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Ultrasheaves and Double Negation

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ULTRASHEAVES AND DOUBLE NEGATION

STEVE AWODEY AND JONAS ELIASSON

Abstract. Moerdijk has introduced a topos of sheaves on a category of filters. Following his suggestion, we prove that its double negation subtopos is the topos of sheaves on the subcategory of ultrafilters - the *ultrasheaves*. We then use this result to establish a double negation translation of results between the topos of ultrasheaves and the topos on filters.

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1. Introduction

In 1993 I. Moerdijk [10] introduced a model of constructive nonstandard arithmetic in the topos $\mathrm{Sh}(\mathbb{F})$, of sheaves on a category of filters for a certain Grothendieck topology J. Further contributions to this model were made by I. Moerdijk and E. Palmgren [11] and Palmgren [13, 14, 15, 16]. A previous work by the second author [4] studies the sheaves on the full subcategory of ultrafilters, $\mathbb{U}$, henceforth called *ultrasheaves*. The resulting topos is Boolean, so its internal logic is no longer intuitionistic, but it is a model of nonstandard set theory. In fact it is a model of Nelsons internal set theory, see Nelson [12], an axiomatization of nonstandard set theory.

The question arises what the exact relationship is between the topos of ultrasheaves, $\mathrm{Sh}(\mathbb{U})$, and $\mathrm{Sh}(\mathbb{F})$? The subcategory $\mathbb{U}$ is "large" in $\mathbb{F}$, in the sense that it is a generating family for $\mathbb{F}$. We also know that "many" sheaves (namely the representable ones) on $\mathbb{F}$ are still sheaves when restricted to $\mathbb{U}$. Moerdijk conjectured that $\mathrm{Sh}(\mathbb{U})$ is the double negation subtopos of $\mathrm{Sh}(\mathbb{F})$ and in this paper we show that this is true.

Given a (intuitionistic) logic one can force it to become classical by adding the law of excluded middle to the assumptions. For a topos of sheaves there is a corresponding transformation, namely adding the double negation topology to the underlying site. Not all of the original sheaves will be sheaves with respect to the new topology, but the internal logic in the resulting topos of sheaves will be classical.

Some previous work has been done on sheaves on filters. In [6], D. P. Ellerman treated ultraproducts as sheaves on the Stone space of ultrafilters, and generalized the construction to other topological spaces. The exact relationship between his work and ours will be investigated

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elsewhere; briefly, there is a connected covering of Ellerman's topos by a slice of the topos $\mathrm{Sh}(\mathbb{F})$. Other work on filter categories has been done by M. Makkai [9], A. M. Pitts [17, 18, 19] and C. Butz [2]. Pitts uses the filter construction on coherent categories to prove completeness and interpolation results. Makkai's topos of types is related to the prime filters in Pitts construction. The precise relation between the two toposes is considered by Butz, who uses filters to construct generic saturated models of intuitionistic first-order theories.

In the second section of this paper we collect some definitions and results we will need subsequently. Then, in the third section, we prove that the topos $\mathrm{Sh}(\mathbb{U})$ is equivalent to a topos of sheaves on $\mathbb{F}$ for a finer topology than J, thereby showing that $\mathrm{Sh}(\mathbb{U})$ is in fact a subtopos of $\mathrm{Sh}(\mathbb{F})$. This is, of course, also useful in a setting (e.g. constructive mathematics) where you want to avoid using ultrafilters.

In the following fourth section we prove that this smaller topos on $\mathbb{F}$ is in fact equivalent to the double negation subtopos of $\mathrm{Sh}(\mathbb{F})$. Finally, in the fifth section we establish a double negation translation of results between $\mathrm{Sh}(\mathbb{U})$ and $\mathrm{Sh}(\mathbb{F})$.

2. Preliminary definitions and results

Definition 2.1. The category $\mathbb{F}$ has as *objects* pairs $(A, \mathcal{F})$, where A is a set and $\mathcal{F}$ a filter on A. The *morphisms* $\alpha : (A, \mathcal{F}) \to (B, \mathcal{G})$ are equivalence classes of partial functions $\alpha : A \to B$ such that

(i) α is defined on some $F \in \mathcal{F}$,
(ii) $\alpha^{-1}(G) \in \mathcal{F}$, for all $G \in \mathcal{G}$.

Two such partial functions $\alpha : F \to B$ and $\alpha' : F' \to B$ are equivalent if there is $E \subseteq F \cap F'$ in $\mathcal{F}$ such that $\alpha|_E = \alpha'|_E$.

A *filter* $\mathcal{F}$ on a set A is a non-empty collection of subsets of A which is closed under intersections and supersets. A maximal filter is called an *ultrafilter*, these filters $\mathcal{F}$ are distinguished by the property that for any subset B of A, either B or the complement of B is in $\mathcal{F}$. This category of filters $\mathbb{F}$ was introduced by V. Koubek and J. Reiterman [7] and studied further by A. Blass [1].

Note that for almost all equivalence classes $\alpha : (A, \mathcal{F}) \to (B, \mathcal{G})$ there is a total continuous function $f : A \to B$ representing α. The only exception is if B is the empty set. Then there is a morphism $\alpha : (A, \mathcal{F}) \to (\emptyset, \{\emptyset\})$ only if the filter $\mathcal{F}$ contains $\emptyset$ (i.e. if $\mathcal{F}$ is improper). In this case, α is the unique such morphism and an isomorphism $\mathcal{F} \cong (\emptyset, \{\emptyset\})$. The filter on the empty set, $(\emptyset, \{\emptyset\})$, is the initial object 0 in $\mathbb{F}$. Terminal object 1 is $(\{0\}, \{\{0\}\})$.

From Koubek and Reiterman [7] we have the following useful characterizations:

Proposition 2.2. *For morphisms* $\alpha : (A, \mathcal{F}) \to (B, \mathcal{G})$ *we have:*

(i) *α is monic if and only if there is an $F \in \mathcal{F}$ such that α is injective on F,*
(ii) *α is epic if and only if $\alpha(F) \in \mathcal{G}$, for all $F \in \mathcal{F}$.*

These characterizations hold true also in $\mathbb{U}$, but the situation is further simplified by the fact that all morphisms in $\mathbb{U}$ are epi, as the reader can check.

Moerdijk (in [10]) defined a subcanonical Grothendieck topology J on $\mathbb{F}$ as follows:

Definition 2.3. A finite family $\{\alpha_i : \mathcal{G}_i \to \mathcal{F}\}_{i=1}^n$ is a *J-covering* if the induced map

$$\mathcal{G}_1 + \ldots + \mathcal{G}_n \to \mathcal{F}$$

is an epimorphism.

Over the resulting site he studied, in particular, the representable sheaves
of the form ${}^*S = \mathrm{Hom}_{\mathbb{F}}(-,(S,\{S\}))$. At any filter
$\mathcal{F}$, ${}^*S(\mathcal{F})$ is the reduced power of S over
$\mathcal{F}$ (for more on reduced powers, and their use in model theory, see for instance C.C. Chang and H.J. Keisler [3]). Thus restricting the underlying category to the full subcategory $\mathbb{U}$ one can study ultrapowers as sheaves.

For the ultrafilters in $\mathbb{F}$ we have the following result from
Palmgren [14]:

Theorem 2.4.
(i) *Any morphism from a proper filter to an ultrafilter is a covering map.*
(ii) *Any cover of an ultrafilter contains a single map covering the ultrafilter.*

The topology induced on $\mathbb{U}$ by $(\mathbb{F}, J)$ is the atomic topology. In Eliasson [4] it is proved that all representable sheaves on $\mathbb{F}$ are still sheaves when restricted to $\mathbb{U}$. Thus the atomic topology is subcanonical.

We now turn our interest to the internal logics of the toposes $\mathrm{Sh}(\mathbb{F})$ and $\mathrm{Sh}(\mathbb{U})$. Since we want $\mathrm{Sh}(\mathbb{F})$ and $\mathrm{Sh}(\mathbb{U})$ to really be toposes, we have to make the category $\mathbb{F}$ into a set. Formally, this is done by introducing a universe of sets into set theory, e.g. V_κ, where κ is an inaccessible cardinal. We will write product and coproduct in Sets instead of small product and small coproduct in V_κ. For more details see Palmgren [14] and Eliasson [4].

Let L be a first order language and $I = \langle S, R_1, R_2, \ldots, f_1, \ldots, c_1, \ldots \rangle$ an L-structure. Let ${}^*I$, the *-transform of I, be the L-structure in $\mathrm{Sh}(\mathbb{U})$ defined as follows:

- Set S: ${}^*S$ the representable sheaf previously defined.

- Constant $s \in S$: ${}^*s$ constant function

$$\lambda x.s \in {}^*S(\mathcal{U}).$$

- Relation $R \subseteq S$: ${}^*R$ subsheaf of ${}^*S$ given at $\mathcal{U}$ by

$$\alpha \in {}^*R(\mathcal{U}) \iff (\exists U \in \mathcal{U})(\forall x \in U)\alpha(x) \in R.$$

- Function $f : T \to S$: ${}^*f$ representable natural transformation from ${}^*T$ to ${}^*S$ given at $\mathcal{U}$ by

$${}^*f_{\mathcal{U}}(\alpha) = \lambda x.f(\alpha(x)).$$

We also define what it means to be *standard* for a $\gamma \in {}^*S(\mathcal{U})$:

- $\mathrm{St}(\gamma)$ if and only if γ is constant on some $U \in \mathcal{U}$.

Thus every L-structure I (in Sets) gives rise to an $L \cup \{\mathrm{St}\}$-structure ${}^*I$ in $\mathrm{Sh}(\mathbb{U})$. With the standard predicate we can use $\mathrm{Sh}(\mathbb{U})$ to model non-standard theories such as Nelsons [12] (see Eliasson [4]).

We have the usual interpretation of the the logical symbols in the two Grothendieck toposes. Below we give the sheaf semantics for $\mathrm{Sh}(\mathbb{U})$ in full detail. For the more complicated case $\mathrm{Sh}(\mathbb{F})$ we refer the reader to Palmgren [14].

Theorem 2.5. *Let $\mathcal{U}$ be an ultrafilter, Φ and Ψ arbitrary formulas and $\alpha \in {}^*T(\mathcal{U})$. Then*

(i) $\mathcal{U} \Vdash \Phi(\alpha) \wedge \Psi(\alpha)$
if and only if
$\mathcal{U} \Vdash \Phi(\alpha)$ *and* $\mathcal{U} \Vdash \Psi(\alpha)$,

(ii) $\mathcal{U} \Vdash \Phi(\alpha) \vee \Psi(\alpha)$
if and only if
$\mathcal{U} \Vdash \Phi(\alpha)$ *or* $\mathcal{U} \Vdash \Psi(\alpha)$,

(iii) $\mathcal{U} \Vdash \Phi(\alpha) \to \Psi(\alpha)$
if and only if
$\mathcal{U} \Vdash \Phi(\alpha)$ *implies* $\mathcal{U} \Vdash \Psi(\alpha)$,

(iv) $\mathcal{U} \Vdash \neg\Phi(\alpha)$
if and only if
$\mathcal{U} \nVdash \Phi(\alpha)$,

(v) $\mathcal{U} \Vdash (\exists x \in {}^*S)\Phi(\alpha, x)$
if and only if
for some $\beta : \mathcal{V} \to \mathcal{U}$ *and* $\delta \in {}^*S(\mathcal{V})$

$$\mathcal{V} \Vdash \Phi(\alpha \circ \beta, \delta),$$

(vi) $\mathcal{U} \Vdash (\forall x \in {}^*S)\Phi(\alpha, y)$
if and only if
for all $\beta : \mathcal{V} \to \mathcal{U}$ *and* $\delta \in {}^*S(\mathcal{V})$

$$\mathcal{V} \Vdash \Phi(\alpha \circ \beta, \delta).$$

As is evident in the theorem above, the internal logic in $\mathrm{Sh}(\mathbb{U})$ is classical, i.e. the topos is Boolean.

We think of the ultrasheaves as generalized ultrapowers. This is justified by the following generalization to ultrasheaves of Łoś's theorem for ultrapowers.

Theorem 2.6 (Moerdijk). *Let $\mathcal{F}$ be a filter, Θ an L-formula and $\alpha \in {}^*S(\mathcal{F})$. Then*

$$\mathcal{F} \Vdash {}^*\Theta(\alpha) \text{ if and only if } (\exists F \in \mathcal{F})(\forall x \in F)\Theta(\alpha(x)).$$

This result is proved by Moerdijk in [10] for $\mathrm{Sh}(\mathbb{F})$ and by the second author in [4] for $\mathrm{Sh}(\mathbb{U})$. That Łoś's theorem follows from it is proved in Eliasson [5].

3. $\mathrm{Sh}(\mathbb{U})$ is equivalent to a topos of sheaves on $\mathbb{F}$

We will study the topos $\mathrm{Sh}(\mathbb{U})$ of ultrasheaves and its relation to sheaves on the category $\mathbb{F}$ of filters. For clarity let A be the atomic topology on $\mathbb{U}$. We first define a new topology J_∞ on $\mathbb{F}$.

Definition 3.1. A basis for the *J_∞-topology* are small families $\{\alpha_i : \mathcal{F}_i \to \mathcal{F}\}_{i\in I}$ (for any set I) such that the induced morphism

$$\coprod_{i\in I} \mathcal{F}_i \to \mathcal{F}$$

is epic.

Note that from Blass [1] we know that the category $\mathbb{F}$ has all coproducts. Now the following theorem holds:

Theorem 3.2. $\mathrm{Sh}(\mathbb{U}, A) \cong \mathrm{Sh}(\mathbb{F}, J_\infty)$.

To prove the theorem we will need three lemmas.

Lemma 3.3. $(\mathbb{F}, J_\infty)$ *is a subcanonical site.*

Proof. Any epi in $\mathbb{F}$ is regular [10, Lemma 1.2]. Hence the covering map $\coprod_{i\in I} \mathcal{F}_i \to \mathcal{F}$ is regular, and the topology subcanonical. □

Lemma 3.4. *The collection of ultrafilters in $\mathbb{F}$ generates $\mathbb{F}$.*

See Eliasson [4, Prop. 2.2] for a proof, the details of which also imply the following.

Lemma 3.5. *Every object in $\mathbb{F}$ is covered (in the sense of J_∞) by objects in $\mathbb{U}$.*

Now the theorem follows by the *Comparison Lemma* (see, for instance, Mac Lane and Moerdijk [8]). It gives that the restriction $\mathbf{Sets}^{\mathbb{F}^{\mathrm{op}}} \to \mathbf{Sets}^{\mathbb{U}^{\mathrm{op}}}$ induces an equivalence of categories $\mathrm{Sh}(\mathbb{U}, A) \cong \mathrm{Sh}(\mathbb{F}, J_\infty)$.

4. $\mathrm{Sh}(\mathbb{U})$ IS THE DOUBLE NEGATION SUBTOPOS OF $\mathrm{Sh}(\mathbb{F}, J)$

In this section we prove that $\mathrm{Sh}(\mathbb{U})$ is the double negation subtopos of $\mathrm{Sh}(\mathbb{F}, J)$. Instead of working with sheaves for the Grothendieck topology J we will work with the (equivalent) Lawvere-Tierney topology j on $\mathbf{Sets}^{\mathbb{F}^{\mathrm{op}}}$.

A presheaf F in $\mathbf{Sets}^{\mathbb{F}^{\mathrm{op}}}$ is a *j-sheaf*, with respect to a topology j, if for every dense monomorphism $m : A \to E$ in $\mathbf{Sets}^{\mathbb{F}^{\mathrm{op}}}$, every map $A \to F$ extends uniquely along m to a map $E \to F$.

We will prove that the $j_{\neg\neg}$-sheaves are the same as the j_∞-sheaves in two steps. First we prove that a subpresheaf of a representable sheaf is dense with respect to the topology $j_{\neg\neg}$ if and only if the $\neg\neg$-closure of it is j-dense. Then we prove that the latter are exactly the dense subobjects with respect to j_∞. Note that it is enough to prove this for subobjects of representable sheaves.

We will prove both lemmas working with sieves on a filter, rather than in the Heyting algebra of subobjects. So, we will list some sieve formulations of topological and algebraical concepts.

- A sieve on $\mathcal{F}$ is a subpresheaf $A \rightarrowtail \mathbf{y}(\mathcal{F})$.
- The j-closure of A, which is the sheafification of A, is the set:

$$\begin{aligned}\overline{A}^j =& \{h : \mathcal{G} \to \mathcal{F} \mid h^*A \in J(\mathcal{G})\} \\ =& \{h : \mathcal{G} \to \mathcal{F} \mid \exists \{g_i : \mathcal{G}_i \to \mathcal{G}\}_{i=1}^n \in J(\mathcal{G}) \\ & \text{such that } h \circ g_i \in A,\ i = 1, \dots n\}.\end{aligned}$$

- A is j-dense if and only if A is a J-covering sieve of $\mathcal{F}$.
- If B is also a sieve on $\mathcal{F}$ then

$$\begin{aligned}A \Rightarrow B = \{h : \mathcal{G} \to \mathcal{F} \mid & \forall g : \mathcal{H} \to \mathcal{G} \\ & h \circ g \in A \Rightarrow h \circ g \in B\},\end{aligned}$$

  which is a sieve on $\mathcal{F}$.

We know that the double negation closure of a subpresheaf A, $\neg\neg A$, is $(A \Rightarrow 0) \Rightarrow 0$ and this can be calculated as

$$\neg\neg A = \{h : \mathcal{G} \to \mathcal{F} \mid \forall g : \mathcal{H} \to \mathcal{G}\, \exists f : \mathcal{H}' \to \mathcal{H} \text{ such that } h \circ g \circ f \in A\}.$$

Moreover, from Mac Lane and Moerdijk [8, VI Lemma 1.2], we have that the double negation (in $\mathrm{Sh}(\mathbb{F}, J)$) of a j-sheaf E is $(E \Rightarrow \overline{0}^j) \Rightarrow \overline{0}^j$.

Here $\overline{0}^j$ is the sheafification of the empty presheaf which is isomorphic to $\mathbf{y}(0)$, the initial object in $\mathrm{Sh}(\mathbb{F}, J)$. To be precise, as a subobject of $\mathbf{y}(\mathcal{F})$:

$$\overline{0}^j(\mathcal{G}) = \begin{cases} \{!_{\mathcal{F}} \circ f\} & \text{if } \mathcal{G} \text{ is improper (i.e. isomorphic to 0),} \\ \emptyset & \text{if } \mathcal{G} \text{ is proper.} \end{cases}$$

Here $f : \mathcal{G} \to 0$ is an isomorphism.

We will prove that, for a subpresheaf $A \to \mathbf{y}(\mathcal{F})$ of a j-sheaf $\mathbf{y}(\mathcal{F})$ we have:

$$(\overline{A}^j \Rightarrow \overline{0}^j) \Rightarrow \overline{0}^j = 1_{\mathbf{y}(\mathcal{F})} \text{ if and only if } \overline{(A \Rightarrow 0) \Rightarrow 0}^j = 1_{\mathbf{y}(\mathcal{F})}.$$

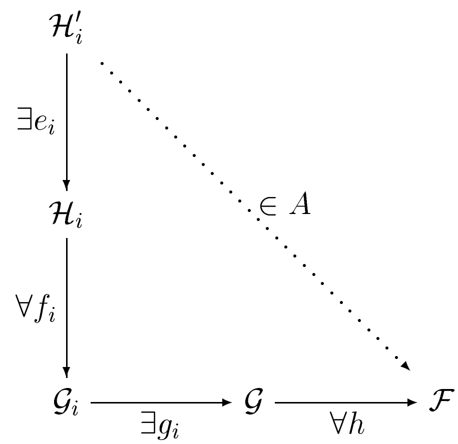
Assume that $\mathcal{F}$ is proper.

The righthand side then says that for all $h : \mathcal{G} \to \mathcal{F}$ we have $h \in \overline{(A \Rightarrow 0) \Rightarrow 0}^j$. Hence $\forall h : \mathcal{G} \to \mathcal{F}\ \exists \{g_i : \mathcal{G}_i \to \mathcal{G}\}_{i=1}^n \in J(\mathcal{G})$ such that $h \circ g_i \in \neg\neg A$ for all $i = 1, \dots, n$.

Hence we get the following condition:

$$\forall h : \mathcal{G} \to \mathcal{F}\, \exists \{g_i : \mathcal{G}_i \to \mathcal{G}\}_{i=1}^n \in J(\mathcal{G}) \text{ such that, for any } i \in \{1, \dots, n\}, \\ \forall f_i : \mathcal{H}_i \to \mathcal{G}_i\, \exists e_i : \mathcal{H}_i' \to \mathcal{H}_i \text{ such that } h \circ g_i \circ f_i \circ e_i \in A. \quad (1)$$

We illustrate this in a commutative diagram:



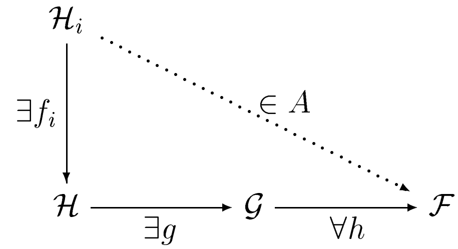
Remember that $\{g_i : \mathcal{G}_i \to \mathcal{G}\}_{i=1}^n$ is a J-cover of $\mathcal{G}$ and that $\mathcal{F}$ is proper.

If $\mathcal{F}$ is proper the lefthand side is equivalent to $\overline{A}^j \Rightarrow \overline{0}^j \leq \overline{0}^j$. We will study it pointwise, at a filter $\mathcal{G}$, i.e. $(\overline{A}^j \Rightarrow \overline{0}^j)(\mathcal{G}) \leq \overline{0}^j(\mathcal{G})$. We see that $(\overline{A}^j \Rightarrow \overline{0}^j)(\mathcal{G}) = \{h : \mathcal{G} \to \mathcal{F} \mid \forall g : \mathcal{H} \to \mathcal{G}\ h \circ g \in \overline{A}^j(\mathcal{H}) \Rightarrow h \circ g \in \overline{0}^j(\mathcal{H})\}$. If the filter $\mathcal{G}$ is proper, then $\overline{0}^j(\mathcal{G})$ is empty, and hence $(\overline{A}^j \Rightarrow \overline{0}^j)(\mathcal{G})$ is also empty.

For proper $\mathcal{G}$ we then have that:

$$\forall h : \mathcal{G} \to \mathcal{F}\, \exists g : \mathcal{H} \to \mathcal{G} \text{ such that } h \circ g \notin \overline{0}^j(\mathcal{H}) \\ \text{and } \exists \{f_i : \mathcal{H}_i \to \mathcal{H}\}_{i=1}^n \in J(\mathcal{H}) \text{ such that,} \\ \text{for any } i \in \{1, \dots, n\}, h \circ g \circ f_i \in A. \quad (2)$$

We illustrate this case too, with a commutative diagram:



Here we assume that $\mathcal{G}$ and $\mathcal{F}$ are proper.

Lemma 4.1. *For a subpresheaf $A \to \mathbf{y}(\mathcal{F})$ of a j-sheaf $\mathbf{y}(\mathcal{F})$ we have:*

$$(\overline{A}^j \Rightarrow \overline{0}^j) \Rightarrow \overline{0}^j = 1_{\mathbf{y}(\mathcal{F})} \text{ if and only if } \overline{(A \Rightarrow 0) \Rightarrow 0}^j = 1_{\mathbf{y}(\mathcal{F})}.$$

Proof. If the filter $\mathcal{F}$ is improper then $\mathbf{y}(\mathcal{F})$ is isomorphic to its subsheaf $\overline{0}^j$. But both $(\overline{A}^j \Rightarrow \overline{0}^j) \Rightarrow \overline{0}^j$ and $\overline{(A \Rightarrow 0) \Rightarrow 0}^j$ are j-sheaves and, thus, greater than or equal to $\overline{0}^j$. Hence both sides of the equation are true, and therefore equivalent.

Now assume $\mathcal{F}$ is proper. Then we have the descriptions ((1) and (2) above) of the left- and righthand sides of the relation, and the scene is set for proving the equivalence:

"$\Longrightarrow$": Note that it is enough to find a cover on $\mathcal{F}$ (because of the stability of the topology J). Let $\mathcal{F}$ be covered by the identity $1_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$. Take any $f : \mathcal{G} \to \mathcal{F}$ and prove that there is an $e : \mathcal{H} \to \mathcal{G}$ such that $1_{\mathcal{F}} \circ f \circ e \in A$.

If $\mathcal{G}$ is improper then $f : \mathcal{G} \to \mathcal{F}$ is already in A, and you can take e to be the identity. If $\mathcal{G}$ is proper then by assumption, given $f : \mathcal{G} \to \mathcal{F}$, there is $g : \mathcal{H} \to \mathcal{G}$ and $f_1 : \mathcal{H}_1 \to \mathcal{H}$ such that $f \circ g \circ f_1 \in A$. Hence, let $e = g \circ f_1$.

"$\Longleftarrow$": Take any $h : \mathcal{G} \to \mathcal{F}$. If $\mathcal{G}$ is improper prove that $(\overline{A}^j \Rightarrow \overline{0}^j)(\mathcal{G}) \leq \overline{0}^j(\mathcal{G})$. Note that $(\overline{A}^j \Rightarrow \overline{0}^j)(\mathcal{G})$ contains at most one map, since there are only one map $h : \mathcal{G} \to \mathcal{F}$. But this map $!_{\mathcal{F}} \circ f : \mathcal{G} \to \mathcal{F}$ (where $f : \mathcal{G} \to 0$ isomorphism) is in $(\overline{A}^j \Rightarrow \overline{0}^j)(\mathcal{G})$ since, for any $g : \mathcal{H} \to \mathcal{G}$, $(!_{\mathcal{F}} \circ f) \circ g =!_{\mathcal{F}} \circ (f \circ g) \in \overline{0}^j(\mathcal{H})$.

If $\mathcal{G}$ is proper then find a $g : \mathcal{H} \to \mathcal{G}$ and a cover $\{f_i\}$ of $\mathcal{H}$ such that $h \circ g \circ f_i \in A$. By assumption there are $g_1 : \mathcal{G}_1 \to \mathcal{G}$ and $e_1 : \mathcal{H}_1 \to \mathcal{G}_1$ such that $h \circ g_1 \circ 1_{\mathcal{G}_1} \circ e_1 \in A$. Let $g = g_1 \circ e_1$ and the identity $1_{\mathcal{H}_1} : \mathcal{H}_1 \to \mathcal{H}_1$ be a covering. Then we have $h \circ g \circ 1_{\mathcal{H}_1} \in A$. □

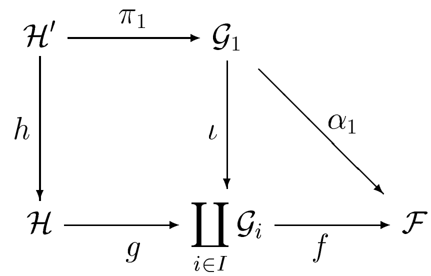
Our second lemma proves that the righthand side in the lemma above is equivalent to being j_∞-dense.

Lemma 4.2. *A subpresheaf A of a j-sheaf $\mathbf{y}(\mathcal{F})$ is j_∞-dense if and only if $\neg\neg A$ is j-dense.*

Proof. If the filter $\mathcal{F}$ is improper then $\mathbf{y}(\mathcal{F})$ is isomorphic to its subsheaf $\overline{0}^j$. But both $\overline{A}^{j_\infty}$ and $\overline{(A \Rightarrow 0) \Rightarrow 0}^j$ are j-sheaves and, thus, greater than or equal to $\overline{0}^j$. Hence both conditions stated in the lemma are true, and therefore equivalent. Now assume $\mathcal{F}$ is proper.

"$\Longrightarrow$": Take $\{\alpha_i : \mathcal{G}_i \to \mathcal{F}\}_{i \in I}$ a J_∞-covering in A. Prove that the induced map $f : \coprod_{i \in I} \mathcal{G}_i \to \mathcal{F}$ is in $\neg\neg A$. Take any $g : \mathcal{H} \to \coprod_{i \in I} \mathcal{G}_i$. Consider $\iota : \mathcal{G}_1 \to \coprod_{i \in I} \mathcal{G}_i$ (observe that we have $f \circ \iota = \alpha_1$).

Next take the pullback of $g : \mathcal{H} \to \coprod_{i \in I} \mathcal{G}_i$ and $\iota : \mathcal{G}_1 \to \coprod_{i \in I} \mathcal{G}_i$. Call the pullback $\mathcal{H}'$ and the projection on $\mathcal{H}$, $h : \mathcal{H}' \to \mathcal{H}$ as indicated in the diagram.

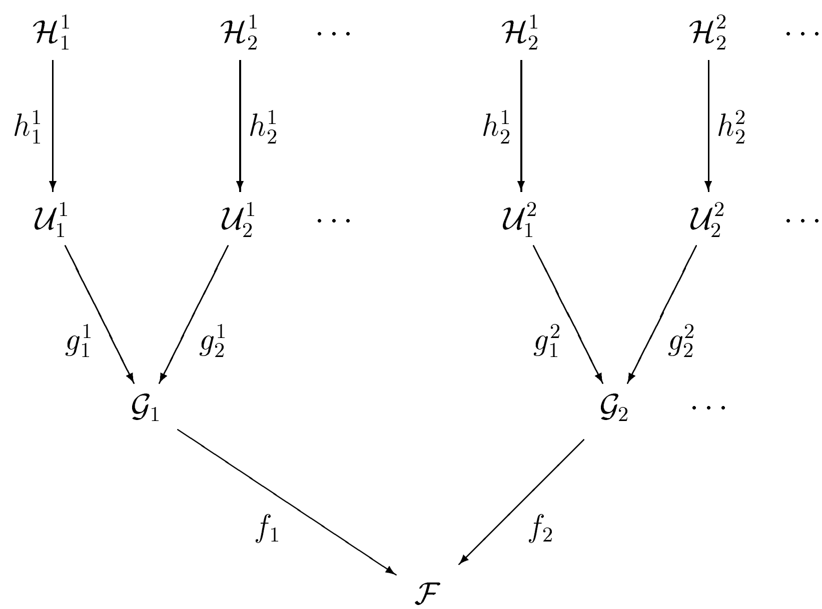


Then we have $f \circ (g \circ h) = (f \circ \iota) \circ \pi_1 = \alpha_1 \circ \pi_1 \in A$, since A is a sieve and $\alpha_1 \in A$. But $f : \coprod_{i\in I} \mathcal{G}_i \to \mathcal{F}$ is an epimorphism, and hence a J-covering of $\mathcal{F}$.

"$\Longleftarrow$": Take $\{f_i : \mathcal{G}_i \to \mathcal{F}\}_{i=1}^{n}$ a J-covering in $\neg\neg A$. We know that for every $\mathcal{G}_i$ there is a J_∞-covering of ultrafilters $\{g_j^i : \mathcal{U}_j^i \to \mathcal{G}_i\}_{j\in I_i}$ (Lemma 3.5).

Now since f_i is in $\neg\neg A$ there are $h_j^i : \mathcal{H}_j^i \to \mathcal{U}_j^i$, for $i = 1, \ldots, n$, $j \in I_i$, such that $f_i \circ g_j^i \circ h_j^i \in A$. But the families $\{f_i\}$ and $\{g_j^i\}$ are jointly epimorphic and the h_j^i's are epimorphisms (since $\mathcal{U}_j^i$ is an ultrafilter) and, hence, the family $\{f_i \circ g_j^i \circ h_j^i : \mathcal{H}_j^i \to \mathcal{F}\}$ is jointly epimorphic and a J_∞-covering of $\mathcal{F}$.

The proof is illustrated in this commutative diagram.



□

By Lemma 4.2 we have that a subpresheaf A of a representable sheaf is j_∞-dense if and only if its $\neg\neg$-closure is j-dense. By Lemma 4.1 we have that the $\neg\neg$-closure is j-dense if and only if the j-closure of A is $\neg\neg$-dense (in $\mathrm{Sh}(\mathbb{F}, J)$). Hence, maps from j_∞-dense subobjects of a sheaf extends to F if and only if maps from $j_{\neg\neg}$-dense subobjects extends to F. This gives that $\mathrm{Sh}_{\neg\neg}(\mathbb{F}, J) \cong \mathrm{Sh}(\mathbb{F}, J_\infty)$. Together with the result from section 3 we get the desired result:

Theorem 4.3. *A presheaf F is in $\mathrm{Sh}_{\neg\neg}(\mathbb{F}, J)$ if and only if it is in $\mathrm{Sh}(\mathbb{F}, J_\infty)$, and $\mathrm{Sh}(\mathbb{F}, J_\infty)$ is equivalent to $\mathrm{Sh}(\mathbb{U})$, thus*

$$\mathrm{Sh}_{\neg\neg}(\mathbb{F}, J) \cong \mathrm{Sh}(\mathbb{U}).$$

As was pointed out to us by an anonymous referee, there is the following elegant conceptual alternative to the foregoing pedestrian proof. The topos $\mathrm{Sh}(\mathbb{F}, J_\infty)$ is Boolean, since it is equivalent to the Boolean topos $\mathrm{Sh}(\mathbb{U})$, by Theorem 3.2. It thus suffices to show that the inclusion $\mathrm{Sh}(\mathbb{F}, J_\infty) \to \mathrm{Sh}(\mathbb{F}, J)$ is dense, i.e. that the direct image functor preserves the initial object 0. But this can be checked directly from the definitions of the topologies J and J_∞.

5. The double negation translation

In this section we show how the previous result can be used to transfer the truth of formulas between the toposes $\mathrm{Sh}(\mathbb{U})$ of ultrasheaves and $\mathrm{Sh}(\mathbb{F})$ of sheaves on filters. Since the methods are standard we will omit the proofs.

Between $\mathrm{Sh}(\mathbb{U})$ and $\mathrm{Sh}(\mathbb{F})$ there is a geometric morphism

$$\mathrm{Sh}(\mathbb{U}) \underset{i}{\overset{a}{\rightleftarrows}} \mathrm{Sh}(\mathbb{F})$$

consisting of the factors sheafification (with respect to the topology $\neg\neg$) $a : \mathrm{Sh}(\mathbb{F}) \to \mathrm{Sh}(\mathbb{U})$ and inclusion $i : \mathrm{Sh}(\mathbb{U}) \to \mathrm{Sh}(\mathbb{F})$.

For any ultrasheaf F in $\mathrm{Sh}(\mathbb{F})$ we then have the corresponding maps

$$\mathrm{Sub}_{\neg\neg}(F) \underset{i}{\overset{a}{\rightleftarrows}} \mathrm{Sub}(F).$$

The sheafification of such subobjects now corresponds to closure with respect to the double negation topology, previously written $\neg\neg(\cdot) : \mathrm{Sub}(F) \to \mathrm{Sub}_{\neg\neg}(F)$. The inclusion map of course acts as the identity on the closed subobjects of F.

Given a first order formula $\Theta(\alpha)$, with a free variable α of a sort (interpreted as) the ultrasheaf F, the interpretations of $\Theta(\alpha)$ in $\mathrm{Sub}_{\neg\neg}(F)$ and $\mathrm{Sub}(F)$ will in general be different, since the interpretations of the logical symbols are different in the two toposes.

The first translation is from classical to intuitionistic logic, and it takes the form of a double negation translation. Let superscript $(\cdot)'$ denote the usual double negation translation (see for instance [20]). Then:

Theorem 5.1. *Let $\Theta(\alpha)$ be a first order formula with a free variable of a sort interpreted as an ultrasheaf F. Then, if $\Theta(\alpha)$ is true in ultrasheaves* $\mathrm{Sh}(\mathbb{U})$, *its double negation translation $\Theta'(\alpha)$ is true in sheaves on filters* $\mathrm{Sh}(\mathbb{F})$.

In the other direction we have the following result:

Theorem 5.2. *Let $\Theta(\alpha)$ be a first order formula with a free variable of a sort interpreted as an ultrasheaf F. Assume, moreover, that $\Theta(\alpha)$ is without universal quantifiers and has double negation stable predicates. Then, if $\Theta(\alpha)$ is true in* $\mathrm{Sh}(\mathbb{F})$, *then $\Theta(\alpha)$ is also true in ultrasheaves* $\mathrm{Sh}(\mathbb{U})$.

Of course, classically any formula is equivalent to a formula without universal quantifiers, so we have as an easy corollary:

Corollary 5.3. *For every first order formula $\Theta(\alpha)$ with a free variable of a sort interpreted as an ultrasheaf F and with double negation stable predicates, there is a classically equivalent formula $\Theta^+(\alpha)$ such that if $\Theta^+(\alpha)$ is true in* $\mathrm{Sh}(\mathbb{F})$ *then $\Theta(\alpha)$ is true in ultrasheaves* $\mathrm{Sh}(\mathbb{U})$.

Theorem 5.2 cannot be extended to include universal quantifiers, as can be seen by considering the following fact. In Moerdijk [10] it is shown that

$$\mathrm{Sh}(\mathbb{F}) \models \neg(\forall x \in {}^*\mathbb{N})[\mathrm{St}(x) \vee \neg\mathrm{St}(x)].$$

Note that ${}^*\mathbb{N}$ is an ultrasheaf, and the standard predicate St is double negation stable.

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