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Jeff A. Heisserman
Carnegie Mellon University

Carnegie Mellon University.Engineering Design Research Center.

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J.A. Heisserman

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Jeff A. Heisserman
Engineering Design Research Center
& Department of Architecture
Carnegie Mellon University
Pittsburgh, Pennsylvania 15213

Abstract

The Euler-Poincaré equation, $v - e + f = 2(s - g)$, relates the numbers of topological elements of 2-manifold surfaces. Here v, e, f, s, and g refer to the numbers of vertices, edges, faces, shells (surfaces) and handles. However, the equation does not correctly relate the elements of non-manifold surfaces, and specifically the boundaries of r-sets.

We introduce a generalized form of the Euler-Poincaré equation

$$v' - e' + f = 2(s' - g)$$

which relates the number of vertex uses, edge uses, faces, shell uses, and handles of both 2-manifold surfaces and non-manifold surfaces of r-sets. In addition, I introduce the equation

$$(v' - v) - (e' - e) - (s' - s) = g' - c$$

which relates the number of vertex uses, vertices, edge uses, edges, shell uses, and shells to the number of non-manifold handles and non-manifold chambers.

We present proofs of the correctness of the new equations.

Introduction

The Euler-Poincaré equation,

$$v - e + f = 2(s - g), \tag{1}$$

relates the numbers of topological elements of 2-manifold surfaces. Here v, e, f, s, and g refer to the numbers of vertices, edges, faces, shells (surfaces) and handles. However, the equation does not correctly relate the elements of nonmanifold surfaces, and specifically the boundaries of r-sets [4].

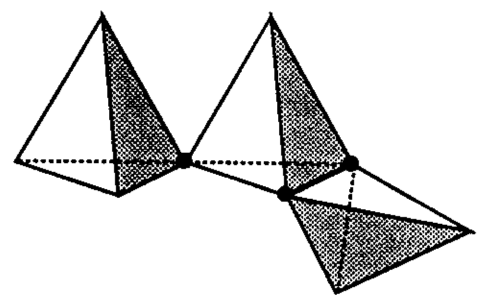


Figure 1: A nonmanifold solid.

We can see this clearly by a simple example. The nonmanifold solid in Figure 1 may be formed by joining two tetrahedra at a single vertex, and a third tetrahedron at an edge. The heavy solid vertices and thick edges in the figure indicate vertices and edges with nonmanifold neighborhoods. The solid has 9 vertices, 17 edges, 12 faces, one (nonmanifold) shell, and zero handles. Clearly,

$$9 - 17 + 12 \neq 2(1 - 0).$$

What then is the relation between the elements of regular solids?

We introduce a generalized form of the Euler-Poincaré equation

$$v' - e' + f = 2(s' - g) \tag{2}$$

which relates the number of vertex uses, edge uses, faces, shell uses, and (manifold) handles of both 2-manifold surfaces and nonmanifold surfaces of r-sets. In addition, we introduce the equation

$$(v' - v) - (e' - e) - (s' - s) = g' - c \tag{3}$$

which relates the number of vertex uses, vertices, edge uses, edges, shell uses, and shells to the number of nonmanifold handles and nonmanifold chambers.

To understand these equations, we need definitions of the new terms. We introduce definitions of vertex uses, edge uses, shell uses, nonmanifold handles, and nonmanifold chambers.

Vertex Uses

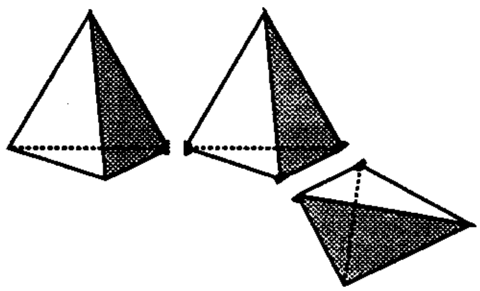


Figure 2: Vertex uses.

A ***vertex use***, denoted v', is a 2-manifold use of a vertex. This is a use of a vertex with respect to a shell. For example, the three nonmanifold vertices of the solid in Figure 1 each have two vertex uses. These as shown in Figure 2. Any vertex on a 2-manifold surface will have a single vertex use. A vertex with a nonmanifold neighborhood will have multiple vertex uses.

Edge Uses

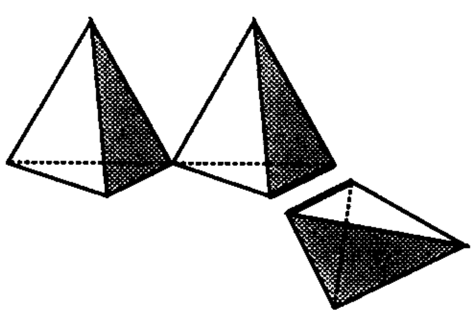


Figure 3: Edge uses.

An *edge use*, denoted e', is a 2-manifold use of an edge, and corresponds to a pair of incident faces with respect to that edge. The two edge uses of the nonmanifold (thickened) edge of the solid in Figure 1 are shown in Figure 3. Any edge on a 2-manifold surface will have a single pair of incident faces, thus a single edge use. An edge with a nonmanifold neighborhood will have multiple edge uses.

Shell Uses

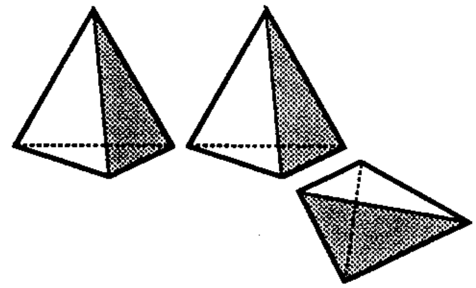


Figure 4: Shell uses.

A ***shell use***, denoted s', is a 2-manifold piece of a connected surface. We can separate the shell uses of a surface by splitting apart the vertices and edges with nonmanifold neighborhoods. The three shell uses of the nonmanifold solid in Figure 1 are illustrated in Figure 4. We can define a shell use as a partition of the faces, edge uses, and vertex uses of a shell (surface) such that one can traverse from any face, edge use, or vertex use of that partition to any other by their adjacencies without crossing any faces, edge uses, or vertex uses of any other partition. A 2-manifold shell will have a single shell use, and a 2-manifold solid will have an equal number of shells and shell uses. A nonmanifold solid may have more shell uses than shells.

Nonmanifold Handles

We can define the first Betti number (connectivity number) of a 2-manifold surface (our definition is adapted from [3]), in order to determine its number of handles and genus. Using a variation of this definition, we can define the connectivity number for a nonmanifold surface. Next, we introduce the concept of a *nonmanifold handle*, and use the new connectivity number to determine the number of nonmanifold handles and the genus of a nonmanifold solid.

Definition 1 *The first Betti number h_1 is the maximum size of a set of closed curves that taken together can be drawn on a surface that will not divide the surface into two or more pieces. This is also known as the connectivity number of the surface.*

For any nonmanifold solid, we can create a 2-manifold surface by offsetting the surface of the solid by an arbitrarily small amount ϵ. By choosing a sufficiently small ϵ, we can guarantee that we do not create additional handles or nonmanifold conditions from self-intersection of the surface. By offsetting the surface, we are able to "thicken" the vertices and edges that have nonmanifold neighborhoods, thereby eliminating these nonmanifold conditions. This is used to define the connectivity number for nonmanifold solids.

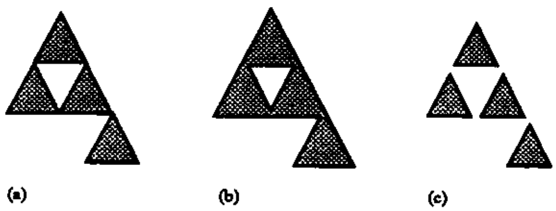


Figure 5: A solid, its offset and minus offset solids.

Definition 2 *The manifold connectivity number h_1 of the surface of a nonmanifold solid is the maximum size of a set of closed curves that taken together can be drawn on a 2-manifold minus offset surface of that solid (offset from the complement of the solid) that will not divide the surface into two or more pieces.*

Definition 3 *The nonmanifold connectivity number h'_1 of the surface of a nonmanifold solid is the maximum size of a set of closed curves that taken together can be drawn on a 2-manifold offset surface of that solid that will not divide the surface into two or more pieces.*

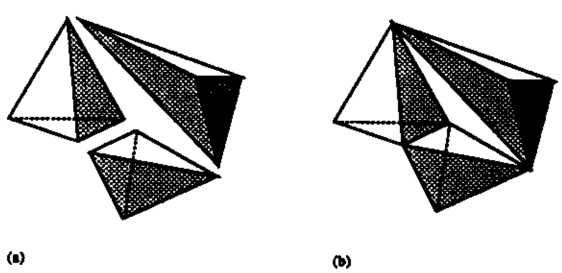


Figure 6: A solid with a nonmanifold handle.

A *nonmanifold handle*, denoted g', is a "pinched" handle. That is, it is connected through one or more vertices or edges that have nonmanifold neighborhoods. Three tetrahedra may be combined to form a nonmanifold handle as is illustrated in Figure 6. The resulting solid has one nonmanifold handle that is pinched in three places. A (manifold) handle has no vertices or edges with nonmanifold neighborhoods.

This then gives us the ability to define nonmanifold handles and the genus of a nonmanifold solid.

Definition 4 *The number of nonmanifold handles, denoted g', is $g' = h'_1/2 - h_1/2$.*

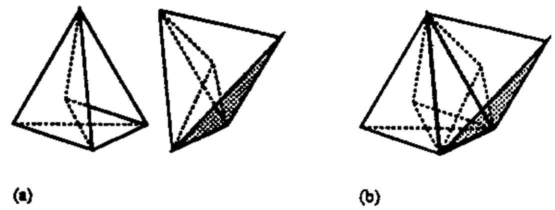


Figure 7: A solid with a nonmanifold chamber.

Nonmanifold Chambers

The last nonmanifold element that appears in our equations is the *nonmanifold chamber*. This is a bubble or space that is trapped when two surfaces (or two portions of the same surface) are joined together at a loop of edges with nonmanifold neighborhoods. For example, a nonmanifold chamber is is formed by joining the two modified tetrahedra of Figure 7. The loop of edges divides the surface of the solid into two portions: an inside surface of the solid (forming the chamber), and an outside surface of the solid. (Topologically speaking, the portion of the surface considered as the chamber is arbitrary.) We denote the number of nonmanifold chambers of a solid by c.

Proof of Correctness

Finally, we are ready to explain the relationship between these elements of nonmanifold solids.

Theorem 1 *The Euler equation for regular solids is*

$$v' - e' + f = 2(s' - g)$$

where v', e', and s' are the numbers of vertex uses, edge uses, and shell uses, respectively.

Proof. We will systematically decompose the nonmanifold solid into its 2-manifold components. We begin by decomposing the edges with nonmanifold neighborhoods. For each edge with a nonmanifold neighborhood, we split the edge, introducing a new edge adjacent to the two vertices of the original edge and to one pair of adjacent faces. This will remove one edge use from the original edge, and will create a new edge with a single edge use. The total number of edge uses is unchanged. We repeat this process until no more edges have multiple edge uses, and $e' = e$.

Next, we decompose the vertices with nonmanifold neighborhoods. We split a vertex with a nonmanifold neighborhood and introduce a new vertex adjacent to one of the cycles of edges and faces. This will remove one vertex use from the original vertex, and will create a new vertex with a single vertex use. The total number of vertex uses does not change. We will have one of three cases:

1. If the original vertex and the new vertex are on the same shell use, we remove a nonmanifold handle. This does not affect the number of shells or shell uses.
2. If the original vertex and the new vertex are on different shell uses, but remain on the same shell, then we remove a nonmanifold handle. This does not affect the number of shells or shell uses.
3. If the original vertex and the new vertex are on different shell uses and separated sets of connected shell uses, then we split the shell. This partitions the shell uses into two separate shells, each with connected shell uses, and creating a new shell. The total number of shell uses is unchanged.

We repeat this process until no more vertices have multiple vertex uses, and $v' = v$. By our process of creating new shells, we will also have $s' = s$.

Once we have completed this process, the solid is decomposed into 2-manifold components and $v' = v$, $e' = e$, and $s' = s$. For these 2-manifold components, we know that

$$v - e + f = 2(s - g).$$

Therefore,

$$v' - e' + f = 2(s' - g).\square$$

Theorem 2 *The number of nonmanifold handles is determined by*

$$(v' - v) - (e' - e) - (s' - s) = g' - c.$$

Proof. Following our previous proof, we can confirm the relation of nonmanifold handles and chambers to other elements as we decompose the solid. First, we decompose the edges with nonmanifold neighborhoods. For each edge with a nonmanifold neighborhood, we split the edge, introducing a new edge adjacent to the two vertices of the original edge and to one pair of adjacent faces. This will remove one edge use from the original edge, and will create a new edge with a single edge use, thus $e' - e$ will be reduced by one. In addition, we will have one of two cases:

1. Splitting the edge removes a nonmanifold edge that is part of a cycle of nonmanifold edges, thereby destroying a nonmanifold chamber. Both c and $e' - e$ are reduced by one, and g' is unchanged.
2. Splitting the edge creates a nonmanifold handle. Then, $e' - e$ is reduced by one, and and g' is increased by one.

We repeat this process until no more edges have multiple edge uses. Therefore, $e' - e = 0$ and $c = 0$.

We next decompose the vertices with nonmanifold neighborhoods. We split a vertex with a nonmanifold neighborhood and introduce a new vertex adjacent to one of the cycles of edges and faces. This will remove one vertex use from the original vertex, and will create a new vertex with a single vertex use. The total number of vertex uses remains invariant. We will have one of three cases:

1. If the original vertex and the new vertex are on the same shell use, then the number of nonmanifold handles will be reduced by one. Therefore, the additional vertex balances the removed nonmanifold handle.
2. If the original vertex and the new vertex are on different shell uses, but remain on the same shell, then the number of nonmanifold handles will be reduced by one. Therefore, the additional vertex balances the removed nonmanifold handle.
3. If the original vertex and the new vertex are on different shell uses and different sets of connected shell uses, then we split the shell. This partitions the shell uses into two separate shells, each with connected shell uses, and creating a new shell. Therefore, the additional vertex cancels the additional shell.

We repeat this process until no more vertices have multiple vertex uses, and the solid is decomposed into

2-manifold components. Therefore, $v' - v = 0$, and $s' - s = 0$. Since we know that 2-manifold solids have no nonmanifold handles, it is clear that that

$$(v' - v) - (e' - e) - (s' - s) = g' - c.$$

The above operations are invertible. Thus the equation is preserved when applying these operations in either the forward or inverse directions. Using the 2-manifold components as the base case, we can construct any nonmanifold solid from its 2-manifold components. Therefore, by induction, we know that our equation holds for all nonmanifold solids.□

Examples

We can now see that our solid in Figure 1 satisfies this relation, since

$$v' - e' + f = 2(s' - g)$$
$$12 - 18 + 12 = 2(3 - 0).$$
$$(v' - v) - (e' - e) - (s' - s) = g' - c$$
$$(12 - 9) - (18 - 17) - (3 - 1) = 0 - 0$$

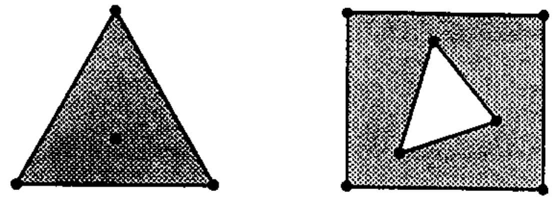
We can also verify some additional examples.

The solid in Figure 6(b) is composed of three tetrahedra. The first two tetrahedra are joined at two vertices and an edge, and the third is joined to the first two at two vertices. This solid has four vertices with nonmanifold neighborhoods and one edge with a nonmanifold neighborhood.

$$v' - e' + f = 2(s' - g)$$
$$12 - 18 + 12 = 2(3 - 0)$$
$$(v' - v) - (e' - e) - (s' - s) = g' - c$$
$$(12 - 8) - (18 - 17) - (3 - 1) = 1 - 0.$$

The solid in Figure 7(b) is composed of two manifold solids joined at three vertices and three edges. This solid has three vertices with nonmanifold neighborhoods. It also has three edges with nonmanifold neighborhoods in a cycle, forming a nonmanifold chamber.

$$v' - e' + f = 2(s' - g)$$
$$10 - 18 + 12 = 2(2 - 0)$$
$$(v' - v) - (e' - e) - (s' - s) = g' - c$$
$$(10 - 7) - (18 - 15) - (2 - 1) = 0 - 1.$$



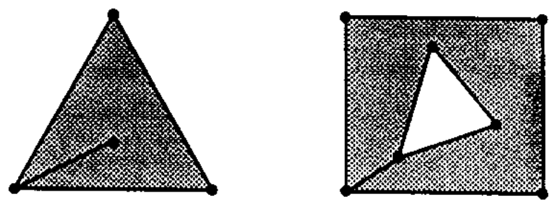
(a) (b)

Figure 8: Faces with multiple loops.

Multiple Face Boundaries

The Euler-Poincaré formula developed is restricted to faces represented as simple polygons. That is, a face may have only one boundary. We would like to represent faces with multiple boundaries, as shown in Figure 8.

Allowing faces with multiple boundaries requires an additional modification of the Euler-Poincaré formula.



(a) (b)

Figure 9: Simple faces.

A modification of the Euler-Poincaré formula was introduced by Braid, Hillyard, and Stroud [1] to additional face boundaries, which they denoted *rings*. Edges may be added to a face representation in order to connect the face boundaries. These edges are known as *artifact* or *bridge* edges. Artifact edges are added to the faces of Figure 8 as shown in Figure 9. The modified form of the Euler-Poincaré formula adds an artifact edge for each ring, denoted r,

$$v - (e + r) + f = 2(s - g).$$

We can allow faces with multiple boundaries in the generalized Euler-Poincaré formula by adding artifact edges. This extends the generalized Euler-Poincaré formula in a straightforward manner,

$$v' - (e' + r) + f = 2(s' - g).$$

We should also consider how multiple face loops affect the second equation. An artifact edge will always have exactly one edge use. Thus the equation

$$(v' - v) - (e' - e) - (s' - s) = g' - c$$

is unaffected by the addition of multiple face boundaries.

It is sometimes convenient to consider the total numbers of face boundaries, or *loops*, instead of the additional face boundaries, or rings. We can express the relationship between these three quantities,

$$r = (l - f),$$

where r, l, and f are the numbers of rings, loops, and faces.

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