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Dedicated to George Nemhauser for his 70th birthday

Abstract

In this paper we consider a semi-infinite relaxation of mixed integer linear programs. We show that minimal valid inequalities for this relaxation correspond to maximal lattice-free convex sets, and that they arise from nonnegative, piecewise linear, positively homogeneous, convex functions.

1 Introduction

Consider a mixed integer linear program (IP)

$$\begin{array}{llll} \min & cx & & \\ & Ax & = & b \\ & x_j & \in & \mathbb{Z} \quad \text{for } j = 1, \ldots, p, \\ & x_j & \geq & 0 \quad \text{for } j = 1, \ldots, n, \end{array}$$

where $p \leq n$, the matrix $A \in \mathbb{Q}^{m \times n}$, the row vector $c \in \mathbb{Q}^n$, the column vector $b \in \mathbb{Q}^m$ are data, and $x \in \mathbb{R}^n$ is a column vector of variables. We assume that A has full row rank m.

A common approach to solve (IP) is to first solve the linear programming relaxation (LP) obtained by ignoring the integrality restrictions on x. Consider the corresponding optimal simplex tableau, where B and J denote the sets of basic and nonbasic variables respectively.

$$x_i = f_i + \sum_{j \in J} r^j x_j \quad \text{for } i \in B. \tag{1}$$

We have $f \geq 0$. If $f_i \in \mathbb{Z}$ for all $i \in B \cap \{1, \ldots, p\}$, then the basic solution $x_i = f_i$ for all $i \in B$ and $x_i = 0$ otherwise, is an optimal solution of (IP). On the other hand, if $f_i \notin \mathbb{Z}$ for

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some $i \in B \cap \{1, \ldots, p\}$, the above basic solution is not feasible to (IP) and one may want to generate one or several cutting planes that cut it off while preserving all the feasible solutions of (IP). Different strategies have been proposed for generating cutting planes. For example, Balas [2] introduced *intersection cuts* obtained by intersecting the rays $f + \alpha r^j$, $\alpha \geq 0$, with a convex set whose interior contains f but no integral point. Most general purpose cutting planes used in state-of-the-art solvers are obtained by generating a linear combination of the original constraints $Ax = b$, and by applying integrality arguments to the resulting equation (Gomory's Mixed Integer Cuts [12] , MIR inequalities [17] and split cuts [6] are examples). Recently, there has been an interest in cutting planes that cannot be deduced from a single equation, but can be deduced by integrality arguments involving two equations (Dey and Richard [8], Andersen, Louveaux, Weismantel and Wolsey [1], Gomory [15]).

Gomory [13] introduced the *corner polyhedron*, obtained from the constraints of (IP) by dropping the nonnegativity restriction on all the basic variables x_i, $i \in B$, in (1). Note that we can now drop the constraints $x_i = f_i + \sum_{j \in J} r^j x_j$ for $i \in B \cap \{p+1, \ldots, n\}$ since these variables x_i only appear in one equation and no other constraint. Interestingly, the cutting planes mentioned above are valid not only for (IP) but also for the corner polyhedron. In this paper, we make one further relaxation: We drop the integrality restriction on all the nonbasic variables x_j, $j \in J$. We are left with a system of the form

$$\begin{array}{rcl} x & = & f + \sum_{j=1}^{k} r^j s_j \\ x & \in & \mathbb{Z}^q \\ s & \geq & 0 \end{array} \tag{2}$$

where we now denote by s the nonbasic variables and by x the remaining basic variables. We will keep this notation in the remainder of the paper. This relaxation of (IP) is denoted by $R_f(r^1, \ldots, r^k)$ where $f, r^1, \ldots, r^k \in \mathbb{Q}^q$. Such a relaxation was considered in [1] for the case $q = 2$. The main result of [1] is a characterization of the polyhedron $\text{conv}(R_f(r^1, \ldots, r^k))$ by split inequalities and intersection cuts arising from triangles and quadrilaterals.

Gomory and Johnson [14] suggested relaxing the k-dimensional space of variables $s = (s_1, \ldots, s_k)$ to an infinite-dimensional space, where the variables s_r are defined for any $r \in \mathbb{Q}^q$. We get the *semi-infinite relaxation* R_f

$$\begin{array}{rcl} x & = & f + \sum r s_r \\ x & \in & \mathbb{Z}^q \\ s & \geq & 0 \text{ has a finite support.} \end{array} \tag{3}$$

By $s \geq 0$ *has a finite support* we mean that $s_r > 0$ for a finite number of $r \in \mathbb{Q}^q$. A *feasible solution* of R_f is a vector (x, s) that satisfies the three conditions (3). We say that a linear inequality is *valid* for R_f if it is satisfied by all its feasible solutions. The polyhedron $\text{conv}(R_f(r^1, \ldots, r^k))$ is the face of $\text{conv}(R_f)$ obtained by setting $s_r = 0$ for all $r \in \mathbb{Q}^q \setminus \{r^1, \ldots, r^k\}$. Any valid inequality for (3) yields a valid inequality for (2), and thus for (IP) as well, by simply restricting it to the space $r^1, \ldots, r^k$.

In the remainder, we assume $f \notin \mathbb{Z}^q$. Thus the basic solution $x = f$, $s = 0$ is not feasible for R_f. In this paper, we study valid linear inequalities for R_f that cut off this infeasible basic solution. These inequalities can be stated in terms of the variables s only:

$$\sum \psi(r)s_r \geq 1 \tag{4}$$

where $\psi : \mathbb{Q}^q \to \mathbb{R} \cup \{+\infty\}$. Since we are only interested in solutions s with a finite support, we assume in this paper that the sum in (4) is only taken over those vectors r such that $s_r > 0$. In particular, for any vector r such that $\psi(r) = +\infty$ and $s_r = 0$, the convention is that $\psi(r)s_r = 0$.

Our main interest in this paper is in *minimal* valid inequalities for R_f, namely valid inequalities $\sum \psi(r)s_r \geq 1$ such that there is no valid inequality $\sum \psi'(r)s_r \geq 1$ where $\psi' \leq \psi$ and $\psi'(r) < \psi(r)$ for at least one $r \in \mathbb{Q}^q$. The reason for this interest is that the linear inequalities needed in any characterization of R_f are $s \geq 0$ and minimal valid inequalities. We show that, for a minimal valid inequality $\sum \psi(r)s_r \geq 1$, the function ψ is nonnegative, piecewise linear, convex, and positively homogeneous (namely $\psi(\lambda r) = \lambda\psi(r)$ for any scalar $\lambda \in \mathbb{Q}_+$ and $r \in \mathbb{Q}^q$). The function ψ is not always continuous or finite. However, when it is finite, we show that the piecewise linear function ψ has at most 2^q pieces.

We make use of the following theorem of Lovász about maximal lattice-free convex sets. A convex set S is *lattice-free* if it does not have any integral point in its interior. However S may contain integral points on its boundary.

Theorem 1.1. (Lovász [16]) *A maximal lattice-free convex set in $\mathbb{R}^n$ is either an irrational hyperplane (i.e. it cannot be written in the form $\alpha_1 x_1 + \ldots + \alpha_n x_n = \beta$ with $\alpha \in \mathbb{Q}^n$), or a full-dimensional polyhedron P of the form $P = K + L$ where K is a polytope with $1 \leq dimK \leq n$, and L is a rational linear space (i.e. L is generated by rational vectors $v^1, \ldots, v^t \in \mathbb{Q}^n$, where $t + dimK = n$). Every facet of P contains an integral point in its relative interior.*

When $t \geq 1$ in Theorem 1.1, the polyhedron P is called a *cylinder over K*.

A theorem of Doignon [10], Bell [5] and Scarf [19] implies that maximal lattice-free convex sets in $\mathbb{R}^n$ are polyhedra with at most 2^n facets.

Let $\psi : \mathbb{Q}^q \to \mathbb{R} \cup \{+\infty\}$ be a minimal valid function for R_f. Define

$$B_\psi := \{x \in \mathbb{Q}^q : \ \psi(x - f) \leq 1\}. \tag{5}$$

Let $\text{cl}(B_\psi)$ denote the topological closure of B_ψ in $\mathbb{R}^q$. In this paper we prove the following theorem.

Theorem 1.2. *Let $f \in \mathbb{Q}^q \setminus \mathbb{Z}^q$. A minimal valid function ψ for R_f is nonnegative, piecewise linear, positively homogeneous and convex. Furthermore the set $cl(B_\psi)$ is a full-dimensional maximal lattice-free convex set containing f. Conversely, for any full-dimensional maximal lattice-free convex set $B \subset \mathbb{R}^q$ containing f, there exists a minimal valid function ψ for R_f such that $cl(B_\psi) = B$, and when f is in the interior of B, this function is unique.*

The remainder of this paper is dedicated to proving this theorem. This result has been applied recently by Espinoza [11] in a computational study of multi-row cuts for (IP), by Cornuéjols and Margot [7] in characterizing the facets of R_f and $R_f(r^1, \ldots, r^k)$ when $q = 2$, and by Dey and Wolsey [9] in their recent study of the corner polyhedron when $q = 2$. See also Zambelli [20] and Basu, Bonami, Cornuéjols and Margot [3].

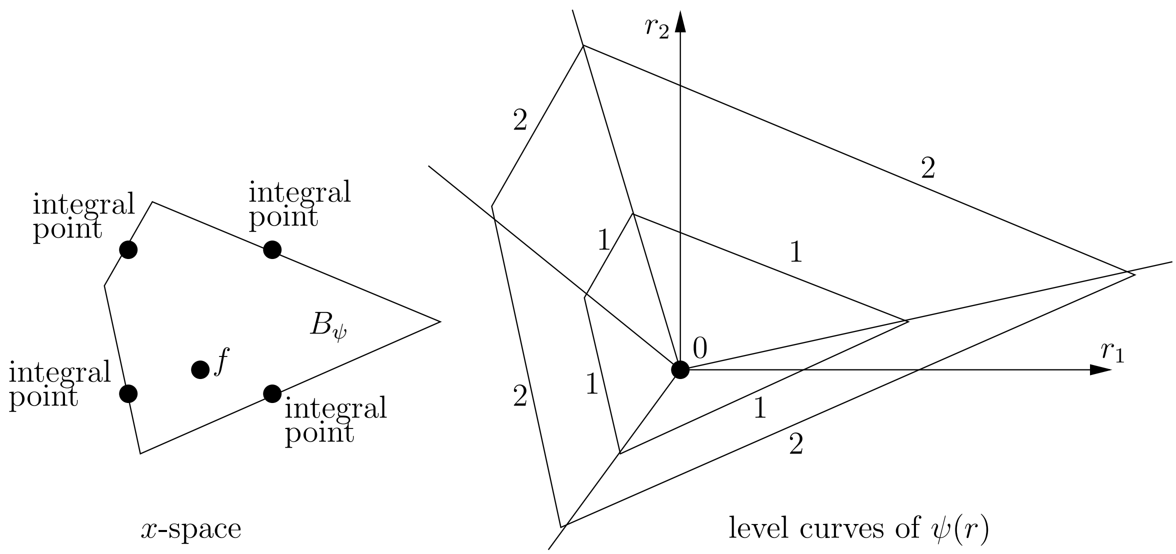

Figure 1: A maximal lattice-free convex set and corresponding function ψ in $\mathbb{R}^2$

2 Minimal Valid Inequalities

Let $f \in \mathbb{Q}^q$. We consider the semi-infinite integer programming problem R_f defined by (3) where we assume $f \notin \mathbb{Z}^q$. Note that $R_f \neq \emptyset$ since $x = 0$, $s_r = 1$ for $r = -f$ and $s_r = 0$ otherwise, is a feasible solution of (3). Any valid inequality for R_f that cuts off the infeasible solution $x = f, s = 0$ can be written as

$$\sum \psi(r)s_r \geq 1. \tag{6}$$

We say that the function $\psi : \mathbb{Q}^q \to \mathbb{R} \cup \{+\infty\}$ is *valid* if the corresponding inequality (6) is satisfied by every feasible solution of R_f, i.e. by every $s \geq 0$ with a finite support such that $f + \sum rs_r \in \mathbb{Z}^q$.

We assume that there exists at least one feasible solution of R_f such that $\sum \psi(r)s_r < +\infty$ (otherwise the function ψ is uninteresting). When $\psi(r) = +\infty$ and $s_r = 0$, we define $\psi(r)s_r = 0$.

Lemma 2.1. *If the function ψ is valid, then $\psi \geq 0$.*

Proof. Suppose $\psi(\tilde{r}) < 0$ for some $\tilde{r} = \begin{pmatrix} \frac{p_1}{D} \\ \dots \\ \frac{p_q}{D} \end{pmatrix}$ where $p_1, \dots, p_q \in \mathbb{Z}$ and $D \in \mathbb{Z}_+$ is a common denominator.

Let $(\bar{x}, \bar{s})$ be a feasible solution of R_f such that $\sum \psi(r)\bar{s}_r < +\infty$. Let $(\tilde{x}, \tilde{s})$ be defined by $\tilde{s}_{\tilde{r}} := \bar{s}_{\tilde{r}} + MD$ where M is a positive integer, $\tilde{s}_r := \bar{s}_r$ for $r \neq \tilde{r}$, and $\tilde{x} := f + \sum r\tilde{s}_r$.

The point $(\tilde{x}, \tilde{s})$ is a feasible solution of R_f since $\tilde{x} = \bar{x} + MD\tilde{r} \in \mathbb{Z}^q$. We have $\sum \psi(r)\tilde{s}_r = \sum \psi(r)\bar{s}_r + \psi(\tilde{r})MD$. Choose the integer M large enough, namely $M > \frac{\sum \psi(r)\bar{s}_r - 1}{D|\psi(\tilde{r})|}$. Then $\sum \psi(r)\tilde{s}_r < 1$, contradicting the fact that $(\tilde{x}, \tilde{s})$ is feasible. $\square$

A valid function ψ is *minimal* if there is no valid function ψ' such that $\psi' \leq \psi$ and $\psi'(r) < \psi(r)$ for at least one $r \in \mathbb{Q}^q$.

Lemma 2.2. *If ψ is a minimal valid function, then $\psi(0) = 0$.*

Proof. If $(\bar{x}, \bar{s})$ is a feasible solution in R_f, then so is $(\bar{x}, \tilde{s})$ defined by $\tilde{s}_r := \bar{s}_r$ for $r \neq 0$, and $\tilde{s}_0 = 0$. Therefore, if ψ is valid, then ψ' defined by $\psi'(r) = \psi(r)$ for $r \neq 0$ and $\psi'(0) = 0$ is also valid. Since ψ is minimal, it follows that $\psi(0) = 0$. □

A function $g : \mathbb{Q}^q \to \mathbb{R} \cup \{+\infty\}$ is *subadditive* if $g(a) + g(b) \geq g(a + b)$ for all $a, b \in \mathbb{Q}^q$.

Lemma 2.3. *If ψ is a minimal valid function, then ψ is subadditive.*

Proof. Let $r^1, r^2 \in \mathbb{Q}^q$. Define the function ψ' as follows.

$$\psi'(r) := \begin{cases} \psi(r^1) + \psi(r^2) & \text{if } r = r^1 + r^2 \\ \psi(r) & \text{if } r \neq r^1 + r^2. \end{cases}$$

We will show that ψ' is valid. Consider any $(\bar{x}, \bar{s}) \in R_f$. Define $(\bar{x}, \tilde{s})$ as follows

$$\tilde{s}_r := \begin{cases} \bar{s}_{r^1} + \bar{s}_{r^1+r^2} & \text{if } r = r^1 \\ \bar{s}_{r^2} + \bar{s}_{r^1+r^2} & \text{if } r = r^2 \\ 0 & \text{if } r = r^1 + r^2 \\ \bar{s}_r & \text{otherwise.} \end{cases}$$

Using the definitions of ψ' and $\tilde{s}$, it is easy to verify that

$$\sum_r \psi'(r)\bar{s}_r = \sum_r \psi(r)\tilde{s}_r. \tag{7}$$

Furthermore we have $\bar{x} = f + \sum r\bar{s}_r = f + \sum r\tilde{s}_r$. Since $\bar{x} \in \mathbb{Z}^q$ and $\tilde{s} \geq 0$, this implies that $(\bar{x}, \tilde{s}) \in R_f$.

Since ψ is valid, this implies $\sum_r \psi(r)\tilde{s}_r \geq 1$. Therefore, by (7), $\sum \psi'(r)\bar{s}_r \geq 1$. Thus ψ' is valid. Since ψ is minimal, we get $\psi(r^1) + \psi(r^2) = \psi'(r^1 + r^2) \geq \psi(r^1 + r^2)$. □

A function $g : \mathbb{Q}^q \to \mathbb{R} \cup \{+\infty\}$ is *positively homogeneous* if $g(\lambda a) = \lambda g(a)$ for all $\lambda \in \mathbb{Q}_+$ and $a \in \mathbb{Q}^q$.

Lemma 2.4. *If ψ is a minimal valid function, then ψ is positively homogeneous.*

Proof. Let $\tilde{r} \in \mathbb{Q}^q$ and $\lambda \in \mathbb{Q}_+$. We will show $\psi(\lambda\tilde{r}) = \lambda\psi(\tilde{r})$. This holds when $\lambda = 0$. Therefore we assume now $\lambda > 0$.

Define the function ψ' as follows.

$$\psi'(r) := \begin{cases} \frac{1}{\lambda}\psi(\lambda\tilde{r}) & \text{if } r = \tilde{r} \\ \psi(r) & \text{otherwise.} \end{cases}$$

We will show that ψ' is valid. Consider any $(\bar{x}, \bar{s}) \in R_f$. Define $(\bar{x}, \tilde{s})$ as follows

$$\tilde{s}_r := \begin{cases} \bar{s}_{\lambda\tilde{r}} + \frac{1}{\lambda}\bar{s}_{\tilde{r}} & \text{if } r = \lambda\tilde{r} \\ 0 & \text{if } r = \tilde{r} \\ \bar{s}_r & \text{otherwise.} \end{cases}$$

Using the definition of ψ' and $\tilde{s}$, it is easy to verify that

$$\sum \psi'(r)\bar{s}_r = \sum_r \psi(r)\tilde{s}_r. \tag{8}$$

Furthermore we have $\bar{x} = f + \sum r\bar{s}_r = f + \sum r\tilde{s}_r$. Since $\bar{x} \in \mathbb{Z}^q$ and $\tilde{s} \geq 0$, this implies that $(\bar{x}, \tilde{s}) \in R_f$.

Since ψ is valid, this implies $\sum_r \psi(r)\tilde{s}_r \geq 1$. Therefore, by (8), $\sum \psi'(r)\bar{s}_r \geq 1$. Thus ψ' is valid. Since ψ is minimal, we get

$$\psi(\lambda\tilde{r}) = \lambda\psi'(\tilde{r}) \geq \lambda\psi(\tilde{r}). \tag{9}$$

Setting $\tilde{v} = \lambda\tilde{r}$ and $\mu = \frac{1}{\lambda}$, (9) becomes

$$\psi(\tilde{v}) \geq \frac{1}{\mu}\psi(\mu\tilde{v}). \tag{10}$$

Since (9) holds for every $\tilde{r} \in \mathbb{Q}^q$ and $\lambda \in \mathbb{Q}_+$, (10) holds for every $\tilde{v} \in \mathbb{Q}^q$ and $\mu \in \mathbb{Q}_+$. (9) and (10) imply $\psi(\lambda\tilde{r}) = \lambda\psi(\tilde{r})$. □

Corollary 2.5. *If ψ is a minimal valid function, then ψ is convex.*

Proof. Let $r^1, r^2 \in \mathbb{Q}^q$ and $0 < t < 1$ rational. Then, by Lemmas 2.3 and 2.4,
$t\psi(r^1) + (1-t)\psi(r^2) = \psi(tr^1) + \psi((1-t)r^2) \geq \psi(tr^1 + (1-t)r^2)$. □

Lemma 2.6. *Let ψ be a nonnegative, positively homogeneous, subadditive function. Then ψ is valid for R_f if and only if $\psi(x-f) \geq 1$ for all $x \in \mathbb{Z}^q$.*

Proof. If $\psi(\bar{x} - f) < 1$ for some $\bar{x} \in \mathbb{Z}^q$, set $\bar{s}_r = 1$ for $r = \bar{x} - f$ and $\bar{s} = 0$ otherwise. Then $(\bar{x}, \bar{s}) \in R_f$ and $\sum \psi(r)\bar{s}_r = \psi(\bar{x} - f) < 1$, showing that ψ is not valid for R_f.

Conversely, assume $\psi(x-f) \geq 1$ for all $x \in \mathbb{Z}^q$. For any $(x,s) \in R_f$ we have $\sum rs_r = x - f$. Thus $\psi(\sum rs_r) = \psi(x-f)$. By subadditivity and positive homogeneity of ψ

$$\sum \psi(r)s_r \geq \psi(\sum rs_r) = \psi(x-f) \geq 1.$$

Therefore ψ is valid for R_f. □

Example 2.7. *Let $f \in \mathbb{Q}^q$ and assume $0 < f_i < 1$ for some $i = 1, \ldots, q$. Fix such an index i and let r_i denote the ith component of vector $r \in \mathbb{Q}^q$. Define ψ_i as follows.*

$$\psi_i(r) := \begin{cases} \frac{r_i}{1-f_i} & \text{if } r_i \geq 0 \\ \frac{-r_i}{f_i} & \text{if } r_i \leq 0. \end{cases}$$

The corresponding inequality $\sum \psi_i(r)s_r \geq 1$ is the Gomory mixed integer cut [12] obtained from row i of (3). Equivalently, it is the simple split inequality [6] obtained from the disjunction $x_i \leq 0$ or $x_i \geq 1$.

Example 2.8. *In $\mathbb{R}^q$, suppose $f_i = \frac{1}{2}$ for $i = 1, \ldots, q$. Define $\psi(r) := \frac{2}{q}(|r_1| + \ldots + |r_q|)$. The corresponding inequality $\sum \psi(r)s_r \geq 1$ is an intersection cut [2] obtained from the octahedron Ω_f centered at f with vertices $f \pm \frac{q}{2}e^i$, where e^i denotes the ith unit vector. Ω_f has 2^q facets, each of which contains a 0,1 point in its center.*

As one would expect, a minimal valid inequality may be implied by a linear combination of other minimal valid inequalities. This is the case here. Indeed, the above intersection cut from the octahedron is implied by the q split inequalities $\psi_i(r) := 2|r_i|$ for $i = 1, \ldots, q$ (see Example 2.7 above) since $\psi = \sum_{i=1}^{q} \frac{1}{q}\psi_i$ and $\sum \psi_i(r)s_r \geq 1$ for $i = 1, \ldots, q$ imply $\sum \psi(r)s_r \geq 1$.

A minimal valid function ψ may not be continuous nor finite, as shown by the following example.

Example 2.9. *In $\mathbb{R}^2$, let $f := \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$. Define ψ as follows.*

$$\psi\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} := \begin{cases} r_2 & \text{if } r_2 > 0 \\ 2|r_1| & \text{if } r_2 = 0 \\ +\infty & \text{if } r_2 < 0. \end{cases}$$

It is easy to verify that this function is valid using Lemma 2.6. Indeed $\psi(x - f) \geq 1$ for all $x \in \mathbb{Z}^2$ and equality holds when $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} i \\ 1 \end{pmatrix}$ for $i \in \mathbb{Z}$.

Next we prove the minimality of ψ. Let φ be a minimal valid function such that $\varphi \leq \psi$. By Lemma 2.6 $\varphi(x - f) \geq 1$ for all $x \in \mathbb{Z}^q$. In particular, for any point $\bar{x} \in \mathbb{Z}^q$ such that $\psi(\bar{x} - f) = 1$, the inequality $\varphi \leq \psi$ implies that we also have

$$\varphi(\bar{x} - f) = 1. \tag{11}$$

Applying (11) to $\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and to $\bar{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and using the positive homogeneity of φ, we get $\varphi(r) = \psi(r)$ for all $r \in \mathbb{Q}^2$ such that $r_2 = 0$.

Applying (11) to $\bar{x} = \begin{pmatrix} i \\ 1 \end{pmatrix}$ for $i \in \mathbb{Z}$ and using the convexity of φ, we get $\varphi(x - f) = 1$ for all $x = \begin{pmatrix} \alpha \\ 1 \end{pmatrix}$ for $\alpha \in \mathbb{Q}$. By positive homogeneity of φ, we get $\varphi(r) = \psi(r)$ for all $r \in \mathbb{Q}^2$ such that $r_2 > 0$.

Consider now a vector $r \in \mathbb{Q}^2$ such that $r_2 < 0$. By convexity of φ we have $\frac{1}{2}\psi\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \frac{1}{2}\psi\begin{pmatrix} M - r_1 \\ -r_2 \end{pmatrix} \geq \psi\begin{pmatrix} \frac{M}{2} \\ 0 \end{pmatrix}$. Thus $\psi\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \geq M - |r_2|$. When M goes to $+\infty$, this implies $\psi\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = +\infty$.

Therefore $\varphi = \psi$, showing that ψ is minimal.

3 Maximal lattice-free convex sets

Let $f \in \mathbb{Q}^q \setminus \mathbb{Z}^q$ and let $\psi : \mathbb{Q}^q \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Define

$$B_\psi := \{x \in \mathbb{Q}^q : \ \psi(x - f) \leq 1\}. \tag{12}$$

We are interested in the properties of B_ψ when ψ is a minimal valid function for R_f. For the function ψ_i of Example 2.7 (the Gomory mixed integer cut), B_{ψ_i} is the set of rational points satisfying $0 \le x_i \le 1$. In Example 2.8, B_ψ is the set of rational points in the octahedron Ω_f. For the function ψ of Example 2.9, B_ψ contains all the rational points in the band $0 < x_2 \le 1$ and in the segment $x_2 = 0$, $0 \le x_1 \le 1$, but the two half-lines $x_2 = 0, x_1 < 0$, $x_2 = 0, x_1 > 1$ are not in B_ψ.

For a set $S \in \mathbb{R}^q$, let $\mathrm{cl}(S)$ denote its topological closure. Consider a closed set $S \in \mathbb{R}^q$, i.e. $\mathrm{cl}(S) = S$. A point $x \in \mathbb{R}^q$ is in the *interior* of S if there exists a ball of positive radius centered at x contained in S. Thus, if S has a nonempty interior, it is full-dimensional. The *boundary* of S is the set of points of S that are not in its interior. The set S is said to be *lattice-free* if it contains no integral point in its interior. Note that a lattice-free set may contain integral points on its boundary. We will show that $\mathrm{cl}(B_\psi)$ is a full-dimensional maximal lattice-free convex set. By Theorem 1.1 this implies that $\mathrm{cl}(B_\psi)$ is a polyhedron, and by a theorem of Doignon [10], Bell [5] and Scarf [19], it follows that $\mathrm{cl}(B_\psi)$ has at most 2^q facets.

Lemma 3.1. *Let ψ be a minimal valid function for R_f. Then cl(B_ψ) is a lattice-free convex set in $\mathbb{R}^q$. Furthermore $f \in B_\psi$ and, if ψ is finite over $\mathbb{Q}^q$, then cl(B_ψ) contains f in its interior.*

Proof. The fact that B_ψ is convex over $\mathbb{Q}^q$ follows from Corollary 2.5 and Definition (12). Therefore $\mathrm{cl}(B_\psi)$ is convex over $\mathbb{R}^q$.

$f \in B_\psi$ since $\psi(0) = 0$.

Let $\bar{x} \in \mathbb{Z}^q$. By Lemma 2.6, $\psi(\bar{x} - f) \ge 1$. If, in addition, $\bar{x} \in B_\psi$, this implies $\psi(\bar{x} - f) = 1$. Suppose $\bar{x}$ is in the interior of $\mathrm{cl}(B_\psi)$. Then, starting from f on the straight line passing through $\bar{x}$, there exists a point $\bar{\bar{x}}$ that is beyond $\bar{x}$ but still in B_ψ. By positive homogeneity, we have $\psi(\bar{\bar{x}} - f) > 1$, which contradicts $\bar{\bar{x}} \in B_\psi$. Therefore $\mathrm{cl}(B_\psi)$ is lattice-free.

Now assume $\psi(r) < +\infty$ for all $r \in \mathbb{Q}^q$. For each unit vector e^i, the positive homogeneity of ψ implies the existence of $\lambda_i > 0$ such that $\psi(\lambda_i e^i) \le 1$. Similarly, for $i = 1, \dots, q$, choose any $\mu_i > 0$ such that $\psi(\mu_i(-e^i)) \le 1$. Then the convex hull of the $2q$ points $f + \lambda_i e^i$, $f - \mu_i e^i$ is contained in $\mathrm{cl}(B_\psi)$ by convexity and it contains f in its interior. Thus f is in the interior of $\mathrm{cl}(B_\psi)$. □

Lemma 3.2. *Let ψ and ψ' be two convex functions from $\mathbb{Q}^q$ to $\mathbb{R} \cup \{+\infty\}$. Then $\psi \le \psi'$ if and only if $B_{\psi'} \subseteq B_\psi$. Furthermore if the inclusion cl($B_{\psi'}$) $\subset$ cl(B_ψ) is strict, there exists $r \in \mathbb{Q}^q$ such that $\psi(r) < \psi'(r)$.*

Proof. The first statement is immediate from the definition (12) of B_ψ. Assume the inclusion $\mathrm{cl}(B_{\psi'}) \subset \mathrm{cl}(B_\psi)$ is strict. Then there exists $\bar{x} \in B_\psi \setminus B_{\psi'}$. This implies $\psi(r) \le 1 < \psi'(r)$ for $r = f - \bar{x}$. □

3.1 Maximal lattice-free convex sets with f in the interior

Definition 3.3. *Let B be a full-dimensional maximal lattice-free convex set in $\mathbb{R}^q$ and let $f \in \mathbb{Q}^q$ be a point in its interior. B is a polyhedron by Theorem 1.1. The* recession cone *of B is the set of vectors $r \in \mathbb{R}^q$ such that the whole ray $f + \lambda r$, $\lambda \ge 0$, is contained in B.*

Define the function $\psi_B : \mathbb{Q}^q \to \mathbb{R}$ as follows. Set $\psi_B(r) = 0$ for any vector $r \in \mathbb{Q}^q$ in the recession cone of B. For any $r \in \mathbb{Q}^q$ that is not in the recession cone of B, set $\psi_B(r) = \frac{1}{\lambda}$ where $\lambda > 0$ is the scalar for which the point $f + \lambda r$ is on the boundary of B.

Observe that the function ψ_B defined in 3.3 is nonnegative, positively homogeneous, and satisfies $B_{\psi_B} = B \cap \mathbb{Q}^q$. The next lemma shows that it is a minimal valid function for R_f. Theorem 3.5 will show that, conversely, all finite minimal valid functions for R_f are of this form.

Lemma 3.4. *Let B be any full-dimensional maximal lattice-free convex set in $\mathbb{R}^q$. If B contains $f \in \mathbb{Q}^q$ in its interior, the function ψ_B defined in 3.3 is a minimal valid function for R_f such that $cl(B_{\psi_B}) = B$.*

Proof. The fact that $\text{cl}(B_{\psi_B}) = B$ follows from $B_{\psi_B} = B \cap \mathbb{Q}^q$. We now show that ψ_B is a minimal valid function for R_f.

Claim 1: ψ_B is subadditive.

The proof uses the convexity of B and the positive homogeneity of ψ_B.

Let $a, b \in \mathbb{Q}^q$. Suppose first that neither a nor b is in the recession cone of B. Set $\alpha > 0$ to be the scalar such that $\psi_B(\alpha a) = 1$. Similarly set $\beta > 0$ such that $\psi_B(\beta b) = 1$. Then $f + \alpha a, f + \beta b \in B$. By convexity of B, for any $0 \leq \lambda \leq 1$ we have $f + \lambda\alpha a + (1 - \lambda)\beta b \in B$. Therefore

$$\lambda\psi_B(\alpha a) + (1 - \lambda)\psi_B(\beta b) = 1 \geq \psi_B(\lambda\alpha a + (1 - \lambda)\beta b). \tag{13}$$

Set $\lambda := \frac{\beta}{\alpha+\beta}$ in (13). We get, by positive homogeneity of ψ_B

$$\psi_B(a) + \psi_B(b) \geq \psi_B(a + b). \tag{14}$$

If both a, b are in the recession cone of B, then $a + b$ also is and again (14) holds. So we may assume that b is in the recession cone of B but not a. Choose $\alpha > 0$ such that $\psi_B(\alpha a) = 1$. Then $\alpha a \in B$ and, since b is in the recession cone of B, we also have $\alpha a + \alpha b \in B$. Thus $\psi_B(\alpha(a + b)) \leq 1$. Now (14) holds since $\psi_B(\alpha a) + \psi_B(\alpha b) = 1$ and ψ_B is positively homogeneous.

Using Lemma 2.6 and Claim 1, we get that ψ_B is valid.

Claim 2: ψ_B is minimal.

Suppose not. Let ψ be a minimal valid function for R_f such that $\psi \leq \psi_B$ and $\psi(\bar{r}) < \psi_B(\bar{r})$ for some $\bar{r} \in \mathbb{Q}^q$. Positive homogeneity of ψ_B and ψ implies that there exist $\mu > \lambda > 0$ such that $\psi_B(\lambda\bar{r}) = 1$ and $\psi(\mu\bar{r}) \leq 1$. Let $\bar{x} = f + \lambda\bar{r}$ and let $\bar{\bar{x}}$ be a point strictly between $\bar{x}$ and $f + \mu\bar{r}$. Then $\psi_B(\bar{\bar{x}} - f) > 1$, which implies $\bar{\bar{x}} \notin B$. But $\psi(\bar{\bar{x}} - f) < 1$, which implies $\bar{\bar{x}}$ is in the interior of $\text{cl}(B_\psi)$. It follows that B is strictly contained in $\text{cl}(B_\psi)$. By Lemma 3.1, $\text{cl}(B_\psi)$ is a lattice-free convex set. This contradicts the assumption that B is a maximal lattice-free convex set. Thus ψ_B is a minimal valid function for R_f. □

Theorem 3.5. *Let $f \in \mathbb{Q}^q \setminus \mathbb{Z}^q$. If ψ is a finite minimal valid function for R_f, then ψ is a nonnegative, positively homogeneous, piecewise linear, convex function with at most 2^q pieces. Furthermore ψ can be extended to $\mathbb{R}^q$ into a continuous function.*

Proof. Let ψ be a finite minimal valid function for R_f. From the results of Section 2, we know that ψ is nonnegative, positively homogeneous, and convex. We will now show that ψ is piecewise linear.

Let B_ψ be the set defined in (12). By Lemma 3.1, $\text{cl}(B_\psi)$ is a lattice-free convex set, and f is in its interior. Thus $\text{cl}(B_\psi)$ is full-dimensional. Suppose $\text{cl}(B_\psi)$ is not maximal and let B be a maximal lattice-free convex set such that $\text{cl}(B_\psi) \subset B$. Then by Lemma 3.4 there exists a minimal valid function ψ_B such that $\text{cl}(B_{\psi_B}) = B$. By Lemma 3.2, $\psi_B \leq \psi$ and there exists r such that $\psi_B(r) < \psi(r)$, contradicting the minimality of ψ. Thus $\text{cl}(B_\psi)$ is a maximal lattice-free convex set.

By Theorem 1.1 every maximal lattice-free convex set is a polyhedron and by a theorem of Doignon [10], Bell [5] and Scarf [19], this polyhedron has at most 2^q facets. The proof of the upper bound on the number of facets is simple and elegant: By Theorem 1.1, each facet F contains an integral point x^F in its relative interior. If there are more than 2^q facets, two integral points x^F and $x^{F'}$ must be identical modulo 2. Then their middle point $\frac{1}{2}(x^F + x^{F'})$ is integral and interior, contradiction.

Consider a facet F of B_ψ. We have $\psi(x - f) = 1$ for all $x \in F \cap \mathbb{Q}^q$. By positive homogeneity (Lemma 2.4), ψ is linear in the cone $\{r \in \mathbb{Q}^q : \ r = \lambda(x - f) \text{ with } \lambda \geq 0, x \in F\}$. Since the union of these cones over all facets of B_ψ covers $\mathbb{Q}^q$, the function ψ is piecewise linear with at most 2^q pieces.

Since the linear extension of ψ to $\mathbb{R}^q$ is continuous in each cone and the values match at the boundary of the cones, the last statement of the theorem follows. □

Even though a minimal valid function ψ is only defined over rational vectors (and f is a rational vector), the facets of the corresponding polyhedron $\text{cl}(B_\psi)$ may be defined by irrational hyperplanes [7].

3.2 Maximal lattice-free convex sets with f on the boundary

To complete the proof of Theorem 1.2, we consider the case of a minimal valid function ψ that is not finite everywhere, i.e. $\psi(r) = +\infty$ for some $r \in \mathbb{Q}^q$. This corresponds to a set B_ψ where f is on the boundary of $\text{cl}(B_\psi)$. One could argue that this case is unimportant from a practical point of view since Zambelli [20] showed recently that every minimal valid inequality for $R_f(r^1, \ldots, r^k)$ can be derived from a function ψ where f is in the interior of $\text{cl}(B_\psi)$. The minimal valid inequalities for R_f are more complicated. We present the result for the sake of completeness.

In this section, we need to define lattice-free convex sets in affine subspaces of $\mathbb{R}^q$. A point $x \in \mathbb{R}^q$ is in the *relative interior* of a convex S if there exists a ball B of positive radius centered at x such that $B \cap \text{aff}(S)$ is contained in S, where $\text{aff}(S)$ denotes the affine hull of S. The *boundary* of S is the set of points of S that are not in its relative interior. The set S is said to be *lattice-free* if it contains no integral point in its relative interior.

Definition 3.6. *Let B be a full-dimensional maximal lattice-free convex set in $\mathbb{R}^q$ and let $f \in \mathbb{Q}^q$ be a point on the boundary of B. B is a polyhedron by Theorem 1.1. Let $\mathcal{F}$ denote the family of all faces of B that contain f and have dimension at least one. In particular, $B \in \mathcal{F}$. Define $M_B := B$. We will define sets M_F for $F \in \mathcal{F} \setminus \{B\}$ starting from faces F of dimension $\dim(B) - 1$ and then recursively decreasing the dimension by 1. For each*

$F \in \mathcal{F} \setminus \{B\}$, let $\mathcal{F}_F$ denote the set of all faces $G \neq F$ of B that contain the face F, and let M_F be a maximal lattice-free convex subset of $F \cap (\bigcap_{G \in \mathcal{F}_F} M_G)$ that contains f. A slight generalization of Theorem 1.1 shows that a maximal lattice-free convex subset of a polyhedron is a polyhedron (this is easy to show when the polyhedron is rational; see [4] for a proof in the general case). Thus M_F is a polyhedron.

Define the function $\psi_{\mathcal{F}} : \mathbb{Q}^q \to \mathbb{R} \cup \{+\infty\}$ as follows. $\psi_{\mathcal{F}}(0) = 0$. For each $r \in \mathbb{Q}^r \setminus \{0\}$, let $F \in \mathcal{F}$ be the face of lowest dimension such that the ray $R := \{x = f + \lambda r : \ \lambda \geq 0\}$ goes through the relative interior of F. If there is no such face or if $R \cap M_F = \{f\}$, set $\psi_{\mathcal{F}}(r) = +\infty$. If r is in the recession cone of M_F, set $\psi_{\mathcal{F}}(r) = 0$. Otherwise, set $\psi_{\mathcal{F}}(r) = \frac{1}{\lambda}$ where λ is the positive scalar for which the point $f + \lambda r$ is on the boundary of M_F.

Example 3.7. *If B is the set $0 \leq x_2 \leq 1$ in $\mathbb{R}^2$, and $f := \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$, then $\mathcal{F}$ consists of B and its face F defined by the line $x_2 = 0$. The set M_F is the segment $0 \leq x_1 \leq 1, x_2 = 0$, which is the unique maximal lattice-free convex set contained in F that contains f. The corresponding function $\psi_{\mathcal{F}}$ is the function ψ of Example 2.9.*

Lemma 3.8. *The function $\psi_{\mathcal{F}}$ defined in 3.6 is a valid function for R_f.*

Proof. **Claim:** $\psi_{\mathcal{F}}$ is subadditive.

Let $a, b \in \mathbb{Q}^q$. The result holds if $\psi_{\mathcal{F}}(a) = +\infty$ or $\psi_{\mathcal{F}}(b) = +\infty$, so we assume $\psi_{\mathcal{F}}(a) < +\infty$ and $\psi_{\mathcal{F}}(b) < +\infty$. Let $F_a \in \mathcal{F}$ be the face of lowest dimension such that the ray $R_a := \{x = f + \lambda a : \lambda \geq 0\}$ goes through the relative interior of F_a. Define F_b and R_b similarly, as well as F_{a+b} and R_{a+b}. Note that both F_a and F_b are faces of F_{a+b}.

Suppose first that neither a is in the recession cone of M_{F_a} nor b is in the recession cone of M_{F_b}. Let $\alpha > 0$ and $\beta > 0$ be such that $\psi_{\mathcal{F}}(\alpha a) = 1$ and $\psi_{\mathcal{F}}(\beta b) = 1$ respectively. Thus $f + \alpha a \in M_{F_a}$ and $f + \beta b \in M_{F_b}$. Since $M_{F_{a+b}}$ is a convex set and $M_{F_a} \subseteq M_{F_{a+b}}$, $M_{F_b} \subseteq M_{F_{a+b}}$, it follows that, for any $0 \leq \lambda \leq 1$ the point $\lambda(f + \alpha a) + (1-\lambda)(f + \beta b)$ belongs to $M_{F_{a+b}}$. Therefore $\psi_{\mathcal{F}}(\lambda \alpha a + (1-\lambda)\beta b) \leq 1$. Thus we have

$$\lambda \psi_{\mathcal{F}}(\alpha a) + (1-\lambda)\psi_{\mathcal{F}}(\beta b) = 1 \geq \psi_{\mathcal{F}}(\lambda \alpha a + (1-\lambda)\beta b). \quad (15)$$

Set $\lambda := \frac{\beta}{\alpha + \beta}$ in (15). We get, by positive homogeneity of $\psi_{\mathcal{F}}$

$$\psi_{\mathcal{F}}(a) + \psi_{\mathcal{F}}(b) \geq \psi_{\mathcal{F}}(a+b). \quad (16)$$

If both a is in the recession cone of M_{F_a} and b is in the recession cone of M_{F_b}, then $a+b$ is in the recession cone of $M_{F_{a+b}}$ and again (16) holds. So we may assume that b is in the recession cone of M_{F_b} and that a is not in the recession cone of M_{F_a}. Choose $\alpha > 0$ such that $\psi_{\mathcal{F}}(\alpha a) = 1$. Then $\alpha a \in M_{F_a}$ and, since b is in the recession cone of M_{F_b}, we have $\alpha a + \alpha b \in M_{Fa+b}$. Thus $\psi_{\mathcal{F}}(\alpha(a+b)) \leq 1$. Now (16) holds since $\psi_{\mathcal{F}}(\alpha a) + \psi_{\mathcal{F}}(\alpha b) = 1$ and $\psi_{\mathcal{F}}$ is positively homogeneous. This proves the claim.

Lemma 2.6 and the construction of Definition 3.6 imply that $\psi_{\mathcal{F}}$ is valid. □

Theorem 3.9. *If ψ is a minimal valid function for R_f, then ψ is a nonnegative, piecewise linear, positively homogeneous, convex function.*

Proof. Let B_ψ be the corresponding convex set as defined in (12). By Lemma 3.1 $\text{cl}(B_\psi)$ is a lattice-free convex set in $\mathbb{R}^q$ and $f \in B_\psi$.

First we prove that $\text{cl}(B_\psi)$ is a maximal lattice-free convex set in $\mathbb{R}^q$. Suppose not and let B be a maximal lattice-free convex set that contains $\text{cl}(B_\psi)$. Then there is a point b in the relative interior of B that is not in $\text{cl}(B_\psi)$. Let $D := \text{conv}(\text{cl}(B_\psi) \cup \{b\})$. Define ψ_D as follows. For any $r \in \mathbb{Q}^q$ such that the ray $R_r := \{x = f + \lambda r : \lambda \geq 0\}$, does not go through the relative interior of D, define $\psi_D(r) = \psi(r)$. If the ray R_r goes through the relative interior of D and is in the recession cone of D, set $\psi_D(r) = 0$. Otherwise let $\psi_D(r) = \frac{1}{\lambda}$ where $\lambda > 0$ is the scalar such that the point $f + \lambda r$ is on the boundary of D. The function ψ_D is nonnegative and positively homogeneous. Next we show that ψ_D is subadditive. Let $a, b \in \mathbb{Q}^q$. If neither R_a, R_b goes through the relative interior of D, $\psi_D(a) + \psi_D(b) = \psi(a) + \psi(b) \geq \psi(a+b) \geq \psi_D(a+b)$. If both R_a, R_b go through the relative interior of D, the convexity of D and positive homogeneity of ψ_D imply the subadditivity of ψ_D. Similarly when R_a goes through the relative interior of D but R_b does not. Thus $\text{cl}(B_\psi)$ is a maximal lattice-free convex set in $\mathbb{R}^q$.

By Theorem 1.1, $\text{cl}(B_\psi)$ is a polyhedron. Suppose first that $\text{cl}(B_\psi)$ is a hyperplane. Since $B_\psi \in \mathbb{Q}^q$, this implies that $\text{cl}(B_\psi)$ is a rational hyperplane, contradicting Theorem 1.1. Therefore $\text{cl}(B_\psi)$ is a full-dimensional polyhedron. If f is in the interior of $\text{cl}(B_\psi)$, the theorem follows from the results in Section 3.1. Therefore we assume now that the point f lies on the boundary of $\text{cl}(B_\psi)$. Let $\mathcal{F}$ be the family of faces of $\text{cl}(B_\psi)$ that contain f and have dimension at least one. For each $F \in \mathcal{F}$, define

$$\Phi_F := \text{cl}(\{x \in \mathbb{Q}^q \cap F : \ \psi(x - f) \leq 1\}.$$

Let $\mathcal{F}_F$ denote the set of all faces $G \neq F$ of B that contain F. Note that $\Phi_F \subseteq \bigcap_{G \in \mathcal{F}_F} \Phi_G$ since ψ is a convex function. Furthermore Φ_F is a lattice-free convex subset of F. We claim that Φ_F is a maximal lattice-free convex subset of $F \cap (\bigcap_{G \in \mathcal{F}} \Phi_G)$ for all $F \in \mathcal{F}$. Suppose not. Then there exist maximal lattice-free convex sets $M_F \subseteq F \cap (\bigcap_{G \in \mathcal{F}_F} M_G)$ such that $\Phi_F \subseteq M_F$ for all $F \in \mathcal{F}$ and the inclusion $\Phi_F \subseteq M_F$ is strict for at least one face $F \in \mathcal{F}$. Thus the function $\psi_{\mathcal{F}}$ defined in 3.6 satisfies $\psi_{\mathcal{F}} \leq \psi$ and $\psi_{\mathcal{F}}(r) < \psi(r)$ for at least one $r \in \mathbb{Q}^q$. By Lemma 3.8, $\psi_{\mathcal{F}}$ is valid for R_f, but this contradicts the minimality of ψ. Thus Φ_F is a maximal lattice-free convex subset of F for all $F \in \mathcal{F}$. These sets are polyhedra [4]. Thus ψ is a piecewise linear function. $\square$

4 Conclusion

Most cutting planes used in integer programming can be viewed in the context of Gomory's corner polyhedron. A natural first step in studying the corner polyhedron is to investigate the semi-infinite relaxation R_f. This paper establishes a close connection between minimal valid inequalities for R_f and maximal lattice-free convex sets in the space of the integer variables. We use this connection to prove that minimal valid inequalities for R_f are nonnegative, positively homogeneous, piecewise linear, convex functions. Interest in cutting planes from two rows of a simplex tableau was initiated by Dey and Richard [8], Andersen, Louveaux, Weismantel and Wolsey [1], and Gomory [15]. Our results on minimal valid inequalities for R_f have been used in several recent publications (Espinoza [11], Dey and Wolsey [9], Cornuéjols and Margot [7], Zambelli [20]). Further investigations are under way.

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