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Calculation and Strategy in the Equation Solving Tutor

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Abstract

This paper examines performance on an intelligent tutoring system designed to teach students in a first-year algebra class to solve simple linear equations. We emphasize the effects of requiring students to complete low-level arithmetic operations on higher-level strategic decisions. On $aX+b=c$ problems, students who were required to perform arithmetic became less likely to solve such problems by first dividing by a than students who were not required to perform the arithmetic required to carry out the operation. The shift away from this strategy is in keeping with the predictions of ACT-R. We discuss these results in terms of the educational implications of providing computational tools to students learning basic mathematics.

Introduction

The availability of low-cost calculators has changed the way mathematics is taught. Where arithmetic skills such as long division were once given strong emphasis in the early grades, they are now seen as less important, on the assumption that students in higher-level mathematics courses will have access to calculators. A similar transformation can be expected in high school algebra classes as low-cost calculators capable of complex symbol manipulation become available (Fey, 1989). The reaction to these changes has been mixed. On the one hand, many educators welcome the availability of such tools, believing that they will allow students to concentrate on more strategic problem-solving skills and the conceptual structure of the domain rather than being held back by the requirement that they master the rather low-level skills now being performed by calculators. On the other hand, many educators worry that students cannot really understand mathematical problem-solving skills without a firm grounding in the basic skills of arithmetic and symbol manipulation. Without knowledge of the basic skills, the argument goes, students' problem solving becomes an exercise in applying a meaningless (to the student) algorithm.

Our focus here is on a more subtle effect of the loss of the use of low-level skills on higher-level tasks. We look at how the requirement that students perform arithmetic computations affects their strategic planning when they solve equations. Our data shows that students who were released from the requirement to perform arithmetic computations when they solve problems of the form $aX+b=c$ were more likely to divide by the coefficient a as a first step to solving the problem. Furthermore, we show that students who were so biased were at a disadvantage when asked to solve such problems in a context where they were required to perform the arithmetic calculations. The disadvantage results not from an inability to complete the arithmetic but from a learned bias to use a strategy that is less effective in the new context.

Our data is taken from performance on an intelligent tutoring system designed to teach students in a first-year algebra class to solve simple linear equations. The tutor provides assistance intermediate between unassisted equation solving with pencil and paper and a more powerful system in which students can solve equations simply by pushing a "solve" key. In some conditions, students are required to complete the arithmetic operations required to solve problems. In other conditions, the computer-calculates these operations for the students.

Overview of the Tutor

The Equation Solving Tutor (EST) is one of a class of ACT-based model-tracing tutors developed using the Tutor Development Kit (Anderson and Pelletier, 1991). Such tutors contain an expert system capable of solving problems in the domain. When students perform an action, the tutor checks to see if the action is one that the expert system would take in this situation. If so, the student proceeds uninterrupted. If not, the tutor signals an error. Errors can be signaled in one of two ways. In "immediate feedback" mode, the tutor beeps when it detects an error, and it removes the offending action. In "flag feedback" mode, the tutor beeps but the erroneous action remains. The action is printed in a different style than correct actions, so that students can immediately identify their error. None of the analyses presented here were affected by feedback mode, so we combine data from the two modes.

At any time, the student can ask for help, which the tutor gives by examining the action the expert system would take in the student's situation. Some additional rules, called "buggy rules," are used to identify and give immediate remediation for common student errors. The tutor tracks the student's competence on a number of underlying skills (corresponding to rules in the expert system), so that it can present problems tailored to address weaknesses in the student's understanding (Corbett and Anderson, 1992; Anderson and Corbett, 1993).

The tutor presents students with a linear equation which they are to solve (see Figure 1). A menu offers them four choices for transforming the equation: "Add to both sides," "Subtract from both sides," "Divide both sides" and "Multiply both sides." After picking any of these options, the student is prompted for a value to add, subtract, multiply or divide. Three additional options are available for simplifying an equation: "Combine like terms," "Reduce fractions," and "Multiply through." Students need to qualify these actions by indicating whether the simplification should be performed on the left side, right side or both sides of the equation. In addition to these items, there are menu options which can be used to distribute, convert fractions to decimals, indicate completion of the problem, undo the last step or ask for help.

Students

Students taking Algebra I in three of the Pittsburgh Public Schools used the tutor. 146 students at Langley High School used the tutor for four consecutive class periods in December of 1993. 204 students at Brashear and Carrick High Schools used the tutor for four class periods in March and April of 1994. Since the Langley students used the tutor earlier in the school year, and since there were some modifications to the tutor before it was used at Brashear and Carrick (noted below), we report results from Langley separately from those from the other schools. At all schools, the classes emphasized algebraic problem solving rather than the kind of symbol manipulation being taught by the EST, so the majority of students' class instruction in equation solving was with EST.

Calculation Conditions

The requirement that students perform arithmetic calculations in the context of solving problems varied between classes. The two main calculation conditions will be referred to as "computer-calculates" and "student-calculates." In the computer-calculates condition, students needed to specify the strategic actions required to solve the equation, but they did not have to compute the results of this action. For example, if the equation was $X=10/3-2/3$, the student could pick "Combine like terms [right side]" and the computer would respond with $X=8/3$. In contrast, in the student-calculates condition, after specifying "Combine like terms [right side]", the student would be prompted with $X=RIGHT$. The student then needed to click on the word "RIGHT" and type the resulting right side.

At Brashear and Carrick High Schools, there were two computer-calculates conditions. In the "student simplifies" variant, the computer carried out calculations without simplifying the equation. For example, if a student indicated that the tutor should "Subtract 3 from both sides" of the equation $X+3=10$, the tutor would respond with $X+3-3=10-3$. The student then needed to pick "Combine like terms [both sides]" to see the final equation, $X=7$. In the "tutor simplifies" variant, the computer responded to the "Subtract 3 from both sides" command with the final equation. At Langley High School, only the "student simplifies" variant

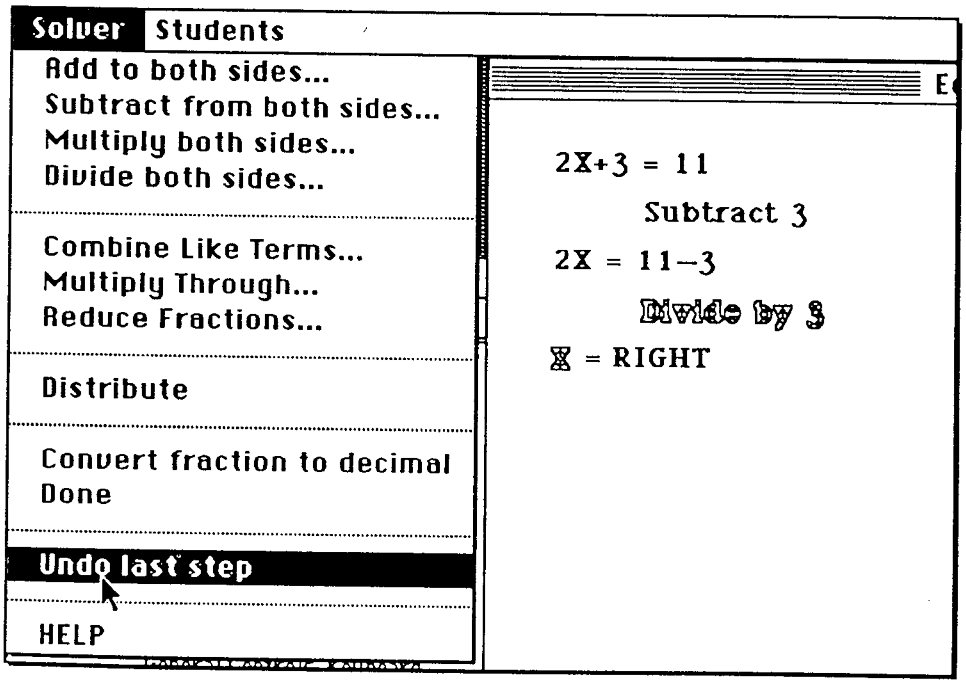

Figure 1: The menu and main window of EST in calculate, flag-feedback mode. The highlighted actions have been identified as errors by the tutor

of the computer-calculates condition was used. Since our interest here is in the difference between the student-calculates and computer-calculates conditions, in most cases, our data combines the "tutor simplifies" and "student simplifies" variants.

In the student-calculates condition, students could (but did not have to) enter a simplified version of the equation. For example, a student in the student-calculates condition, after subtracting 4 from both sides of $X+4=10$ would be prompted with *LEFT=RIGHT*. The student could then click on the word LEFT and enter X and click on the word RIGHT and enter 6. In practice, the vast majority of students chose to enter the simplified version of the equation. Choosing to subtract 4 from both sides will be referred to as a strategic action. Entering the results "X" and "6" will be referred to as calculations.

Students in the student-calculates condition were able to use a standard (non-symbolic) calculator if they wished. Thus, the difference between the calculation conditions was not so much that students in the student-calculates condition needed to rely on their arithmetic skills but that they needed to understand not only which strategic actions were appropriate but how to carry out these actions. For example, a student in the student-calculates condition might know that subtracting 4 is the appropriate action in $X+4=10$ but may not understand the way that subtracting 4 transforms the equation. In the computer-calculates condition, the student can give the "subtract 4" instruction and let the computer show the result. In the student-calculates condition, the student needs to understand the effect of that action.

Procedure

Students were administered a paper-based pretest by their teachers the day before starting the tutor[1]. In their first day using the tutor, the teacher guided students through the first two or three problems. After that, they were free to work at their own pace. Although students were encouraged to ask the tutor for help when they had problems, the teacher was available to help students individually. Following the four class days on the tutor, a paper-based posttest identical in format to the pretest was given. Students were allowed to use non-symbolic calculators on the pre- and posttests.

Curriculum

Students progressed from easier to more difficult problems. In each section, students had to complete a small number of required problems. If a student did not demonstrate mastery of the skills being traced in the section (as determined by the knowledge tracing algorithm described in Corbett and Anderson, 1992), additional problems targeted at the deficient skill were given until mastery was demonstrated.

Since the amount of time students spent on the tutor was limited, students progressed to different levels of achievement. The curriculum used at Brashear and Carrick High school differed from that used at Langley, primarily in that problems presented in the first section at Langley were split into three sections at Brashear and Carrick. The curriculum is summarized in Table 1.

Predictions

The contrast between the calculation conditions provides us with a chance to see whether the requirement that student perform arithmetic calculations affects decisions at the strategic level. Consider the equation $3X+4=10$. In order to solve for X, we need to isolate the X term (by subtracting 4 or adding -4) and remove the coefficient of X (by dividing by 3 or multiplying by 1/3). Typically, students isolate the X term first, perhaps because doing so avoids creating the intermediate fractions 4/3 and 10/3. This can be an important consideration for students who have difficulty manipulating fractions. In the computer-calculates condition, however, there is little incentive to isolate the X term first, since the computer will take care of combining the fractions.

According to the ACT-R theory (Anderson, 1990), productions are selected based on their expected utility. This utility, in turn, depends on an estimate of the probability that the production will succeed and an estimate of the probability that the goal will be reached, given that the production succeeds. Thus, the theory predicts that students choosing between the "subtract first" and the "divide first" strategy will consider both the probability that they will be able to specify the first operation and the probability that, if they specify that operation correctly, they will still be able to correctly solve the problem.

This leads to four predictions:

1. Although students may be initially biased to use the subtract-first strategy, this bias should be no stronger in the student-calculates condition than in the computer-calculates condition.

2. Since the "divide-first" strategy creates fractions (with which students typically have trouble), students in the student-calculates condition will have difficulty completing problems which they have started with that strategy. In the computer-calculates condition, in contrast, the presence of fractions causes no special difficulties. Thus, students in this condition will likely succeed with the divide-first strategy.

3. Students in the student-calculates condition will be unlikely to return to the divide-first strategy on subsequent problems, since they have experienced failure with the strategy. Students in the computer-calculates condition will return to the divide-first strategy, since they are successful with it.

4. Students in the computer-calculates condition who are successful using the divide-first strategy will be less successful on the paper-based posttest, where the strategy is less effective.

[1]Due to scheduling problems, three of the classes at Langley High School did not take pre- or posttests. Taking absenteeism in other classes into account, data from the pre- and posttests is based on 70 students at Langley and 127 students at Brashear and Carrick.

Results

Initial Bias

In prior instruction on equation solving, students have likely been told to use the subtract-first strategy. Thus, we should expect to see many students use that strategy exclusively. If students understand the way that the effectiveness of the divide-first strategy varies with calculation condition, we might expect students in the computer-calculates condition to try the strategy more often than those in the student-calculates condition.

In order to assess the relative bias against the divide-first strategy in the two conditions, we measured the percentage of students using the divide-first strategy on at least one problem. As shown in Table 2, there was no initial bias at Langley (χ^2(1)<1), but at Brashear and Carrick, students in the computer-calculates condition were more likely to try the divide-first strategy, χ^2(1)=20.0, p<.01. Since Brashear and Carrick students were tested later in the year, it may be the case that they already had enough experience with these types of problems to understand the implications of the divide-first strategy. However, overall, as predicted, subjects showed a bias against the divide-first strategy, and this bias was stronger in the calculate condition.

Table 2: Percentage of student using divide-first strategy on at least one problem

School	Student-calculates	Computer-calculates
Langley	24	29
Brashear/Carrick	19	49

Effectiveness of *divide-first* strategy

We consider a strategy to be effective if a student is able to completely solve a problem following that strategy. A failure of the strategy is recorded if, for example, a student starting by dividing by the coefficient of X and then selected "undo last step" and continued by subtracting the constant and later dividing by the coefficient of X. As expected, students in the "no-calculate" condition were much more successful in the divide-first strategy than students in the "calculate" condition (at Langley, χ^2(1)=55.5; at Brashear and Carrick, χ^2(1)<56.7, both p<.01).

Table 3: Percentage of problems started with divide-first strategy that were completed with that strategy

School	Student-calculates	Computer-calculates
Langley	32	84
Brashear/Carrick	50	97

Table 1: Curriculum used with the EST

Section	Example problems
1. Addition and subtraction, integers	13 = -11+X Y+4=20
2. Multiplication and division, integers dividing evenly *(included in section 1 at Langley)*	-Y=20 25=5X -6Y=36
3. Multiplication and division, integers not dividing evenly *(included in section 1 at Langley)*	4Y=27 3X=20
4. Single-transformation, decimals	Y-2.5 = 11.13 -3.9 = 8.43X
5. Two transformations, integers	-6Y-11 = -10 -8 = -92+Y
6. Two transformations, decimals	-3.08+83.09Y = 8.28 36.65 = -49.53-0.59Y
7. Variables on both sides, integers and decimals *(at Langley, fractions problems were included)*	-4-11Y = -7-22Y -9.47+7.88Y = 88.49-3.11Y
8. Single-transformation, fractions *(not used at Langley)*	4Y/3 = 20 Y+1/2=30
9. Two transformations, fractions *(at Langley, this section came before the "variables on both sides" section)*	-1/3 = -4Y/7+7/12 -3 = 11+8Y/5
10. Distribution, decimals, fractions and integers	8.61 = -12.36(-1.69X-43.33) 7 = 63(-7+71X)
11. Variable in the denominator of a fraction, integers, fractions and decimals	15.85/(92.85Y) = 55.94 -8/(11Y) = -6/5

Subsequent use of *divide-first* strategy

The ACT-R theory predicts that students who have failed to solve a problem with the divide-first strategy will be unlikely to return to this strategy. As Table 4 shows, this prediction is confirmed. Note that following the divide-first strategy in the calculate condition requires (at least) two separate productions: one to select the action to perform ("divide by the coefficient") and another to perform the division (e.g. divide 3X by 3 to get X). Students' change of strategy is not based their failure to correctly select the action; in fact, it is by the correct specification of this action that we classify students as having started to solve with the divide-first strategy. Rather, the students' failure is in the later calculation production. Nevertheless, the next time students are in this situation, they are less likely to use the divide-by-coefficient *selection* production. In ACT-R's terms, the estimate of the selection production's success increases (since it was successfully executed), but the estimate of the probability of reaching the goal, given the production's success, decreases because of later difficulties in performing the division.

Table 4: Number of times students return to divide-first strategy after using it once

School	Student-calculates	Computer-calculates
Langley	0.36	2.06
Brashear/Carrick	0.86	4.89

Performance on the posttest

Finally, we predict that students in the no-calculate condition who tried the divide-first strategy on the tutor will perform worse on the posttest than students in the calculate condition who tried the divide-first strategy. This is because students in the no-calculate condition will be encouraged to use the strategy on the posttest while subjects in the calculate condition will be discouraged from using the strategy. We also assume that the strategy will be relatively ineffective on the paper-based posttest (as it was in the "calculate" condition on the tutor). The posttest data for students who attempted the divide-first strategy are presented in Table 5. An analysis of variance shows no effect in the Langley data, $F(1,32)<1$. The Brashear and Carrick data show a significant effect, $F(1,52)=5.31$, $p<.05$. At Brashear and Carrick, students in the calculate condition who tried the divide-first strategy (and likely failed at the strategy) benefited more from the tutor than students who never tried it at all (34% vs. 17%). At Langley, students who never tried the divide-first strategy benefited more from the tutor (a 38% improvement) than students who tried it in either condition.

Table 5: Percent improvement from pre- to posttest for students who tried the divide-first strategy

School	Student-calculates	Computer-calculates
Langley	9	12
Brashear/Carrick	34	13

Discussion

This study used data from a computer-based tutoring system to explore how the requirement that students perform arithmetic calculations while solving equations affects the high-level strategies they follow. Our results show a shift in strategy that is consistent with the predictions of the ACT-R theory. The shift is somewhat unusual in that it depends on a decrease in the probability of reaching the goal following the successful application of a production, rather than a shift away from using the production following a failure of the production itself.

While our data shows that students who were biased towards the divide-first strategy were at a disadvantage on a paper-based posttest, we must be cautious about making any conclusions based on these data about the proper use of technology in the mathematics classroom. Overall, we found no significant differences due to calculation condition in posttest scores (F's at both Langley and Brashear and Carrick were less than 1.0). Freeing students from having to perform arithmetic calculations has some benefits, as well. Students in the computer-calculates condition advanced further in the curriculum in their four days on the tutor (significantly at Brashear and Carrick: $F(1,147) = 44.3$, $p<.01$; non-significantly at Langley). We may also question whether performance on a paper-based test constitutes a reasonable measure of performance in a world where powerful calculators are becoming more and more available. Finally, while we have demonstrated that the computer-calculates condition biased some students to use a less-effective strategy on paper, we have not shown that they suffer in their understanding. The fact that students at Brashear and Carrick who used the divide-first strategy and failed tended to perform better on the posttest than those who never tried it provides an intriguing hint, however. It is possible that, through their failure, they gained a better understanding of how the different strategies work. We are currently conducting research to explore this possibility.

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