A Counterexample to a Conjecture of Gomory and Johnson

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Abstract

In *Mathematical Programming* 2003, Gomory and Johnson conjecture that the facets of the infinite group problem are always generated by piecewise linear functions. In this paper we give an example showing that the Gomory-Johnson conjecture is false.

1 Introduction

Let $f \in]0,1[$ be given. Consider the following infinite group problem (Gomory and Johnson [3]) with a single equality constraint and a nonnegative integer variable s_r associated with each real number $r \in]0,1[$.

$$\begin{array}{rcll} \sum_{r\in]0,1[} r s_r & = & f & \\ s_r & \in & \mathbb{Z}_+ & \forall r \in]0,1[\\ s & & \text{has finite support,} & \end{array} \tag{1}$$

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where additions are performed modulo 1. Vector s has *finite support* if $s_r \neq 0$ for a finite number of distinct $r \in]0,1[$.

Gomory and Johnson [4] say that $\pi: [0,1[\rightarrow \mathbb{R}$ is a *valid function* with rhs element f if π is continuous and nonnegative, $\pi(0) = 0$, $\pi(f) = 1$ and every solution of (1) satisfies

$$\sum_{r \in]0,1[} \pi(r) s_r \geq 1.$$

The continuity assumption is critical and will be discussed in Section 6.

In [4], Gomory and Johnson also define the notion of facet for the infinite group problem (1). Intuitively, a facet is a valid inequality whose contact with the convex hull of solutions of (1) is maximal. For any valid function π, let $P(\pi)$ be the set of solutions of (1) that satisfy $\sum_{r \in]0,1[} \pi(r) s_r \geq 1$ at equality. A valid function π defines a *facet* if there is no other valid function π^* such that $P(\pi^*) \supsetneq P(\pi)$. Gomory and Johnson [4] gave many examples of facets that are *piecewise linear*, namely there are finitely many values $0 = r_0 < r_1 < \ldots < r_k = 1$ such that the function is of the form $\pi(r) = a_j r + b_j$ in interval $[r_{j-1}, r_j[$, for $j = 1, \ldots k$. The *slopes* of a piecewise linear function are the different values of a_j for $j = 1, \ldots k$.

In this paper we settle the following conjecture made by Gomory and Johnson in [4] about facets of (1).

Conjecture 1.1 (Facet Conjecture)**.** *If π is a facet of (1), then π is piecewise linear.*

We show that the above conjecture is false by exhibiting a facet of (1) which is not piecewise linear. This is in contrast with the result of Borozan and Cornuéjols [1] showing that the facets of a continuous version of (1) are always piecewise linear.

In earlier work, Gomory and Johnson [3] emphasized extreme functions rather than facets. A valid function π is *extreme* if it cannot be expressed as a convex combination of two distinct valid functions. It follows from the definition that facets are extreme. Therefore our counterexample also provides an extreme function that is not piecewise linear.

2 Preliminaries

A valid function $\pi: [0,1[\rightarrow \mathbb{R}$ is *minimal* if there is no valid function π' such that $\pi'(a) \leq \pi(a)$ for all $a \in [0,1[$ and the inequality is strict for at least one a.

When convenient, we will extend the domain of definition of the function π to the whole real line $\mathbb{R}$ by making the function periodic: $\pi(x) = \pi(x+k)$ for any $x \in [0,1[$ and $k \in \mathbb{Z}$.

A function $\pi: \mathbb{R} \rightarrow \mathbb{R}$ is *subadditive* if for every $a, b \in \mathbb{R}$

$$\pi(a+b) \leq \pi(a) + \pi(b).$$

Given $f \in]0,1[$, a function $\pi: [0,1[\rightarrow \mathbb{R}$ is *symmetric* if for every $a \in [0,1[$

$$\pi(a) + \pi(f-a) = 1.$$

Gomory and Johnson prove the following results in [3].

Theorem 2.1 (Minimality Theorem)**.** *Let $\pi: [0,1[\rightarrow \mathbb{R}$ be such that $\pi(0) = 0$ and $\pi(f) = 1$. A necessary and sufficient condition for π to be valid and minimal is that π is continuous, subadditive and symmetric.*

Any facet of (1) is minimal. Therefore if π is a facet of (1), then π is subadditive and symmetric.

The following theorem, due to Gomory and Johnson [3], gives a class of facets.

Theorem 2.2. *Let $\pi : [0,1[\to \mathbb{R}$ be a minimal valid function that is piecewise linear. If π has only two slopes, then π is a facet.*

Let $E(\pi)$ denote the set of all possible equalities $\pi(u_1) + \pi(u_2) = \pi(u_1 + u_2)$ that are satisfied by π. Here u_1 and u_2 are any real numbers. The following theorem is proved in Gomory and Johnson [4].

Theorem 2.3 (Facet Theorem)**.** *Let π be a minimal valid function. If there is no valid function that is a solution to $E(\pi)$ other than π itself, then π is a facet.*

The following fact is well-known and will be useful in our arguments.

Fact 2.4. *If π_1 and π_2 are subadditive, then $\pi_1 + \pi_2$ is subadditive.*

Fact 2.5. *Let π be a subadditive function and define $\pi'(x) = \alpha\pi(\beta x)$ for some constants $\alpha > 0$ and β. Then π' is subadditive.*

Proof. $\pi'(a+b) = \alpha\pi(\beta(a+b)) \le \alpha(\pi(\beta a) + \pi(\beta b)) = \pi'(a) + \pi'(b)$. □

A function π' defined as $\pi'(x) = \alpha\pi(\beta x)$ will be referred to as a *scaling* of π.

To prove our result we need the following lemma, due to Gomory and Johnson [4].

Lemma 2.6 (Interval Lemma)**.** *Let $\pi : [0,1[\to \mathbb{R}$ be a continuous function. Let $U = [u_1, u_2]$, $V = [v_1, v_2]$, and $U + V = [u_1 + v_1, u_2 + v_2]$ be three intervals of the real line such that $u_1 < u_2$ and $v_1 < v_2$. If, whenever $u \in U$ and $v \in V$, we have $\pi(u) + \pi(v) = \pi(u+v)$, then the graph of π above U, V, and $U+V$ is a straight line with some constant slope s.*

3 The construction

We first define a sequence of valid functions $\psi_i : [0,1[\to \mathbb{R}$ that are piecewise linear, and then consider the limit ψ of this sequence. We will then show that ψ is a facet but not piecewise linear.

Let $0 < \alpha < 1$. ψ_0 is the triangular function given by

$$\psi_0(x) = \begin{cases} \frac{1}{\alpha}x & 0 \le x \le \alpha \\ \frac{1-x}{1-\alpha} & \alpha \le x < 1. \end{cases}$$

(Notice that the corresponding inequality $\sum_{r\in[0,1[} \psi_0(r)s_r \ge 1$ defines the Gomory mixed-integer inequality.)

We first fix a nonincreasing sequence of positive real numbers ϵ_i, for $i = 1, 2, 3, \ldots$, such that $\epsilon_1 \le 1 - \alpha$ and

$$\sum_{i=1}^{+\infty} 2^{i-1}\epsilon_i < \alpha. \tag{2}$$

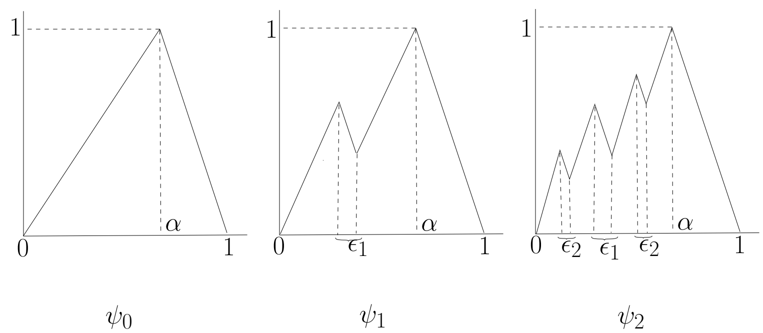


Figure 1: First two steps in the construction of the limit function

For example, $\epsilon_i = \alpha(\frac{1}{4})^i$ is such a sequence when $0 < \alpha \leq \frac{4}{5}$. The upper bound of $\frac{4}{5}$ on α is implied by the fact that $\epsilon_1 \leq 1 - \alpha$.

We construct ψ_{i+1} from ψ_i by modifying each segment with positive slope in the graph of ψ_i as follows.

For every maximal (with respect to set inclusion) interval $[a, b] \subseteq [0, \alpha]$ where ψ_i has constant positive slope we replace the line segment from $(a, \psi_i(a))$ to $(b, \psi_i(b))$ with the following three segments.

- The segment connecting $(a, \psi_i(a))$ and $(\frac{(a+b)-\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) + \frac{\epsilon_{i+1}}{2(1-\alpha)})$,
- The segment connecting $(\frac{(a+b)-\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) + \frac{\epsilon_{i+1}}{2(1-\alpha)})$ and $(\frac{(a+b)+\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) - \frac{\epsilon_{i+1}}{2(1-\alpha)})$,
- The segment connecting $(\frac{(a+b)+\epsilon_{i+1}}{2}, \psi_i(\frac{a+b}{2}) - \frac{\epsilon_{i+1}}{2(1-\alpha)})$ and $(b, \psi_i(b))$.

Figure 1 shows the transformation of ψ_0 to ψ_1 and ψ_1 to ψ_2.

The function ψ which we show to be a facet but not piecewise linear is defined as the limit of this sequence of functions, namely

$$\psi(x) = \lim_{i \to \infty} \psi_i(x) \tag{3}$$

This limit is well defined when (2) holds, as shown in Section 5.

In the next section we show that each function ψ_i is well defined and is a facet. In Section 5 we analyze the limit function ψ, showing that it is well defined, is a facet, but is not piecewise linear.

4 Analysis of the function ψ_i

Fact 4.1. *For $i \geq 0$, ψ_i is a continuous function which is piecewise linear with 2^i pieces with positive slope and 2^i pieces with negative slope. Furthermore:*

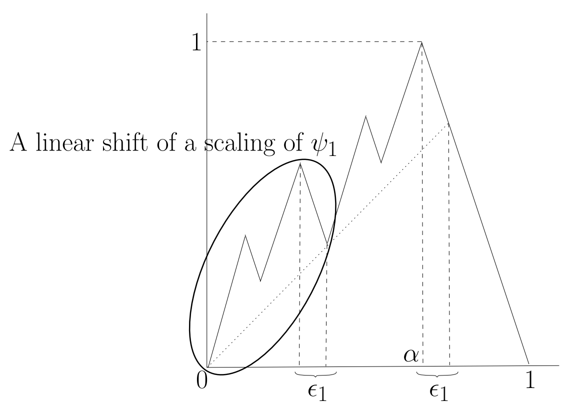
1. *There is one negative slope interval of length $1 - \alpha$ and there are 2^{k-1} negative slope intervals of length ϵ_k for $k = 1, \ldots, i$;*

2. *The negative slope pieces have slope* $-\frac{1}{1-\alpha}$*;*
3. *Each positive slope interval has length* $\frac{\gamma_i}{2^i}$*, where* $\gamma_i = \alpha - \sum_{k=1}^{i} 2^{k-1}\epsilon_k$*;*
4. *The positive slope pieces have slope* $\frac{1-\gamma_i}{(1-\alpha)\gamma_i}$*;*
5. *The function* ψ_i *is well-defined.*

Proof. The fact that ψ_i is a continuous function which is piecewise linear with 2^i pieces with positive slope and 2^i pieces with negative slope, and Facts 1. and 2. are immediate by construction.
Therefore the sum of the lengths of the negative slope intervals is $1-\alpha+\sum_{k=1}^{i} 2^{k-1}\epsilon_k$. Since ψ_i contains 2^i positive slope intervals with the same length, this proves 3.
The total decrease of ψ_i in $[0,1]$ is $\frac{-1}{1+\alpha}(1-\gamma_i)$. Since ψ_i is continuous, piecewise linear, all positive slope intervals have the same slope and $\psi_i(0) = \psi_i(1) = 0$, then a positive slope interval has slope $\frac{1-\gamma_i}{(1-\alpha)\gamma_i}$ and this proves 4.
Finally, by (2), $\gamma_i > 0$ for every $i \geq 0$, thus ψ_i is a well-defined function.

□

We now demonstrate that each function ψ_i is subadditive. Note that the function ψ_i depends only upon the choice of parameters $\alpha, \epsilon_1, \epsilon_2, \dots, \epsilon_i$. It will sometimes be convenient to denote the function ψ_i by $\psi_i^{\alpha,\epsilon_1,\epsilon_2,\dots,\epsilon_i}$ in this section.



ψ_2

Figure 2: Illustrating the proof of subadditivity of ψ_2

The key observation is the following lemma. Figure 2 illustrates this for the function $\psi_2^{\alpha,\epsilon_1,\epsilon_2}$.

Lemma 4.2. *For* $x \in [0, \alpha+\epsilon_1]$ *and* $i \geq 1$, $\psi_i^{\alpha,\epsilon_1,\epsilon_2,\dots,\epsilon_i}(x) = \lambda x + \mu \psi_{i-1}^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1},\frac{2\epsilon_2}{\alpha+\epsilon_1},\frac{2\epsilon_3}{\alpha+\epsilon_1},\dots,\frac{2\epsilon_i}{\alpha+\epsilon_1}}\left(\frac{2x}{\alpha+\epsilon_1}\right)$, *where*

$$\lambda = \frac{1-\alpha-\epsilon_1}{(\alpha+\epsilon_1)(1-\alpha)} \text{ and } \mu = \frac{\epsilon_1}{(\alpha+\epsilon_1)(1-\alpha)}.$$

Proof. Notice that λ is the slope of the line passing through the points $(0,0)$ and $(\frac{\alpha+\epsilon_1}{2}, \psi_1^{\alpha,\epsilon_1}(\frac{\alpha+\epsilon_1}{2})) = (\frac{\alpha+\epsilon_1}{2}, \frac{1-\epsilon_1}{2})$.

For $x \in [0, \alpha+\epsilon_1]$, let $\phi_i(x) = \psi_i^{\alpha,\epsilon_1,\epsilon_2,\dots,\epsilon_i}(x) - \lambda x$. Notice that the graph of ϕ_1 in the interval $[0, \alpha+\epsilon_1]$ is comprised of two identical triangles, one with basis $[0, \frac{\alpha+\epsilon_1}{2}]$ and apex $(\frac{\alpha-\epsilon_1}{2}, \mu)$ and the other with basis $[\frac{\alpha+\epsilon_1}{2}, \alpha+\epsilon_1]$ and apex (α, μ), where $\mu = \psi_1(\frac{\alpha-\epsilon_1}{2}) - \lambda(\frac{\alpha-\epsilon_1}{2})$. Therefore, for $x \in [0,1[$,

$$\mu^{-1}\phi_1\left(\frac{(\alpha+\epsilon_1)x}{2}\right) = \psi_0^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1}}(x)$$

thus $\phi_1(x) = \mu\psi_0^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1}}(\frac{2x}{\alpha+\epsilon_1})$ because $\phi_1(x) = \phi_1(x + \frac{\alpha+\epsilon_1}{2})$ for every $x \in [0, \frac{\alpha+\epsilon_1}{2}[$.

Assume by induction that $\phi_i(x) = \phi_i(x + \frac{\alpha+\epsilon_1}{2})$ for every $x \in [0, \frac{\alpha+\epsilon_1}{2}[$, and that, for $x \in [0,1[$, $\mu^{-1}\phi_i\left(\frac{(\alpha+\epsilon_1)x}{2}\right) = \psi_{i-1}^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1}, \frac{2\epsilon_2}{\alpha+\epsilon_1}, \dots, \frac{2\epsilon_i}{\alpha+\epsilon_1}}(x)$.

Notice that, by Fact 4.1, the slope of ψ_i in the intervals of positive slope is always greater than λ, hence ϕ_i has positive slope exactly in the same intervals where ψ_i has positive slope.

Therefore, by construction of $\psi_{i+1}^{\alpha,\epsilon_1,\dots,\epsilon_{i+1}}$, the function ϕ_{i+1} is obtained from ϕ_i by replacing each maximal positive slope segment $[(a, \phi_i(a)), (b, \phi_i(b))]$ with:
- the segment connecting $(a, \phi_i(a))$ and $(\frac{(a+b)-\epsilon_{i+1}}{2}, \phi_i(\frac{a+b}{2}) + \frac{\epsilon_{i+1}}{2(1-\alpha)})$,
- the segment connecting $(\frac{(a+b)-\epsilon_{i+1}}{2}, \phi_i(\frac{a+b}{2}) + \frac{\epsilon_{i+1}}{2(1-\alpha)})$ and $(\frac{(a+b)+\epsilon_{i+1}}{2}, \phi_i(\frac{a+b}{2}) - \frac{\epsilon_{i+1}}{2(1-\alpha)})$,
- the segment connecting $(\frac{(a+b)+\epsilon_{i+1}}{2}, \phi_i(\frac{a+b}{2}) - \frac{\epsilon_{i+1}}{2(1-\alpha)})$ and $(b, \phi_i(b))$.

Thus, by induction, $\phi_{i+1}(x) = \phi_{i+1}(x + \frac{\alpha+\epsilon_1}{2})$ for every $x \in [0, \frac{\alpha+\epsilon_1}{2}[$ and, for $x \in [0,1[$, we have that $\mu^{-1}\phi_{i+1}\left(\frac{(\alpha+\epsilon_1)x}{2}\right) = \psi_i^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1}, \frac{2\epsilon_2}{\alpha+\epsilon_1}, \dots, \frac{2\epsilon_i}{\alpha+\epsilon_1}, \frac{2\epsilon_{i+1}}{\alpha+\epsilon_1}}(x)$. Therefore, for $x \in [0, \alpha+\epsilon_1[$, we have $\phi_{i+1}(x) = \mu\psi_i^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1}, \frac{2\epsilon_2}{\alpha+\epsilon_1}, \dots, \frac{2\epsilon_{i+1}}{\alpha+\epsilon_1}}(\frac{2x}{\alpha+\epsilon_1})$. □

Remark 4.3. *Given any $0 < \alpha < 1$ and any nonincreasing sequence of positive real numbers ϵ_i satisfying (2) and $\epsilon_1 \le 1-\alpha$, let $\alpha' = \frac{\alpha-\epsilon_1}{\alpha+\epsilon_1}$, $\epsilon'_i = \frac{2\epsilon_{i+1}}{\alpha+\epsilon_1}$, $i \ge 1$. Then $\epsilon'_1 \le 1-\alpha'$, $\{\epsilon'_i\}$ is a nonincreasing sequence, and $\sum_{i=1}^{+\infty} 2^{i-1}\epsilon'_i < \alpha'$.*

We next prove that each ψ_i is a nonnegative function.

Fact 4.4. $\psi_i^{\alpha,\epsilon_1,\dots\epsilon_i}(x) \ge 0$ *for all x, and for all parameters such that $\epsilon_1 \le 1-\alpha$ and ϵ_i is a nonincreasing sequence.*

Proof. The proof is by induction on i. ψ_0 is nonnegative by definition.

Consider ψ_{i+1}. Clearly $\psi_{i+1}(x) \ge 0$ for $x \in [\alpha+\epsilon_1, 1[$, since $\psi_{i+1}(x) = \psi_0(x)$ in this interval. Note that in Lemma 4.2, $\lambda \ge 0$ because $1-\alpha \ge \epsilon_1$. So, when $x \in [0, \alpha+\epsilon_1]$ Lemma 4.2 implies that ψ_{i+1} is nonnegative, because ψ_i is nonnegative. Note that the parameters for ψ_i also satisfy the hypothesis by Remark 4.3, so we can use the induction hypothesis. □

Lemma 4.5. *Given any $0 < \alpha < 1$ and any nonincreasing sequence of positive real numbers ϵ_i satisfying (2) and $\epsilon_1 \le 1-\alpha$, the function $\psi_i^{\alpha,\epsilon_1,\epsilon_2,\dots,\epsilon_i}$ is subadditive for all i.*

Proof. The proof is by induction. ψ_0^{α} is subadditive, since it is a valid and minimal Gomory function. By the induction hypothesis, $\psi_k^{\alpha,\epsilon_1,\ldots,\epsilon_k}$ is subadditive and we wish to show this implies that $\psi_{k+1}^{\alpha,\epsilon_1,\ldots,\epsilon_{k+1}}$ is subadditive.

By Remark 4.3 and induction, the function $\psi_k^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1},\frac{2\epsilon_2}{\alpha+\epsilon_1},\frac{2\epsilon_3}{\alpha+\epsilon_1},\ldots,\frac{2\epsilon_{k+1}}{\alpha+\epsilon_1}}$ is subadditive.

We define the function $\psi_k'(x) = \mu\psi_k^{\frac{\alpha-\epsilon_1}{\alpha+\epsilon_1},\frac{2\epsilon_2}{\alpha+\epsilon_1},\frac{2\epsilon_3}{\alpha+\epsilon_1},\ldots,\frac{2\epsilon_{k+1}}{\alpha+\epsilon_1}}(\frac{2x}{\alpha+\epsilon_1})$ where μ is defined in the statement of Lemma 4.2. Note that ψ_k' has a period of $\frac{\alpha+\epsilon_1}{2}$ and ψ_k' is subadditive by Fact 2.5.

In the remaining part of the proof, we will not need the extended notation $\psi_k^{\alpha,\epsilon_1,\epsilon_2,\ldots,\epsilon_k}$ and we shall refer to this function as simply ψ_k. We now prove the subadditivity of ψ_{k+1} assuming the subadditivity of ψ_k, i.e. $\psi_{k+1}(a+b) \leq \psi_{k+1}(a) + \psi_{k+1}(b)$. Assume without loss of generality that $a \leq b$. We then have the following two cases.

Case 1 : b is in the range $[0, \alpha+\epsilon_1]$.
If $a+b \in [0,\alpha+\epsilon_1]$, $\psi_{k+1}(a+b) = \psi_k'(a+b) + \lambda(a+b)$ by Lemma 4.2. Now Fact 2.4 shows that ψ_{k+1} is subadditive.
If $a+b$ is in the range $[\alpha+\epsilon_1, 1]$, then $\psi_{k+1}(a+b) \leq \lambda(a+b)$. On the other hand, $\psi_{k+1}(a) \geq \lambda a$ and $\psi_{k+1}(b) \geq \lambda b$, hence $\psi_{k+1}(a+b) \leq \psi_{k+1}(a) + \psi_{k+1}(b)$.
If $a+b$ is greater than 1, then $(a+b)\text{mod } 1 < \alpha+\epsilon_1$. Let $x = \alpha+\epsilon_1 - b$ and $y = 1-\alpha-\epsilon_1$. So $(a+b)\text{mod } 1 = a-x-y$. Then

$$\begin{aligned}\psi_{k+1}(a)+\psi_{k+1}(b) &= \psi_k'(a)+\psi_k'(b)+\lambda a+\lambda b \\ &\geq \psi_k'(a+b)+\lambda a+\lambda b \qquad ((a+b)\text{mod } 1 < \alpha+\epsilon_1) \\ &= \psi_k'(a-x)+\lambda a+\lambda b \qquad (a-x = (a+b)-(\alpha+\epsilon_1))\end{aligned} \tag{4}$$

because ψ_k' has period $\alpha+\epsilon_1$. Also,

$$\begin{aligned}\psi_{k+1}(a+b) &= \psi_{k+1}((a-x)+(1-y)) \\ &\leq \psi_{k+1}(a-x)+\psi_{k+1}(1-y) \qquad ((a+b)\text{mod } 1,\ a-x,\ 1-y \text{ are in } [0,\alpha+\epsilon_1]) \\ &= \psi_{k+1}(a-x)+\tfrac{1-\alpha-\epsilon_1}{1-\alpha} \qquad (\text{All negative slopes in } \psi_{k+1} \text{ are } -\tfrac{1}{1-\alpha}) \\ &= \psi_{k+1}(a-x)+\lambda(b+x) \qquad (\text{by definition of } x, y) \\ &= \psi_k'(a-x)+\lambda(a-x)+\lambda(b+x) \qquad (\text{by Lemma 4.2 because } 0 \leq a-x \leq \alpha+\epsilon_1) \\ &= \psi_k'(a-x)+\lambda a+\lambda b.\end{aligned} \tag{5}$$

From (4) and (5) we get that $\psi_{k+1}(a)+\psi_{k+1}(b) \geq \psi_{k+1}(a+b)$.

Case 2 : b is in the range $[\alpha+\epsilon_1, 1]$

If $a+b$ is also in the range $[\alpha+\epsilon_1, 1]$, then $\psi_{k+1}(a+b) \leq \psi_{k+1}(b)$. Therefore, $\psi_{k+1}(a+b) \leq \psi_{k+1}(a)+\psi_{k+1}(b)$.

Now we consider the case where $a+b > 1$. Since every line segment with negative slope in the graph of ψ_{k+1} has slope $-\frac{1}{1-\alpha}$, then in the range $[0,a]$ the line of slope $-\frac{1}{1-\alpha}$ passing through $(a, \psi_{k+1}(a))$ lies above the graph of ψ_{k+1}. Formally, for every $x \in [0,a]$,

$$-\frac{1}{1-\alpha}x + \psi_{k+1}(a) + \frac{a}{1-\alpha} \geq \psi_{k+1}(x). \tag{6}$$

Now

$$\psi_{k+1}(a)+\psi_{k+1}(b) = \psi_{k+1}(a)+\frac{1-b}{1-\alpha} \geq -\frac{a+b-1}{1-\alpha}+\psi_{k+1}(a)+\frac{a}{1-\alpha} \geq \psi_{k+1}(a+b-1)$$

where the first equality is because $\psi_{k+1}(b) = \frac{1}{1-\alpha}(1-b)$, and the last inequality follows by (6). Therefore, we get $\psi_{k+1}(a) + \psi_{k+1}(b) \geq \psi_{k+1}(a+b-1) = \psi_{k+1}(a+b)$. $\square$

Fact 4.6. *$\psi_i(x)$ is a symmetric function.*

Proof. It is straightforward to show that ψ_0 is symmetric. Notice that, by construction, the function $\psi_{i+1} - \psi_i$ satisfies

$$(\psi_{i+1} - \psi_i)(x) + (\psi_{i+1} - \psi_i)(\alpha - x) = 0.$$

Therefore, if ψ_i is symmetric, also ψ_{i+1} is symmetric. $\square$

Theorem 4.7. *For $i \geq 0$, the function ψ_i is a facet.*

Proof. Since ψ_i is a function that is continuous, piecewise linear, subadditive, symmetric and has only two slopes, then, by Theorems 2.1 and 2.2, ψ_i is a facet. $\square$

5 Analysis of the limit function

Recall that ψ is the function defined by

$$\psi(x) = \lim_{i \to \infty} \psi_i(x)$$

for every $x \in [0, 1[$.

Fact 4.1 implies the following.

Fact 5.1. *Let $\gamma = \alpha - \sum_{i=1}^{+\infty} 2^{i-1}\epsilon_i$. Then $\gamma < \gamma_i$ and the value s_i of the positive slope in ψ_i is bounded above by $\frac{1-\gamma}{(1-\alpha)\gamma}$.*

We can now show the following lemma.

Lemma 5.2. *For any x, the sequence $\{\psi_i(x)\}_{i=1,2,3,\ldots}$ is a Cauchy sequence, and therefore it converges. Moreover, the sequence of functions $\{\psi_i\}_{i=1,2,3,\ldots}$ converges uniformly to ψ.*

Proof. From Fact 4.1 the intervals with positive slope in ψ_i have length $\frac{\gamma_i}{2^i}$. Note that $|\psi_i(x) - \psi_{i+1}(x)| \leq s_{i+1}\frac{\gamma_i}{2^i}$ since the values of the two functions match at the ends of the positive-slope intervals of ψ_i.

By Fact 5.1, $s_{i+1} \leq \frac{1-\gamma}{(1-\alpha)\gamma}$ and we know that $\gamma_i < \alpha$. So $|\psi_i(x) - \psi_{i+1}(x)| \leq C\frac{1}{2^i}$ where $C = \alpha\frac{1-\gamma}{(1-\alpha)\gamma}$. Therefore, $|\psi_n(x) - \psi_m(x)| \leq \sum_{i=n}^{m-1} C\frac{1}{2^i}$ if $n < m$. We can bound this expression using

$$\sum_{i=n}^{m-1} C\frac{1}{2^i} \leq \sum_{i=n}^{\infty} C\frac{1}{2^i} = C\frac{1}{2^{n-1}}$$

This implies that the sequence is Cauchy and hence convergent. Moreover, since the bound on $|\psi_n(x) - \psi_m(x)|$ does not depend on x, the above argument immediately implies that the sequence of functions ψ_i converges uniformly to ψ. $\square$

This also implies the following corollary.

Corollary 5.3. *The function ψ is continuous.*

Proof. ψ_i is continuous for each $i \in \{1, 2, 3, \ldots\}$ by construction. Since this sequence of functions converges uniformly to ψ, ψ is continuous [5]. $\square$

Fact 5.4. *The function ψ is not piecewise linear.*

Proof. Let I be any interval where ψ_i has a negative slope. Note that $\psi(x) = \psi_i(x)$ for all $x \in I$. Furthermore, if $[a_j, b_j]$ and $[a_{j+1}, b_{j+1}]$ are consecutive intervals with negative slope in ψ_i, then $\psi_i(b_j) < \psi_i(a_{j+1})$. It follows that there is no constant K such that ψ has K piecewise linear segments. $\square$

Lemma 5.5. *The function ψ is subadditive.*

Proof. For every $a, b \in [0, 1[$, we have

$$\begin{aligned}
\psi(a+b) &= \lim_{i\to\infty} \psi_i(a+b) \\
&\leq \lim_{i\to\infty} (\psi_i(a) + \psi_i(b)) \\
&= \lim_{i\to\infty} \psi_i(a) + \lim_{i\to\infty} \psi_i(b) \\
&= \psi(a) + \psi(b).
\end{aligned}$$

$\square$

Lemma 5.6. *The function ψ is symmetric.*

Proof. For every $x \in [0, 1[$, we have

$$\begin{aligned}
\psi(x) + \psi(\alpha - x) &= \lim_{i\to\infty} \psi_i(x) + \lim_{i\to\infty} \psi_i(\alpha - x) \\
&= \lim_{i\to\infty} (\psi_i(x) + \psi_i(\alpha - x)) \\
&= \lim_{i\to\infty} 1 && \text{(By symmetry of } \psi_i\text{)} \\
&= 1
\end{aligned}$$

$\square$

By Theorem 2.1 (using Corollary 5.3 and Lemmas 5.5 and 5.6) ψ is valid and minimal. We finally show that the function ψ is a facet.

Theorem 5.7. *The function ψ is a facet for problem (1).*

Proof. We will use the Facet theorem (Theorem 2.3). We show that if a valid function is a solution to the set of equalities $E(\psi)$, then it coincides with ψ everywhere.

Consider any valid function ϕ that is a solution to the set of equalities $E(\psi)$. Therefore, if $\psi(u) + \psi(v) = \psi(u+v)$, then $\phi(u) + \phi(v) = \phi(u+v)$.

Let $S \subseteq [0, 1[$ be defined as the union of all intervals over which ψ has negative slope. We will show the following two facts:

(i) S is dense in $[0, 1[$,

(ii) $\phi(x) = \psi(x)$ for all $x \in S$.

Since ϕ and ψ are both continuous and by (i) and (ii) they coincide on a dense subset of the unit interval, they must be equal everywhere on the unit interval [5], thus showing that ψ is a facet, by Theorem 2.3.

We first show (i). Let $a \in [0, 1[$. We need to show that, for any $\delta > 0$, there exists $b \in S$ such that $|a - b| < \delta$. Choose i such that $\frac{1}{2^i} < \delta$. If ψ_i has negative slope in a, then $a \in S$ and we are done. Thus a is in a positive slope interval of ψ_i. By Fact 4.1.3, such an interval has length $\frac{\gamma_i}{2^i}$, hence there exists a point b in a negative slope interval of ψ_{i+1}, and thus in S, such that $|a - b| \leq \frac{\gamma_i}{2^i} < \delta$.

Finally we show (ii). For any interval $[a, b]$ over which the graph of ψ has a negative slope, consider the following intervals : $U = [(a + b)/2, b]$, $V = [1 - ((b - a)/2), 1]$ and therefore $U + V = [a, b]$. It is easy to see that $\psi(u) + \psi(v) = \psi(u + v)$ for $u \in U$, $v \in V$. This implies $\phi(u) + \phi(v) = \phi(u + v)$. Now Lemma 2.6 (the Interval Lemma) implies that ϕ are straight lines over U, V and $U + V$.

We now use an inductive argument to prove that not only do the slopes of ψ and ϕ coincide on intervals where the slope of ψ is negative, in fact $\psi(x) = \phi(x)$ for all x in these intervals.

Every segment s with negative slope in ψ also appears in ψ_i for some i. Let $index(s)$ be the least such i. We prove that $\psi(x) = \phi(x)$ for every s with negative slope by induction on $index(s)$. $\phi(\alpha) = \psi(\alpha) = 1$ and $\phi(0) = \psi(0) = 0$ since ϕ is assumed to be a valid inequality. This implies that ϕ is the same as ψ in the range $[\alpha, 1]$. This proves the base case of the induction.

By the induction hypothesis, we assume the claim is true for negative-slope segments s with $index(s) = k$. Consider all negative-slope segments s with $index(s) = k + 1$. Amongst these consider the segment s_c which is closest to the origin. Let the midpoint of this segment be m. We know that $2m$ is the start of a negative-slope segment s' in ψ with $index(s') = k$. By construction, $\psi(m) + \psi(m) = \psi(2m)$. So $\phi(m) + \phi(m) = \phi(2m)$. From the induction hypothesis, we know that $\psi(2m) = \phi(2m)$ and so $\phi(m) = \frac{1}{2}\phi(2m) = \frac{1}{2}\psi(2m) = \psi(m)$.

Now consider any other negative-slope segment s with $index(s) = k + 1$ and let its midpoint be m_s. Note that $m_s + m$ is the start of a negative-slope segment s' with $index(s') = k$. So

$$\phi(m_s + m) = \psi(m_s + m) \tag{7}$$

because of the inductive hypothesis. Note that $\psi(m_s + m) = \psi(m_s) + \psi(m)$ by construction. So, $\phi(m_s + m) = \phi(m_s) + \phi(m)$. Since we showed that $\phi(m) = \psi(m)$, (7) implies that $\phi(m_s) = \psi(m_s)$. Since the values coincide at the midpoints of these segments and the slopes of the segments are the same, $\phi(x) = \psi(x)$ for any x in the domain of these segments.

□

6 Conclusion

The definition of valid function given by Gomory and Johnson [4] requires continuity. The continuity assumption was not present in the 1972 paper [3]. If we drop this assumption in the definition of valid function, then there are extreme functions that are not continuous, as shown by Dey et al. [2]. The continuity assumption is crucial in our proof, when showing that ψ is a facet. Indeed, we use it twice. First when invoking the Interval lemma. Second when showing that the set of equalities $E(\psi)$ has only ψ as a solution: We show that for any other valid inequality ϕ satisfying $E(\psi)$ the functions ψ and ϕ coincide on a dense subset of the unit interval, and thus they coincide everywhere by continuity of ϕ and ψ.

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