. By uniform continuity of $\tilde{f}$, there exists an $\epsilon = \epsilon(\delta) > 0$ such that

$$\left|\tilde{f}(x,y) - \tilde{f}(x,z)\right| \le \delta \tag{4.7}$$

for all $x \in \Omega'$ and for all $y, z \in \overline{B(0,1)}$ with $|y - z| < \epsilon$. However, if $|y - z| \ge \epsilon$, then $\frac{|y-z|^2}{\epsilon^2} \ge 1$ so that by boundedness of $\tilde{f}$ we have

$$\left|\tilde{f}(x,y) - \tilde{f}(x,z)\right| \le 2M \le 2M\frac{|y-z|^2}{\epsilon^2} \tag{4.8}$$

Combining equations (4.7) and (4.8) and defining $C_\delta := \frac{2M}{\epsilon^2}$ yields inequality (4.6).

Let $\varphi : \Omega \to \overline{B(0,1)} \subset \mathbb{R}^m$ be continuous, to be chosen later. To prove (4.3), we write

$$\begin{aligned}
&\left| \int_{\Omega'} f\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right)\, d|\lambda_n| - \int_{\Omega'} f\left(x, \frac{d\lambda}{d|\lambda|}(x)\right)\, d|\lambda| \right| \\
&\leq \left| \int_{\Omega'} \tilde{f}\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right)\, d|\lambda_n| - \int_{\Omega'} \tilde{f}\left(x, \varphi(x)\right)\, d|\lambda_n| \right| \\
&\quad + \left| \int_{\Omega'} \tilde{f}\left(x, \varphi(x)\right)\, d|\lambda_n| - \int_{\Omega'} \tilde{f}\left(x, \varphi(x)\right)\, d|\lambda| \right| \\
&\quad + \left| \int_{\Omega'} \tilde{f}\left(x, \varphi(x)\right)\, d|\lambda| - \int_{\Omega'} \tilde{f}\left(x, \frac{d\lambda}{d|\lambda|}(x)\right)\, d|\lambda| \right| \\
&=: I + II + III.
\end{aligned}$$

We have that II goes to zero by applying the convergence result found in equation (4.2) with $A = \Omega'$ and $\psi = \tilde{f}\left(x, \varphi(x)\right)$. As for I and III, by (4.6) we can bound

$$\begin{aligned}
I + III &\leq \int_{\Omega'} \left(C_\delta \left| \frac{d\lambda_n}{d|\lambda_n|}(x) - \varphi(x) \right|^2 + \delta \right)\, d|\lambda_n| \\
&\quad + \int_{\Omega'} \left(C_\delta \left| \frac{d\lambda}{d|\lambda|}(x) - \varphi(x) \right|^2 + \delta \right)\, d|\lambda| \\
&\leq \int_{\Omega'} \left(2C_\delta \left(1 - \frac{d\lambda_n}{d|\lambda_n|}(x) \cdot \varphi(x)\right) + \delta \right)\, d|\lambda_n| \\
&\quad + \int_{\Omega'} \left(2C_\delta \left(1 - \frac{d\lambda}{d|\lambda|}(x) \cdot \varphi(x)\right) + \delta \right)\, d|\lambda|,
\end{aligned}$$

where in the last inequality we have used the fact that $\frac{d\lambda}{d|\lambda|}(x), \frac{d\lambda_n}{d|\lambda_n|}(x) \in S^{m-1}$ and $|\varphi| \leq 1$. Letting $n \to \infty$, and again applying the convergence results (4.1) and (4.2), we have

$$\begin{aligned}
I + III &\leq 2 \int_{\Omega} \left(2C_\delta \left(1 - \frac{d\lambda}{d|\lambda|}(x) \cdot \varphi(x)\right) + \delta \right)\, d|\lambda| \\
&= 2\delta|\lambda|(\Omega) + 4C_\delta \int_{\Omega} \left(1 - \frac{d\lambda}{d|\lambda|}(x) \cdot \varphi(x)\right)\, d|\lambda|.
\end{aligned}$$

First choosing $\delta > 0$ small, and then choosing φ close to $\frac{d\lambda}{d|\lambda|}$ (since $\frac{d\lambda}{d|\lambda|} \in L^1(\Omega, |\lambda|)$, and using the density result in Chapter 2, Theorem 102), the result is demonstrated. $\square$

Remark 157. *We can now establish the equivalence of the hypotheses of Theorem 14 and Theorem 16. That* $\lambda_n \overset{*}{\rightharpoonup} \lambda$ *in* $(C_b(\Omega;\mathbb{R}^m))'$ *and (1.20) imply (1.22) is relatively straightforward, since weak-star convergence in* $(C_b(\Omega;\mathbb{R}^m))'$ *is stronger than weak-star convergence in* $(C_0(\Omega;\mathbb{R}^m))'$*, and applying Theorem 14, we conclude that (1.21) holds for* $f(x,z) = |z|$*. Conversely, assuming (1.22), we have that (1.20) holds for* $g(x,z) = |z|$*, and given* $\phi \in C_b(\Omega;\mathbb{R}^m)$*, we may apply Theorem 16 to the function* $f(x,z) = \phi(x)\cdot z$ *to prove weak-star convergence in* $(C_b(\Omega;\mathbb{R}^m))'$*.*

4.2 PROOF OF THEOREM 17

To simplify the proof, we proceed in two steps, first assuming the additional hypothesis $f(x,0) = 0$ for all $x \in \Omega$ (which is true if f is real-valued by positive 1-homogeneity), and then proceeding to the general case.

Theorem 158. *Let* $\Omega \subset \mathbb{R}^N$ *be open and* $\lambda_n, \lambda \in M_b(\Omega;\mathbb{R}^m)$*; if* $\lambda_n \overset{*}{\rightharpoonup} \lambda$ *in* $(C_0(\Omega;\mathbb{R}^m))'$*, then*

$$\liminf_{n\to\infty} \int_\Omega f\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right)\, d|\lambda_n| \geq \int_\Omega f\left(x, \frac{d\lambda}{d|\lambda|}(x)\right)\, d|\lambda|$$

for every lower semicontinuous function $f : \Omega\times\mathbb{R}^m \to [0,\infty]$*, positively 1-homogeneous and convex in the second variable such that* $f(x,0) = 0$ *for all* $x \in \Omega$*.*

Proof. Since we have assumed $f(x,0) = 0$, we have that f is real-valued and non-negative and hence can apply Theorem 140 from Chapter 2, Convexity to represent f as

$$f(x,z) = \sup_i b_i(x)\cdot z, \tag{4.9}$$

where $b_i : \Omega \to \mathbb{R}^m$ are bounded and continuous. Without loss of generality we may pass to a subsequence and assume that

$$\liminf_{n\to\infty} \int_\Omega f\left(x, \frac{d\lambda_n}{d|\lambda_n|}\right)\, d|\lambda_n| = \lim_{n\to\infty} \int_\Omega f\left(x, \frac{d\lambda_n}{d|\lambda_n|}\right)\, d|\lambda_n| < \infty, \tag{4.10}$$

as otherwise there is nothing to prove. Possibly passing to a further subsequence, by Theorem 28, there exists a positive Radon measure $\nu \in M_b(\Omega)$ such that

$$f\left(x, \frac{d\lambda_n}{d|\lambda_n|}\right)\ d|\lambda_n| \overset{*}{\rightharpoonup} \nu \text{ in } (C_0(\Omega))'$$

as $n \to \infty$. We claim it is enough to show that

$$\frac{d\nu}{d|\lambda|}(x_0) \geq f\left(x_0, \frac{d\lambda}{d|\lambda|}(x_0)\right) \text{ for } |\lambda| \text{ a.e. } x_0 \in \Omega. \tag{4.11}$$

If we can prove (4.11), then by Theorems 78 and 82 we can write

$$\nu = \frac{d\nu}{d|\lambda|}|\lambda| + \nu_s,$$

where $\nu_s \geq 0$ (since f, and in turn ν, are non-negative), and we have the following inequalities

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega f\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right)\ d|\lambda_n| \geq \nu(\Omega) &\geq \int_\Omega \frac{d\nu}{d|\lambda|}(x)\ d|\lambda| \\ &\geq \int_\Omega f\left(x, \frac{d\lambda}{d|\lambda|}(x)\right)\ d|\lambda|.\end{aligned}$$

By the Theorem 87 (Besicovitch derivation theorem), we have that for $|\lambda|$ a.e. $x \in \Omega$

$$\frac{d\nu}{d|\lambda|}(x) = \lim_{\epsilon\to 0} \frac{\nu(Q(x,\epsilon))}{|\lambda|(Q(x,\epsilon))} < \infty, \tag{4.12}$$

where as introduced in Chapter 2, $Q(x, \epsilon)$ is the cube centered at x with side length ϵ. Additionally, by Theorem 88 (Lebesgue differentiation theorem) we have that $|\lambda|$ a.e. $x \in \Omega$ is a Lebesgue point of $\frac{d\lambda}{d|\lambda|}$ with respect to the measure $|\lambda|$. Thus, let x_0 be a Lebesgue point of $\frac{d\lambda}{d|\lambda|}$ with respect to the measure $|\lambda|$ such that (4.12) holds. Since ν and $|\lambda|$ are Radon measures, Theorem 41 implies that we may choose a sequence $\epsilon_k \to 0^+$ such that $\nu(\partial Q(x_0, \epsilon_k)) = 0$ and $|\lambda|(\partial Q(x_0, \epsilon_k)) = 0$. Therefore, combining equation (4.12) with (4.1) and (4.9) we have that

$$\begin{aligned}\frac{d\nu}{d|\lambda|}(x_0) &= \lim_{k\to\infty} \frac{\nu(Q(x_0,\epsilon_k))}{|\lambda|(Q(x_0,\epsilon_k))} \\ &= \lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{|\lambda|(Q(x_0,\epsilon_k))} \int_{Q(x_0,\epsilon_k)} f\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right)\ d|\lambda_n| \\ &\geq \liminf_{k\to\infty}\liminf_{n\to\infty} \frac{1}{|\lambda|(Q(x_0,\epsilon_k))} \int_{Q(x_0,\epsilon_k)} b_i(x) \cdot \frac{d\lambda_n}{d|\lambda_n|}(x)\ d|\lambda_n| \\ &= \liminf_{k\to\infty} \frac{1}{|\lambda|(Q(x_0,\epsilon_k))} \int_{Q(x_0,\epsilon_k)} b_i(x) \cdot \frac{d\lambda}{d|\lambda|}(x)\ d|\lambda|,\end{aligned}$$

where we have used the weak-star convergence $\lambda_n \overset{*}{\rightharpoonup} \lambda$ in $(C_0(\Omega;\mathbb{R}^m))'$. By the continuity of b_i, for every $\eta > 0$ we have that

$$\frac{1}{|\lambda|(Q(x_0,\epsilon_k))}\int_{Q(x_0,\epsilon_k)} |b_i(x) - b_i(x_0)|\ d|\lambda| \leq \eta,$$

whenever k is sufficiently large. Thus, we have that

$$\begin{aligned}\lim_{k\to\infty} \frac{1}{|\lambda|(Q(x_0,\epsilon_k))}\int_{Q(x_0,\epsilon_k)} b_i(x)\cdot\frac{d\lambda}{d|\lambda|}(x)\ d|\lambda| \\ = \lim_{k\to\infty} \frac{1}{|\lambda|(Q(x_0,\epsilon_k))}\int_{Q(x_0,\epsilon_k)} b_i(x_0)\cdot\frac{d\lambda}{d|\lambda|}(x)\ d|\lambda| \\ = b_i(x_0)\cdot\frac{d\lambda}{d|\lambda|}(x_0),\end{aligned}$$

and combining this with the above, we have

$$\frac{d\nu}{d|\lambda|}(x_0) \geq b_i(x_0)\cdot\frac{d\lambda}{d|\lambda|}(x_0). \tag{4.13}$$

Finally, taking the supremum over i and using equation (4.9), we obtain the inequality (4.11), and the result is demonstrated. □

We now remove the hypothesis that $f(x,0) = 0$ for all $x \in \Omega$, with some subtle analysis of the set of $x \in \Omega$ such that $f(x,0) = 0$.

Proof. Define the set

$$C := \{x \in \Omega : f(x,0) = 0\},$$

and note that by lower semicontinuity of f, C is a closed set. We will show that without loss of generality, the complement of C has $|\lambda|$ measure zero, which combined with a representation for f on C similar to the one used in the proof of Theorem 158 will yield the result. Thus, we claim that $|\lambda|(\Omega\setminus C) = 0$. To see this, note that assumption (4.10) implies that for n large, say $n \geq n_0$,

$$f\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right) < \infty \text{ for } |\lambda_n| \text{ a.e. } x \in \Omega.$$

Fix $n \geq n_0$ and let $x \in \Omega$ be such that $f\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right) < \infty$. Applying positive 1-homogeneity and using lower semicontinuity of f, we have that

$$0 \leq f(x,0) \leq \lim_{t\to 0^+} f\left(x, t\frac{d\lambda_n}{d|\lambda_n|}(x)\right) = \lim_{t\to 0^+} tf\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right) = 0.$$

Thus,

$$f(x,0) = 0 \text{ for } |\lambda_n| \text{ a.e. } x \in \Omega,$$

which combined with the weak-star convergence $\lambda_n \overset{*}{\rightharpoonup} \lambda$ in $(C_0(\Omega;\mathbb{R}^m))'$ implies

$$0 \leq \int_\Omega f(x,0)\ d|\lambda| \leq \liminf_{n\to\infty} \int_\Omega f(x,0)\ d|\lambda_n| = 0,$$

so

$$f(x,0) = 0 \text{ for } |\lambda| \text{ a.e. } x \in \Omega.$$

By Theorem 140, we may represent $f : C \times \mathbb{R}^m \to [0,\infty)$ as

$$f(x,z) = \sup_i b_i(x) \cdot z,$$

where $b_i : C \to \mathbb{R}^m$ are bounded and continuous. Now, since C is closed, by the Tietze extension theorem (see [36], Theorem 4.16) we may extend b_i to $\tilde{b}_i : \Omega \to \mathbb{R}^m$ such that $\tilde{b}_i$ are still bounded and continuous. But then examining the blowup argument in the previous proof under these modifications, for any $x_0 \in C$ and n large we have

$$\begin{aligned}
\frac{d\nu}{d|\lambda|}(x_0) &= \lim_{k\to\infty} \frac{\nu(Q(x_0,\epsilon_k))}{|\lambda|(Q(x_0,\epsilon_k))} \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{|\lambda|(Q(x_0,\epsilon_k))} \int_{Q(x_0,\epsilon_k)\cap C} f\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right)\ d|\lambda_n| \\
&\geq \liminf_{k\to\infty} \liminf_{n\to\infty} \frac{1}{|\lambda|(Q(x_0,\epsilon_k))} \int_{Q(x_0,\epsilon_k)\cap C} \tilde{b}_i(x) \cdot \frac{d\lambda_n}{d|\lambda_n|}(x)\ d|\lambda_n| \\
&= \liminf_{k\to\infty} \liminf_{n\to\infty} \frac{1}{|\lambda|(Q(x_0,\epsilon_k))} \int_{Q(x_0,\epsilon_k)} \tilde{b}_i(x) \cdot \frac{d\lambda_n}{d|\lambda_n|}(x)\ d|\lambda_n| \\
&= \liminf_{k\to\infty} \frac{1}{|\lambda|(Q(x_0,\epsilon_k))} \int_{Q(x_0,\epsilon_k)} \tilde{b}_i(x) \cdot \frac{d\lambda}{d|\lambda|}(x)\ d|\lambda|,
\end{aligned}$$

where we have used twice the $|\lambda_n|$ negligibility of the complement of C for n large. However, this again says that

$$\frac{d\nu}{d|\lambda|}(x_0) \geq \tilde{b}_i(x_0) \cdot \frac{d\lambda}{d|\lambda|}(x_0) = b_i(x_0) \cdot \frac{d\lambda}{d|\lambda|}(x_0),$$

since $x_0 \in C$ and $\tilde{b}_i$ is an extension of b_i. This inequality is similar to (4.13) in Theorem 158, and we follow the remainder of the argument of Theorem 158, along with the $|\lambda|$ negligibility of the complement of C to reach the desired conclusion.

To prove the last claim of the theorem, assume that $\lambda_n \overset{*}{\rightharpoonup} \lambda$ in $(C_b(\Omega;\mathbb{R}^m))'$ and let f be as in the final part of the statement. Consider the function

$$g(x,z) := f(x,z) - b(x)\cdot z \geq 0.$$

Applying the first part of the proof to the function g, we have

$$\begin{aligned}
\liminf_{n\to\infty} \int_\Omega f\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right)\, d|\lambda_n| - \int_\Omega b(x)\cdot \frac{d\lambda}{d|\lambda|}(x)\, d|\lambda| \\
= \liminf_{n\to\infty} \int_\Omega g\left(x, \frac{d\lambda_n}{d|\lambda_n|}(x)\right)\, d|\lambda_n| \\
\geq \int_\Omega g\left(x, \frac{d\lambda}{d|\lambda|}(x)\right)\, d|\lambda| \\
= \int_\Omega f\left(x, \frac{d\lambda}{d|\lambda|}(x)\right)\, d|\lambda| - \int_\Omega b(x)\cdot \frac{d\lambda}{d|\lambda|}(x)\, d|\lambda|.
\end{aligned}$$

This concludes the proof. □

Remark 159. *Under the hypotheses of Theorem 15, we have that (1.24) holds. We obtain this using Theorem 140 to conclude that for f real-valued, we have*

$$f(x,z) \geq b(x)\cdot z$$

for some $b \in [C(\Omega)]^m$. To show that $b \in C_b(\Omega;\mathbb{R}^m)$, we combine this lower bound with the upper bound $f(x,z) \leq C|z|$ at $z = b(x)$. Then we have

$$|b(x)|^2 \leq f(x, b(x)) \leq C|b(x)|,$$

which implies that $|b(x)| \leq C$.

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