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Martin-Löf Complexes

Steve Awodey
Carnegie Mellon University, awodey@cmu.edu

P. Hofstra

M. Warren

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MARTIN-LÖF COMPLEXES

S. AWODEY, P. HOFSTRA, AND M. A. WARREN

Dedicated to Per Martin-Löf on the occasion of his retirement.

ABSTRACT. In this paper we define Martin-Löf complexes to be algebras for monads on the category of (reflexive) globular sets which freely add cells in accordance with the rules of intensional Martin-Löf type theory. We then study the resulting categories of algebras for several theories. Our principal result is that there exists a cofibrantly generated Quillen model structure on the category of 1-truncated Martin-Löf complexes and that this category is Quillen equivalent to the category of groupoids. In particular, 1-truncated Martin-Löf complexes are a model of homotopy 1-types. In order to establish these facts we give a proof-theoretic analysis, using a modified version of Tait's logical predicates argument, of the propositional equality classes of terms of identity type in the 1-truncated theory.

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1. INTRODUCTION

This paper pursues a surprising connection between Geometry, Algebra, and Logic that has only recently come to light, in the form of an interpretation of the constructive type theory of Martin-Löf into homotopy theory, resulting in new examples of certain algebraic structures which are important in topology. This fascinating connection is currently under investigation from several different perspectives ([1, 20, 6, 13, 3, 5]), and these preliminary results confirm the significance of the link. Some of these results will be surveyed in this brief introduction in order to position the present work in its context; especially for the reader coming from one field or the other, a brief summary is given of the essential concepts from the different subjects involved.

Martin-Löf type theory [14] is a formal system originally intended to provide a rigorous framework in which to develop constructive mathematics. At heart,

it is a calculus for reasoning about dependent types and terms, and equality between those. Under the Curry-Howard correspondence, one may identify types with propositions, and terms with proofs. Viewed in this manner, the system can be shown to be at least as strong as second-order logic, and it is also known to interpret constructive set theory. Indeed, Martin-Löf type theory has been used successfully to formalize parts of constructive mathematics, such as pointless topology (constructive locale theory). Moreover, it has been employed as a framework for the development of programming languages as well, a task for which it is especially well-suited in virtue of its combination of expressive strength and desirable proof-theoretic properties. (See the textbook [15] for a discussion.)

The type theory has two variants: an intensional, and an extensional version. The difference between them lies mainly in the treatment of equality. In the intensional version (with which we are mainly concerned in the present work), one has two different kinds of equality: the first kind is called **definitional equality**, and behaves much like equality between terms in the simply-typed lambda-calculus, or any other conventional equational theory. The second kind is a more subtle relation, called **propositional equality**, which, under the Curry-Howard correspondence, represents the equality formulas of first-order logic. Specifically, given two terms a, b of the same type A , one may form a new type $\mathbf{Id}_A(a, b)$, which we think of as the proposition that a and b are equal; a term of this type thus represents a proof of the proposition that a equals b (hence the name “propositional equality”).

When a and b are definitionally equal, then (since they can be freely substituted for each other) they are also propositionally equal, in the sense that the type $\mathbf{Id}_A(a, b)$ is inhabited by a term; but the converse is generally not true, at least in the intensional version of the theory. In the extensional version, by contrast, the two notions of equality are forced by an additional rule to coincide. As a consequence, the extensional version of the theory is essentially a dependent type theory with a standard, extensional equality relation. As is well-known, however, the price one pays for this simplification is a loss of desirable proof-theoretic properties, such as strong normalization and decidable equality of terms.

In the intensional version with which we shall be concerned here, it can be shown that the identity types $\mathbf{Id}_A(a, b)$ satisfy certain conditions which were observed by Hofmann and Streicher in [8] to be analogous to the groupoid laws. Specifically, the posited reflexivity of propositional equality produces identity proofs $\mathbf{r}(a) : \mathbf{Id}_A(a, a)$ for any term $a : A$, playing the role of a unit arrow for a ; and when $f : \mathbf{Id}_A(a, b)$ is an identity proof, then (corresponding to the symmetry of identity) there also exists a proof $f^{-1} : \mathbf{Id}_A(b, a)$, to be thought of as the inverse of f ; finally, when $f : \mathbf{Id}_A(a, b)$ and $g : \mathbf{Id}_A(b, c)$ are identity proofs, then (corresponding to transitivity) there is a new proof $(g \cdot f) : \mathbf{Id}_A(a, c)$, thought of as the composite of f and g . Moreover, this structure on each type A can be shown to satisfy the usual groupoid laws, but only **up to propositional equality**. We shall return to this point below.

1.1. Groupoid semantics. A good notion of a model for the extensional theory is due to Seely [16], who showed that one can interpret type dependency in locally cartesian closed categories in a very natural way. (There are certain coherence issues related to this semantics, prompting a later refinement, but this need not concern us here.) Of course, intensional type theory may also be interpreted this way in lcccs, but then the interpretation of the identity types necessarily becomes trivial.

The first non-trivial semantics for intensional type theory were developed by Hofmann and Streicher [8] using **groupoids**, which are categories in which every arrow is an isomorphism. The category of groupoids is not locally cartesian closed, and the model employs certain fibrations (equivalently, groupoid-valued functors) to model type dependency. Intuitively, the identity type over a groupoid G is interpreted as the groupoid of arrows $G^{\rightarrow} \rightarrow G \times G$, so that an identity proof $f : \text{Id}_A(a, b)$ becomes an arrow $f : a \rightarrow b$ in G . The interpretation no longer validates extensionality, since there can be different elements a, b related by non-identity arrows $f : a \rightarrow b$. Indeed, there may be many different such arrows $f, g, \dots : a \rightarrow b$; however, unlike in the type theory, these cannot in turn be further related by identity terms of higher type $\vartheta : \text{Id}_{\text{Id}_A}(f, g)$, since a (conventional) groupoid has no non-trivial higher-dimensional structure. Thus the groupoid semantics validates a certain truncation principle, stating that all higher identity types are trivial—a form of extensionality one dimension up. In particular, the groupoid laws for the identity types are strictly satisfied in these models, rather than holding only up to propositional equality.

This situation has led to the use of the higher-dimensional analogues of groupoids, as formulated in category theory, in order to provide models admitting non-trivial higher identity types. Such higher groupoids have been studied extensively in homotopy theory in recent years, since they occur naturally as the (higher) fundamental groupoids of spaces (see below). In this direction, Warren [20] has generalized the groupoid model of [8] to strict ω -groupoids, thereby showing that the type theory truly possesses non-trivial higher-dimensional structure. Along similar lines, Garner [6] has used a 2-dimensional notion of fibration to model intensional type theory, and shown that when various truncation axioms are added the theory is sound and complete with respect to this semantics.

1.2. Homotopy theory. In homotopy theory one is concerned with spaces and continuous mappings up to homotopy; a homotopy between continuous maps $f, g : X \rightarrow Y$ is a continuous map $\vartheta : X \times [0, 1] \rightarrow Y$ satisfying $\vartheta(x, 0) = f(x)$ and $\vartheta(x, 1) = g(x)$. Such a homotopy ϑ can be thought of as a “continuous deformation” of f into g , determining a higher-dimensional arrow $\vartheta : f \rightarrow g$. As already suggested, one also considers homotopies between homotopies, referred to as higher homotopies. Algebraic invariants, such as homology or the fundamental group, are homotopy-invariant, in that any two spaces which are homotopy-equivalent must have the same invariants.

When we consider a space X , the paths in X , the homotopies between paths, and all higher homotopies, we obtain a structure called the **fundamental weak ω -groupoid of X** . We can truncate this structure by considering only paths up to homotopy, and this yields the usual fundamental groupoid of the space. This truncation is evidently analogous to adding to our type theory axioms of extensionality above the first identity type. Indeed, this is more than a mere analogy: these constructions, including the basic assignment of fundamental groupoids to objects, are special cases of a common, general construction that can be described abstractly in axiomatic homotopy theory. The central concept is that of a **Quillen model category**, which captures axiomatically some of the essential features of homotopy of topological spaces, enabling us to “do homotopy” in different mathematical settings, and to express the fact that two categories carry the same homotopical

information, even if they are not equivalent in the ordinary sense. The basic result of Awodey and Warren in [1] (see also [20]) is that it is possible to model the type theory in any Quillen model category which is well-behaved in certain ways (essentially using just the basic notion of a weak factorization system). In this interpretation, one uses path objects to model identity types in a non-trivial way, recovering the groupoid model as a special case. This suggests that intensional type theories are a sort of internal language of (certain kinds of) model categories. Indeed, in [5] it is shown that the type theory itself carries a natural such homotopy structure (i.e. a weak factorization system), so that the theory is not only sound but also complete with respect to such abstract homotopical semantics.

Thus we are justified in thinking of types in the intensional theory as **spaces**. From this point of view, the terms of the type A are the points of the “space” A , the identity type $\text{Id}_A(a, b)$ represents the collection of paths from a to b , and the higher identities are homotopies between paths, homotopies between homotopies of paths, et cetera. The fact that paths and homotopies do not form a groupoid, but only a groupoid up to homotopy, is of course precisely the same observation as the fact that the identity types only satisfy the groupoid laws up to propositional equality. This parallel between type theory and homotopy theory, which was first pointed out by Moerdijk a few years ago, has now been made precise by the recognition that both cases are instances of one and the same abstract axiomatic theory.

In particular, it has been shown independently by Lumsdaine [13] and Van den Berg and Garner [3] that the tower of identity types over any fixed base type A in the intensional theory indeed gives rise to a certain infinite dimensional categorical structure called a weak ω -groupoid. In fact, something apparently stronger is shown, namely that at every type the type theory already hosts an internal model of such a higher category. The next step in exploring the connection between type theory and topology is to investigate the relationship between type theoretic “truncation” (i.e. higher-dimensional extentionality principles) and topological “truncation” of the higher fundamental groups. Spaces for which the homotopy type is already completely determined by the fundamental groupoid are called **homotopy 1-types**, or simply 1-types. More generally, one has n -types, which are thought of as spaces which have no homotopical information above dimension n . One of the goals of homotopy theory is to obtain good models of homotopy n -types. For example, the category of groupoids is Quillen equivalent to the category of 1-types and therefore the corresponding homotopy categories (obtained by inverting weak equivalences) are equivalent; in this precise sense, groupoids are said to model homotopy 1-types (for more on homotopy types see [2]).

1.3. Contributions of this paper. The current paper aims at further investigation of the relationship between type theory and homotopy theory, but in a way that is somewhat different than the work already mentioned. First of all, our objective is not to give a new semantics, although it will become apparent that in fact implicitly we do obtain a possibility of having one. Secondly, we are not mainly interested in the possibility of obtaining higher-dimensional structures from type theories. Rather, we are interested in understanding the limitations of this process. Thirdly, we wish to make another connection between model categories and type theory, namely by showing that a category of suitably truncated type theories gives a model of the homotopy 1-types. It is our hope that this picture can then be extended to higher dimensions.

Our first goal is to show how every extension of type theory gives rise to a monad on the category of globular sets. Intuitively, the monad associated to a theory freely adds cells to a globular set in accordance with the rules of the type theory. For example, the monad will formally add composites and inverses for all cells of the globular set; however, even in dimension 0 it adds much more than just these formal composites; it also produces a plethora of new cells which we here call **doppelgänger**s. For every such monad we may consider its category of algebras: these we refer to as **Martin-Löf complexes** (or **ML-complexes**), and these are the main objects of study of the paper.

The theories which we shall consider arise from basic intensional Martin-Löf type theory having dependent sums and products as well as a natural numbers object. (The latter plays no conceptual role in this paper but because of its importance in virtually every application of the theory to mathematics and computer science we thought it important to show that our results are not affected by its presence.) We shall then consider extensions of this basic theory obtained by adding truncation axioms, which effectively trivialize the higher identity types above a fixed dimension. Using these theories we get a hierarchy of categories of Martin-Löf complexes, and in this paper we shall investigate the first two dimensions in detail.

The 0-dimensional case is relatively straightforward — we shall prove here that the monad on globular sets is idempotent and is in fact isomorphic to the connected components functor, so that its category of algebras (the 0-dimensional ML-complexes) is simply the category of sets.

Matters become more interesting in dimension 1. Towards an analysis of 1-dimensional ML-complexes we first observe, using the Hofmann-Streicher groupoid semantics, that every ML-complex has an underlying groupoid, and that there is a canonical comparison functor between the underlying groupoid of a free ML-complex and the free groupoid on the same globular set. This functor is not an isomorphism of groupoids, because the free ML-complex is, intuitively speaking, much larger due to all the doppelgänger s produced by the theory. The main technical difficulty then is to prove that there is still an equivalence of groupoids between the two. To this end, we employ a Tait-style computability predicate argument [19], which allows us to prove that every term of the theory represents, up to propositional equality, an object or morphism of the free groupoid. This essentially shows that even though the theory forces the existence of many more objects and arrows than needed to form the free groupoid, it does not force anything which is undesirable from a homotopical point of view. The proof also relies on a careful analysis of elimination terms for identity types, and in particular exploits the idea of identity types acting on contexts.

Once this key result is in place, we turn to an analysis of the category of 1-dimensional ML-complexes as a whole. To start, we set up an adjunction between this category and the category of groupoids. This adjunction is not an equivalence: a ML-complex structure on a globular set carries essentially more information than a groupoid structure. We can, however, make use of the adjunction by transferring along it the standard Quillen model structure on the category of groupoids [11], turning the category of 1-dimensional ML-complexes into a cofibrantly generated model category. Doing so requires a detailed analysis of colimits in the category of ML-complexes, and to this end we adopt a more model-theoretic perspective, viewing ML-complexes as “classifying complexes” for their own internal language.

This allows us to regard instead the category of theories and compute colimits there, which is technically easier.

Finally, we prove that the adjunction between groupoids and 1-dimensional ML-complexes is in fact a Quillen equivalence. Because the categories of groupoids and that of homotopy 1-types are Quillen equivalent, this makes precise in which sense the 1-truncated version of the type theory models homotopy 1-types. It also explains why the groupoid semantics is adequate from a homotopical point of view, but is still incomplete because it lacks the possibility (which is present in ML-complexes) of handling different interpretations for doppelgänger terms.

1.4. Plan of the paper. In Section 2 we recall the basics of Martin-Löf type theory as well as several facts about globular sets which will be required later. We also fix notation (some of which is non-standard). The reader who is familiar with this material should feel free to skip ahead. Section 3 describes the construction of monads on the category of reflexive globular sets coming from type theories. We then define the categories $\mathbf{MLC}\mathbf{x}$ and $\mathbf{MLC}\mathbf{x}_n$ of Martin-Löf complexes and n -truncated Martin-Löf complexes as the Eilenberg-Moore categories of the monads T_ω and T_n , respectively, generated by suitable theories.

In Section 4 we study 0-truncated Martin-Löf complexes. In particular, we show that the category $\mathbf{MLC}\mathbf{x}_0$ is equivalent to the category of sets and, moreover, that if X is a reflexive globular set, then $T_0(X)$ is the set of connected components of X . Even the proofs of these eminently plausible results are a bit more subtle than one might at first expect and their discussion in Section 4 serves as a reasonable “warm-up” for the later sections. One of the principal difficulties one faces when proving results about Martin-Löf complexes is that the type theory also adds, in addition to composition and inverses, the doppelgänger terms mentioned earlier. In our analysis of $\mathbf{MLC}\mathbf{x}_0$ then it is necessary to develop techniques for proving that such terms are essentially harmless. The principal technique, introduced in Section 4, we employ is a modified version of the Tait-style [19] logical-predicates argument. Section 5 begins with a proof that every 1-truncated Martin-Löf complex can be equipped with the structure of a groupoid. We then explicitly introduce the modified logical-predicates technique, called here the method of *relevant types and terms*, and developing several of the technical results on it which are required later. The basic idea behind this approach is that the *relevant terms* of identity type should be representatives of elements of the free groupoid on the underlying reflexive globular set and the principal result (Proposition 5.13) is that all terms of appropriate type are propositionally equal to relevant terms. The proof of Proposition 5.13 relies in particular on an analysis, up to propositional equality, of the elimination “ J -terms” for identity types via the action of maps $\{0, 1\} \rightarrow \{0, 1\}$ on those contexts which depend on identity types. This analysis is given in Sections 5.4 and 5.7. An immediate consequence of Proposition 5.13 is that the free groupoid $\mathcal{F}(G)$ on a reflexive globular set is equivalent to the induced groupoid structure on $T_1(G)$. This is Theorem 5.14. Finally, in Section 6 we develop the general categorical properties of the category $\mathbf{MLC}\mathbf{x}_1$. In particular we prove as Theorem 6.22 that $\mathbf{MLC}\mathbf{x}_1$ is a bicomplete category with a Quillen closed model structure. This model structure is obtained via an adjunction with the category of groupoids via Crans’s Transfer Theorem [4]. Finally, we have, using our earlier results on relevant terms, that the adjunction between $\mathbf{MLC}\mathbf{x}_1$ and the category of groupoids is a Quillen equivalence (Corollary 6.23).

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2. BACKGROUND

The purpose of this section is to provide the reader with a brief introduction to Martin-Löf type theory. We begin by giving a quick exposition of the main features of the most basic version of the theory we shall be concerned with. In particular we explain the different kinds of judgements of the system, dependent products and sums, identity types and the notion of propositional equality. We also use this as an opportunity to fix some notation and terminology, in particular concerning identity types.

We assume that the reader is somewhat familiar with at least simple type theory. For more background on (dependent) type theory we refer to the textbook [10]. The reader who is more familiar with higher-dimensional category theory or homotopy theory might also consult [1] for a “homotopical” view of type theory.

In the last subsection we introduce the basic categorical structures used in the paper, namely globular sets. A more detailed exposition of globular sets may be found in [17], or the textbook [12].

2.1. Type dependency, contexts and judgements. Type dependency means that types may depend on variables of other types; for example one can have a type $T(x)$ depending on a variable x of type S . Such a type $T(x)$ is often thought of as being indexed by the type S . To illustrate this, suppose that we let S denote the type of rings; then the type $T(x)$ of modules depends on, or varies over, the type of rings.

One may then substitute a term a of type S into the type $T(x)$, as to obtain a new type $T(a)$. In the above example, $T(a)$ would be the type of modules over the ring a .

The fact that types may depend on terms has two obvious consequences: first, one can no longer, as in simple type theory, separate the formation of types and that of terms into two inductive definitions; rather, types and terms are derived simultaneously. Second, the notion of a **variable context** also needs to take dependency into account. Explicitly, this means that a variable context Γ is now an ordered sequence of variable declarations $\Gamma = (x_1 : T_1, \dots, x_k : T_k)$, where each type T_i may only depend on the variables declared earlier, i.e. on x_1, \dots, x_{i-1} . For example,

$$x : S, y : S, z : T(x), v : R(x, y, z)$$

is a legitimate variable context, but

$$x : S, y : s, v : R(x, y, z), z : T(x)$$

is not, because of the fact that R depends on z , which hasn't been declared yet. Throughout, we shall always assume that contexts are well-formed in this sense.

Thus the theory is concerned with types and terms in context, and with equalities between such types and terms. Formally, statements about these are called **judgements**, and these come in four kinds:

$$\Gamma \vdash T : \text{type}$$

This judgement states that T is a type, possibly depending on the variables declared in the context Γ .

$$\Gamma \vdash \tau : T$$

This judgement states that τ is a term of type T , where both τ and the type T may depend on the variables from Γ .

$$\Gamma \vdash T = S : \mathbf{type}$$

This judgement states that T and S are (definitionally) equal types.

$$\Gamma \vdash \tau = \tau' : T$$

This judgement states that τ and τ' are (definitionally) equal terms of type T .

In the theory, such judgements are derived from axioms using inference rules. These derivations (which may formally be regarded as finite trees suitably labelled by judgements and inference rules) are the main objects of study. Below we shall discuss several of the rules which may be used to derive new judgements from old; the axioms typically include judgements stating the existence of certain basic types and terms.

When the context plays no role in a judgement or rule of the theory, we shall usually omit it altogether.

2.2. Definitional equality. The notion of equality here is the standard one, but often the qualifier **definitional** is used to distinguish it from the different notion of propositional equality, to be discussed below. The rules governing the behaviour of definitional equality are as expected. Apart from the rules expressing that definitional equality is an equivalence relation, there are rules which force that it is a congruence with respect to substitution into types and terms:

$$\frac{\vdash a = b : A \quad x : A \vdash B(x) : \mathbf{type}}{\vdash B(a) = B(b) : \mathbf{type}}$$

$$\frac{\vdash a = b : A \quad x : A \vdash f(x) : B(x)}{\vdash f(a) = f(b) : B(a)}$$

$$\frac{\vdash A = B : \mathbf{type} \quad \vdash a : A}{\vdash a : B}$$

The first rule states that substituting equal terms into a type results in equal types; the second states the same, but now for substitution into terms; the last rule states that equal types are inhabited by the same terms. A complete set of rules for definitional equality may be found in the appendix.

2.3. Dependent products and sums. There are several ways to construct new types from old. For each new type one specifies three things: an introduction rule which generates new terms of the type; an elimination rule which shows how general terms of the new type may be used; and a conversion rule which governs the interaction between the two.

We now discuss the formation of dependent products and sums. Given a type $B(x)$ depending on a variable x of type A , we may form the type $\prod_{x:A} B(x)$, to be thought of as the type of sections of $B(x)$ over A . The rules are as follows:

$$\frac{x : A \vdash B(x) : \mathbf{type}}{\vdash \prod_{x:A} B(x) : \mathbf{type}} \quad \prod \text{ formation}$$

$$\frac{x : A \vdash f(x) : B(x)}{\vdash \lambda_{x:A}.f(x) : \prod_{x:A} B(x)} \text{ } \prod \text{ introduction}$$

$$\frac{\vdash f : \prod_{x:A} B(x) \quad \vdash a : A}{\vdash \mathbf{app}(f, a) : B(a)} \text{ } \prod \text{ elimination}$$

$$\frac{\vdash \lambda_{x:A}.f(x) : \prod_{x:A} B(x) \quad \vdash a : A}{\vdash \mathbf{app}(\lambda_{x:A}.f(x), a) = f(a) : B(a)} \text{ } \prod \text{ conversion}$$

Thus an introduction term of type $\prod_{x:A} B(x)$ is a lambda expression, thought of as an operation assigning to each $x : A$ a value $f(x) : B(x)$. A general term f of type $\prod_{x:A} B(x)$ may be applied to a term a of type A , as to return a term $\mathbf{app}(f, a)$ of type $B(a)$. Finally, the conversion rule, commonly known as **beta-conversion**, allows us to reduce $\mathbf{app}(\lambda_{x:A}.f(x), a)$ to $f(a)$. In the case where the type $B(x)$ does not depend on the variable x , we shall often write B^A for the type $\prod_{x:A} B(x)$.

Similarly, the theory admits formation of **dependent sum types** $\sum_{x:A} B(x)$. The rules are:

$$\frac{x : A \vdash B(x) : \mathbf{type}}{\vdash \sum_{x:A} B(x) : \mathbf{type}} \text{ } \sum \text{ formation}$$

$$\frac{\vdash a : A \quad \vdash b : B(a)}{\vdash \mathbf{pair}(a, b) : \sum_{x:A} B(x)} \text{ } \sum \text{ introduction}$$

$$\frac{\vdash p : \sum_{x:A} B(x) \quad x : A, y : B(x) \vdash \psi(x, y) : C(\mathbf{pair}(x, y))}{\vdash \mathbf{R}([x : A, y : B(x)]\psi(x, y), p) : C(p)} \text{ } \sum \text{ elimination}$$

$$\frac{\vdash a : A \quad \vdash b : B(a) \quad x : A, y : B(x) \vdash \psi(x, y) : C(\mathbf{pair}(x, y))}{\vdash \mathbf{R}([x : A, y : B(x)]\psi(x, y), \mathbf{pair}(a, b)) = \psi(a, b) : C(\mathbf{pair}(a, b))} \text{ } \sum \text{ conversion}$$

The notation $[x : A, y : B(x)]$ indicates that the variables x and y are formally bound in the term. Using these rules, we may define projection terms by letting

$$\pi_0(p) = \mathbf{R}([x : A, y : B(x)]x, p), \quad \pi_1(p) = \mathbf{R}([x : A, y : B(x)]y : B(p))$$

in

$$\frac{\vdash p : \sum_{x:A} B(x) \quad x : A, y : B(x) \vdash x : A}{\vdash \pi_0(p) : A} \text{ } \sum \text{ elimination}$$

$$\frac{\vdash p : \sum_{x:A} B(x) \quad x : A, y : B(x) \vdash y : B(x)}{\vdash \pi_1(p) : B(\pi_0(p))} \text{ } \sum \text{ elimination}$$

The projection terms $\pi_0(p)$ and $\pi_1(p)$ then satisfy the conversion rules

$$\pi_i(\mathbf{pair}(a, b)) = \begin{cases} a & \text{if } i = 0 \\ b & \text{if } i = 1. \end{cases}$$

We point out that we do not adopt the η -rule for sums

$$\frac{\vdash p : \sum_{x:A} B(x)}{\vdash p = \mathbf{pair}(\pi_0(p), \pi_1(p)) : \sum_{x:A} B(x)}$$

but that it can easily be proved that every term of type $\sum_{x:A} B(x)$ is *propositionally* equal to a pair term (see the discussion of identity types below for what this means).

In some treatments a different formulation of the rules for sum types is used, taking the projection terms as primitive. In the presence of the η -rule both formulations are equivalent, but without the η -rule this latter approach is strictly weaker (see [7]).

2.4. Identity types. Let A be a type. For every pair of terms a, b of type A we may form a new type $\underline{A}(a, b)$. This type is thought of as the type of proofs of the fact that a and b are equal. A term $\tau : \underline{A}(a, b)$ is sometimes referred to as a **propositional identity proof**. It is important to note that the existence of such a proof term does not necessarily imply that $a = b$ in the definitional sense of equality discussed above. From a more geometric perspective one may think of a propositional equality as a **homotopy** between a and b (see [1]). This explains why we sometimes use the notation $a \simeq b$ to indicate the existence of a propositional identity between a and b . We also point out that it is perhaps more common to denote the identity type $\underline{A}(a, b)$ by $\mathbf{Id}_A(a, b)$, but we have chosen to adopt a notation more suggestive of hom-sets.

The formation rule for the identity types is thus as follows (omitting contexts for simplicity)

$$\frac{\vdash a, b : A}{\vdash \underline{A}(a, b) : \mathbf{type}} \text{ Id formation}$$

where we write $a, b : A$ as an abbreviation for the two judgements $a : A$ and $b : A$. Then, there are the introduction and elimination rules:

$$\frac{\vdash a : A}{\mathbf{r}(a) : \underline{A}(a, a)} \text{ Id introduction}$$

$$\frac{\begin{array}{l} x : A, y : A, z : \underline{A}(x, y) \vdash B(x, y, z) : \mathbf{type} \\ x : A \vdash \varphi(x) : B(x, x, \mathbf{r}(x)) \\ \vdash f : \underline{A}(a, b) \end{array}}{\vdash \mathbf{J}_{[x,y:A,z:\underline{A}(x,y)]B(x,y,z)}([x:A]\varphi(x), a, b, f) : B(a, b, f)} \text{ Id elimination}$$

The introduction term $\mathbf{r}(a)$ is called the **reflexivity term**; it witnesses the fact that $a \simeq a$. The elimination rule is a bit more involved. What we start with is first of all a type A (which is referred to as the **type over which the elimination occurs**), and an identity proof $f : \underline{A}(a, b)$ (this is the term which is being eliminated). Next we need a type $B(x, y, z)$ (called the **pattern type**) and a term $\varphi(x)$ of type $B(x, x, \mathbf{r}(x))$ which intuitively witnesses the fact that the pattern type is inhabited in the trivial instance where we substitute a reflexivity term. Given all of this, we may form a new term $\mathbf{J}([x : A]\varphi(x), a, b, f)$ of type $B(a, b, f)$. One way to think of this \mathbf{J} -term is as the result of expanding the term $\varphi(x)$ using the propositional equality $f : \underline{A}(a, b)$. This viewpoint will be developed in more detail later on.

Note also that the variables x, y, z in the elimination rule need not necessarily occur in the type $B(x, y, z)$, and similarly that x need not occur in $\varphi(x)$. Also,

it may happen that the term f (and possibly also a, b) are themselves variable, in which case the J -term depends on those variables.

Finally, there is a conversion rule:

$$\frac{\vdash a : A}{\vdash \mathbf{J}_{[x,y:A,z:\underline{A}(x,y)]B(x,y,z)}([x : A]\varphi(x), a, a, \mathbf{r}(a)) = \varphi(a) : B(a, a, \mathbf{r}(a))} \text{Id conversion}$$

Thus, using a trivial identity proof $\mathbf{r}(a)$ to build a J -term does simply give back $\varphi(a)$.

To illustrate the use of the rules for derivations of judgements, we give an example of a derivation which shows that the result of applying a term to two propositionally equal terms results in propositionally equal terms.

Example 2.1. Let $\tau \simeq \tau' : \prod_{v:S} T(v)$, and let $\sigma : S$ be derivable. Then $\mathbf{app}(\tau, \sigma) \simeq \mathbf{app}(\tau', \sigma)$ is also derivable. Indeed, consider the following derivation:

$$\frac{\begin{array}{l} x, y : \prod_{v:S} T(v), z : \prod_{v:S} T(v)(x, y) \vdash \underline{T(\sigma)}(\mathbf{app}(x, \sigma), \mathbf{app}(y, \sigma)) : \text{type} \\ x : \prod_{v:S} T(v) \vdash \mathbf{r}(\mathbf{app}(x, \sigma)) : \underline{T(\sigma)}(\mathbf{app}(x, \sigma), \mathbf{app}(x, \sigma)) \\ \vdash f : \prod_{v:S} T(v)(\tau, \tau') \end{array}}{\vdash \mathbf{J}([x : \prod_{v:S} T(v)]\mathbf{r}(\mathbf{app}(x, \sigma)), \tau, \tau', f) : \underline{T(\sigma)}(\mathbf{app}(\tau, \sigma), \mathbf{app}(\tau', \sigma))} \text{Id elim.}$$

Here, f is a term witnessing the propositional identity $\tau \simeq \tau'$. Of course, the two other premises have to be derived as well, but this is straightforward.

Similarly we may derive from $\sigma \simeq \sigma'$ that $\mathbf{app}(\tau, \sigma) \simeq \mathbf{app}(\tau, \sigma')$.

2.5. Natural numbers. So far we have discussed only methods to construct new types and terms from ones already present. It is common to introduce as a basic type the type \mathbf{N} of natural numbers, and to add axioms

$$\frac{}{0 : \mathbf{N}} \text{N introduction (i)}$$

$$\frac{n : \mathbf{N}}{\mathbf{S}(n) : \mathbf{N}} \text{N introduction (ii)}$$

which allow us to construct the standard numerals. Since the type of natural numbers will not play a central role in this paper we refer to the appendix for the precise formulation of the elimination rule (expressing the possibility of defining terms by recursion) and the conversion rules. We do point out however that aside from the standard numerals the theory may prove the existence of other, non-standard, numerals as well.

2.6. Theories and extensions. We shall denote by \mathbb{T}_ω the system having all of the above constructors and rules, including those for the type of natural numbers (for a complete description see the appendix). By a **type theory** we shall mean any extension of the basic system \mathbb{T}_ω obtained by adding axioms and possibly also inference rules. The axioms are judgements which may assert the existence of basic types or terms, or may assert the equality between certain types or terms. Possible additional inference rules include the so-called **truncation-** and **reflection rules**, which express triviality of certain identity types. See Section 3 for a discussion of these rules.

Given two type theories \mathbb{T} and \mathbb{T}' , we say that \mathbb{T}' is an **extension** of \mathbb{T} when every judgement which is derivable in \mathbb{T} is also derivable in \mathbb{T}' . Notation: $\mathbb{T} \subseteq \mathbb{T}'$. Thus by our definitions, \mathbb{T}_ω is the smallest type theory.

We will often be interested in extensions which are obtained by adding only axioms of an equational nature. A set E of pairs of terms of some theory \mathbb{T} is **admissible** if it satisfies the condition that if $\langle \tau, \tau' \rangle$ is in E , then there exists a context $(x_1 : A_1, \dots, x_n : A_n(x_1, \dots, x_{n-1}))$ together with terms $a_1, a'_1, \dots, a_{n-1}, a'_{n-1}$ such that a_i is related to a'_i in under the equivalence relation generated by E and in \mathbb{T} the judgements $\vdash \tau : A_n(a_1, \dots, a_{n-1})$ and $\vdash \tau' : A_n(a'_1, \dots, a'_{n-1})$ are derivable. An extension \mathbb{T}' of \mathbb{T} is then **equational** if it is obtained by adding to \mathbb{T} only axioms

$$\vdash \tau = \tau' : T$$

for $\langle \tau, \tau' \rangle$ in an admissible set of pairs of terms with $\tau : T$ in \mathbb{T} . Note that we only permit equational extensions where the equations are between *closed* terms. However, it is possible to extend this notion of equational extension to include also equations between open terms, but this direction is not pursued here.

2.7. Expressions. Because the types and terms of such theories are defined simultaneously, in order to formally specify the syntax of the theory it is convenient to first define inductively a class of *expressions* — which need not satisfy any typing conventions — from which the genuine syntactical data of the theory is then extracted via the rules given above (and stated in full in Appendix A). For example, in order to formally define the theory \mathbb{T}_ω we first fix a countable set V of (untyped) variables and then define the class of **expressions of \mathbb{T}_ω** , denoted $\text{Exp}(\mathbb{T}_\omega)$, by

- v is in $\text{Exp}(\mathbb{T}_\omega)$, for any v in V ;
- 0 and \mathbb{N} are in $\text{Exp}(\mathbb{T}_\omega)$;
- $\mathbf{S}(p)$, $\mathbf{app}(p, q)$, $\mathbf{pair}(p, q)$, $\mathbf{r}(p)$, $\mathbf{rec}(p, q, r)$, $\mathbf{R}(p, q)$, $\mathbf{J}(p, q, r, s)$ and $\lambda_{p.q}.r$ are in $\text{Exp}(\mathbb{T}_\omega)$ when p, q, r and s are;
- $\underline{p}(q, r)$, $\prod_{p:q} r$ and $\sum_{p:q} r$ are in $\text{Exp}(\mathbb{T}_\omega)$ when p, q and r are.

Thus, the expressions are generated by applying all term- and type constructors without regard for well-typedness. The derivation rules of the type theory may then be regarded as carving out from this set of all expressions those which are well-formed and well-typed. The syntax of other the theories extending \mathbb{T}_ω that we consider later is similarly specified in this way with the evident modifications to the definition of the expressions. Moreover, because the expressions are inductively generated it follows that the sets of the form $\text{Exp}(-)$ possess an obvious universal property.

2.8. Context morphisms. Recall that if Γ and

$$\Delta = (x_1 : A_1, \dots, x_n : A_n(x_1, \dots, x_{n-1}))$$

are contexts, then a **context morphism** $a : \Gamma \rightarrow \Delta$ is a sequence of terms

$$\Gamma \vdash a_1 : A_1, \Gamma \vdash a_2 : A_2(a_1), \dots, \Gamma \vdash a_n : A_n(a_1, \dots, a_{n-1}).$$

There is a category of contexts with arrows the context morphisms (cf. [7]). We will only be employing the terminology of context morphisms very briefly in Section 5.4.

2.9. Globular sets. Globular sets are structures which form the basis for several definitions of higher dimensional category. One way to think of a globular set is as a higher dimensional graph: not only are there vertices and edges between the vertices, but one has edges between edges, and so on. Formally, a globular set G is a tuple $(G_n, s_n, t_n)_{n \in \mathbb{N}}$, where each G_n is a set, and where $s_n, t_n : G_{n+1} \rightarrow G_n$ are functions subject to the **globular identities**

$$d_n d_{n+1} = d_n s_{n+1}; \quad s_n s_{n+1} = s_n d_{n+1} \quad (1)$$

for $d = s, t$. Elements of G_n are referred to as **n-cells**, and are said to have **dimension n**. The maps s_n and t_n are called **source** and **target** maps, respectively.

If G is a globular set for which $G_n = \emptyset$ for all $n > 1$, then we may simply regard G as a (directed) graph. If there exist elements of higher dimension, then the globular identities ensure that the source $s_n(x)$ and target $t_n(x)$ for such an n -dimensional edge are a parallel pair of edges of dimension $n - 1$.

Because it is often convenient, given a n -cell α of a globular set G , to be able to refer to the result of iteratively taking the source or target of α we introduce the notation α_0^j, α_1^j for these corresponding j -cells. Explicitly, for $0 \leq j \leq n - 1$,

$$\alpha_i^j := \begin{cases} s_j \circ \cdots \circ s_{(n-1)}(\alpha) & \text{if } i = 0 \\ t_j \circ \cdots \circ t_{(n-1)}(\alpha) & \text{if } i = 1. \end{cases}$$

By the globular identities, α_0^j and α_1^j are the only elements of G_j which are obtainable from α by applying the source and target maps.

A globular set G is said to be **reflexive** if it comes equipped with a family of maps $i_n : G_n \rightarrow G_{n+1}$, such that

$$t_n i_n = 1 = s_n i_n \quad (2)$$

We think of $i_n(x)$ as the identity edge from x to itself. In this paper we shall be working with reflexive globular sets only. For readability we often omit the dimension from the source, target and identity maps of a globular set.

A **morphism of globular sets** $f : G \rightarrow H$ is a family of functions $f_n : G_n \rightarrow H_n$ which commute with the source and target maps. Globular sets and their morphisms form a category denoted **GSet**. For reflexive globular sets we also require that the f_n commute with the identity maps; this gives a category **rGSet**.

There is a functor $\Delta : \mathbf{Set} \rightarrow \mathbf{rGSet}$ which takes a set A to the constant globular set with $\Delta(A)_n = A$. A globular set which is isomorphic to one of the form $\Delta(A)$ will be called **constant**. The functor Δ has a left adjoint $\pi_0 : \mathbf{rGSet} \rightarrow \mathbf{Set}$; this functor assigns to a globular set G its set of connected components

$$\pi_0(G) = G_0 / \sim$$

where the equivalence relation on 0-dimensional elements is generated by

$$x \sim y \Leftrightarrow \exists f \in G_1. s(f) = x, t(f) = y.$$

We may express this as a (reflexive) coequalizer diagram:

$$G_1 \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} G_0 \twoheadrightarrow \pi_0(G).$$

The composite $\Delta\pi_0 : \mathbf{rGSet} \rightarrow \mathbf{rGSet}$ is an idempotent monad, to which we refer as **0-truncation**. Often it will be convenient to identify the essential image of this functor (the constant globular sets) with the category of sets.

We may also truncate a globular set at dimension 1: in this case we replace the category of sets by the category **rGraph** of directed reflexive graphs. There is a functor **rGSet** \rightarrow **rGraph** which assigns to a globular set G the graph whose vertex set is G_0 and whose edge set is G_1/\sim , where two edges h, k satisfy $h \sim k$ if there is an $\alpha \in G_2$ with $s(\alpha) = h, t(\alpha) = k$.

In the other direction, any directed reflexive graph G gives a globular set with which is the same as G in dimensions 0 and 1, and is degenerate above dimension 1. The composite functor **rGSet** \rightarrow **rGraph** \rightarrow **rGSet** will be called **1-truncation**, and a globular set in the essential image of this functor will be said to be **1-truncated**. We shall often identify the subcategory of 1-truncated globular sets with the category of graphs.

3. THE MARTIN-LÖF COMPLEX MONAD

The goal of this section is to state the formal definition of Martin-Löf complexes. Because Martin-Löf complexes are defined to be algebras for a monad on the category of reflexive globular sets the principal matter addressed here is the construction of the appropriate monad. The monad for the theory \mathbb{T}_ω obtained by the construction below essentially corresponds to the monad obtained from the operadic constructions due to van den Berg and Garner [3] and Lumsdaine [13], who show that the algebras are weak omega-groupoids. It is worth emphasizing that, because the converse seems not to hold, the problem of determining precisely the higher-dimensional structure of the algebras for these monads remains open. It is to the solution of this problem that the results of the present paper contribute.

Because we will be interested in algebras for the monad generated by theories, such as the theories \mathbb{T}_n described in Section 3.5 below, which extend \mathbb{T}_ω the description of the monad involved in the definition of Martin-Löf complexes will be described for an arbitrary extension of \mathbb{T}_ω . As such, throughout this section \mathbb{T} is assumed to be an arbitrary theory extending \mathbb{T}_ω . Finally, observe that although we choose to work with reflexive globular sets, the construction of the monad can be modified to yield a corresponding monad for globular sets.

3.1. Notation for iterated identity types and other conventions. In order to most efficiently (and readably) state some of the additional principles for identity types that we consider it is useful to introduce notation for iterated identity types. Fixing a type A together with terms $a, b : A$ in some ambient context, we introduce the (at this stage superfluous) notation

$$\begin{aligned} \underline{A}^0 &:= A, \text{ and} \\ \underline{A}^1(a, b) &:= \underline{A}(a, b). \end{aligned}$$

In general, assuming given terms

$$\vdash a_{n+1}, b_{n+1} : \underline{A}^n(a_1, b_1; \cdots; a_n, b_n),$$

we define

$$\underline{A}^{n+1}(a_1, b_1; \cdots; a_n, b_n; a_{n+1}, b_{n+1}) := \underline{A}^n(a_1, b_1; \cdots; a_n, b_n)(a_{n+1}, b_{n+1}).$$

In the sequel we will be dealing extensively with sets of terms from various theories extending \mathbb{T}_ω . We adopt the convention that such terms are always assumed to be identified modulo definitional equality and α -equivalence.

As a notational convenience we adopt the convention of, given a reflexive globular set $G = (G_n)_{n \geq 0}$, writing G for the set $\sum_{n \geq 0} G_n$.

3.2. The reflexive globular set generated by a type. Fix a type A in \mathbb{T} . It is possible that A is a type in context, yet we will assume that A is a type in the empty context. The case where the context is non-empty is obtained in essentially the same way, and so this is a reasonable simplification. We will now construct a reflexive globular set denoted by $\mathbf{\Gamma}(A)_{\mathbb{T}}$ and called the **reflexive globular set generated by A (in \mathbb{T})**. When the theory \mathbb{T} is fixed we will omit the subscript and write simply $\mathbf{\Gamma}(A)$. This construction will be carried out in such a way that the following conditions are satisfied:

- (1) Each element of $\mathbf{\Gamma}(A)_n$ is a tuple of $(2n + 1)$ elements of the set of terms of \mathbb{T} .
- (2) If both $(\vec{\alpha}; \beta)$ and $(\vec{\alpha}; \beta')$ are in $\mathbf{\Gamma}(A)_n$, then $\vdash \underline{A}^{n+1}(\vec{\alpha}; \beta, \beta') : \mathbf{type}$ is derivable in \mathbb{T} .
- (3) The source and target maps $s, t : \mathbf{\Gamma}(A)_{n+1} \rightarrow \mathbf{\Gamma}(A)_n$ must send a tuple $(\alpha_0, \dots, \alpha_{2n})$ to $(\alpha_0, \dots, \alpha_{2n-2})$ and $(\alpha_0, \dots, \alpha_{2n-3}, \alpha_{2n-1})$, respectively.

We begin by defining

$$\mathbf{\Gamma}(A)_0 := \{a \mid \vdash a : A\},$$

$$\mathbf{\Gamma}(A)_1 := \{(a_0, a_1; \alpha) \mid a_0, a_1 \in \mathbf{\Gamma}(A)_0 \text{ and } \vdash \alpha : A(a_0, a_1)\},$$

and the maps $s, t : \mathbf{\Gamma}(A)_1 \rightarrow \mathbf{\Gamma}(A)_0$ are simply the projections π_0, π_1 sending $(a_0, a_1; \alpha)$ to a_0 and a_1 , respectively. Assuming $\mathbf{\Gamma}(A)$ has been constructed up to stage n , we define $\mathbf{\Gamma}(A)_{n+1}$ to be the following set

$$\{(\vec{\alpha}; \beta_0, \beta_1; \gamma) \mid (\vec{\alpha}; \beta_i) \in \mathbf{\Gamma}(A)_n \text{ for } i = 0, 1, \text{ and } \vdash \gamma : \underline{A}^{n+1}(\vec{\alpha}; \beta_0, \beta_1)\}.$$

The source and target maps $s, t : \mathbf{\Gamma}(A)_{n+1} \rightarrow \mathbf{\Gamma}(A)_n$ are given by the projections

$$(\vec{\alpha}; \beta_0, \beta_1; \gamma) \longmapsto (\vec{\alpha}; \beta_i),$$

for $i = 0$ and $i = 1$, respectively.

Lemma 3.1. *Given an extension \mathbb{T} of \mathbb{T}_ω and a (closed) type A of \mathbb{T} , the graded set $\mathbf{\Gamma}(A)$ described above is a reflexive globular set.*

Proof. The maps $i : \mathbf{\Gamma}(A)_n \rightarrow \mathbf{\Gamma}(A)_{n+1}$ are obtained using reflexivity terms. The equations for reflexive globular sets are then readily verified. \square

3.3. The type theory associated to a reflexive globular set. Not only does every type A give rise to a reflexive globular set, but also every reflexive globular set G gives rise to a type theory $\mathbb{T}[G]$.

Definition 3.2. Given a reflexive globular set G , the type theory $\mathbb{T}[G]$ **generated by G** (or **\mathbb{T} with G adjoined**) is obtained by augmenting \mathbb{T} with the following additional symbols and rules:

- A basic type $\vdash \ulcorner G \urcorner$;
- Basic terms $\vdash \ulcorner g \urcorner : \ulcorner G \urcorner$, for each vertex $g \in G_0$;
- Basic terms $\vdash \ulcorner f \urcorner : \ulcorner G \urcorner(\ulcorner g \urcorner, \ulcorner h \urcorner)$, for each element $f \in G_1$ with $s(f) = g$ and $t(f) = h$;

- Basic terms

$$\vdash \ulcorner \alpha \urcorner : \ulcorner G \urcorner^n (\ulcorner \alpha_0^0 \urcorner, \ulcorner \alpha_1^0 \urcorner; \ulcorner \alpha_0^1 \urcorner, \ulcorner \alpha_1^1 \urcorner; \dots; \ulcorner \alpha_0^{n-1} \urcorner, \ulcorner \alpha_1^{n-1} \urcorner) \quad (3)$$

where α_i^j for $i = 0, 1$ and $0 \leq j \leq n - 1$ are as defined in Section 2.9, for each $\alpha \in G_n$;

- New conversion rules:

$$\ulcorner i(\alpha) \urcorner = \mathbf{r}(\ulcorner \alpha \urcorner) : \ulcorner G \urcorner^{n+1}(\dots; \ulcorner \alpha \urcorner, \ulcorner \alpha \urcorner)$$

for every $\alpha \in G_n$.

xs

Remark. As a matter of notation, we write $\Gamma \vdash_G \mathcal{J}$ to indicate that the judgement $\Gamma \vdash \mathcal{J}$ is derivable in $\mathbb{T}[G]$. Finally, we also write Exp_G instead of the more cumbersome $\text{Exp}(\mathbb{T}[G])$. Also, when no confusion will result, we identify the symbol $\ulcorner \tau \urcorner$ with τ itself. E.g., we write $f : \underline{G}(g, h)$ instead of the more cumbersome $\ulcorner f \urcorner : \ulcorner G \urcorner(\ulcorner g \urcorner, \ulcorner h \urcorner)$.

In subsequent sections it will be convenient to have at our disposal techniques for constructions maps between the sets of expressions of one type theory $\mathbb{T}[G]$ and another $\mathbb{T}[H]$, for G and H globular sets. Along these lines, we make the following observation.

Lemma 3.3. *Given globular sets G and H , any function*

$$G \xrightarrow{\varphi} \text{Exp}_H$$

has a unique extension $\hat{\varphi} : \text{Exp}_G \rightarrow \text{Exp}_H$, commuting with the operations from which the expressions are formed, such that the following diagram of sets commutes:

$$\begin{array}{ccc} \text{Exp}_G & \xrightarrow{\hat{\varphi}} & \text{Exp}_H \\ & \swarrow i_G & \nearrow \varphi \\ & G & \end{array}$$

where i_G is the map sending $g \in G_n$ to $\ulcorner g \urcorner$.

Note that the basic type $\ulcorner G \urcorner$ is sent by the extension $\hat{\varphi}$ to $\ulcorner H \urcorner$. Of course, depending on the nature of φ the extension $\hat{\varphi}$ may or may not preserve derivable judgements. Such a $\hat{\varphi}$ will, however, commute with substitution. I.e., if $e(x)$ is an expression of $\mathbb{T}[G]$ with x free, then, for any other expression f ,

$$\hat{\varphi}(e)[\hat{\varphi}(f)/x] = \hat{\varphi}(e[f/x]). \quad (4)$$

3.4. The induced monad on globular sets. We will now see that composing the foregoing processes

$$G \mapsto \mathbb{T}[G], \text{ and}$$

$$A : \text{type} \mapsto \mathbf{\Gamma}(A),$$

yields a monad T on the category \mathbf{rGSet} of reflexive globular sets. Given a globular set G ,

$$T(G) := \mathbf{\Gamma}(\ulcorner G \urcorner). \quad (5)$$

Suppose given a map $\varphi : G \rightarrow H$ of globular sets. To see that this assignment is in fact functorial we begin by noting that, by Lemma 3.3, the map

$$g \longmapsto \ulcorner \varphi(g) \urcorner$$

for $g \in G_n$, possesses a canonical extension $\varphi_* : \text{Exp}_G \rightarrow \text{Exp}_H$. I.e., in the notation of Lemma 3.3,

$$\varphi_* := \widehat{i_H \circ \varphi}.$$

In order to be able to use φ_* to define the action of T on arrows we must first verify that it preserves derivable judgements, where the action of φ_* extends to judgements in the obvious manner.

Lemma 3.4. *Suppose \mathcal{J} is a judgement derivable in $\mathbb{T}[G]$, then $\varphi_*(\mathcal{J})$ is derivable in $\mathbb{T}[H]$.*

Proof. The proof is a straightforward induction on the structure of derivations $\vdash_G \mathcal{J}$. For example, suppose \mathcal{J} is the conclusion $\Gamma \vdash_G \lambda_{x:A}.b(x) : \prod_{x:A}.B(x)$ of the introduction rule for dependent products. Then we have by the induction hypothesis that

$$\varphi_*(\Gamma), x : \varphi_*(A) \vdash_H \varphi_*(b(x)) : \varphi_*(B(x)).$$

Applying the introduction rule in $\mathbb{T}[H]$ yields the appropriate judgement since

$$\varphi_*\left(\prod_{x:A}.B(x)\right) = \prod_{x:\varphi_*(A)}. \varphi_*(B)(x),$$

by definition of φ_* . The only case which merits special attention are those judgements of the form (3) which occur as axioms of $\mathbb{T}[G]$. Such judgements are preserved by the fact that φ is a map of globular sets. \square

Lemma 3.5. *The assignment (5) is functorial $T : \mathbf{rGSet} \rightarrow \mathbf{rGSet}$.*

Proof. Let

$$T(\varphi)(\alpha_0, \alpha_1, \dots, \alpha_{2n}) := (\varphi_*(\alpha_0), \varphi_*(\alpha_1), \dots, \varphi_*(\alpha_{2n})),$$

for $\vec{\alpha}$ in $T(G)_{n+1}$. That this definition makes sense follows from Lemma 3.4 and the definition of φ_* . Trivially, $T(1_G) = 1_{T(G)}$. To see that T is well behaved with respect to composition it suffices to show that, when given $\psi : H \rightarrow I$, we have

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_*.$$

For this we observe that on the generators $g \in G_n$,

$$\psi_*(\varphi_*(\ulcorner g \urcorner)) = \psi_*(\ulcorner \varphi(g) \urcorner) = \ulcorner \psi \circ \varphi(g) \urcorner = (\psi \circ \varphi)_*(\ulcorner g \urcorner).$$

\square

As a notational convenience we will often write elements $\vec{\alpha} \in T(G)$ in terms of their boundaries. I.e., we write $\vec{\alpha} = (\alpha_0^0, \alpha_1^0; \dots; \alpha_0^{n-1}, \alpha_1^{n-1}; \alpha)$ instead of $(\alpha_0, \alpha_1, \dots, \alpha_{2n})$.

Proposition 3.6. *$T : \mathbf{rGSet} \rightarrow \mathbf{rGSet}$ is the functor part of a monad.*

Proof. Given a globular set G , the unit $\eta_G : G \rightarrow T(G)$ is the “insertion of generators” defined by setting

$$\eta_G(g) := (\ulcorner g_0^{0\urcorner}, \ulcorner g_1^{0\urcorner}, \dots, \ulcorner g^\urcorner)$$

for $g \in G_n$. This is a globular map which is natural in G by definition.

For the multiplication $\mu_G : T^2G \rightarrow TG$ we begin by defining $\tau_G : \text{Exp}_{TG} \rightarrow \text{Exp}_G$ to be the canonical extension, which exists by Lemma 3.3, of the assignment $\pi : TG \rightarrow \text{Exp}_G$ given by

$$(\alpha_0^0, \alpha_1^0; \dots; \alpha_0^{n-1}, \alpha_1^{n-1}; \alpha) \mapsto \alpha,$$

where $\vec{\alpha}$ is in $(TG)_n$. That is, $\tau_G = \hat{\pi}$ is the canonical extension such that

$$\begin{array}{ccc} \text{Exp}_{TG} & \xrightarrow{\tau_G} & \text{Exp}_G \\ & \swarrow i_{TG} & \nearrow \pi \\ & TG & \end{array}$$

commutes. As such, given $\vec{\alpha}$ in $(TG)_n$ as above,

$$\tau_G(\ulcorner \vec{\alpha}^\urcorner) = \alpha,$$

where this definition makes sense because α is itself a term of $\mathbb{T}[G]$. We would like to show that τ_G preserves derivable judgements. As in the proof of Lemma 3.4, the non-trivial step is to verify that the axioms added in the formation of $\mathbb{T}[TG]$ are preserved. That is, where $\vec{\alpha}$ is as above, we need to show that

$$\vdash_{TG} \ulcorner \vec{\alpha}^\urcorner : \ulcorner TG^\urcorner^n (\ulcorner \vec{\alpha} \urcorner_0^{0\urcorner}, \dots, \ulcorner \vec{\alpha} \urcorner_1^{n-1\urcorner})$$

implies the corresponding judgement in $\mathbb{T}[G]$. But, we have that

$$\tau_G(\ulcorner \vec{\alpha} \urcorner_j^i) = \alpha_j^i. \quad (6)$$

Thus, we must show that

$$\vdash_G \alpha : \ulcorner G^\urcorner^n (\alpha_0^0, \dots, \alpha_1^{n-1}).$$

However, this is a trivial consequence of the fact that $\vec{\alpha}$ is an element of $(TG)_n$. Therefore τ_G preserves derivable judgements and we may define

$$\mu_G(\vec{\beta}_0^0, \dots, \vec{\beta}_1^{n-1}, \beta) := (\tau_G(\beta_0^0), \dots, \tau_G(\beta_1^{n-1}), \tau_G(\beta)),$$

for $\vec{\beta}$ in $(T^2G)_n$. Since τ_G preserves valid judgements this gives a globular map which is natural in G .

To see that the first unit law for monads is satisfied, let $\vec{\alpha}$ in $(TG)_n$ be given as above. Then

$$\begin{aligned} \mu_G \circ \eta_{TG}(\vec{\alpha}) &= \mu_G(\ulcorner \vec{\alpha} \urcorner_0^{0\urcorner}, \dots, \ulcorner \vec{\alpha} \urcorner_1^{n-1\urcorner}, \ulcorner \vec{\alpha} \urcorner) \\ &= \left(\tau_G(\ulcorner \vec{\alpha} \urcorner_0^{0\urcorner}), \dots, \tau_G(\ulcorner \vec{\alpha} \urcorner_1^{n-1\urcorner}), \tau_G(\ulcorner \vec{\alpha} \urcorner) \right) \\ &= \vec{\alpha}, \end{aligned}$$

where the final equation is by (6). For the other unit law, observe that the following diagram commutes:

$$\begin{array}{ccccc}
 \mathbf{Exp}_G & \xrightarrow{(\eta_G)_*} & \mathbf{Exp}_{TG} & \xrightarrow{\tau_G} & \mathbf{Exp}_G \\
 & \swarrow i_G & \nwarrow i_{TG} & & \nearrow \pi \\
 & & G & \xrightarrow{\eta_G} & TG
 \end{array}$$

Thus, $\tau_G \circ (\eta_G)_* \circ i_G = i_G$ and, by Lemma 3.3, $\mu_G \circ T(\eta_G) = 1_{TG}$.

Next, to see that the multiplication law is satisfied it suffices to prove that

$$\tau_G \circ \tau_{TG} = \tau_G \circ (\mu_G)_*. \quad (7)$$

Given $\vec{\beta} = (\beta_0^0, \beta_1^0; \dots; \beta_0^{n-1}, \beta_1^{n-1}; \beta)$ in $(T^2G)_n$ we have

$$\begin{aligned}
 \tau_G \circ \tau_{TG}(\ulcorner \vec{\beta} \urcorner) &= \tau_G(\beta) \\
 &= \tau_G\left(\ulcorner (\tau_G(\beta_0^0), \tau_G(\beta_1^0), \dots, \tau_G(\beta)) \urcorner\right) \\
 &= \tau_G(\ulcorner \mu_G(\vec{\beta}) \urcorner) \\
 &= \tau_G \circ (\mu_G)_*(\ulcorner \vec{\beta} \urcorner).
 \end{aligned}$$

Thus, by Lemma 3.3, (7) holds. \square

Example 3.7. Suppose g is a vertex of G , then $\ulcorner (\ulcorner g \urcorner) \urcorner$ is likewise a vertex of T^2G . The multiplication μ_G acts on such a vertex by removing the outermost $\ulcorner - \urcorner$. I.e.,

$$\mu_G(\ulcorner (\ulcorner g \urcorner) \urcorner) = \ulcorner g \urcorner.$$

Similarly, if f is in G_n , then

$$\mu_G(\ulcorner (\ulcorner f_0^0 \urcorner, \ulcorner f_1^0 \urcorner, \dots, \ulcorner f_1^{n-1} \urcorner, \ulcorner f \urcorner) \urcorner) = (\ulcorner f_0^0 \urcorner, \dots, \ulcorner f \urcorner).$$

The action of μ_G on composite terms (constructed out of the basic terms of $\mathbb{T}[TG]$ using the rules of \mathbb{T}) is then to go through the term recursively removing occurrences of $\ulcorner - \urcorner$. Thus, the unit acts by adding $\ulcorner - \urcorner$ and the multiplication acts by removing it.

3.5. Martin-Löf complexes and other categories of algebras. It is possible to extend Proposition 3.6 by allowing the extension \mathbb{T} of \mathbb{T}_ω employed in the construction to vary. We denote by $\mathbf{Ext}(\mathbb{T}_\omega)$ the category of all extensions of \mathbb{T}_ω . I.e., the objects of $\mathbf{Ext}(\mathbb{T}_\omega)$ are dependent type theories extending \mathbb{T}_ω (where we only allow those extensions obtained by the addition of set-many new symbols and rules). A morphism $\mathbb{T} \rightarrow \mathbb{T}'$ in $\mathbf{Ext}(\mathbb{T}_\omega)$ is an inclusion of theories (i.e., such a morphism exists whenever \mathbb{T}' extends \mathbb{T}). We also denote by $\mathbf{Mon}(\mathbf{rGSet})$ the category of monads on \mathbf{rGSet} (regarded as monoids in $[\mathbf{rGSet}, \mathbf{rGSet}]$).

Lemma 3.8. *The construction of a monad on \mathbf{rGSet} from an extension of \mathbb{T}_ω from Section 3.4 gives the action on objects of a functor*

$$\mathcal{T} : \mathbf{Ext}(\mathbb{T}_\omega) \longrightarrow \mathbf{Mon}(\mathbf{rGSet}).$$

Proof. Assume given theories \mathbb{T} and \mathbb{T}' in $\mathbf{Ext}(\mathbb{T}_\omega)$ such that \mathbb{T}' is an extension of \mathbb{T} . We will now describe the induced natural transformation $\xi : T \rightarrow T'$, where we write T and T' as abbreviations for $\mathcal{T}(\mathbb{T})$ and $\mathcal{T}(\mathbb{T}')$, respectively. Given a

reflexive globular set G and an element $\vec{\alpha} = (\alpha_0^0, \dots, \alpha_1^{n-1}, \alpha)$ of $T(G)_n$, we note that since \mathbb{T}' extends \mathbb{T} it follows that each component of the list $\vec{\alpha}$ is also a term of $\mathbb{T}'[G]$. Moreover, all of these terms necessarily possess the appropriate boundaries so that $\vec{\alpha}$ is also an element of $T'(G)_n$. As such, we may simply define $(\xi_G)_n : T(G)_n \rightarrow T'(G)_n$ to be the map which sends any $\vec{\alpha}$ as above to itself (now regarded as a list of terms from $\mathbb{T}'[G]$). This clearly describes a map of reflexive globular sets which is clearly ξ is natural and that it commutes with the multiplication and unit maps for T and T' . Finally, it is trivial to see that, with this definition \mathcal{T} is functorial. \square

The specific extensions of \mathbb{T}_ω to which we would like to apply Lemma 3.8 are obtained by augmenting \mathbb{T}_ω by axioms that force the identity types to be trivial once they have been iterated sufficiently many times. To begin with, recall that the **reflection rule** for identity types is the principle which states that all identity types are trivial in the sense that

$$\frac{\vdash a, b : A \quad \vdash p : \underline{A}(a, b)}{\vdash a = b : A} \text{Reflection}$$

Higher-dimensional generalizations of this rule are then given by “truncating” the identity types only after they have been iterated a certain number of times. Explicitly, the n -**truncation rule** is stated as follows:

$$\frac{\vdash a_{n+1}, b_{n+1} : \underline{A}^n(a_1, b_1; \dots; a_n, b_n) \quad \vdash p : \underline{A}^{n+1}(a_1, b_1; \dots; a_{n+1}, b_{n+1})}{\vdash a_{n+1} = b_{n+1} : \underline{A}^n(a_1, b_1; \dots; a_n, b_n)} \text{TR}_n$$

With these rules at our disposal we are able to describe the type theories extending \mathbb{T}_ω with which we will be concerned. Explicitly, for $n \geq 0$, the theory \mathbb{T}_n is defined to be the result of adding to \mathbb{T}_ω the (instances of the) principle TR_n . These theories then arrange themselves according to the following hierarchy of theories:

$$\mathbb{T}_\omega \subseteq \dots \subseteq \mathbb{T}_{n+1} \subseteq \mathbb{T}_n \subseteq \dots \subseteq \mathbb{T}_1 \subseteq \mathbb{T}_0, \quad (8)$$

since TR_m clearly implies TR_n , when $m < n$. The theory \mathbb{T}_0 is also known as *extensional type theory* as contrasted with the *intensional type theory* \mathbb{T}_ω .

Definition 3.9. Denote by T_ω the monad $\mathcal{T}(\mathbb{T}_\omega)$. A reflexive globular set G is a **Martin-Löf complex** (or **ML-complex**) if it is an algebra for T_ω . We write **MLCx** for the Eilenberg-Moore category consisting of T_ω -algebras and homomorphisms thereof. Similarly, we denote by **MLCx_n** the category of T_n -algebras for $n = 0, 1, 2, \dots$, where T_n denotes the monad $\mathcal{T}(\mathbb{T}_n)$.

Corresponding to the hierarchy of theories (8) we obtain, by Lemma 3.8, the following sequence of inclusions of categories:

$$\text{MLCx}_0 \longrightarrow \dots \longrightarrow \text{MLCx}_n \longrightarrow \text{MLCx}_{n+1} \longrightarrow \dots \longrightarrow \text{MLCx}_\omega$$

and it is our goal to understand how these categories relate to the hierarchy of categories of homotopy types discussed in Section 1.

3.6. Connection between truncation and other rules. The truncation principles TR_n are related to several other type theoretic principles which we employ occasionally in the sequel. For example consider the following n -dimensional generalization of the principle of **(definitional) uniqueness of identity proofs**:

$$\frac{\vdash a_{n+1}, b_{n+1} : \underline{A}^n(a_1, b_1; \dots; a_n, b_n)}{\vdash a_{n+1} = b_{n+1} : \underline{A}^n(a_1, b_1; \dots; a_n, b_n)} \text{UIP}_n$$

The question whether UIP_1 — or the variant where the definitional equality occurring in the conclusion is replaced by a propositional equality — is derivable in \mathbb{T}_ω was one of the motivations for the original groupoid model due to Hofmann and Streicher [8]. In particular, the groupoid model shows that neither UIP_1 nor the propositional version are derivable in \mathbb{T}_ω .

Another related principle is the n -dimensional **ordinary unit principle**

$$\frac{\vdash a_{n+1} : \underline{A}^n(a_1, b_1; \dots; a_n, b_n) \quad \vdash p : \underline{A}^{n+1}(a_1, b_1; \dots; a_{n+1}, a_{n+1})}{\vdash p = \mathbf{r}(a_{n+1}) : \underline{A}^{n+1}(a_1, b_1; \dots; a_{n+1}, a_{n+1})} \text{OUP}_n$$

Whereas the uniqueness of identity proofs principles can be thought of as requiring that the identity types are preorders above a given dimension, the ordinary unit rules indicate that all loops (above certain dimensions) are necessarily identities.

The truncation and ordinary unit principles have been considered previously by Garner in [6] and by Warren in [20]. The relation between the truncation, uniqueness of identity proofs and ordinary unit principles are clarified in the following lemma (the idea for the proof of which comes essentially from results, which are not “stratified” in the way considered here, from [18]).

Lemma 3.10. *Assuming the rules of \mathbb{T}_ω and the usual rules for identity types, the following implications hold:*

- (1) TR_n implies OUP_n .
- (2) TR_n implies UIP_{n+1} .
- (3) UIP_n implies TR_n .

for $n \geq 0$.

Proof. For (1), let a term a_{n+1} of type $\underline{A}^n(a_1, b_1; \dots; a_n, b_n)$ and a “loop” p of type $\underline{A}^{n+1}(a_1, b_1; \dots; a_{n+1}, a_{n+1})$ be given. Then, by TR_n it suffices to show that

$$\vdash p \simeq \mathbf{r}(a_{n+1}) : \underline{A}^{n+1}(a_1, b_1; \dots; a_{n+1}, a_{n+1}).$$

To this end, define the type

$$D(x, y) := \underline{A}^{n+2}(y, \mathbf{r}(x))$$

in the context $(x : \underline{A}^n(a_1, b_1; \dots; a_n, b_n), y : \underline{A}^{n+1}(a_1, b_1; \dots; a_n, b_n; x, x))$. Clearly,

$$x : \underline{A}^n(a_1, b_1; \dots; a_n, b_n) \vdash \mathbf{r}(\mathbf{r}(x)) : D(x, \mathbf{r}(x)),$$

and therefore the elimination rule yields the required term of type

$$D(a_{n+1}, p) = \underline{A}^{n+2}(p, \mathbf{r}(a_{n+1})).$$

Note that the particular form of the eliminaton rule used here, which is essentially Streicher’s K elimination rule, applies in this case because we are assuming TR_n and whenever we are given $z : \underline{A}^{n+1}(a_1, b_1; \dots; x, y)$ it therefore follows that x and y are definitionally equal (for more on the K rules we refer the reader to [18]).

Suppose, for the proof of (2), that we are given terms a_{n+2} and b_{n+2} of type $\underline{A}^{n+1}(a_1, b_1; \dots; a_{n+1}, b_{n+1})$. Then, by TR_n , $a_{n+1} = b_{n+1}$. By (1) it follows that OUP_n holds and therefore we obtain

$$a_{n+2} = \mathbf{r}(a_{n+1}) = b_{n+2},$$

as required.

Finally, (3) holds trivially. \square

4. DOPPELGÄNGERS AND T_0 -ALGEBRAS

Our purpose in this section is to characterize the category $\mathbf{MLC}\mathbf{x}_0$ of algebras for the monad T_0 by proving that it is equivalent to the category of sets. This result will in fact be a consequence of the work done in the next section, but the 0-dimensional case is already instructive and provides us with an opportunity to introduce some ideas and concepts which will be put to work in a more complicated setting in the 1-dimensional case.

We begin by discussing the reason why the results are nontrivial by explaining the various ways in which the type theory $\mathbb{T}_0[G]$ proves the existence of infinitely many duplicates of all of the vertices, edges, and higher edges of the globular set G . These duplicates (here called doppelgängers) must all be shown to be propositionally equal to an element in the original globular set G .

Next, we establish the characterization of the T_0 -algebras in a number of steps, making use of the set-theoretic interpretation of extensional type theories and a form of the technique of logical predicates due originally to Tait [19]. We will concentrate on stating the main concepts and theorems and omit some of the detailed proofs, allowing the reader to follow the line of argument.

4.1. Doppelgängers. Fix a globular set G and consider the type theory $\mathbb{T}_\omega[G]$ (or any extension of it). It is clear from the definition of the theory $\mathbb{T}_\omega[G]$ that every vertex $a \in G_0$ is represented as a term in $\mathbb{T}_\omega[G]$, namely $a : G$. (We shall, as before, not distinguish between an actual element in G and its “name” in the type theory.) Similarly, every 1-dimensional edge $f \in G_1$ is represented by $f : \underline{G}(a, b)$, where $s(f) = a, t(f) = b$, and so on in higher dimensions. One might, at first sight, conjecture that these are the only judgements of this form, i.e. that whenever $\mathbb{T}_\omega[G]$ derives $\tau : G$ for a closed term τ , then τ must be an element of G already, and whenever $\mathbb{T}_\omega[G]$ derives $\sigma : \underline{G}(a, b)$ then $\sigma \in G_1$ already. However, things are more complicated than that, due to the elimination rule for identity types.

Suppose, for example, that we have $a, b, c \in G_0$ and a non-reflexivity term $f : a \rightarrow b$ in G_1 . Now we can consider the following derivation:

$$\begin{array}{c} x : G, y : G, z : \underline{G}(x, y) \vdash G : \mathbf{type} \\ x : G \vdash c : G \\ \vdash f : \underline{G}(a, b) \\ \hline \vdash \mathbf{J}([x : G]c, a, b, f) : G \end{array} \text{ Id elimination}$$

This creates a new term of type G which we denote by $c\langle f \rangle$; we call it the **doppelgänger** of c (at f). This term is not definitionally equal to any of a, b, c . However, it is propositionally equal to c : this we can see from the derivation

$$\begin{array}{c} x : G, y : G, z : \underline{G}(x, y) \vdash \underline{G}(c, \mathbf{J}([v : G]c, x, y, z)) : \mathbf{type} \\ x : G \vdash \mathbf{r}(c) : \underline{G}(c, \mathbf{J}([v : G]c, x, x, \mathbf{r}(x))) \\ \vdash f : \underline{G}(a, b) \\ \hline \vdash \mathbf{J}([x : G]c, a, b, f) : \underline{G}(c, \mathbf{J}([v : G]c, a, b, f)) \end{array} \text{ Id elimination}$$

showing that there is a term witnessing $c \simeq c\langle f \rangle$ (note that by the conversion rule the second premise reduces to $x : G \vdash \mathbf{r}(c) : \underline{G}(c, c)$ so that the trivial term is well-defined).

Of course, this idea works in general: given any term $\tau : T$ and any (non-reflexivity) identity proof $f : \underline{A}(a, b)$ we may form

$$\tau\langle f \rangle := J([x : A]\tau, a, b, f) : T$$

and then show that $\tau \simeq \tau\langle f \rangle$.

There are other ways to create doppelgänger: consider again $f : a \rightarrow b \in G_1$ and form

$$f^\sharp := J_{[x,y:G,z:\underline{G}(x,y)]G}([x : G]x, a, b, f) : G.$$

This term is a new vertex which is homotopic to both a and b (again this is proved by defining a suitable witness using the J-rule).

Yet another possibility is to construct

$$f^\flat := J_{[x,y:G,z:\underline{G}(x,y)]\underline{G}(x,y)}([x : G]\mathbf{r}(x), a, b, f) : \underline{G}(a, b),$$

which turns out to be homotopic to f .

While in the above examples of doppelgänger it is easy to show that each of the newly created terms is, up to homotopy, equal to a basic term coming from the original globular set, it is not clear why this would always be the case, i.e. why for *every* term derivable in $\mathbb{T}_\omega[G]$ there is a suitable homotopy. Moreover, it will be seen in the next section that the elimination rule for identity types does in certain instances give genuinely new terms which are not homotopic to any basic term (namely, the formal composites which are used to give the Martin-Löf complexes their categorical structure).

4.2. T_0 -Algebras. We now study the category of algebras $\mathbf{MLC}\mathbf{x}_0$ for the monad T_0 . We fix a reflexive globular set G , and consider $T_0(G)$, the free algebra on G .

Lemma 4.1. *The reflexive globular set $T_0(G)$ is constant.*

Proof. Since the theory $\mathbb{T}_0[G]$ satisfies the reflection rule, it follows that any term $\tau : \underline{G}^n(a, b)$ is definitionally equal to a reflexivity term (see Subsection 3.6 above). Hence for $n > 0$, the elements of $T_0(G)_n$ are all degenerate, and the globular set $T_0(G)$ is completely determined by its vertices. \square

Thus in order to characterize the globular set $T_0(G)$, it suffices to understand the set $T_0(G)_0$ of its vertices. Recall from the construction of the monad T_0 that the elements of $T_0(G)_0$ are equivalence classes of closed terms $\tau : G$, where two of these are identified if the theory proves that they are definitionally equal. We begin by noting that there is a canonical map from $\pi_0(G)$ to $T_0(G)_0$, induced by the coequalizer

$$\begin{array}{ccc} G_1 & \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} & G_0 & \xrightarrow{e} & \pi_0(G) \\ & & \searrow \eta_0 & & \downarrow p \\ & & & & T_0(G)_0 \end{array} \quad (9)$$

Here, the map η_0 is the component of the unit $\eta : G \rightarrow T_0(G)$ at dimension 0. For every $f \in G_1$ with $s(f) = a, t(f) = b$ there is an axiom $f : \underline{G}(a, b)$ in $\mathbb{T}_0[G]$; by truncation this forces $a = b$ in the theory, and hence a and b are identified as well in $T_0(G)$; hence $\eta_0 s = \eta_0 t$.

We would like to show that p is a bijection; this would prove that T_0 is isomorphic to the (idempotent) monad $\Delta\pi_0$ on \mathbf{rGSet} , and in particular it would follow that the category of T_0 -algebras is just the category of sets.

The first step in proving this is to exploit the fact that extensional ML type theories may be modelled in locally cartesian closed categories (see the original work of Seely [16], or the expository texts [10, 7]). In particular, these theories may be soundly interpreted in the category of sets. More concretely, the theory $\mathbb{T}_0[G]$ has the following set-theoretic model: interpret the basic type G by the set $\pi_0(G)$, and interpret the basic terms $a : G$ by the element $[a]$, the connected component of a in G .

Lemma 4.2. *The above interpretation extends to a model of $\mathbb{T}_0[G]$ in the category of sets.*

Proof. We need only verify the new axioms and the new conversion rule of the theory; if these are valid under the interpretation then the result follows by soundness. By construction, the judgements $a : G$ for $a \in G$ are valid. The identity types $\underline{G}(a, b)$ will be interpreted in a degenerate way, namely as the emptyset when $[a] \neq [b]$ and as the one element set when $[a] = [b]$. Thus if we have an element $f \in G_1$ with $s(f) = a$ and $t(f) = b$, then the interpretation of f may be taken to be $[a] = [b]$, since the reflection rule allows us to derive $a = b$ from the axiom $f : \underline{G}(a, b)$. Similar reasoning works to show that the term judgements associated to higher cells of G are soundly interpreted. Finally, the new conversion rule $i(a) = \mathbf{r}(a)$ holds under the interpretation since both sides of the equation will be interpreted as $[a]$. \square

The soundness of this interpretation guarantees that the map p is injective: indeed, given two connected components $[a]$ and $[b]$ of G , suppose that $p[a] = p[b]$. Then $T_0[G]$ proves that $a = b$. But then this equation should hold in the model $\pi_0(G)$, i.e. $[a] = [b]$ as elements of $\pi_0(G)$.

4.3. Relevant types and terms. More involved is the proof that the comparison map $p : \pi_0(G) \rightarrow T_0(G)_0$ is also surjective. For this, we need to show that for any judgement $\vdash \tau : G$ in $\mathbb{T}_0[G]$ there is a vertex $a \in G_0$ for which $\tau \simeq a$. Therefore we need to understand which closed terms of type G can be derived in the theory. As discussed in the previous section, we have to worry about doppelgänger, and in particular we have to prove that all doppelgänger are homotopic to a basic vertex.

In order to establish these facts, we use an appropriate form of the logical predicates argument to prove a more general statement about a much wider class of terms.

Definition 4.3. The collection of **relevant types** of the theory $\mathbb{T}_0[G]$ is inductively defined as follows:

- the basic type G is relevant;
- $\prod_{x:A} T(x)$ is relevant if $x : A \vdash T(x) : \mathbf{type}$ is;
- $\sum_{x:A} T(x)$ is relevant if at least one of A and $T(x)$ is relevant.

If a type is not relevant then it will be called **irrelevant**.

Next, we define a class of terms called **relevant terms**.

Definition 4.4. First, let τ be a term of type T in the empty context. Then τ is **quasi-relevant** whenever one of the following holds:

- T is irrelevant;
- $T = G$ and τ is propositionally equal to a basic term;
- $T = \prod_{x:A} S(x)$ is relevant, and τ is propositionally equal to a term τ' for which $\mathbf{app}(\tau', \sigma)$ is quasi-relevant for each quasi-relevant closed term $\sigma : A$;
- $T = \sum_{x:A} S(x)$ is relevant and τ is propositionally equal to a term σ for which both terms $\pi_0(\sigma)$ and $\pi_1(\sigma)$ are quasi-relevant.

Quasi-relevant terms of relevant type will be called **relevant**.

Next, a (possibly open) term $\Gamma \vdash \tau : T$ in context Γ , where T is relevant, is said to be **relevant** whenever the result of substituting closed quasi-relevant terms for all variables of the context yields a quasi-relevant closed term.

A few features of this definition are worth pointing out. To begin with, it follows from the definition that the class of relevant terms is closed under propositional equality. Next, the definition follows the pattern of Tait's computability predicates, but with two differences: the closure under propositional equality is added at each type, and for open terms we test on all relevant substitution instances instead on just a canonical one. Indeed, types need not be inhabited, and therefore no canonical term may exist.

The key result is now:

Proposition 4.5. *Every term of relevant type is relevant.*

We will not prove this result here since it will follow from a more general statement in the next section. Essentially the result can be obtained from an induction on derivations, but some of the steps are not straightforward and require a finer analysis of the interplay between the rules for identity types and the other types. See the next section for details. By definition of closed relevant terms, the above proposition implies the following corollary:

Corollary 4.6. *If $\tau : G$ is a closed term derivable in $\mathbb{T}_0[G]$ then $\tau \simeq a$ for some basic term a .*

The statement in the corollary immediately gives that the comparison map $p : \pi_0(G) \rightarrow T_0(G)_0$ is surjective.

We summarize the situation in the following theorem:

Theorem 4.7. *The monad T_0 on the category of reflexive globular sets is naturally isomorphic to the monad $\Delta\pi_0$ (the composite of the connected components functor and the constant objects functor). Hence, the category $\mathbf{MLC}\mathbf{x}_0$ of algebras for the monad is equivalent to the category of sets.*

Proof. Only the issue of naturality has not yet been addressed explicitly. It is easily seen that the comparison map $p : \pi_0(G) \rightarrow T_0(G)_0$ is natural in G . Now extend this in the obvious way to higher dimensions to obtain a map $\Delta\pi_0(G) \rightarrow T_0(G)$ of globular sets. Clearly this map is again an isomorphism since all higher dimensions are degenerate. For the same reason, this map is also natural in G . \square

5. T_1 -ALGEBRAS

The aim of this section is to generalize the results described in Section 4 to the case of T_1 -algebras and the category $\mathbf{MLC}\mathbf{x}_1$. Contrary to what one might expect this category is not equivalent to the category of groupoids. However, there is an adjunction

$$\mathbf{MLC}\mathbf{x}_1 \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} \mathbf{Gpd} \quad (10)$$

analogous to the adjunction between topological spaces or, better yet, homotopy 1-types, and groupoids.

Most of this section is devoted to proving that the free T_1 -algebra on a globular set G is a groupoid which is equivalent (but not isomorphic) to the free groupoid on G .

The outline of the proof is roughly analogous to that sketched in Section 4. However, doppelgänger terms become more of a significant problem in this case and a substantial part of the proof is devoted to showing, again using the logical predicates technique, that such terms are ultimately harmless. We emphasize here that all results of Sections 5.3, 5.4, 5.5, 5.6, and 5.7 are valid for arbitrary extensions of \mathbb{T}_ω .

5.1. All T_1 -algebras are groupoids. We begin with the straightforward proof that all T_1 -algebras are in fact groupoids (regarding groupoids themselves as reflexive globular sets in which all n -cells are degenerate for $n \geq 2$). This result follows immediately from the construction of composition and inverse operations — as well as the corresponding propositional equalities witnessing the associativity, unit and inverse laws — by Hofmann and Streicher [8]. However, we will later require some of the details of the proof and we therefore describe the construction explicitly. First, recall that, given any type A together with terms $a, b : A$ and $f : \underline{A}(a, b)$, the **inverse** $f^{-1} : \underline{A}(b, a)$ of f is defined to be the following elimination term:

$$f^{-1} := \mathbf{J}_{[x,y:A,z:\underline{A}(x,y)]\underline{A}(y,x)}([x : A]\mathbf{r}(x), a, b, f).$$

Moreover, when there exists a further propositional equality $g : \underline{A}(b, c)$, the **composite** $(g \cdot f)$ of g with f is defined to be the term $\mathbf{app}(\mathbf{J}(\lambda v.v, b, c, g), f)$, where the J -term here is written in full as

$$\mathbf{J}_{[x,y:A,z:\underline{A}(x,y)]\underline{A}(a,y)\underline{A}(a,x)}([x : A]\lambda v:\underline{A}(a,x).v, b, c, g) : \underline{A}(a, c)^{\underline{A}(a,b)}.$$

We will use these operations on terms of identity type to define the composition and inverses for T_1 -algebras. To this end, let an object G of $\mathbf{MLC}\mathbf{x}_1$ be given with action $\gamma : T_1(G) \rightarrow G$. Of course, we will regard G as a groupoid with objects the vertices of G and arrows the edges of G . Identities are given by the edges of the form $i(a)$ for a a vertex. In order to define composition in G let a composable pair of edges f, g in G be given with

$$a \xrightarrow{f} b \xrightarrow{g} c.$$

By definition, both of these edges (and their endpoints) are represented by corresponding terms $f : \underline{G}(a, b)$ and $g : \underline{G}(b, c)$ in the theory $\mathbb{T}_1[G]$. As such, the composite $(g \cdot f) : \underline{G}(a, c)$, as defined above, exists and we define the result of composing f with g in G to be the edge obtained by applying the action of G to $(g \cdot f)$. I.e.,

$$(g \circ f) := \gamma(\ulcorner a \urcorner, \ulcorner c \urcorner; (\ulcorner g \urcorner \cdot \ulcorner f \urcorner)).$$

This edge possesses the appropriate source and target since γ is an arrow in \mathbf{rGSet} . Likewise, the inverse f^{-1} of f is defined by setting

$$f^{-1} := \gamma(\ulcorner b \urcorner, \ulcorner a \urcorner; \ulcorner f \urcorner^{-1}),$$

where f^{-1} on the right-hand side is the inverse of the term $\ulcorner f \urcorner$, as defined above.

With these definitions, the groupoid laws are a consequence of their up-to propositional equality counterparts (for which see [8]) together with the 1-truncation rule. In this way the unit law is an immediate consequence of the fact that $\mathbf{r}(\ulcorner a \urcorner) = \ulcorner i(a) \urcorner$. For the associativity law, suppose we are given f and g as above together with a further edge $h : c \rightarrow d$ in G . To prove the associative law $h \circ (g \circ f) = (h \circ g) \circ f$ holds it suffices to show that

$$\gamma(\ulcorner h \urcorner \cdot \ulcorner \gamma(\ulcorner g \urcorner \cdot \ulcorner f \urcorner) \urcorner) = \gamma(\ulcorner \gamma(\ulcorner h \urcorner \cdot \ulcorner g \urcorner) \urcorner \cdot \ulcorner f \urcorner), \quad (11)$$

where we have omitted all but the final entries of lists of terms as the missing entries are evident in this case. To see that this is indeed the case observe that the left-hand side of (11) is equal to

$$\begin{aligned} \gamma(\gamma_*(\ulcorner \ulcorner h \urcorner \urcorner) \cdot \gamma_*(\ulcorner \ulcorner g \urcorner \cdot \ulcorner f \urcorner \urcorner)) &= \gamma \circ T_1(\gamma)(\ulcorner \ulcorner h \urcorner \urcorner \cdot \ulcorner \ulcorner g \urcorner \cdot \ulcorner f \urcorner \urcorner) \\ &= \gamma \circ \mu_G(\ulcorner \ulcorner h \urcorner \urcorner \cdot \ulcorner \ulcorner g \urcorner \cdot \ulcorner f \urcorner \urcorner) \\ &= \gamma(\ulcorner h \urcorner \cdot \ulcorner g \urcorner \cdot \ulcorner f \urcorner) \end{aligned}$$

where the penultimate equality is by the multiplication law for actions. By the remarks above, $\ulcorner h \urcorner \cdot \ulcorner g \urcorner \cdot \ulcorner f \urcorner$ is definitionally equal to $(\ulcorner h \urcorner \cdot \ulcorner g \urcorner) \cdot \ulcorner f \urcorner$. A dual calculation reveals that the right-hand side of (11) is equal to $\gamma((\ulcorner h \urcorner \cdot \ulcorner g \urcorner) \cdot \ulcorner f \urcorner)$. Lastly, that f^{-1} is the inverse of f is straightforward using similar reasoning.

It then follows, in particular, that the free T_1 -algebra $T_1(G)$ on G is a groupoid and therefore the unit $\eta_G : G \rightarrow T_1(G)$ extends canonically along the unit η'_G for \mathcal{F} to a morphism of groupoids

$$\begin{array}{ccc} \mathcal{F}(G) & \xrightarrow{\Phi_G} & T_1(G) \\ & \nwarrow \eta'_G \quad \nearrow \eta_G & \\ & G & \end{array}$$

Recall that $\mathcal{F}(G)$ has the same vertices as G , and arrows $a \rightarrow b$ in $\mathcal{F}(G)$ are a zig-zag paths

$$\begin{array}{ccccccc} & & a_1 & & \cdots & & a_n \\ & \swarrow & & \searrow & & \swarrow & \searrow \\ a & & & & a_2 & & \cdots & & a_{n-1} & & b \end{array}$$

of edges in G modulo the evident relations forcing the groupoid laws to hold. The action of the induced functor Φ_G is then to send an equivalence class of such ‘‘formal composites’’ from $\mathcal{F}(G)$ to the term representing the result of taking inverses and composites of its edges. Corresponding to this notion of formal composite or arrow in the free groupoid we have type theoretic formal composites as well.

Definition 5.1. A term τ of type $\underline{G}(a, b)$ in the theory $\mathbb{T}_\omega[G]$ (or any of its extensions) is a **formal composite** if

- τ is a basic term (including reflexivity terms);

- τ is of the form γ^{-1} , for γ a formal composite; or
- τ is of the form $(\delta \cdot \gamma)$, for formal composites γ and δ .

Thus, the action of Φ_G on arrows is always a formal composite.

We would like to show that Φ_G is in fact an equivalence of categories and the remainder of Section 5 is devoted to the proof of the following theorem:

Theorem 5.2. *Given a reflexive globular set G , $\Phi_G : \mathcal{F}(G) \rightarrow T_1(G)$ is an equivalence of categories.*

The first step to proving this theorem is to show that there exists a retraction $\Psi_G : T_1(G) \rightarrow \mathcal{F}(G)$ and this is what we will now prove in Section 5.2.

5.2. Interpretation of $\mathbb{T}_1[G]$ using the free groupoid on G . Let $\mathcal{F}(G)$ denote the free groupoid (regarding the free groupoid monad as a monad on reflexive globular sets) on G . Then $\mathbb{T}_1[G]$ is soundly modelled using groupoids by extending the interpretation from [8] by the following additional clauses:

- The new type $\ulcorner G \urcorner$ is interpreted as the free groupoid on G :

$$\llbracket \ulcorner G \urcorner \rrbracket := \mathcal{F}(G).$$

- The new terms basic $\ulcorner a \urcorner$ of type $\ulcorner G \urcorner$ are interpreted by the objects of $\mathcal{F}(G)$ which they represent:

$$\llbracket \ulcorner a \urcorner \rrbracket := a.$$

- The new basic terms $\ulcorner f \urcorner$ of identity type $\ulcorner G \urcorner(\ulcorner a \urcorner, \ulcorner b \urcorner)$ are likewise interpreted as the arrows they represent

$$\llbracket \ulcorner f \urcorner \rrbracket := f.$$

- If $\ulcorner \alpha \urcorner$ is a new basic term of type $\ulcorner G \urcorner^n(\ulcorner \alpha_0^0 \urcorner, \dots, \ulcorner \alpha_1^{n-1} \urcorner)$, for $n > 1$, then

$$\llbracket \ulcorner \alpha \urcorner \rrbracket := \alpha_0^1.$$

With these definitions, the axioms of $\mathbb{T}_1[G]$ are clearly satisfied. We now remind the reader how the particular kinds of terms we are interested in are interpreted in this model. To begin with recall that the identity type $x, y : \ulcorner G \urcorner \vdash \ulcorner G \urcorner(x, y) : \mathbf{type}$ is interpreted as the functor $I_G : \mathcal{F}(G) \times \mathcal{F}(G) \rightarrow \mathbf{Gpd}$ which sends a pair of objects (a, b) of $\mathcal{F}(G)$ to the discrete groupoid $\mathcal{F}(G)(a, b)$ and which sends an arrow $(\alpha, \beta) : (a, b) \rightarrow (a', b')$ to the functor $\mathcal{F}(G)(a, b) \rightarrow \mathcal{F}(G)(a', b')$ with action $f \mapsto (\beta \circ f \circ \alpha^{-1})$. The extended context $(x, y : \ulcorner G \urcorner, z : \ulcorner G \urcorner(x, y))$ is interpreted as the result of applying the Grothendieck construction $\int I_G$ to I_G . In this instance, $\int I_G$ coincides with the arrow category $\mathcal{F}(G)^\rightarrow$. As such, the elimination data $x : \ulcorner G \urcorner \vdash \varphi(x) : B(x, x, \mathbf{r}(x))$ is interpreted by a functor $\llbracket B \rrbracket : \mathcal{F}(G)^\rightarrow \rightarrow \mathbf{Gpd}$ together with a functor $\llbracket \varphi \rrbracket : \mathcal{F}(G) \rightarrow \int \llbracket B \rrbracket$ such that

$$\begin{array}{ccc} \mathcal{F}(G) & \xrightarrow{\varphi} & \int \llbracket B \rrbracket \\ & \searrow r & \downarrow \pi \\ & & \mathcal{F}(G)^\rightarrow \end{array}$$

commutes. I.e., for an object a of $\mathcal{F}(G)$, $\varphi(a)$ is a tuple composed of $1_a : a \rightarrow a$ together with an object, which we denote by a_φ , of the groupoid $\llbracket B \rrbracket(1_a : a \rightarrow a)$.

For an arrow $\alpha : a \rightarrow a'$ of $\mathcal{F}(G)$, $\varphi(\alpha)$ is then a tuple composed of α itself together with an arrow

$$\llbracket B \rrbracket \left(\begin{array}{ccc} a & \xrightarrow{1_a} & a \\ \alpha \downarrow & & \downarrow \alpha \\ a' & \xrightarrow{1_{a'}} & a' \end{array} \right) (a_\varphi) \xrightarrow{\alpha_\varphi} a'_\varphi$$

in the groupoid $\llbracket B \rrbracket(1_{a'} : a' \rightarrow a')$.

The resulting elimination term $x, y : \ulcorner G \urcorner, z : \ulcorner G \urcorner(x, y) \vdash J(\varphi, x, y, z) : B(x, y, z)$ is interpreted as the section J of the projection $\int \llbracket B \rrbracket \rightarrow \mathcal{F}(G)^\rightarrow$ which sends an object $f : a \rightarrow b$ of $\mathcal{F}(G)^\rightarrow$ to the pair consisting of f and the object

$$\llbracket B \rrbracket \left(\begin{array}{ccc} a & \xrightarrow{1_a} & a \\ 1_a \downarrow & & \downarrow f \\ a & \xrightarrow{f} & b \end{array} \right) (a_\varphi)$$

of $\llbracket B \rrbracket(a, b, f)$. Similarly, the action of J on an arrow

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \alpha \downarrow & & \downarrow \beta \\ a' & \xrightarrow{g} & b' \end{array} \quad (12)$$

from $f : a \rightarrow b$ to $g : a' \rightarrow b'$ in $\mathcal{F}(G)^\rightarrow$ is the pair consisting of the arrow itself together with

$$\llbracket B \rrbracket \left(\begin{array}{ccc} a' & \xrightarrow{1_{a'}} & a' \\ 1_{a'} \downarrow & & \downarrow g \\ a' & \xrightarrow{g} & b' \end{array} \right) (\alpha_\varphi) : \llbracket B \rrbracket \left(\begin{array}{ccc} a & \xrightarrow{1_a} & a \\ \alpha \downarrow & & \downarrow \beta \circ f \\ a' & \xrightarrow{g} & b' \end{array} \right) a_\varphi \longrightarrow \llbracket B \rrbracket \left(\begin{array}{ccc} a' & \xrightarrow{1_{a'}} & a' \\ 1_{a'} \downarrow & & \downarrow g \\ a' & \xrightarrow{g} & b' \end{array} \right) (a'_\varphi)$$

So, for example, given a term $h : \ulcorner G \urcorner(g, g')$ in $\mathbb{T}_1[G]$, consider h^{-1} . The pattern type $B(x, y, z)$ in this instance is $\underline{G}(y, x)$ and $\llbracket B \rrbracket$ is the functor sending $f : a \rightarrow b$ to the discrete groupoid $\underline{G}(b, a)$ and which sends an arrow (12) in $\mathcal{F}(G)^\rightarrow$ to the functor $\lambda_v. \alpha \circ v \circ \beta^{-1} : \underline{G}(b, a) \rightarrow \underline{G}(b', a')$. As such, it is straightforward to verify with the description of the interpretation given above that $\llbracket h^{-1} \rrbracket$ is equal to the inverse $\llbracket h \rrbracket^{-1}$ in $\mathcal{F}(G)$. Similarly, given $f : \ulcorner G \urcorner(a, b)$ and $g : \ulcorner G \urcorner(b, c)$ in $\mathbb{T}_1[G]$, it is straightforward to verify that the interpretation commutes with composition in the sense that $\llbracket \ulcorner g \urcorner \cdot \ulcorner f \urcorner \rrbracket$ is equal to $\llbracket g \rrbracket \circ \llbracket f \rrbracket$ in $\mathcal{F}(G)$. These observations yield the following:

Lemma 5.3. *The assignment $\Psi_G : T_1(G) \rightarrow \mathcal{F}(G)$ which sends an n -cell $\vec{\alpha} = (\alpha_0^0, \dots, \alpha_1^{n-1}, \alpha)$ of $T_1(G)$ to $\llbracket \alpha \rrbracket$ is functorial.*

Proof. By the results of Section 5.1 it follows that $T_1(G)$ is a groupoid in with the result of composing 1-cells $(a, b; f)$ and $(b, c; g)$ is $(a, c; g \cdot f)$. Thus, because the interpretation function commutes with composition it follows that Ψ_G is functorial (that Ψ_G preserves identities is also straightforward). \square

5.3. Relevant types and terms. In this section we introduce the first ingredient for the proof: the relevant types and terms. The second ingredient will be the actions on contexts by endomorphisms of the two-element set introduced in Section 5.4 below. We note that in this section, as well as in Sections 5.4 and 5.7 we make no use of the hypothesis that we are working in $\mathbb{T}_1[G]$ and, in particular, the results of these sections apply to arbitrary extensions of $\mathbb{T}_\omega[G]$ (although the definition of relevant types and terms given below are useful mainly in the 0- and 1-dimensional cases, as will be evident from the definition). While in the 0-dimensional case we considered the class of terms generated (in the sense of logical predicates) by the basic terms $a : G$, we now have to add the basic terms $g : \underline{G}(a, b)$, as well as the formal composites of such basic terms. This results in:

Definition 5.4. The collection of **relevant types** of the theory $T_1[G]$ is inductively defined as follows:

- the basic type G is relevant;
- the type $\underline{G}(p, q)$ is relevant provided p and q are either variables or basic terms;
- $\prod_{x:A} T(x)$ is relevant if $x : A \vdash T(x) : \mathbf{type}$ is;
- $\sum_{x:A} T(x)$ is relevant if at least one of A and $T(x)$ is relevant.

If a type is not relevant then it will be called **irrelevant**.

Next, we define a class of terms called **relevant terms**.

Definition 5.5. First, let τ be a closed term of type T . Then τ is **quasi-relevant** whenever one of the following holds:

- T is irrelevant.
- $T = G$ and τ is propositionally equal to a basic term.
- $T = \underline{G}(a, b)$ and τ is propositionally equal to a formal composite.
- $T = \prod_{x:A} S(x)$ is relevant, and τ is propositionally equal to a term τ' which has the property that $\mathbf{app}(\tau', \sigma)$ is quasi-relevant for each quasi-relevant closed term $\sigma : A$.
- $T = \sum_{x:A} S(x)$ is relevant and τ is propositionally equal to a term σ for which both terms $\pi_0(\sigma)$ and $\pi_1(\sigma)$ are quasi-relevant.

Quasi-relevant terms of relevant type will be called **relevant**.

Next, consider a term-in-context $\Gamma \vdash \tau : T$, where T is relevant. Then $\Gamma \vdash \tau : T$ is said to be relevant (or briefly: τ is relevant if Γ is understood) whenever the result of substituting closed quasi-relevant terms for all variables mentioned in the context yields a relevant closed term.

Thus a term is quasi-relevant if either it is relevant, or it is of irrelevant type. The main interest is in the relevant terms, but the quasi-relevant terms facilitate some of the definitions and proofs because they allow us to avoid unwieldy case distinctions.

We note that the clause for terms of dependent product type is equivalent to: τ is quasi-relevant when $\mathbf{app}(\tau, \sigma)$ is quasi-relevant for each quasi-relevant σ . Indeed, when $\tau \simeq \tau'$ and τ' has this property, then for every σ we have $\mathbf{app}(\tau, \sigma) \simeq \mathbf{app}(\tau', \sigma)$, showing that τ has the property as well.

The following closure properties of relevant terms are virtually immediate from the definition:

Lemma 5.6. *If $x : A \vdash \tau(x) : T$ is relevant and $\sigma : A$ is quasi-relevant then $\tau[\sigma/x]$ is (quasi-)relevant.*

Proof. This is immediate from the fact that relevance of open terms is defined in terms of substitution. \square

Lemma 5.7. *The class of relevant terms (in context) is closed under propositional equality, i.e. if τ is relevant and $\tau \simeq \tau'$ then τ' is relevant as well.*

Proof. We first show that it suffices to prove that the class of closed relevant terms is closed under propositional equality. Suppose that $\Gamma \vdash \tau \simeq \tau' : T$, where T is relevant, and suppose that $\Gamma \vdash \tau : T$ is relevant. We must show that $\Gamma \vdash \tau' : T$ is also relevant. To that end, consider a closed substitution instance $\tau'[\vec{a}/\vec{x}]$ of τ' where all of the \vec{a} are quasi-relevant. Since $\tau[\vec{a}/\vec{x}] \simeq \tau'[\vec{a}/\vec{x}]$ we have reduced the problem to the case of closed terms.

However, this is an easy induction on the structure of relevant types and terms, as closure under propositional equality is built into the definition of closed relevant terms. \square

5.4. Action of identity types on contexts. One elementary consequence of the elimination rule for identity types is the ability to “substitute” one term τ of type $T(a)$ along a propositional identity $f : \underline{A}(a, b)$ to obtain a new term τ' of type $T(b)$. Explicitly, such a result of substituting b for a in τ along the propositional equality f is defined as in the following derivation:

$$\begin{array}{c} x, y : A, z : \underline{A}(x, y) \vdash T(y)^{T(x)} : \mathbf{type} \\ x : A \vdash \lambda_{v:T(x)}.v : T(x)^{T(x)} \\ \vdash f : \underline{A}(a, b) \\ \hline \vdash \mathbf{J}(\lambda_v.v, a, b, f) : T(b)^{T(a)} \qquad \vdash \tau : T(a) \\ \hline \vdash \mathbf{app}(\mathbf{J}(\lambda_v.v, a, b, f), \tau) : T(b) \end{array}$$

Indeed, we have already encountered such a term since the composition of $(g \cdot f)$ of propositional equalities defined in Section 5.1 is of this form. Of course, there is no reason to restrict ourselves to types of $T(x)$ which depend only on a single variable from the type A . Indeed, in order to understand the propositional equality classes of J-terms of relevant type we will require a more general version of this construction where the type T may contain variables of type $x, y : A$ and $z : \underline{A}(x, y)$, as well as further parameter variables which themselves depend on x, y and z .

Remark (Notation). In the following construction we will be dealing extensively with lists of terms and we adopt the convention of employing “vector” notation \vec{v} for such lists. We also will employ notation such as $\Theta(\vec{v})$, when Θ is some given operation on terms, for the list $\Theta(v_1), \Theta(v_2), \dots, \Theta(v_n)$, where lengths of lists are always clear from the context. We will write, given a term

$$\tau : \prod_{x_1:A_1} \prod_{x_2:A_2(x_1)} \cdots \prod_{x_n:A_n(\vec{x})} .T(\vec{x})$$

and terms $a_1 : A_1, \dots, a_n : A_n(a_1, \dots, a_{n-1})$,

$$\mathbf{app}(\tau, a_1, \dots, a_n)$$

for the result of iteratively forming application terms. I.e.,

$$\mathbf{app}(\tau, a_1, \dots, a_n) = \mathbf{app}(\mathbf{app}(\dots \mathbf{app}(\tau, a_1), a_2) \dots), a_n).$$

Assume given a context Δ of the form

$$\Delta := (x_0, x_1 : A, z : \underline{A}(x_0, x_1), v_1 : B_1(x_0, x_1, z), \dots, v_n : B_n(x_0, x_1, z, \vec{v})). \quad (13)$$

In fact, we might even have some other Γ to the left of the declaration $(x_0 : A)$ here, but we prefer to avoid having to write Γ everywhere since it will not enter into the subsequent construction. Now, given a function $\chi : \{0, 1\} \rightarrow \{0, 1\}$ there is an action of χ on the context Δ which yields a new context Δ_χ defined to be

$$(x_0, x_1 : A, z : \underline{A}(x_0, x_1), w_1 : B_1(x_{\chi 0}, x_{\chi 1}, z_\chi), \dots, w_n : B_n(x_{\chi 0}, x_{\chi 1}, z_\chi, \vec{w}))$$

where

$$z_\chi := \begin{cases} z & \text{if } \chi = \lambda_{x \in \{0,1\}}.x \\ z^{-1} & \text{if } \chi = \lambda_{x \in \{0,1\}}.|1-x| \\ \mathbf{r}(x_0) & \text{if } \chi = \lambda_{x \in \{0,1\}}.0 \text{ and} \\ \mathbf{r}(x_1) & \text{if } \chi = \lambda_{x \in \{0,1\}}.1 \end{cases}$$

From the “semantic” perspective, the context Δ_χ is the pullback

$$\begin{array}{ccc} \Delta_\chi & \xrightarrow{\quad} & \Delta \\ \downarrow & & \downarrow \pi \\ (x_0, x_1 : A, z : \underline{A}(x_0, x_1)) & \xrightarrow{\quad} & (x_0, x_1 : A, z : \underline{A}(x_0, x_1)) \end{array}$$

obtained by pulling back the projection $\pi : \Delta \rightarrow (x_0, x_1 : A, z : \underline{A}(x_0, x_1))$ along the horizontal map $(x_0, x_1 : A, z : \underline{A}(x_0, x_1)) \rightarrow (x_0, x_1 : A, z : \underline{A}(x_0, x_1))$ which is:

- the identity when χ is the identity map $\{0, 1\} \rightarrow \{0, 1\}$;
- the context morphism given by

$$\begin{aligned} (x_0, x_1 : A, z : \underline{A}(x_0, x_1)) &\vdash x_i : A \\ (x_0, x_1 : A, z : \underline{A}(x_0, x_1)) &\vdash x_i : A \\ (x_0, x_1 : A, z : \underline{A}(x_0, x_1)) &\vdash \mathbf{r}(x_i) : \underline{A}(x_i, x_i) \end{aligned}$$

- when χ is the constant i -valued function $\{0, 1\} \rightarrow \{0, 1\}$; and
- the context morphism given by

$$\begin{aligned} (x_0, x_1 : A, z : \underline{A}(x_0, x_1)) &\vdash x_1 : A \\ (x_0, x_1 : A, z : \underline{A}(x_0, x_1)) &\vdash x_0 : A \\ (x_0, x_1 : A, z : \underline{A}(x_0, x_1)) &\vdash z^{-1} : \underline{A}(x_1, x_0) \end{aligned}$$

when χ is the non-identity permutation of $\{0, 1\}$.

5.5. Context morphisms induced by the action of identity types. Where Δ and χ are as above, we now describe induced context morphisms

$$\begin{array}{ccc} & \Delta \triangleleft_\chi & \\ \Delta & \xrightarrow{\quad} & \Delta_\chi \\ & \Delta \triangleright_\chi & \end{array}$$

with the properties that on the first three components $x_0, x_1 : A$ and $z : \underline{A}(x_0, x_1)$ of these contexts they are the identity and

$$\Delta \triangleleft_\chi [x/x_0, x/x_1, \mathbf{r}(x)/z] = 1_{\Delta[x/x_0, x/x_1, \mathbf{r}(x)/z]} = \Delta \triangleright_\chi [x/x_0, x/x_1, \mathbf{r}(x)/z].$$

First, we construct $\Delta \triangleleft_\chi$ by induction on $n \geq 1$. The first three components of $\Delta \triangleleft_\chi$ are just x_0, x_1 and z . We indicate the additional components using the following notation

$$\begin{aligned} \Delta \vdash z \triangleleft_\chi v_1 &: B_1(x_{\chi 0}, x_{\chi 1}, z_\chi), \\ \Delta \vdash z \triangleleft_\chi (v_1, v_2) &: B_2(x_{\chi 0}, x_{\chi 1}, z_\chi, z \triangleleft_\chi v_1), \end{aligned}$$

et cetera. When $n = 1$, $z \triangleleft_\chi v_1$ is constructed as follows. First, we form the term

$\mathbf{J}_{[x_0, x_1 : A, z : \underline{A}(x_0, x_1)] B_1(x_{\chi 0}, x_{\chi 1}, z_\chi)^{B_1(x_0, x_1, z)}}(\lambda v.v, x_0, x_1, z) : B_1(x_{\chi 0}, x_{\chi 1}, z_\chi)^{B_1(x_0, x_1, z)}$
in context $(x_0, x_1 : A, z : \underline{A}(x_0, x_1))$. Then, we define $z \triangleleft_\chi v_1$ to be the term

$$\Delta \vdash \mathbf{app}(\mathbf{J}(\lambda v.v, x_0, x_1, z), v_1) : B_1(x_{\chi 0}, x_{\chi 1}, z_\chi).$$

Clearly $\mathbf{r}(x) \triangleleft_\chi v_1 = v_1$ in context $\Delta[x/x_0, x/x_1, \mathbf{r}(x)/z]$. For the induction step, suppose we have defined operations $z \triangleleft_\chi (v_1, \dots, v_i)$, $i = 1, \dots, n$, with the property that in each case

$$\Delta[x/x_0, x/x_1, \mathbf{r}(x)/z] \vdash \mathbf{r}(x) \triangleleft_\chi (v_1, \dots, v_i) = v_i : B_i(x, x, \mathbf{r}(x), \dots).$$

Then to define $z \triangleleft_\chi (v_1, \dots, v_{n+1})$ we first form the term

$$x_0, x_1 : A, z : \underline{A}(x_0, x_1) \vdash \mathbf{J}(\lambda v_1 \dots \lambda v_n.\lambda v.v, x_0, x_1, z) : C(x_0, x_1, z)$$

where $C(x_0, x_1, z)$ is the type

$$\prod_{v_1} \dots \prod_{v_n} B_{n+1}(x_{\chi 0}, x_{\chi 1}, z_\chi, z \triangleleft_\chi v_1, \dots, z \triangleleft_\chi (v_1, \dots, v_n))^{B_{n+1}(x_0, x_1, z, \vec{v})}$$

where $v_i : B_i(x_0, x_1, z, v_1, \dots, v_{n-1})$. Then, $z \triangleleft_\chi (v_1, \dots, v_{n+1})$ is the term

$$\mathbf{app}\left(\mathbf{J}(\lambda v_1 \dots \lambda v_n.\lambda v.v, x_0, x_1, z), v_1, \dots, v_{n+1}\right),$$

which trivially has the property that

$$\Delta[x/x_0, x/x_1, \mathbf{r}(x)/z] \vdash \mathbf{r}(x) \triangleleft_\chi (v_1, \dots, v_{n+1}) = v_{n+1}.$$

Thus, we have completed the construction of $\Delta \triangleleft_\chi$.

The construction of $\Delta \triangleright_\chi$ is essentially the same. For example, it is the identity in the first three components. In the case where $n = 1$ we define the term

$$x_0, x_1 : A, z : \underline{A}(x_0, x_1), w_1 : B_1(x_{\chi 0}, x_{\chi 1}, z_\chi) \vdash z \triangleright_\chi w_1 : B_1(x_0, x_1, z)$$

by setting

$$z \triangleright_\chi w_1 := \mathbf{app}(\mathbf{J}_{[x_0, x_1 : A, z : \underline{A}(x_0, x_1)] B_1(x_0, x_1, z)^{B_1(x_{\chi 0}, x_{\chi 1}, z_\chi)}}(\lambda w.w, x_0, x_1, z), y_1).$$

In the induction step we first form the type $J(\lambda_{w_1} \cdots \lambda_{w_n} \cdot \lambda_w \cdot w, x_0, x_1, z)$ of type $D(x_0, x_1, z)$ where this type is defined to be

$$\prod_{w_1} \cdots \prod_{w_n} B_{n+1}(x_0, x_1, z, z \triangleright_\chi w_1, \dots, z \triangleright_\chi (w_1, \dots, w_n))^{B_{n+1}(x_{\chi 0}, x_{\chi 1}, z_\chi, \vec{w})}$$

and $w_i : B_i(x_{\chi 0}, x_{\chi 1}, z_\chi, w_1, \dots, w_{i-1})$. Then

$$z \triangleright_\chi (w_1, \dots, w_{n+1}) := \mathbf{app} \left(J(\lambda_{w_1} \cdots \lambda_{w_n} \cdot \lambda_w \cdot w, x_0, x_1, z), w_1, \dots, w_{n+1} \right).$$

Thus, we have also constructed $\Delta_{\triangleright_\chi}$.

Remark (Notation). In general, when dealing with terms that are either of the form $z \triangleleft_\chi (v_1, \dots, v_n)$ or $z \triangleright_\chi (w_1, \dots, w_n)$ such that the map χ and the sequences of variables (or terms) (v_1, \dots, v_n) or (w_1, \dots, w_n) are understood, we will omit χ and all but the last element of these sequences. That is, we denote $z \triangleleft_\chi (v_1, \dots, v_n)$ and $z \triangleright_\chi (w_1, \dots, w_n)$ by $z \triangleleft v_n$ and $z \triangleright w_n$ when no confusion will result. Also, it is worth emphasizing here that the notation *should* also include the context Δ to which the entire procedure is applied, but this is not in general necessary. Nonetheless, when we do wish to emphasize the context we write $[\Delta]z \triangleleft v_n$ or $[\Delta]z \triangleright w_n$.

In Section 5.7 below we will restrict our attention exclusively to the case where χ is a constant and accordingly we write $z \triangleleft_i v_n$ or $z \triangleright_i v_n$ for $i \in \{0, 1\}$.

Remark. Although it is convenient to construct the maps $\Delta_{\triangleleft_\chi}$ and $\Delta_{\triangleright_\chi}$ at the level of contexts, we will almost exclusively be concerned with substitution instances of these context morphisms. Accordingly, we employ the same notation for these substitution instances. E.g., given $a, b : A$, $f : \underline{A}(a, b)$ and $c : B_1(a, b, f)$, we will write $f \triangleleft_0 c : B_1(a, a, \mathbf{r}(a))$ and $f \triangleleft_1 c : B_1(b, b, \mathbf{r}(b))$ for the results of substituting into the corresponding context morphisms, for a context Δ as above.

5.6. Associated transition terms. Given Δ and χ as above, we would now like to show that there exist, in addition to the context morphisms $\Delta_{\triangleleft_\chi}$ and $\Delta_{\triangleright_\chi}$ described above, associated “2-cells”

$$\begin{array}{ccc} \Delta & \xrightarrow{\Delta_{\triangleleft_\chi}} & \Delta_\chi \\ & \searrow^{1_\Delta} & \downarrow \kappa \\ & & \Delta \\ & & \uparrow \nu \\ & & \Delta_\chi \\ & \swarrow_{1_{\Delta_\chi}} & \xrightarrow{\Delta_{\triangleleft_\chi}} \end{array}$$

which are, in a suitable sense, weakly invertible. Note though that this is meant principally by way of motivation for we will not here define a general notion of 2-cell or of weakly invertible map; these are the only ones with which we are here concerned. Suppose we are given terms $a_0, a_1 : A, f : \underline{A}(a_0, a_1)$ together with

$$b_1 : B_1(a_0, a_1, f), \dots, b_n : B_n(a_0, a_1, f, \vec{b}), \text{ and} \\ d_1 : B_1(a_{\chi 0}, a_{\chi 1}, f_\chi), \dots, d_n : B_n(a_{\chi 0}, a_{\chi 1}, f_\chi, \vec{d}).$$

Observe that given such terms we obtain

$$f \triangleright (f \triangleleft b_n) : B_n(a_0, a_1, f, f \triangleright (f \triangleleft \vec{b})).$$

and

$$f \triangleleft (f \triangleright d_n) : B_n(a_{\chi 0}, a_{\chi 1}, f_{\chi}, f \triangleleft (f \triangleright \vec{d})).$$

In the general case, where $n > 1$, we define terms

$$\kappa(f) := J(\lambda_{v_1} \cdots \lambda_{v_n}.v_n, a_0, a_1, f) : C(a_0, a_1, f)$$

where the type $C(x_0, x_1, z)$ is defined to be

$$\prod_{v_1} \cdots \prod_{v_{n-1}} B_n(x_0, x_1, z, \vec{v})^{B_n(x_0, x_1, z, z \triangleright (z \triangleleft \vec{v}))}$$

In the same way,

$$\nu(f) := J(\lambda_{w_1} \cdots \lambda_{w_n}.w_n, a_0, a_1, f) : C'(a_0, a_1, f)$$

where the type $C'(x_0, x_1, z)$ is defined to be

$$\prod_{w_1} \cdots \prod_{w_{n-1}} B_n(x_{\chi 0}, x_{\chi 1}, z_{\chi}, \vec{w})^{B_n(x_{\chi 0}, x_{\chi 1}, z_{\chi}, z \triangleleft (z \triangleright \vec{w}))}$$

We refer to the terms $\kappa(f)$ and $\nu(f)$ as **transition terms**. We note that, to be as precise as possible, the notation for these transition terms should also include the context Δ and the map χ . However, these will be evident from the context in each of the cases that we consider.

5.7. The action preserves relevance. The following lemma shows that, up to propositional equality, all J -terms are expressible in the form $f \triangleright_i \tau$, for $i = 0, 1$. We recall that, for $i = 0, 1$, we write $f \triangleleft_i \tau$ and $f \triangleleft_i \tau$ as abbreviations for the terms $f \triangleleft_{\chi} \tau$ and $f \triangleright_{\chi} \tau$, respectively, where χ is the constant function $\lambda_{x \in \{0,1\}}.i$.

Lemma 5.8. *Given a J -term $J(\varphi, a, b, f) : B(a, b, f)$, there exist propositional identities:*

$$f \triangleright_0 \varphi(a) \simeq J(\varphi, a, b, f) \simeq f \triangleright_1 \varphi(b).$$

Proof. The required term witnessing the propositional equality $f \triangleright_0 \varphi(a) \simeq J(\varphi, a, b, f)$ is constructed as follows:

$$\begin{array}{l} x, y : A, z : \underline{A}(x, y) \vdash \underline{B}(x, y, z)(z \triangleright_0 \varphi(x), J(\varphi, x, y, z)) : \mathbf{type} \\ x : A \vdash \mathbf{r}(\varphi(x)) : \underline{B}(x, x, \mathbf{r}(x))(\mathbf{r}(x) \triangleright_0 \varphi(x), J(\varphi, x, x, \mathbf{r}(x))) \\ \hline \vdash f : \underline{A}(a, b) \\ \hline \vdash J(\mathbf{r}(\varphi(x)), a, b, f) : \underline{B}(a, b, f)(f \triangleright_0 \varphi(a), J(\varphi, a, b, f)) \quad \text{Id elimination} \end{array}$$

where the term $\mathbf{r}(\varphi(x))$ has the correct type since both $\mathbf{r}(x) \triangleright_0 \varphi(x)$ and $J(\varphi, x, x, \mathbf{r}(x))$ are definitionally equal to $\varphi(x)$. The construction of the other propositional equality $J(\varphi, a, b, f) \simeq f \triangleright_1 \varphi(b)$ is essentially the same. \square

Intuitively, the term $f \triangleright_0 \varphi(a)$ should be thought of as a representation of the groupoid interpretation of $J(\varphi, a, b, f)$, as described in Section 5.2. As such, the fact that these terms are propositionally equal should come as no surprise. Nonetheless, we emphasize that $f \triangleright_0 \varphi(a)$ is constructed using also the dependent products in an essential way.

With this elementary (but useful) observation at our disposal we will be able to prove that all terms of relevant type are relevant in Section 5.8 below. First though

it is necessary to show that the relevant terms are closed under the formation of elimination terms for identity types.

Lemma 5.9. *Let quasi-relevant terms $a_0, a_1 : A$ and $f : \underline{A}(a_0, a_1)$ be given together with a type $x_0, x_1 : A \vdash T(x_0, x_1)$ which is both basic and relevant. If $\tau : T(a_0, a_1)$ is relevant, then so is $f \triangleleft_i \tau$, for $i = 0, 1$. Similarly, if $\tau' : T(a_i, a_i)$ is relevant, then so is $f \triangleright_i \tau'$ for $i = 0, 1$.*

Proof. First observe that when neither variable x_0 nor x_1 occurs in $T(x_0, x_1)$, we have that $f \triangleleft_i \tau \simeq \tau$. Since τ may itself depend on a_0, a_1 or f it is necessary to exercise some care. I.e., we must observe that

$$\begin{array}{c} x_0, x_1 : A, z : \underline{A}(x_0, x_1) \vdash \prod_{v:T} \underline{T}(v, z \triangleleft_i v) : \mathbf{type} \\ x : A \vdash \lambda_{v:T} \mathbf{r}(v) : \prod_{v:T} \underline{T}(v, \mathbf{r}(x) \triangleleft_i v) \\ \vdash f : \underline{A}(a_0, a_1) \\ \hline \vdash \mathbf{J}(\lambda_{v:T} \mathbf{r}(v), a_0, a_1, f) : \prod_{v:T} \underline{T}(v, f \triangleleft_i v) \quad \text{Id elimination} \quad \vdash \tau : T \\ \hline \vdash \mathbf{app}(\mathbf{J}(\lambda_{v:T} \mathbf{r}(v), a_0, a_1, f), \tau) : \underline{T}(\tau, f \triangleleft_i \tau) \end{array}$$

in order to obtain the required homotopy. A similar construction shows that in this case $\tau' \simeq f \triangleright_i \tau'$. Thus, it remains to consider only those cases, to which we now turn, where at least one of x_0 or x_1 occurs in the type.

Case $T(x_0, x_1) = \underline{G}(x_0, x_1)$: If $i = 0$, then $f \triangleleft_i \tau \simeq (f^{-1} \cdot \tau)$ and is therefore relevant. Similarly, if $i = 1$, then $f \triangleleft_i \tau \simeq (\tau \cdot f^{-1})$.

We will not go through each of these individual cases in detail as they are all essentially the same. However, in order to give some indication of the construction of these propositional identities we consider the case where $i = 1$ in full (it is slightly more complicated than $i = 0$). In this case, we first define the type

$$C(x_0, x_1, z) := \prod_{v:\underline{G}(x_0, x_1)} \underline{G}(x_1, x_1)(v \cdot z^{-1}, z \triangleleft_1 v).$$

Then, $x : G \vdash \lambda_{v:\underline{G}(x,x)} \rho_v : C(x, x, \mathbf{r}(x))$ where ρ_v is the propositional equality witnessing the fact that reflexivity terms are (weak) units on the right for composition. I.e., $\rho_\alpha : \alpha \cdot \mathbf{r}(a) \simeq \alpha$ where a is the domain of an identity proof α . Thus,

$$\mathbf{app}(\mathbf{J}([x : G] \lambda_{v:\underline{G}(x,x)} \rho_v, a_0, a_1, f), \tau) : \underline{G}(a_1, a_1)(\tau \cdot f^{-1}, f \triangleleft_1 \tau),$$

as required.

On the other hand, $f \triangleright_0 \tau' \simeq (f \cdot \tau')$ and $f \triangleright_1 \tau' \simeq (\tau' \cdot f)$.

Case $T(x_0, x_1) = \underline{G}(x_0, v)$ for v basic or a variable: In this case:

$$f \triangleleft_i \tau \simeq \begin{cases} \tau & \text{if } i = 0 \text{ and} \\ (\tau \cdot f^{-1}) & \text{if } i = 1. \end{cases}$$

In this case and each subsequent case it is easily seen that $f \triangleright_i \tau'$ is expressible up to propositional equality as the term obtained by replacing τ with τ' , f with f^{-1} , and f^{-1} with f in the expression of $f \triangleleft_i \tau$ as a relevant term for that case. As such, we omit the explicit statement of these expressions.

Case $T(x_0, x_1) = \underline{G}(v, x_0)$ for v relevant: In this case:

$$f \triangleleft_i \tau \simeq \begin{cases} \tau & \text{if } i = 0 \text{ and} \\ (f \cdot \tau) & \text{if } i = 1. \end{cases}$$

Case $T(x_0, x_1) = \underline{G}(x_1, v)$ for v relevant: In this case:

$$f \triangleleft_i \tau \simeq \begin{cases} (\tau \cdot f) & \text{if } i = 0 \text{ and} \\ \tau & \text{if } i = 1. \end{cases}$$

Case $T(x_0, x_1) = \underline{G}(v, x_1)$ for v relevant: In this case:

$$f \triangleleft_i \tau \simeq \begin{cases} (f^{-1} \cdot \tau) & \text{if } i = 0 \text{ and} \\ \tau & \text{if } i = 1. \end{cases}$$

Case $T(x_0, x_1) = \underline{G}(x_1, x_0)$: In the other case, $f \triangleleft_i \tau \simeq (f \cdot \tau)$. In this case:

$$f \triangleleft_i \tau \simeq \begin{cases} (\tau \cdot f) & \text{if } i = 0 \text{ and} \\ (f \cdot \tau) & \text{if } i = 1. \end{cases}$$

□

Lemma 5.10. *Given a context Δ as in (13) together with a fixed $i = 0, 1$. If the type $B_n(x_0, \dots, v_{n-1})$ is basic, then all of the terms*

$$\begin{aligned} \Delta \vdash z \triangleleft v_n, \\ \Delta_\chi \vdash z \triangleright w_n, \\ x_0, x_1 : A, z : \underline{A}(x_0, x_1) \vdash \kappa(z), \text{ and} \\ x_0, x_1 : A, z : \underline{A}(x_0, x_1) \vdash \nu(z) \end{aligned}$$

are quasi-relevant.

Proof. We consider the two possible cases for the type $B_n(x_0, x_1, z, \vec{v})$. In the first case, it is equal to G and, as is easily verified, we have that $z \triangleleft_i v_n \simeq v_n$. As such, $z \triangleleft_i v_n$ is relevant and similarly $\kappa(z)$ is relevant since it is propositionally equal to $\lambda_{v_1} \cdots \lambda_{v_n}.v_n$.

Next, the case where $B_n(x_0, x_1, z, \vec{v})$ is $\underline{G}(u, v)$ can be split into separate cases, as in the proof of Lemma 5.9, depending on what u and v are. First, if neither u nor v occurs in Δ , then this case is identical to the one just considered. Next, note that, since the type of $z \triangleleft_i v_n$ is assumed to be relevant, it follows that $A = G$ and if u or v occurs in Δ , then it is either x_0 or x_1 . Thus, the remaining cases correspond to those from the proof of Lemma 5.9. Indeed, $z \triangleleft_i v_n$ in these cases is seen to be propositionally equal to the result of λ -abstracting the terms constructed there. For example, when $B_n(x_0, x_1, z, \vec{v}) = \underline{G}(u, x_1)$, for u not in Δ , we have that $z \triangleleft_i v_n = \mathbf{app}(\alpha, \vec{v})$ where α is the auxiliary term

$$\mathbf{J}(\lambda_{v_1} \cdots \lambda_{v_n}.v_n, x_0, x_1, z) : \prod_{v_1} \cdots \prod_{v_{n-1}} \underline{G}(u, x_1)^{\underline{G}(u, x_1)}.$$

Now, when $i = 0$, $\alpha \simeq \lambda_{v_1} \cdots \lambda_{v_n}.(z^{-1} \cdot v_n)$ so that $z \triangleleft_0 v_n \simeq (z^{-1} \cdot v_n)$. Similarly, when $i = 1$, $\alpha \simeq \lambda_{v_1} \cdots \lambda_{v_n}.v_n$ and $z \triangleleft_1 v_n \simeq v_n$. The other cases are likewise obtained by abstraction from the terms constructed in the proof of Lemma 5.9. For the transition term $\kappa(z)$, note that here

$$\kappa(z) = \mathbf{J}_{[x_0, x_1 : G, z]C(x_0, x_1, z)}(\lambda_{v_1} \cdots \lambda_{v_n}.v_n, x_0, x_1, z) : C(x_0, x_1, z)$$

where

$$C(x_0, x_1, z) = \prod_{v_1} \cdots \prod_{v_{n-1}} \underline{G}(u, x_1)^{\underline{G}(u, x_1)}.$$

As such, $\kappa(z) \simeq \lambda_{v_1} \cdots \lambda_{v_n} \cdot v_n$. The same fact, indeed, holds in all of the remaining cases.

An essentially “dual” argument can be given for $z \triangleright w_n$ and $\nu(z)$. \square

Proposition 5.11. *Given a context Δ as in (13), all of the terms*

$$\begin{aligned} \Delta &\vdash z \triangleleft v_n, \\ \Delta_\chi &\vdash z \triangleright w_n, \\ x_0, x_1 : A, z : \underline{A}(x_0, x_1) &\vdash \kappa(z), \text{ and} \\ x_0, x_1 : A, z : \underline{A}(x_0, x_1) &\vdash \nu(z) \end{aligned}$$

are quasi-relevant, for $i = 0, 1$.

Proof. We prove simultaneously all of the claims by induction on the structure of $B_n(x_0, x_1, z, \vec{v})$ as a relevant type. When this type is basic, it is by Lemma 5.10.

Assume given a sequence of quasi-relevant terms $a_0, a_1, f, b_1, \dots, b_{n-1}, b_n$ of the appropriate types and consider the case where $B_n(x_0, x_1, z, \vec{v})$ is a dependent product type,

$$B_n(x_0, x_1, z, \vec{v}) = \prod_{y:S(\dots)} T(x_0, x_1, z, \vec{v}, y),$$

for $T(\dots)$ a relevant type. Given a quasi-relevant term s of type $S(a_i, a_i, \mathbf{r}(a_i), f \triangleleft \vec{b})$, it suffices to show that $\mathbf{app}(f \triangleleft b_n, s)$ is quasi-relevant. By induction hypothesis,

$$f \triangleright s : S(a_0, a_1, f, f \triangleright (f \triangleleft \vec{b}))$$

is quasi-relevant, where we emphasize that this term is obtained via the action $f \triangleright_i s$ for the context

$$(x_0, x_1 : A, z : \underline{A}(x_0, x_1), \dots, v_{n-1} : B_{n-1}(\dots), y : S(x_0, x_1, z, \vec{v})). \quad (14)$$

Thus, also by induction hypothesis, the term

$$\xi := \mathbf{app}(\kappa'(f), b_1, \dots, b_{n-1}, f \triangleright s) : S(a_0, a_1, f, \vec{b})$$

is quasi-relevant, where we write $\kappa'(f)$ for the transition term defined with respect to the context (14).

As such, since b_n is relevant it follows that

$$\mathbf{app}(b_n, \xi) : T(a_0, a_1, f, \vec{b}, \xi)$$

is quasi-relevant and thus, by induction hypothesis, so is

$$f \triangleleft \mathbf{app}(b_n, \xi) : T(a_i, a_i, \mathbf{r}(a_i), f \triangleleft \vec{b}, f \triangleleft \xi),$$

where this is now the action for the extended context

$$(x_0, x_1 : A, z : \underline{A}(x_0, x_1), \dots, v_{n-1} : B_{n-1}(\dots), y : S(\dots), y' : T(x_0, x_1, \dots, y)).$$

Observe that there exists a propositional equality $f \triangleleft \xi \simeq s$ defined by

$$\vartheta := \mathbf{app}(J(\lambda_{v_1} \cdots \lambda_{v_{n-1}} \cdot \lambda_{y'} \cdot \mathbf{r}(y'), a_0, a_1, f), \vec{b}, s)$$

Moreover, since $f \triangleleft \xi$ is neither a variable nor a basic term, the type of ϑ is not relevant. Thus, we may conclude that the result

$$\vartheta \triangleleft_1 \mathbf{app}(b_n, \xi) : T(a_i, a_i, \mathbf{r}(a_i), \dots, s)$$

of substituting along ϑ is quasi-relevant by the induction hypothesis (where the operation \triangleleft_1 here is given with respect to the obvious context). Finally, this term is propositionally equal to $\mathbf{app}(f \triangleleft b_n, s)$ and the latter is therefore quasi-relevant.

To see that $\kappa(f)$ is quasi-relevant it suffices to prove that

$$\mathbf{app}(\kappa(f), \vec{b}, \alpha, s') : T(a_0, a_1, f, \vec{b}, s')$$

is quasi-relevant when

$$\alpha : \prod_{y : S(a_0, a_1, f, f \triangleright (f \triangleleft \vec{b}))} T(a_0, a_1, f, f \triangleright (f \triangleleft \vec{b}), y)$$

and $s' : S(a_0, a_1, f, \vec{b})$ are. Well, using the induction hypothesis, both

$$\mathbf{app}(\alpha, f \triangleright (f \triangleleft s')) : T(a_0, a_1, f, f \triangleright (f \triangleleft \vec{b}), f \triangleright (f \triangleleft s'))$$

and

$$\mathbf{app}(\kappa(f), \vec{b}, \mathbf{app}(\alpha, f \triangleright (f \triangleleft s'))) : T(a_0, a_1, f, \vec{b}, s')$$

are quasi-relevant. Moreover, this is easily seen to be propositionally equal to $\mathbf{app}(\kappa(f), \alpha, s')$.

Next, when $B_n(\dots)$ is a dependent sum type

$$\sum_{y : S(x_0, x_1, z, \vec{v})} T(x_0, x_1, z, \vec{v}, y),$$

we have by definition that b_n is propositionally equal to a term τ with $\pi_0(\tau)$ and $\pi_1(\tau)$ quasi-relevant. Since it follows that $f \triangleleft b_n \simeq f \triangleleft \tau$ it therefore suffices to show that $\pi_0(f \triangleleft \tau)$ and $\pi_1(f \triangleleft \tau)$ are quasi-relevant. First, note that there exists a propositional equality $\pi_0(f \triangleleft \tau) \simeq f \triangleleft \pi_0(\tau)$ given by

$$\zeta := \mathbf{app}(\mathbf{J}([x : A]\lambda_{v_1} \cdots \lambda_{v_{n-1}} \cdot \lambda_u : \sum_{y : S(\dots)} T(\dots) \cdot \mathbf{r}(\pi_0(u)), a_0, a_1, f), \vec{b}, \tau).$$

By induction hypothesis, $f \triangleleft \pi_0(\tau)$ is quasi-relevant and therefore so is $\pi_0(f \triangleleft \tau)$. On the other hand, the induction hypothesis implies that

$$f \triangleleft \pi_1(\tau) : T(a_i, a_i, \mathbf{r}(a_i), f \triangleleft \vec{b}, f \triangleleft \pi_0(\tau))$$

is quasi-relevant. But then, because ζ is of irrelevant type,

$$\zeta \triangleleft_0 (f \triangleleft \tau) : T(a_i, a_i, \mathbf{r}(a_i), f \triangleleft \vec{b}, \pi_0(f \triangleleft \tau))$$

is quasi-relevant, where this action $\zeta \triangleleft_0 \dots$ is simply the result of substitution along the propositional equality ζ and is defined with respect to the obvious context. It is easy to see though that this term is propositionally equal to $\pi_1(f \triangleleft \mathbf{pair}(s, \tau))$ and therefore the latter is quasi-relevant, as required.

In this case, to see that $\kappa(f)$ is relevant, let a quasi-relevant term α of type $B_n(a_0, a_1, f, f \triangleright (f \triangleleft \vec{b}))$ be given. As such, there exists a propositionally equal term τ' with $\pi_0(\tau')$ and $\pi_1(\tau')$ quasi-relevant. Since

$$\mathbf{app}(\kappa(f), \vec{b}, \alpha) \simeq \mathbf{app}(\kappa(f), \vec{b}, \tau'),$$

it then suffices to show that the term on the right-hand side is quasi-relevant. By induction hypothesis, $\mathbf{app}(\kappa'(f), \vec{b}, \pi_0(\tau'))$ is quasi relevant, where $\kappa'(f)$ is the transition term defined with respect to the appropriate context. Thus, $\pi_0(\mathbf{app}(\kappa(f), \dots, \tau'))$ is also quasi-relevant since

$$\pi_0(\mathbf{app}(\kappa(f), \dots, \tau')) \simeq \mathbf{app}(\kappa'(f), \vec{b}, \pi_0(\tau')). \quad (15)$$

Likewise, we have by definition that

$$\pi_1(\tau') : T(a_0, a_1, f, f \triangleright (f \triangleleft \vec{b}), \pi_0(\tau'))$$

is relevant. Moreover, there exists a propositional equality

$$f \triangleright (f \triangleleft \mathbf{app}(\kappa'(f), \vec{b}, \pi_0(\tau'))) \simeq \pi_0(\tau)$$

which we denote by ζ' , where this $\kappa'(f)$ is the transition term defined with respect to the evident context. By the induction hypothesis, and the fact that ζ' is quasi-relevant, the term

$$\zeta' \triangleleft_0 \pi_1(\tau') : T(a_0, a_1, f, f \triangleright (f \triangleleft \vec{b}), f \triangleright (f \triangleleft \mathbf{app}(\kappa'(f), \vec{b}, \pi_0(\tau')))),$$

obtained by substituting along the propositional equality ζ' , is quasi-relevant. Thus, again by induction hypothesis, the term

$$\mathbf{app}(\kappa''(f), \vec{b}, \mathbf{app}(\kappa'(f), \vec{b}, \pi_0(\tau')), \zeta' \triangleleft_0 \pi_1(\tau'))$$

of type $T(a_0, a_1, f, \vec{b}, \mathbf{app}(\kappa'(f), \vec{b}, \pi_0(\tau')))$ is quasi-relevant, where this $\kappa''(f)$ is the transition with respect to the obvious extended context ending with a variable declaration of type $T(\dots)$. Finally, substituting along the propositional equality ϑ' from (15) gives a quasi-relevant term

$$\vartheta' \triangleleft_0 \mathbf{app}(\kappa''(f), \vec{b}, \mathbf{app}(\kappa'(f), \vec{b}, \pi_0(\tau')), \zeta' \triangleleft_0 \pi_1(\tau'))$$

which is propositionally equal to $\pi_1(\mathbf{app}(\kappa(f), \vec{b}, \tau'))$.

The proofs, in each of the cases considered above, that the terms $f \triangleright d_n$ and $\nu(f)$ are quasi-relevant are essentially dual to those given for $f \triangleleft b_n$ and $\kappa(f)$. \square

Corollary 5.12. *Assume given a relevant term $x : A \vdash \varphi(x) : B(x, x, \mathbf{r}(x))$ together with terms $a, b : A$ and $f : \underline{A}(a, b)$ assumed to be relevant if they are of relevant type, then $\mathbb{J}_{[x,y:A,z]B(x,y,z)}([x : A]\varphi(x), a, b, f) : B(a, b, f)$ is relevant when $B(a, b, f)$ is.*

Proof. By Lemma 5.8 it suffices to prove that $f \triangleright_0 \varphi(a)$ is relevant. The claim is then immediate by Proposition 5.11. \square

5.8. Proof that $T_1(G)$ is the free groupoid on G . We are finally in a position to prove the central technical result of the paper:

Proposition 5.13. *Every term of relevant type is relevant.*

Proof. Given a term τ of relevant type T , we prove that τ is relevant by induction on the derivation of τ .

When τ is a lambda abstraction $\lambda_{x:A}.\tau'(x)$ and $T = \prod_{x:A} S(x)$ the induction hypothesis implies that $x : A \vdash \tau'(x) : S(x)$ is relevant. Thus, given a quasi-relevant term $a : A$, $\mathbf{app}(\tau, a) = \tau'(a)$ is also relevant.

If τ is the result $\mathbf{app}(\tau', a)$ of applying the elimination rule for dependent products, then, by induction hypothesis, τ' is relevant. We may assume that τ' is itself a closed term, for otherwise in order to determine whether τ is relevant we would

first substitute quasi-relevant terms for the free variables. Thus, by the clause in the definition of relevant terms governing terms of dependent product type, τ is relevant.

The case where τ is an elimination term $J(\varphi, a, b, f)$ is by the induction hypothesis together with Corollary 5.12.

When τ is a recursion term $\mathbf{rec}(m, c, \gamma) : T(m)$ it follows from Corollary B.4 from Appendix B that m is propositionally equal, via a propositional equality $\vartheta : \mathbb{N}(m, \mathbf{n})$, to a canonical numeral $\mathbf{n} = \mathbf{S}^n(0)$. Now, the recursion term

$$\mathbf{rec}(\mathbf{n}, c, \gamma) = \gamma(\mathbf{n} - 1, \gamma(\mathbf{n} - 2, \dots \gamma(0, c) \dots))$$

is relevant since, by induction hypothesis, $\gamma(x, y)$ is. Moreover, the result

$$\vartheta \triangleleft_0 \mathbf{rec}(\mathbf{n}, c, \gamma) : T(m)$$

is therefore also relevant, since ϑ is of irrelevant type. As such, τ is relevant since $\tau \simeq \vartheta \triangleleft_0 \mathbf{rec}(\mathbf{n}, c, \gamma)$. To see that there exists such a propositional equality, define the term

$$J([x : \mathbb{N}] \lambda_v. \lambda_w. r(\mathbf{rec}(x, v, [x', y'] \mathbf{app}(w, x', y')), m, \mathbf{n}, \vartheta) : B(m, \mathbf{n}, \vartheta)$$

with pattern type $x, y : \mathbb{N}, z : \mathbb{N}(x, y) \vdash B(x, y, z) : \mathbf{type}$ given by

$$\prod_{v:T(0)} \prod_{w:\prod_{u:\mathbb{N}} \prod_{u':T(u)} T(S(u))} \frac{T(x)(\mathbf{rec}(x, v, [x', y'] \hat{w}), z \triangleleft_0 \mathbf{rec}(y, v, [x', y'] \hat{w}))}{T(S(u))},$$

where \hat{w} denotes the term $\mathbf{app}(w, x', y')$. Then the required propositional equality is $\mathbf{app}(J(\lambda_v. \lambda_w. r(\mathbf{rec}(x, v, w)), m, \mathbf{n}, \vartheta), c, \lambda_u. \lambda_{u'}. \gamma(u, u'))$.

Assume τ is a pair term $\mathbf{pair}(a, s)$ for $a : A$ and $s : S(a)$. Since $T = \sum_{x:A} S(x)$ it follows from the induction hypothesis that both a and s are quasi-relevant. Thus, τ is trivially relevant by definition.

When τ is an elimination term $\mathbf{R}(\psi, \tau')$ for dependent sums it follows from the induction hypothesis, and the fact that $x : A, y : S(x) \vdash T(\mathbf{pair}(x, y))$ is a relevant type since $T(\tau')$ is, that $x : A, y : S(x) \vdash \psi(x, y) : T(\mathbf{pair}(x, y))$ is relevant. Moreover, the induction hypothesis also implies that τ' is quasi-relevant and we consider the two cases where τ' is relevant and where it is of irrelevant type separately. In the first case, it follows that there exists a propositional equality $\vartheta : \tau' \simeq \alpha$ such that $\pi_0(\alpha)$ and $\pi_1(\alpha)$ are quasi-relevant. As such, $\psi(\pi_0(\alpha), \pi_1(\alpha))$ is a quasi-relevant term of type $T(\mathbf{pair}(\pi_0(\alpha), \pi_1(\alpha)))$. There is a propositional equality $\mathbf{pair}(\pi_0(\alpha), \pi_1(\alpha)) \simeq \alpha$ given by the elimination term

$$\mathbf{R}([x : A, y : S(x)] \mathbf{r}(\mathbf{pair}(x, y)), \alpha) : \sum_{x:A} S(x)(\mathbf{pair}(\pi_0(\alpha), \pi_1(\alpha)), \alpha),$$

which we denote by ζ . Thus, the term $\zeta \triangleleft_1 \psi(\pi_0(\alpha), \pi_1(\alpha))$ of type $T(\alpha)$ is quasi-relevant by Proposition 5.11. Finally, passing along the propositional equality ϑ gives that

$$\vartheta \triangleleft_0 (\zeta \triangleleft_1 \psi(\pi_0(\alpha), \pi_1(\alpha))) : T(\tau')$$

is quasi-relevant and it is easily seen that this term is propositionally equal to τ . \square

Remark. It is important to note at this point to which specific theories Proposition 5.13 applies. Clearly it applies to $\mathbb{T}_w[G]$ as well as to any theory of the form $\mathbb{T}_n[G]$ for $n \geq 0$. Finally, it also applies to any *equational* extension \mathbb{T} , in the sense of

Section 2.6, of either $\mathbb{T}_\omega[G]$ or $\mathbb{T}_n[G]$ for $n \geq 0$. We will use this fact in Section 6 below.

The main consequence is the following:

Theorem 5.14. *Let G be a globular set. Then the comparison functor $\Phi_G : \mathcal{F}(G) \rightarrow T_1(G)$ from the free groupoid on the underlying graph of G to the free T_1 -algebra on G is an equivalence of groupoids. This equivalence is natural in G .*

Proof. In section 5.1 it has been shown that Φ_G has a retraction; as such, it is faithful. Since the closed relevant terms of type G are, up to propositional equality, exactly the basic terms a for $a \in G_0$ this proves that the comparison functor $\Phi_G : \mathcal{F}(G) \rightarrow T_1(G)$ is essentially surjective. Since the closed relevant terms of type $\underline{G}(a, b)$ are, up to propositional and hence up to definitional equality, exactly the formal composites of basic edges $h \in G_1$, this proves that Φ_G is full on arrows as well. Naturality is straightforward and is left to the reader. \square

Remark. It is also readily seen that Φ_- constitutes a morphism of monads $\mathcal{F}(-) \rightarrow T_1(-)$ in $\mathbf{Mon}(\mathbf{rGSet})$.

6. THE QUILLEN MODEL STRUCTURE ON $\mathbf{MLC}\mathbf{x}_1$

In this section we study the general categorical properties of the category $\mathbf{MLC}\mathbf{x}_1$ of 1-truncated Martin-Löf complexes. Specifically, we establish (Theorem 6.22) the existence of a cofibrantly generated Quillen model structure on the category of such complexes which gives a Quillen equivalence with the category of groupoids. Since groupoids are a model of homotopy 1-types it then follows that 1-truncated Martin-Löf complexes are also a model. The model structure on $\mathbf{MLC}\mathbf{x}_1$ is obtained by transferring, using a theorem due to Crans [4], the model structure on the category of groupoids along the adjunction (10). In order to transfer the model structure it is first necessary to understand the behavior of colimits in $\mathbf{MLC}\mathbf{x}_1$. In order to show that one of the hypotheses of the transfer theorem is satisfied it will be necessary to understand in particular the filtered colimits in $\mathbf{MLC}\mathbf{x}_1$ and in Section 6.1 we prove that $\mathbf{MLC}\mathbf{x}_1$ has all filtered colimits and that these are created by the forgetful functor $\mathbf{MLC}\mathbf{x}_1 \rightarrow \mathbf{rGSet}$. For general colimits — which must also be properly understood in order to apply the transfer theorem — it is convenient to use, instead of $\mathbf{MLC}\mathbf{x}_1$, an equivalent category \mathbf{Th}_1 , which has as objects certain 1-truncated type theories. This category is introduced and proved to be equivalent to $\mathbf{MLC}\mathbf{x}_1$ in Section 6.2. We then explain the construction of general colimits in the category \mathbf{Th}_1 in Section 6.3. The adjunction between \mathbf{Th}_1 and \mathbf{Gpd} , as well as other issues related to the connection between groupoids and Martin-Löf complexes, is then described in Section 6.4. Finally, in Section 6.5 we complete the proof of Theorem 6.22 and its corollaries.

6.1. Filtered colimits in $\mathbf{MLC}\mathbf{x}_1$. The aim of this section is to show that $\mathbf{MLC}\mathbf{x}_1$ has all filtered colimits and that these are created by the forgetful functor $\mathbf{MLC}\mathbf{x}_1 \rightarrow \mathbf{rGSet}$. In order to establish this fact we will show that the functor $T_1 : \mathbf{rGSet} \rightarrow \mathbf{rGSet}$ preserves filtered colimits (i.e., T_1 is a finitary monad). That $\mathbf{MLC}\mathbf{x}_1$ has all filtered colimits which are created by the forgetful functor then follows from the fact that the forgetful functor creates any colimits that are preserved by T_1 . We will use this result in Section 6.5 in order to establish the existence of a Quillen model structure on $\mathbf{MLC}\mathbf{x}_1$.

Assume given a filtered category \mathcal{I} and a functor $A : \mathcal{I} \rightarrow \mathbf{rGSet}$. Denote by A^∞ the colimit of this functor. By definition, an n -cell of A^∞ is an equivalence class $[a]$ of n -cells of the coproduct $\coprod_i A(i)$, where $a \in A(i)$ is equivalent to $a' \in A(j)$ if and only if there exist arrows $\varphi : i \rightarrow k$ and $\varphi' : j \rightarrow k$ in \mathcal{I} such that

$$A(\varphi)(a) = A(\varphi')(a').$$

We would like to prove that

$$T_1(A^\infty) \cong \varinjlim_i T_1(A(i)). \quad (16)$$

This will require an analysis of the valid derivations of the theory $\mathbb{T}_1[A^\infty]$. To begin with, note that $\mathbb{T}_1[A^\infty]$ is obtained by augmenting \mathbb{T}_1 with the new basic type $\ulcorner A^\infty \urcorner$ as well as with basic terms $\ulcorner a \urcorner$ of the appropriate types as described in Section 3.

We would like to introduce some operations for manipulating the syntax of this theory. In particular, fixing any object i of \mathcal{I} , if τ is an expression of the theory $\mathbb{T}_1[A^\infty]$ containing exactly the basic term expressions $\ulcorner [a_1] \urcorner, \dots, \ulcorner [a_n] \urcorner$, where it is possible that $n = 0$, then, given any expressions e_1, \dots, e_n of $\mathbb{T}_1[A(i)]$, we define a new expression

$$\tau\{e_1/\ulcorner [a_1] \urcorner, \dots, e_n/\ulcorner [a_n] \urcorner\} \in \text{Exp}_{A(i)}$$

to be the result of formally substituting e_m for each occurrence of $\ulcorner [a_m] \urcorner$ in τ and similarly replacing each occurrence of $\ulcorner A^\infty \urcorner$ with $\ulcorner A(i) \urcorner$. The following lemma says that it is always possible, given a sequence of terms of identity type in $\mathbb{T}_1[A^\infty]$ to find an index i and replacement terms in $\mathbb{T}_1[A(i)]$ which have the same source and target relations as the original terms. In particular, any formal composite in $\mathbb{T}_1[A^\infty]$ already exists at some stage $\mathbb{T}_1[A(k)]$.

Lemma 6.1. *Given a list $[f_1], \dots, [f_n]$ of 1-cells of A^∞ , there exists an object k of \mathcal{I} and 1-cells f'_1, \dots, f'_n of $A(k)$ such that $f'_m \in [f_m]$, for $m = 1, \dots, n$, and $t(f'_m) = s(f'_k)$ when $t(f_m) = s(f_k)$, for any m, k .*

Proof. A straightforward proof by induction on n using the fact that \mathcal{I} is filtered. \square

Lemma 6.2. *Assume \mathcal{D} is a derivation in $\mathbb{T}_1[A^\infty]$ in which exactly the basic term expressions $\ulcorner [a_1] \urcorner, \dots, \ulcorner [a_n] \urcorner$ occur. Then there exists an object k of \mathcal{I} together with cells b_1, \dots, b_n of $A(k)$ such that $b_m \in [a_m]$ for $m = 1, \dots, n$ and*

$$\mathcal{D}' := \mathcal{D}\{\ulcorner b_1 \urcorner/\ulcorner [a_1] \urcorner, \dots, \ulcorner b_n \urcorner/\ulcorner [a_n] \urcorner\}$$

is a derivation in $\mathbb{T}_1[A(k)]$.

Proof. Use Lemma 6.1, regarding vertices as edges via their identity maps, to choose k together with the required terms b_1, \dots, b_n in such a way that these terms satisfy the appropriate source and target relations.

The proof that \mathcal{D}' is a derivation in $\mathbb{T}_1[A(k)]$ is an induction on \mathcal{D} . The non-trivial case of the axioms of $\mathbb{T}_1[A^\infty]$ follows from the fact that b_1, \dots, b_n have been chosen in a way compatible with source and target maps. Thus, when $\ulcorner [f] \urcorner : \ulcorner A^\infty \urcorner(\ulcorner [a] \urcorner, \ulcorner [b] \urcorner)$ is an axiom, it follows that the chosen representatives $f' \in [f]$, $a' \in [a]$ and $b' \in [b]$ at stage $A(k)$ are such that $f' : a' \rightarrow b'$, and therefore the corresponding judgement is an axiom of $\mathbb{T}_1[A(k)]$. \square

Lemma 6.3. *Given a filtered category \mathcal{I} together with a functor $A : \mathcal{I} \rightarrow \mathbf{rGSet}$, there is an isomorphism (16) of reflexive globular sets.*

Proof. Assume given a reflexive globular set X together with a cocone $x_i : T_1A(i) \rightarrow X$. We now describe the induced map $\xi : T_1A^\infty \rightarrow X$. A vertex of T_1A^∞ is just a term τ of $\mathbb{T}_1[A^\infty]$ and by Lemma 6.2 there exists some object k of \mathcal{I} together with elements b_1, \dots, b_n of $A(k)$ such that

$$\tau' := \tau\{\ulcorner b_1 \urcorner / \ulcorner [a_1] \urcorner, \dots, \ulcorner b_n \urcorner / \ulcorner [a_n] \urcorner\}$$

is a vertex of $T_1A(k)$, where $\ulcorner [a_1] \urcorner, \dots, \ulcorner [a_n] \urcorner$ is a complete list of the basic terms occurring in τ . Thus, we define:

$$\xi(\tau) := x_k(\tau').$$

This definition is independent of the choice of k and b_1, \dots, b_n since given any other choice of k', b'_1, \dots, b'_n and corresponding term τ'' , it follows from the fact that \mathcal{I} is filtered that there exists a further object l and maps $\varphi : k \rightarrow l$ and $\varphi' : k' \rightarrow l$ such that $A(\varphi)(b_m) = A(\varphi')(b'_m)$ for $m = 1, \dots, n$. Then

$$\begin{array}{ccccc} T_1A(k) & \xrightarrow{T_1A(\varphi)} & T_1A(l) & \xleftarrow{T_1A(\varphi')} & T_1A(k') \\ & \searrow x_k & \downarrow x_l & \swarrow x_{k'} & \\ & & X & & \end{array}$$

commutes. Moreover, it is easily seen, using the description of the action of T_1 on arrows from Section 3, that

$$T_1A(\varphi)(\tau') = T_1A(\varphi')(\tau'').$$

This definition extends to give an action of ξ on 1-cells of T_1A^∞ . That ξ is a map of reflexive globular sets then follows from the same argument showing that it is well-defined.

It is straightforward to verify that each x_i can be recovered by precomposing ξ with the map $T_1A(i) \rightarrow T_1A^\infty$. Finally, for uniqueness of ξ , observe that implicit in the construction of ξ above we have proved that for each cell τ of T_1A^∞ there exists some k such that τ is in the image of the map $T_1A(k) \rightarrow T_1A^\infty$. \square

We then have as an immediate consequence of Lemma 6.3 that the forgetful functor $\mathbf{MLC}\mathbf{x}_1 \rightarrow \mathbf{rGSet}$ creates filtered colimits and *a fortiori* $\mathbf{MLC}\mathbf{x}_1$ has all filtered colimits.

6.2. Equivalence with the category of theories. It is sometimes more convenient, when studying Martin-Löf complexes, to deal with theories instead of T_1 -algebras. In order to facilitate this approach it is useful to introduce a category of 1-truncated theories which is equivalent to $\mathbf{MLC}\mathbf{x}_1$. Then constructions can be carried out directly in terms of theories which are often simpler to understand than the corresponding algebras.

Definition 6.4. The category \mathbf{Th}_1 of \mathbb{T}_1 -theories has as objects theories \mathbb{T}' such that there exists a reflexive globular set G and \mathbb{T}' is an equational extension (in the sense of Section 2.6) of $\mathbb{T}_1[G]$ such that the added equations are only between terms of the form $\ulcorner G \urcorner^n(\dots)$ for $n \geq 0$. The morphisms in \mathbf{Th}_1 are **theory morphisms**, where a theory morphism $F : \mathbb{T}' \rightarrow \mathbb{T}''$ is required to be a map

$\text{Exp}(\mathbb{T}') \rightarrow \text{Exp}(\mathbb{T}'')$ of expressions (i.e., it must commute with type and term forming operations and, in particular, it must send the “base-type” $\ulcorner G \urcorner$ of \mathbb{T}' to the corresponding “base-type” of \mathbb{T}'') which preserves valid judgements.

Remark. We will often write objects of \mathbf{Th}_1 in the form $\mathbb{T}_1[G|E]$ where E is the set of equations defining the theory as an extension of $\mathbb{T}_1[G]$.

There is a functor $\mathbf{B}(-) : \mathbf{Th}_1 \rightarrow \mathbf{MLC}\mathbf{x}_1$ which sends a theory $\mathbb{T}' = \mathbb{T}_1[G|E]$ to the reflexive globular set $\mathbf{\Gamma}(G)_{\mathbb{T}'}$. For such a theory, the associated action $\gamma : T_1(\mathbf{B}(\mathbb{T}')) \rightarrow \mathbf{B}(\mathbb{T}')$ is defined first on the level of theories as the canonical extension $\text{Exp}_{\mathbf{B}(\mathbb{T}')} \rightarrow \text{Exp}(\mathbb{T}')$ of the map $\tau \mapsto \tau$. I.e., γ scans a term of $\mathbb{T}_1[\mathbf{B}(\mathbb{T}')]$ for subterms of the form $\ulcorner \tau \urcorner$ and when it finds one it removes the brackets to give τ .

Example 6.5. Given terms $\beta : \ulcorner G \urcorner$, $\tau : \ulcorner G \urcorner(\ulcorner a \urcorner, \ulcorner b \urcorner)$ and $\tau' : \ulcorner G \urcorner(\ulcorner a \urcorner, \beta)$ of \mathbb{T}' , we may form the “twisted composite” term in $\mathbb{T}_1[\mathbf{B}(\mathbb{T}')]$ by first forming the elimination term

$$\begin{array}{c} x, y : \ulcorner \mathbf{B}(\mathbb{T}') \urcorner, z : \ulcorner \mathbf{B}(\mathbb{T}') \urcorner(x, y) \vdash \ulcorner \mathbf{B}(\mathbb{T}') \urcorner(y, \ulcorner \beta \urcorner) \ulcorner \mathbf{B}(\mathbb{T}') \urcorner(x, \ulcorner \beta \urcorner) : \mathbf{type} \\ x : \ulcorner \mathbf{B}(\mathbb{T}') \urcorner \vdash \lambda_{v.v} : \ulcorner \mathbf{B}(\mathbb{T}') \urcorner(\ulcorner \ulcorner b \urcorner \urcorner, x) \ulcorner \mathbf{B}(\mathbb{T}') \urcorner(\ulcorner \ulcorner b \urcorner \urcorner, x) \\ \vdash \ulcorner \tau \urcorner : \ulcorner \mathbf{B}(\mathbb{T}') \urcorner(\ulcorner \ulcorner a \urcorner \urcorner, \ulcorner \ulcorner b \urcorner \urcorner) \\ \hline \vdash \mathbf{J}(\lambda_{v.v}, \ulcorner \ulcorner b \urcorner \urcorner, \ulcorner \ulcorner \beta \urcorner \urcorner, \ulcorner \tau \urcorner) : \ulcorner \mathbf{B}(\mathbb{T}') \urcorner(\ulcorner \ulcorner b \urcorner \urcorner, \ulcorner \ulcorner \beta \urcorner \urcorner) \ulcorner \mathbf{B}(\mathbb{T}') \urcorner(\ulcorner \ulcorner a \urcorner \urcorner, \ulcorner \ulcorner \beta \urcorner \urcorner) \end{array}$$

and then applying this to $\ulcorner \tau' \urcorner$ to obtain

$$\vdash \mathbf{app}(\mathbf{J}(\lambda_{v.v}, \ulcorner \ulcorner a \urcorner \urcorner, \ulcorner \ulcorner b \urcorner \urcorner, \ulcorner \tau \urcorner), \ulcorner \tau' \urcorner) : \ulcorner \mathbf{B}(\mathbb{T}') \urcorner(\ulcorner \ulcorner b \urcorner \urcorner, \ulcorner \ulcorner \beta \urcorner \urcorner).$$

The result of applying γ to this term is then

$$\mathbf{app}(\mathbf{J}(\lambda_{v.v}, \ulcorner a \urcorner, \ulcorner b \urcorner, \tau), \tau').$$

The map γ is seen, by induction on derivations, to preserve valid judgements and therefore gives a map of reflexive globular sets. With this definition, γ clearly satisfies the unit law for an action and the multiplication law is easily seen to be satisfied by observing that γ and μ do essentially the same thing. E.g., given a term of the form

$$\theta(\ulcorner \phi_1(\ulcorner \tau_1^1 \urcorner, \dots, \ulcorner \tau_{n_1}^1 \urcorner) \urcorner, \dots, \ulcorner \phi_m(\ulcorner \tau_1^m \urcorner, \dots, \ulcorner \tau_{n_m}^m \urcorner) \urcorner),$$

where we are regarding $\theta(-)$ and $\phi_i(-)$ as “contexts” (in the sense of “terms with holes in them” and not variable contexts) and explicitly exhibiting all basic terms occurring in the term (and the basic terms occurring one level down in its basic terms themselves). Then the action of $\gamma \circ T_1(\gamma)$ is to first erase “inner” brackets to give

$$\theta(\ulcorner \phi_1(\tau_1^1, \dots, \tau_{n_1}^1) \urcorner, \dots, \ulcorner \phi_m(\tau_1^m, \dots, \tau_{n_m}^m) \urcorner),$$

and then to erase the remaining “outer” brackets, leaving

$$\theta(\phi_1(\tau_1^1, \dots, \tau_{n_1}^1), \dots, \phi_m(\tau_1^m, \dots, \tau_{n_m}^m)).$$

On the other hand, $\gamma \circ \mu$ first erases the “outer” brackets and then the remaining “inner” brackets, which clearly gives the same result. Functoriality of $\mathbf{B}(-)$ is by definition of the algebra actions.

There is also a functor $\mathbf{T}(-) : \mathbf{MLC}\mathbf{x}_1 \rightarrow \mathbf{Th}_1$ which sends a T_1 -algebra $(A, \alpha : T_1(A) \rightarrow A)$ to the theory $\mathbb{T}_1[A|\alpha]$ extending $\mathbb{T}_1[A]$ by equations of the form

$$\ulcorner \alpha(\tau) \urcorner = \tau : \ulcorner A \urcorner^m(\tau_0^0, \tau_1^0, \dots, \tau_1^{n-1})$$

for each n -cell τ of $T_1(A)$, with $n \geq 0$. An algebra homomorphism $f : A \rightarrow B$ induces a corresponding morphism of theories as the extension to expressions of the map $A \rightarrow \text{Exp}(T_1[B|\beta])$ given by $a \mapsto f(a)$. This preserves derivable judgements in virtue of the fact that f is a homomorphism.

When (A, α) is an algebra, there exists an induced map of theories $\pi : T_1[A] \rightarrow \mathbf{T}(A)$ which sends a term τ of $T_1[A]$ to the corresponding term τ of $\mathbf{T}(A)$. We will now prove two technical lemmata (6.6 and 6.7) which will be used to establish the equivalence of $\mathbf{MLC}\mathbf{x}_1$ and \mathbf{Th}_1 below.

Lemma 6.6. *If (A, α) is an object of $\mathbf{MLC}\mathbf{x}_1$, then $\pi : T_1[A] \rightarrow \mathbf{T}(A)$ is surjective in the sense that if a judgement $\vdash \tau : T$ is derivable in $\mathbf{T}(A)$, then there is a judgement $\vdash \tau' : T'$ derivable in $T_1[A]$ such that $\vdash \tau = \tau' : T$ in $\mathbf{T}(A)$.*

Proof. It will suffice to prove the following claims:

- (1) If T is a type of $\mathbf{T}(A)$, then there exists a type T' of $T_1[A]$ such that $T = T'$ in $\mathbf{T}(A)$.
- (2) If $\tau : T$ is derivable in $\mathbf{T}(A)$ and T' is any type in $T_1[A]$ for which $T' = T$ in $\mathbf{T}(A)$, then there exists a term $\tau' : T'$ in $T_1[A]$ such that $\tau = \tau'$ in $\mathbf{T}(A)$.

We prove both (1) and (2) simultaneously by induction on derivations. The non-trivial step is to verify that condition (2) is satisfied when $\tau : T$ is obtained as one of the axioms of $\mathbf{T}(A)$. When $T = \ulcorner A \urcorner$ this is trivial, but when $T = \ulcorner A \urcorner(\ulcorner a \urcorner, \ulcorner b \urcorner)$ and $\tau = \ulcorner f \urcorner$ we must verify that, for any choice of type $\ulcorner A \urcorner(\tau, \tau')$ with $\tau = \ulcorner a \urcorner$ and $\tau' = \ulcorner b \urcorner$ in $\mathbf{T}(A)$, there exists a term of this type which is sent to $\ulcorner f \urcorner$. Here we prove by induction on the derivation of $\tau = \ulcorner a \urcorner$ in $\mathbf{T}(A)$ that there exists a term $\hat{f} : \ulcorner A \urcorner(\tau, \tau')$ with $\pi(\hat{f}) = \ulcorner f \urcorner$. When the equation is obtained by transitivity using $\tau = v$ and $v = \ulcorner a \urcorner$, then, by induction hypothesis, there exist lifts $\rho : \ulcorner A \urcorner(\tau, v)$ of $\mathbf{r}(\ulcorner a \urcorner)$ and $\rho' : \ulcorner A \urcorner(v, \tau')$ of $\ulcorner f \urcorner$. Thus, the composite $\rho' \cdot \rho$ is the required lift of $\ulcorner f \urcorner$. In the case where the equation is by an axiom of the form $\tau = \ulcorner \alpha(\tau) \urcorner$, we first observe that by Proposition 5.13, there exists a basic term $\ulcorner c \urcorner$ and a propositional equality $\chi : \tau \rightarrow \ulcorner c \urcorner$ in $T_1[A]$. Applying α gives an edge $\alpha(\chi) : \alpha(\tau) \rightarrow c$ and therefore a term $\ulcorner \alpha(\chi) \urcorner : \ulcorner A \urcorner(\ulcorner \alpha(\tau) \urcorner, \ulcorner c \urcorner)$. We claim that the composite $\ulcorner \alpha(\chi) \urcorner^{-1} \cdot \chi : \tau \rightarrow \ulcorner \alpha(\tau) \urcorner$ is sent to the identity $\mathbf{r}(\tau)$ in $\mathbf{T}(A)$. To see this, note that by the new axioms of $\mathbf{T}(A)$, $\chi = \ulcorner \alpha(\chi) \urcorner$ and so

$$\ulcorner \alpha(\chi) \urcorner^{-1} \cdot \chi = T_1(\alpha)(\ulcorner \chi \urcorner^{-1} \cdot \ulcorner \chi \urcorner) = T_1(\alpha)(\mathbf{r}(\ulcorner \tau \urcorner)) = \mathbf{r}(\ulcorner \alpha(\tau) \urcorner)$$

which is equal to $\mathbf{r}(\tau)$. Now, the induction hypothesis gives a term $\rho : \ulcorner A \urcorner(\ulcorner \alpha(\tau) \urcorner, \tau')$ which is equal in $\mathbf{T}(A)$ to $\ulcorner f \urcorner$. As such, $\rho \cdot \ulcorner \alpha(\chi) \urcorner^{-1} \cdot \chi : \ulcorner A \urcorner(\tau, \tau')$ is equal in $\mathbf{T}(A)$ to $\ulcorner f \urcorner$, as required. The remaining cases are straightforward. As such, we have completed the proof for step (2) in the case where the term is an axiom.

In each possible remaining cases the proof is essentially the same. As such, we state here only the case for the elimination rule for identity types. In this case, we are given that a judgement $J(\varphi, a, b, f) : B(a, b, f)$ is derivable in $\mathbf{T}(A)$. By the induction hypothesis for (1) there exists a type $x, y : A', z : \underline{A}'(x, y) \vdash B'(x, y, z) : \text{type}$ in $T_1[A]$ which is sent by π to $B(x, y, z)$. By the induction hypothesis for (2), there exist terms $\varphi'(x) : B'(x, x, r(x))$ and $f' : \underline{A}'(a', b')$ which are sent to $\varphi(x)$ and f , respectively. Thus, in $T_1[A]$ we may form the term $J(\varphi', a', b', f') : B'(a', b', f')$ and this term is sent to $J(\varphi, a, b, f)$, as required. \square

Remark. Note that most of the work to prove Lemma 6.6 is devoted to verifying step (2), stated in the proof, for axioms. This is the step which will in general fail for other extensions of $\mathbb{T}_1[A]$.

Lemma 6.7. *If τ and τ' are terms in $T_1(A)$ such that $\tau = \tau'$ in $\mathbf{T}(A)$, then $\gamma(\tau) = \gamma(\tau')$.*

Proof. In order to be able to prove the claim by induction on the derivation of $\tau = \tau'$ it is necessary to reformulate the claim a bit. Namely, we prove by induction on derivations that if $\tau(x_1, \dots, x_n)$ and $\tau'(x_1, \dots, x_n)$ are terms in context Γ of a type of the form $\ulcorner A \urcorner^n(\dots)$, for $n \geq 0$, such that $\tau(x_1, \dots, x_n) = \tau'(x_1, \dots, x_n)$, then, for any terms a_1, \dots, a_n such that $\tau(a_1, \dots, a_n)$ is in $T_1(A)$, $\gamma(\tau(a_1, \dots, a_n)) = \gamma(\tau'(a_1, \dots, a_n))$. Formulated in this way, all of the basic cases are trivial. The only case which remains is where the equation is one of the new axioms of $\mathbb{T}_1[A|\alpha]$. In this case, we have that τ' is of the form $\ulcorner \gamma(\tau) \urcorner$. Thus, $\gamma(\tau') = \gamma(\ulcorner \gamma(\tau) \urcorner)$ which is, since γ is an action, equal to $\gamma(\tau)$. \square

Proposition 6.8. *The functors $\mathbf{B}(-)$ and $\mathbf{T}(-)$ give an equivalence of categories $\mathbf{MLC}\mathbf{x}_1 \simeq \mathbf{Th}_1$.*

Proof. The natural isomorphism $\varepsilon_A : \mathbf{B}(\mathbf{T}(A)) \rightarrow A$, for (A, α) a T_1 -algebra, is defined as follows. Given a term τ in $\mathbf{B}(\mathbf{T}(A))$, there exists, by Lemma 6.6, a corresponding term τ' in $\mathbb{T}_1[A]$ such that $\tau = \tau'$ in the theory $\mathbf{T}(A)$. Then $\varepsilon(\tau)$ is defined to be $\alpha(\tau')$. That this definition is independent of the choice of representative τ' is by Lemma 6.7. The inverse of ε_A is the obvious map which sends a cell a of A to $\ulcorner a \urcorner$. Clearly, $\varepsilon_A \circ \varepsilon_A^{-1}$ is the identity. On the other hand, given τ in $\mathbf{B}(\mathbf{T}(A))$, $\varepsilon_A^{-1} \circ \varepsilon_A(\tau)$ is the term $\ulcorner \gamma(\tau') \urcorner$, where τ' is as above. But, by the axioms of $\mathbf{T}(A)$, this is equal to τ' which is equal, since we are now in $\mathbf{T}(A)$, to τ . These maps are natural since given an algebra homomorphism $f : A \rightarrow B$ and $\varepsilon_B^{-1} \circ f$ sends an element a of A to $\ulcorner f(a) \urcorner$ which is the same as $\mathbf{B}(\mathbf{T}(f))(\ulcorner a \urcorner)$ by definition.

Likewise, given a theory $\mathbb{T}' = \mathbb{T}_1[G|E]$, the map $\eta_{\mathbb{T}'} : \mathbb{T}' \rightarrow \mathbf{T}(\mathbf{B}(\mathbb{T}'))$ is obtained as the extension to expressions of the assignment $g \mapsto \ulcorner g \urcorner$. That $\eta_{\mathbb{T}'}$ preserves valid judgements is proved by induction on derivations. The case of axioms of the form $\ulcorner g \urcorner : \ulcorner G \urcorner$ are trivial. To see that equations in E are preserved, first observe that, by definition of the action $\gamma : T_1(\mathbf{B}(\mathbb{T}')) \rightarrow \mathbf{B}(\mathbb{T}')$, it follows that any term in $\mathbf{T}(\mathbf{B}(\mathbb{T}'))$ of the form

$$\theta(\ulcorner \phi_1(\ulcorner g_1^1 \urcorner, \dots, \ulcorner g_{n_1}^1 \urcorner) \urcorner, \dots, \ulcorner \phi_m(\ulcorner g_1^m \urcorner, \dots, \ulcorner g_{n_m}^m \urcorner) \urcorner), \quad (17)$$

where again the θ and ϕ_i are “contexts” as above, is equal in $\mathbf{T}(\mathbf{B}(\mathbb{T}'))$ to the term

$$\ulcorner \theta(\phi_1(\ulcorner g_1^1 \urcorner, \dots, \ulcorner g_{n_1}^1 \urcorner), \dots, \phi_m(\ulcorner g_1^m \urcorner, \dots, \ulcorner g_{n_m}^m \urcorner)) \urcorner$$

obtained by removing all of the outer brackets and replacing them with a single bracket around the entire term. Thus, given an equation $\chi = \chi'$ in E with

$$\begin{aligned} \chi &= \tau(\ulcorner g_1 \urcorner, \dots, \ulcorner g_n \urcorner), \text{ and} \\ \chi' &= \tau'(\ulcorner g'_1 \urcorner, \dots, \ulcorner g'_m \urcorner), \end{aligned}$$

we have that $\eta_{\mathbb{T}'}(\chi)$ is the term

$$\begin{aligned} \tau(\ulcorner g_1 \urcorner, \dots, \ulcorner g_n \urcorner) &= \ulcorner \tau(\ulcorner g_1 \urcorner, \dots, \ulcorner g_n \urcorner) \urcorner \\ &= \ulcorner \tau'(\ulcorner g'_1 \urcorner, \dots, \ulcorner g'_m \urcorner) \urcorner \\ &= \tau'(\ulcorner g'_1 \urcorner, \dots, \ulcorner g'_m \urcorner) \end{aligned}$$

which is $\eta_{\mathbb{T}'}(\chi')$. On the other hand, the inverse $\eta_{\mathbb{T}}^{-1}$ is defined as the extension to expressions of the assignment $\tau \mapsto \tau$. Again, the proof that this preserves valid judgements is by induction on derivations. The new equations of $\mathbf{T}(\mathbf{B}(\mathbb{T}'))$ are trivially preserved though in virtue of the definition of $\eta_{\mathbb{T}'}^{-1}$. It is a trivial consequence of the definition that $\eta_{\mathbb{T}}^{-1} \circ \eta_{\mathbb{T}'}$ is the identity. On the other hand, given a term (17) the result of applying $\eta_{\mathbb{T}'} \circ \eta_{\mathbb{T}}^{-1}$ is

$$\theta(\phi_1(\ulcorner g_1^1 \urcorner, \dots, \ulcorner g_{n_1}^1 \urcorner), \dots, \phi_m(\ulcorner g_1^m \urcorner, \dots, \ulcorner g_{n_m}^m \urcorner))$$

which is equal to (17) in virtue of the defining axioms of $\mathbf{T}(\mathbf{B}(\mathbb{T}'))$. Naturality of these isomorphisms is straightforward. \square

Finally, we relate a useful fact about the objects of \mathbf{Th}_1 that will be employed in the sequel.

Lemma 6.9. *If $\mathbb{T}_1[G|E]$ is an object of \mathbf{Th}_1 , then there exists a model $\llbracket - \rrbracket_{E,0}$ of $\mathbb{T}_1[G|E]$ such that, for any terms $\tau, \tau' : \ulcorner G \urcorner$, if $\llbracket \tau \rrbracket_{E,0}$ is equal to $\llbracket \tau' \rrbracket_{E,0}$, then $\tau \simeq \tau'$ is derivable in $\mathbb{T}_1[G|E]$.*

Proof. Let \equiv_E be the equivalence relation on the set G_0 of vertices of G generated by identifying elements which are either in the same path component or identified by an equation in E . Then we interpret $\llbracket \ulcorner G \urcorner \rrbracket_{E,0}$ as the set G_0 / \equiv_E of equivalence classes. Basic terms of type $\ulcorner G \urcorner$ are interpreted as their equivalence classes and basic terms of the higher-dimensional identity types are interpreted as the equivalence classes of their 0-dimensional endpoints as in Section 4.2. The other operations of the theory are also interpreted as usual in a **Set** based model as described in Section 4.2. Now, assume given terms τ and τ' of type $\ulcorner G \urcorner$ which are interpreted in the same way. By Proposition 5.13 it follows that each of these terms are propositionally equal to basic terms $\ulcorner a \urcorner$ and $\ulcorner b \urcorner$, respectively. By definition of the interpretation, it follows that these basic terms are also interpreted in the same way. In particular, $a \equiv_E b$. Finally, by definition of the equivalence relation it is clear that a term $\phi : \ulcorner \underline{G} \urcorner(\ulcorner a \urcorner, \ulcorner b \urcorner)$ exists. Thus, $\tau \simeq \tau'$, as required. \square

6.3. Construction of colimits. Next we would like to construct coequalizers in \mathbf{Th}_1 . To this end assume given a parallel pair of arrows

$$\mathbb{T}_1[H|F] \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} \mathbb{T}_1[G|E] \quad (18)$$

in \mathbf{Th}_1 . We then define a theory $\mathbb{T}_1[G|E, f = g]$ by extending $\mathbb{T}_1[G|E]$ with axioms of the form

$$\vdash f(\tau) = g(\tau) : \ulcorner \underline{G} \urcorner^n(f(\tau)_0^0, \dots, f(\tau)_1^{n-1})$$

for τ any term of $\mathbb{T}_1[H|F]$ of type $\ulcorner \underline{H} \urcorner^n(\tau_0^0, \dots, \tau_1^{n-1})$ for $n \geq 0$. There exists an induced quotient map $q : \mathbb{T}_1[G|E] \rightarrow \mathbb{T}_1[G|E, f = g]$ which sends a term τ to itself.

Lemma 6.10. *Given a parallel pair of arrows (18) in \mathbf{Th}_1 , the coequalizer of f and g is $\mathbb{T}_1[G|E, f = g]$.*

Proof. Suppose given a theory $\mathbb{T}' = \mathbb{T}_1[K|I]$ together with a map $k : \mathbb{T}_1[G|E] \rightarrow \mathbb{T}'$ such that $k \circ f = k \circ g$. Define a map $\hat{k} : \mathbb{T}_1[G|E, f = g] \rightarrow \mathbb{T}'$ first on the level of expressions as the extension of the map $a \mapsto k(\ulcorner a \urcorner)$. This extension has the property that $\hat{k} \circ q = k$ in virtue of the fact that, as a map of theories, k commutes with the expression forming operations as well and they are equal on basic terms of $\mathbb{T}_1[G|E]$. We now verify that \hat{k} preserves valid judgements. The proof is by induction on derivations and the only cases which do not follow from the fact that k is a morphism of theories are those axioms of the form $\ulcorner f(\tau) \urcorner = \ulcorner g(\tau) \urcorner$ for τ a term of $\mathbb{T}_1[H|F]$. Since all of the maps under consideration commute with the expression forming operations it suffices to note that if $\ulcorner a \urcorner$ is a basic term of $\mathbb{T}_1[H|F]$, then $\hat{k}(\ulcorner f(a) \urcorner)$ is just $k \circ f(\ulcorner a \urcorner)$ which is equal to $\hat{k}(\ulcorner g(a) \urcorner)$ by hypothesis. Finally, to see that \hat{k} is the canonical map $\mathbb{T}_1[G|E, f = g] \rightarrow \mathbb{T}'$ with $\hat{k} \circ q = k$ we observe that any other map h with this property necessarily sends a basic term $\ulcorner a \urcorner$ of $\mathbb{T}_1[G|E, f = g]$ to $k(\ulcorner a \urcorner)$ which is equal to $\hat{k}(\ulcorner a \urcorner)$. \square

Coproducts are constructed using a similar approach. Namely, given an I -indexed family of theories $\mathbb{T}^i = \mathbb{T}_1[G_i|E_i]$, where we assume for the sake of notational simplicity that the globular sets G_i are pairwise disjoint, we define a new type theory $\mathbb{T}_1[\coprod_i G_i | (\mathcal{E}_i)_i]$ extending $\mathbb{T}_1[\coprod_i G_i]$, where this is the coproduct of reflexive globular sets, by adding all of the equations from the sets of equations E_i . I.e., if an equation $\tau = \tau'$ is in E_i for some index i , then we add the corresponding axioms $\tau = \tau'$, which makes sense because any judgement derivable in \mathbb{T}^i is also derivable in the new theory. The proof of the following lemma is entirely analogous to that of Lemma 6.10 above and is therefore left to the reader:

Lemma 6.11. *Given an I -indexed family of theories $\mathbb{T}^i = \mathbb{T}_1[G_i|E_i]$ in \mathbf{Th}_1 , the coproduct $\coprod_i \mathbb{T}^i$ exists and is the theory $\mathbb{T}_1[\coprod_i G_i | (\mathcal{E}_i)_i]$.*

We mention here several useful facts about the theory $\coprod_i \mathbb{T}^i$.

Lemma 6.12. *Given an I -indexed family of theories \mathbb{T}^i , the following hold:*

- (1) *If $\ulcorner a \urcorner$ and $\ulcorner b \urcorner$ are basic terms of $\coprod_i \mathbb{T}^i$ such that $\ulcorner a \urcorner = \ulcorner b \urcorner$ is derivable in $\coprod_i \mathbb{T}^i$, then there exists a unique index i such that both $\ulcorner a \urcorner$ and $\ulcorner b \urcorner$ are basic terms of \mathbb{T}^i and $\ulcorner a \urcorner = \ulcorner b \urcorner$ is derivable in \mathbb{T}^i .*
- (2) *If $\ulcorner f \urcorner : \ulcorner \coprod_i G_i \urcorner(\ulcorner a \urcorner, \ulcorner b \urcorner)$ in $\coprod_i \mathbb{T}^i$, then there exists a unique index i such that $\ulcorner f \urcorner : \ulcorner G_i \urcorner(\ulcorner a \urcorner, \ulcorner b \urcorner)$ in $\mathbb{T}_1[G_i|E_i]$.*

Proof. For (1), we note that, using the interpretation $\llbracket - \rrbracket_{(\mathcal{E}_i)_i, 1}$ of $\coprod_i \mathbb{T}^i$ described in Lemma 6.16 we have by definition of the interpretation that the equivalence class $\llbracket \ulcorner a \urcorner \rrbracket_{(\mathcal{E}_i)_i, 0}$ interpreting a term of basic type necessarily consists solely of elements of the same summand G_i . Thus, that both terms are from the same summand is by Soundness. That $\ulcorner a \urcorner = \ulcorner b \urcorner$ is derivable in \mathbb{T}^i follows from Lemma 6.16 and the fact that, for basic terms from the same summand G_i , if $a \equiv_{(\mathcal{E}_i)_i, 1} b$, then $a \equiv_{\mathcal{E}_i, 1} b$, by definition of these equivalence relations. (2) is then by similar reasoning since $\llbracket \ulcorner f \urcorner \rrbracket_{(\mathcal{E}_i)_i, 1}$ is necessarily from the same summand as its domain and codomain. \square

Proposition 6.13. *The category $\mathbf{MLC}\mathbf{x}_1$ is bicomplete and the forgetful functor $U : \mathbf{MLC}\mathbf{x}_1 \rightarrow \mathbf{rGSet}$ preserves filtered colimits.*

Proof. Since $\mathbf{MLC}\mathbf{x}_1$ is monadic over \mathbf{rGSet} it follows that it is complete. By Lemmata 6.10 and 6.11 it follows that $\mathbf{MLC}\mathbf{x}_1$ is cocomplete. The forgetful functor preserves filtered colimits by Lemma 6.3. \square

6.4. Connection with groupoids. Theorem 5.14 shows that free T_1 -algebras are, up to equivalence, free groupoids. This might lead one to conjecture that the category of T_1 -algebras is equivalent to the category of groupoids. However, that is not the case. Let us first consider the comparison functor

$$\mathbf{MLC}\mathbf{x}_1 \xrightarrow{\Phi} \mathbf{Gpd}.$$

We recall from Section 5.1 that this functor regards a T_1 -algebra as a groupoid. I.e., in particular it sends a T_1 -algebra $(G, \gamma : T_1(G) \rightarrow G)$ to

$$\mathcal{F}(G) \xrightarrow{\Phi_G} T_1(G) \xrightarrow{\gamma} G$$

which is then an algebra structure for the free groupoid monad (we do not distinguish notationally between groupoids and 1-truncated globular sets whose underlying graphs are groupoids).

Thus every T_1 -algebra G has an underlying groupoid $\Phi(G)$ - we might call it the **fundamental groupoid** of G . However, the following example makes clear that different algebras may have the same underlying groupoid.

Example 6.14. Consider the following groupoid G : it has two objects a and b , exactly one arrow $f : a \rightarrow b$ and its inverse $g : b \rightarrow a$. We may define a T_1 -algebra structure $\gamma : T_1(G) \rightarrow G$ on G as follows: on objects, γ is defined by

$$\gamma(v) = \begin{cases} a & \text{if } \vdash v = \ulcorner a \urcorner : G \text{ is derivable} \\ b & \text{otherwise} \end{cases}$$

Thus, all doppelgängers of vertices are sent to b . On 1-cells we define:

$$\gamma(w) = \begin{cases} f & \text{if } \gamma(s(w)) = a, \gamma(t(w)) = b \\ g & \text{if } \gamma(s(w)) = b, \gamma(t(w)) = a \\ 1_a & \text{if } \gamma(s(w)) = a = \gamma(t(w)) \\ 1_b & \text{if } \gamma(s(w)) = b = \gamma(t(w)) \end{cases}$$

It is readily seen that this is a map of globular sets. To see that it is a T_1 -algebra, we remark that the unit law is trivially satisfied because the algebra map sends any generator $\ulcorner v \urcorner$ of $T_1(G)$ to v . For the associativity law, consider an element τ of $T_1^2(G)_0$; this is a term of the theory $\mathbb{T}_1[T_1(G)]$, which is generated by basic terms of the form $\ulcorner \sigma \urcorner$, where σ is a term of the theory $\mathbb{T}_1[G]$. Note that on the one hand

$$(\gamma \circ T_1\gamma)(\tau) = a \Leftrightarrow T_1\gamma(\tau) = \ulcorner a \urcorner \Leftrightarrow \tau = \ulcorner \ulcorner a \urcorner \urcorner,$$

while on the other hand

$$(\gamma \circ \mu)(\tau) = a \Leftrightarrow \mu(\tau) = \ulcorner a \urcorner \Leftrightarrow \tau = \ulcorner \ulcorner a \urcorner \urcorner,$$

showing that both maps agree in dimension 0. To show that they agree in dimension 1 as well, one reasons in a similar fashion.

But clearly by symmetry there is another algebra structure on G , call it δ , defined by sending all doppelgängers to a instead of b . The identity map $G \rightarrow G$ is, however, not a map of T_1 -algebras. Indeed, any map of T_1 -algebras commutes with the formation of doppelgängers; for example, if k is a map of algebras then $T_1(k)$ must send the doppelgänger $a\langle f \rangle$ to $k(a)\langle k(f) \rangle$, and hence we must have $k\gamma(a\langle f \rangle) = \delta k(a)\langle k(f) \rangle$, which is impossible if k is the identity. For the same reason the only other possible map of groupoids, which interchanges a and b , cannot be a map of T_1 -algebras.

Thus T_1 -algebras carry more information than their underlying groupoid, and this information tells us how the formal composites and doppelgänger are interpreted. The fact that non-isomorphic algebras may have the same fundamental groupoid is of course the analogue of the fact that non-homeomorphic topological spaces may have the same fundamental groupoid.

In addition, the above example shows that Φ is not a full functor. (However, it is easily seen to be faithful.)

Proposition 6.15. *The functor $\Phi : \mathbf{MLC}\mathbf{x}_1 \rightarrow \mathbf{Gpd}$ has a left adjoint.*

Proof. It is convenient to work with the category \mathbf{Th}_1 of theories in place of $\mathbf{MLC}\mathbf{x}_1$. In this case, $\Phi : \mathbf{Th}_1 \rightarrow \mathbf{Gpd}$ sends a theory $\mathbb{T}' = \mathbb{T}_1[G|E]$ to the resulting groupoid structure on $\Gamma(G)_{\mathbb{T}'}$ with composition of arrows $\phi : \tau \rightarrow \tau'$ and $\psi : \tau' \rightarrow \tau''$ given by

$$\psi \circ \phi := \psi \cdot \phi.$$

On the other hand, every groupoid G determines a theory $\Psi(G) = \mathbb{T}_1[G|\circ]$ which extends $\mathbb{T}_1[G]$ by adding axioms of the form

$$\vdash \ulcorner g \circ f \urcorner = \ulcorner g \urcorner \cdot \ulcorner f \urcorner : \ulcorner \underline{G} \urcorner(\ulcorner a \urcorner, \ulcorner c \urcorner)$$

for arrows $f : a \rightarrow b$ and $g : b \rightarrow c$ in G . The left-adjoint $\Psi : \mathbf{Gpd} \rightarrow \mathbf{Th}_1$ then has the obvious action on arrows.

The unit of the adjunction is given at a groupoid G by the map $\eta_G : G \rightarrow \Phi\Psi(G)$ which sends an object a of G to the object $\ulcorner a \urcorner$ of $\Phi\Psi(G)$ and an arrow f to $\ulcorner f \urcorner$. It is functorial in virtue of the additional axioms of $\mathbb{T}_1[G|\circ]$.

Likewise, given a theory $\mathbb{T}' = \mathbb{T}_1[G|E]$, the counit $\varepsilon_{\mathbb{T}'} : \Psi\Phi(\mathbb{T}') \rightarrow \mathbb{T}'$ is defined as the extension to expressions of the assignment $\tau \mapsto \tau$. That this gives a map which preserves valid judgements is by an induction on derivations. In particular, we must verify that if $\phi : \tau \rightarrow \tau'$ and $\psi : \tau' \rightarrow \tau''$ are arrows in $\Phi(\mathbb{T}')$, then the judgement $\vdash \ulcorner \psi \circ \phi \urcorner = \ulcorner \psi \urcorner \cdot \ulcorner \phi \urcorner$ of $\Psi\Phi(\mathbb{T}')$ is preserved by $\varepsilon_{\mathbb{T}'}$. This is the case since $\varepsilon_{\mathbb{T}'}(\ulcorner \psi \circ \phi \urcorner)$ is equal to $\psi \cdot \phi$ which is also the result of applying $\varepsilon_{\mathbb{T}'}$ to $\ulcorner \psi \urcorner \cdot \ulcorner \phi \urcorner$. With these definitions it is easily verified that η and ε are natural transformations which satisfy the equations for an adjunction. \square

As we mentioned above, Φ is not full and therefore this adjunction is not an equivalence of categories. We see below that it is however a Quillen equivalence. First we collect a useful 1-dimensional analogue of Lemma 6.9.

Lemma 6.16. *If $\mathbb{T}_1[G|E]$ is an object of \mathbf{Th}_1 , then there exists a model $\llbracket - \rrbracket_{E,1}$ of $\mathbb{T}_1[G|E]$ such that:*

- (1) *Given basic terms $\ulcorner a \urcorner$ and $\ulcorner b \urcorner$ of type $\ulcorner G \urcorner$, if $\llbracket \ulcorner a \urcorner \rrbracket_{E,1}$ is equal to $\llbracket \ulcorner b \urcorner \rrbracket_{E,1}$, then $\ulcorner a \urcorner = \ulcorner b \urcorner$ is derivable in $\mathbb{T}_1[G|E]$.*
- (2) *Given basic terms $\ulcorner a \urcorner, \ulcorner b \urcorner : \ulcorner G \urcorner$ and terms $\tau, \tau' : \ulcorner \underline{G} \urcorner(\ulcorner a \urcorner, \ulcorner b \urcorner)$, if $\llbracket \tau \rrbracket_{E,1}$ is equal to $\llbracket \tau' \rrbracket_{E,1}$, then $\tau = \tau'$ is derivable in $\mathbb{T}_1[G|E]$.*

Proof. Let $\equiv_{E,1}$ be the congruence on the groupoid $\Phi(\mathbb{T}_1[G|E])$ generated by the equations in E and interpret $\ulcorner G \urcorner$ as the groupoid $\Phi(\mathbb{T}_1[G|E]) / \equiv_{E,1}$ with terms interpreted as their equivalence classes and the other operations interpreted as in the Hofmann-Streicher [8] model summarized in Section 5.2. This is clearly a model of the theory. For (1) we note that if $\ulcorner a \urcorner$ and $\ulcorner b \urcorner$ are given the same interpretation, then $a \equiv_{E,1} b$. But, by definition of $\equiv_{E,1}$, this implies that $a = b$ is derivable. For

(2), given τ and τ' as in the statement, it follows from Proposition 5.13 that they are both formal composites. Moreover, formal composites are interpreted by $\llbracket - \rrbracket_{E,1}$ as the equivalence classes of their corresponding composites in the groupoid and therefore, regarded as genuine composites in the groupoid $\Phi(\mathbb{T}_1[G|E])$, $\tau \equiv_{E,1} \tau'$. By definition of $\equiv_{E,1}$ it then follows that $\tau = \tau'$. \square

The following lemma sheds some more light on the functor Φ , but will not be used further.

Lemma 6.17. *If (A, α) is a T_1 -algebra, then*

$$\Phi(T_1^2 A, \mu_{T_1 A}) \begin{array}{c} \xrightarrow{\mu_A} \\ \xrightarrow{T_1(\alpha)} \end{array} \Phi(T_1 A, \mu_A) \xrightarrow{\alpha} \Phi(A, \alpha)$$

is a coequalizer in \mathbf{Gpd} .

Proof. Suppose given a functor $m : \Phi(T_1 A) \rightarrow M$ in \mathbf{Gpd} such that $m \circ \mu_A = m \circ T_1(\alpha)$. Define $\tilde{m} : \Phi(A) \rightarrow M$ by setting $\tilde{m}(a) := m(\ulcorner a \urcorner)$ for a any cell of $\Phi(A)$. This gives a map of reflexive globular sets which is also a functor since, given $f : a \rightarrow b$ and $g : b \rightarrow c$ in $\Phi(A)$, $\tilde{m}(g \circ f)$ is equal to $m(\ulcorner \alpha(\ulcorner g \urcorner \cdot \ulcorner f \urcorner) \urcorner)$ by definition of \tilde{m} and composition in $\Phi(A)$. This in turn is equal to

$$\begin{aligned} m \circ T_1(\alpha)(\ulcorner \ulcorner g \urcorner \cdot \ulcorner f \urcorner \urcorner) &= m \circ \mu_A(\ulcorner \ulcorner g \urcorner \cdot \ulcorner f \urcorner \urcorner) \\ &= m(\ulcorner \ulcorner g \urcorner \cdot \ulcorner f \urcorner \urcorner) \end{aligned}$$

which is the same as $m(\ulcorner g \urcorner \circ \ulcorner f \urcorner)$ where the composition is now in $\Phi(T_1 A)$. Finally, by functoriality of m this is $m(\ulcorner g \urcorner) \circ m(\ulcorner f \urcorner)$, as required. That \tilde{m} is the canonical map $\Phi(A) \rightarrow M$ with $\tilde{m} \circ \alpha = m$ follows from the fact that if n is any other functor for which this equation holds, we have

$$n(a) = n(\alpha \circ \eta_A(a)) = m(\ulcorner a \urcorner) = \tilde{m}(a),$$

for a any cell of $\Phi(A)$. \square

6.5. The induced model structure. In this section we will employ Crans's Theorem [4] for transferring cofibrantly generated model structures along an adjunction to prove that there exists a cofibrantly generated model structure on \mathbf{Th}_1 and therefore also on \mathbf{MLCx}_1 . It will then be a consequence of the construction that the adjunction between the category of groupoids and \mathbf{Th}_1 is in fact a Quillen equivalence. We refer the reader to [9] for further details regarding Quillen model categories.

We begin by defining the relevant classes of maps in \mathbf{Th}_1 .

Definition 6.18. The weak equivalences, fibrations and cofibrations in \mathbf{Th}_1 are defined as follows:

- $f : A \rightarrow B$ in \mathbf{Th}_1 is a **weak equivalence** if and only if $\Phi(f)$ is an equivalence of groupoids;
- $f : A \rightarrow B$ is a **fibration** if and only if $\Phi(f)$ is an isofibration of groupoids (i.e., given an arrow $\varphi : b' \rightarrow b$ in $\Phi(B)$ such that $b = \Phi(f)(a)$, there exists an arrow $\varphi' : a' \rightarrow a$ in $\Phi(A)$ with $\Phi(f)(\varphi') = \varphi$); and
- $f : A \rightarrow B$ is a **cofibration** if and only if it has the left-lifting property with respect to maps which are simultaneously fibrations and weak equivalences.

Recall Crans's transfer theorem:

Theorem 6.19 (Crans [4]). *Given cocomplete categories \mathcal{C} and \mathcal{D} with finite limits and an adjunction $F \dashv G$ between them such that $F : \mathcal{C} \rightarrow \mathcal{D}$, if \mathcal{C} is a λ -cofibrantly generated model category for λ and infinite regular cardinal, if*

- (1) *F preserves λ -smallness; and*
- (2) *if an arrow $f : a \rightarrow b$ of \mathcal{D} is in the saturated class generated by maps of the form $F(g)$ for g a trivial cofibration in \mathcal{C} , then $G(g)$ is a weak equivalence;*

then there is a model structure on \mathcal{D} obtained by defining a map f to be a weak equivalence or fibration if $G(f)$ is a weak equivalence or fibration in \mathcal{C} , respectively, and to be a cofibration if it has the left lifting property with respect to maps which are simultaneously fibrations and weak equivalences.

We will apply this theorem to the adjunction

$$\begin{array}{ccc} & \Psi & \\ \text{MLCx}_1 & \begin{array}{c} \longleftarrow \\ \perp \\ \longrightarrow \end{array} & \mathbf{Gpd} \\ & \Phi & \end{array}$$

described in Section 6.4 above. First we will prove that the components of the unit of this adjunction are all weak equivalences in \mathbf{Gpd} . This fact will be used to establish the model structure and also to show that the model structure on \mathbf{Th}_1 is Quillen equivalent to the model structure on \mathbf{Gpd} .

Lemma 6.20. *If G is a groupoid, then the unit $\eta_G : G \rightarrow \Phi \circ \Psi(G)$ is a categorical equivalence.*

Proof. By definition, $\Phi \circ \Psi(G)$ is the groupoid obtained from the theory $\mathbb{T}_1[G|\circ]$ as described above. Because $\mathbb{T}_1[G|\circ]$ is obtained as an equational extension of $\mathbb{T}_1[G]$, in the sense of Section 2.6, it follows that Proposition 5.13 applies to this theory. Thus we may argue as in the proof of Theorem 5.14 to see that η_G is full and essentially surjective on objects. Indeed, η_G is also faithful for roughly the same reason. Explicitly, there is a model of the theory $\mathbb{T}_1[G|\circ]$ obtained from the groupoid model of Hofmann and Streicher [8], as described in Section 5.2, by interpreting the type $\ulcorner G \urcorner$ as the groupoid G . It is then easily seen that this interpretation satisfies the additional axioms of $\mathbb{T}_1[G|\circ]$. Accordingly, there exists a retraction $\Phi \circ \Psi(G) \rightarrow G$ in \mathbf{Gpd} obtained sending a term τ of $\mathbb{T}_1[G|\circ]$ to its interpretation in this model. Thus, the unit is faithful. \square

Lemma 6.21. *All arrows of \mathbf{Th}_1 in the saturated class generated by those homomorphisms of the form $\Psi(g)$ for $g : G \rightarrow H$ a trivial cofibration in \mathbf{Gpd} are weak equivalences as defined above.*

Proof. Assume given a morphism $f : \mathbb{T}' \rightarrow \mathbb{T}''$ of theories in the saturated class generated by those $\Psi(g)$ with g a trivial cofibration of groupoids. We prove by induction on the structure of f as a member of the saturated class that it is a weak equivalence. First, observe that if $g : G \rightarrow H$ is a trivial cofibration, then it necessarily has a retraction $g' : H \rightarrow G$ which is its quasi-inverse. The property of having a retraction is preserved by Ψ and is stable under all of the generating operations for the formation of the saturated class. As such, we may always assume that maps in this class are split monomorphisms. In particular, if $f : \mathbb{T}' \rightarrow \mathbb{T}''$ is in the saturated class, then $\Phi(f)$ is necessarily faithful.

In the base case, where $f = \Psi(g)$ for $g : G \rightarrow H$ a trivial cofibration in \mathbf{Gpd} , because

$$\begin{array}{ccc} G & \xrightarrow{\eta_G} & \Psi \circ \Phi(G) \\ g \downarrow & & \downarrow \Psi \circ \Phi(g) \\ H & \xrightarrow{\eta_H} & \Psi \circ \Phi(G) \end{array}$$

commutes and, by Lemma 6.20, the two units are also weak equivalences, it follows from “three-for-two” that $\Psi \circ \Phi(g)$ is also a weak equivalence in \mathbf{Gpd} .

Next assume f is of the form $\prod_i f_i : \prod_i \mathbb{T}'_i \rightarrow \prod_i \mathbb{T}''_i$ with each f_i a weak equivalence. To see that $\Psi(\prod_i f_i)$ is essentially surjective on objects note that, by Proposition 5.13, any vertex τ of $\Psi(\prod_i \mathbb{T}''_i)$ is propositionally equal to a basic term $\ulcorner b \urcorner$ with b in some \mathbb{T}''_i . Since f_i is essentially surjective on objects it follows that there exists some $\ulcorner a \urcorner$ in \mathbb{T}'_i with $\ulcorner b \urcorner \simeq \ulcorner f_i(a) \urcorner$. Thus, τ is isomorphic in $\Psi(\prod_i \mathbb{T}''_i)$ to $\ulcorner f_i(a) \urcorner$ which is in the image of $\Psi(\prod_i f_i)$, as required. To see that $\Psi(\prod_i f_i)$ is full note that it suffices, in virtue of Proposition 5.13, to prove that it is full on basic terms. I.e., if $\tau : \ulcorner f_i(a) \urcorner \rightarrow \ulcorner f_j(b) \urcorner$ in $\Psi(\prod_i \mathbb{T}''_i)$, then there exists some $\hat{\tau} : \ulcorner a \urcorner \rightarrow \ulcorner b \urcorner$ in $\prod_i \mathbb{T}'_i$ such that $\prod_i f_i(\hat{\tau}) = \tau$. Given such a term τ it follows from Proposition 5.13 that τ is a formal composite and we may prove the claim by induction on the structure of τ as a formal composite. If τ is itself a basic term $\ulcorner g \urcorner$, then it follows from Lemma 6.12 that $i = j$ and there exists a canonical summand \mathbb{T}''_i such that $\ulcorner g \urcorner : \ulcorner f_i(a) \urcorner \rightarrow \ulcorner f_i(b) \urcorner$ in \mathbb{T}''_i . Since f_i is full it follows that $\ulcorner g \urcorner = \hat{g}$ for $\hat{g} : \ulcorner a \urcorner \rightarrow \ulcorner b \urcorner$ a term of \mathbb{T}'_i with $f_i(\hat{g}) = \ulcorner g \urcorner$. Thus, $\prod_i f_i(\hat{g}) = \ulcorner g \urcorner$ as required. Next, if τ is a composite $(v' \cdot v)$ for $v : \ulcorner f_i(a) \urcorner \rightarrow \ulcorner c \urcorner$ and $v' : \ulcorner c \urcorner \rightarrow \ulcorner f_i(b) \urcorner$, then it follows from the fact that f_i is essentially surjective on objects that there exists some $\ulcorner d \urcorner$ and a term $\phi : \ulcorner c \urcorner \rightarrow \ulcorner f_i(d) \urcorner$. Thus, since f_i is full there exist lifts $\hat{v} : \ulcorner a \urcorner \rightarrow \ulcorner d \urcorner$ and $\hat{v}' : \ulcorner d \urcorner \rightarrow \ulcorner b \urcorner$ of $\phi \cdot v$ and $v' \cdot \phi^{-1}$, respectively. Thus, since f_i is a map of theories and therefore preserves composition and inverses, $\hat{v}' \cdot \hat{v}$ is the required lift of τ . Finally, the case where τ is an inverse is by a similar argument.

Next, let theories $\mathbb{T}' = \mathbb{T}_1[G|E]$, $\mathbb{S} = \mathbb{T}_1[H|F]$ and $\mathbb{S}' = \mathbb{T}_1[K|F']$ and suppose $f : \mathbb{T}' \rightarrow \mathbb{T}''$ is obtained as the pushout of a map $f' : \mathbb{S} \rightarrow \mathbb{S}'$ with the property in question as indicated in the following diagram:

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{g'} & \mathbb{T}' \\ f' \downarrow & & \downarrow f \\ \mathbb{S}' & \xrightarrow{g} & \mathbb{T}'' \end{array}$$

Then \mathbb{T}'' is the theory $\mathbb{T}_1[K+G|F', E, f' = g']$ where the theory is defined by adding the evident axioms to $\mathbb{T}_1[K+G]$. Given a basic term $\ulcorner c \urcorner$ in \mathbb{T}'' , either $\ulcorner c \urcorner$ is a basic term of either \mathbb{S}' or \mathbb{T}' . If it is in \mathbb{T}' , then we are done. On the other hand, if $\ulcorner c \urcorner$ is in \mathbb{S}' , then it follows from the induction hypothesis that there is a propositional equality $\varphi : f'(\tau) \rightarrow \ulcorner c \urcorner$ in \mathbb{S}' , for τ a term of \mathbb{S} . Thus, $g(\varphi) : f \circ g'(\tau) \rightarrow \ulcorner c \urcorner$ in \mathbb{T}'' and $\Psi(f)$ is essentially surjective on objects since, by Proposition 5.13 every object of $\Psi(\mathbb{T}'')$ is isomorphic to a basic term. To see that $\Psi(f)$ is full it again suffices to prove this for basic terms. I.e., assume given a term $\tau : \ulcorner a \urcorner \rightarrow \ulcorner b \urcorner$ in \mathbb{T}'' with $\ulcorner a \urcorner$ and $\ulcorner b \urcorner$ terms from \mathbb{T}' (recall that in this situation f just sends a term of \mathbb{T}' to itself

in \mathbb{T}''). By Proposition 5.13, τ is a formal composite and we prove the claim by induction on its structure as such. First, when τ is a basic term $\ulcorner h \urcorner : \ulcorner a \urcorner \rightarrow \ulcorner b \urcorner$. In this case we have by definition that either $\ulcorner h \urcorner$ is a term of \mathbb{S}' or of \mathbb{T}' , although it may have a different type in either theory. Assume $\ulcorner h \urcorner : \ulcorner c \urcorner \rightarrow \ulcorner d \urcorner$ in \mathbb{S}' with $\ulcorner c \urcorner = \ulcorner a \urcorner$ and $\ulcorner d \urcorner = \ulcorner b \urcorner$ in \mathbb{T}'' . Using the interpretation of \mathbb{T}' from Lemma 6.16 it follows that $a \equiv c$ for the corresponding equivalence relation. Thus, since $\ulcorner a \urcorner$ is a basic term of \mathbb{T}' and $\ulcorner c \urcorner$ is a basic term of \mathbb{S}' , there exist terms $\ulcorner e_1 \urcorner, \ulcorner e_2 \urcorner, \dots, \ulcorner e_n \urcorner$ such that

$$\begin{aligned} \ulcorner c \urcorner &= f'(\ulcorner e_1 \urcorner) \\ g'(\ulcorner e_1 \urcorner) &= g'(\ulcorner e_2 \urcorner) \\ f'(\ulcorner e_2 \urcorner) &= f'(\ulcorner e_3 \urcorner) \\ &\vdots \\ f'(\ulcorner e_{n-1} \urcorner) &= f'(\ulcorner e_n \urcorner) \\ g'(\ulcorner e_n \urcorner) &= \ulcorner a \urcorner \end{aligned}$$

Similarly, there exist terms $\ulcorner e'_1 \urcorner, \dots, \ulcorner e'_m \urcorner$ satisfying the analogous conditions and in particular such that $f'(\ulcorner e'_1 \urcorner) = \ulcorner d \urcorner$ and $g'(\ulcorner e'_m \urcorner) = \ulcorner b \urcorner$. Thus, by induction hypothesis, there exists a term $\hat{h} : \ulcorner e_1 \urcorner \rightarrow \ulcorner e'_1 \urcorner$ in \mathbb{S} such that $f'(\hat{h}) = \ulcorner h \urcorner$. Thus, $g'(\hat{h}) : \ulcorner a \urcorner \rightarrow \ulcorner b \urcorner$ is a term of \mathbb{T}' and $g'(\hat{h}) = \ulcorner h \urcorner$ in \mathbb{T}'' .

On the other hand, if $\ulcorner h \urcorner : \ulcorner c \urcorner \rightarrow \ulcorner d \urcorner$ is in \mathbb{T}' , then there exist terms $\ulcorner e_1 \urcorner, \dots, \ulcorner e_n \urcorner$ and $\ulcorner e'_1 \urcorner, \dots, \ulcorner e'_m \urcorner$ such that

$$\begin{aligned} \ulcorner c \urcorner &= g'(\ulcorner e_1 \urcorner) \\ f'(\ulcorner e_1 \urcorner) &= f'(\ulcorner e_2 \urcorner) \\ g'(\ulcorner e_2 \urcorner) &= g'(\ulcorner e_3 \urcorner) \\ &\vdots \\ f'(\ulcorner e_{n-1} \urcorner) &= f'(\ulcorner e_n \urcorner) \\ g'(\ulcorner e_n \urcorner) &= \ulcorner a \urcorner \end{aligned}$$

and analogously for the terms $\ulcorner e'_i \urcorner$ so that, in particular, $g'(\ulcorner e'_1 \urcorner) = \ulcorner d \urcorner$ and $g'(\ulcorner e'_m \urcorner) = \ulcorner b \urcorner$. Since f' is full there exist terms $\phi_i : e_{i+1} \rightarrow e_i$ for $1 \leq i \leq n-1$ such that $f'(\phi_i) = \mathbf{r}(f'(\ulcorner e_i \urcorner))$. Similarly, we have $\phi'_i : e'_i \rightarrow e'_{i+1}$ with $f'(\phi'_i) = \mathbf{r}(f'(\ulcorner e'_i \urcorner))$. As such, in \mathbb{T}' we have the term

$$g'(\phi'_{n-1}) \circ \dots \circ g'(\phi'_1) \circ \ulcorner h \urcorner \circ g'(\phi_1) \circ \dots \circ g'(\phi_{n-1}) : \ulcorner a \urcorner \rightarrow \ulcorner b \urcorner$$

which is then easily seen to be equal to $\ulcorner h \urcorner$ in \mathbb{T}'' , as required. As such, we have completed the proof of the case where τ is a basic term. The cases where τ is a composite or symmetry term are straightforward generalizations of this case. Thus, we have proved that $\coprod_i f_i$ is a weak equivalence.

Finally, the case where f is a transfinite composition (which we may take to be, in this case, simply a countable colimit) $f : A_0 \rightarrow \varinjlim_i A_i$ is straightforward using the induction hypothesis and the fact that such colimits are filtered and, by Lemma 6.3 are created by the forgetful functor $\mathbf{MLC}\mathbf{x}_1 \rightarrow \mathbf{rGSet}$. \square

With Lemma 6.21 at our disposal we are now in a position to establish our main theorem regarding the category \mathbf{Th}_1 .

Theorem 6.22. *There is a model structure on \mathbf{Th}_1 in which the weak equivalences, fibrations and cofibrations are as defined above.*

Proof. By Lemma 6.21 the second condition of Crans's Transfer Theorem (stated as 6.19 above) is satisfied. The first condition is by the fact that the diagram

$$\begin{array}{ccc} \mathbf{Th}_1 & \xrightarrow{\Phi} & \mathbf{Gpd} \\ & \searrow & \swarrow \\ & \mathbf{rGSet} & \end{array}$$

commutes and the forgetful functors (those with codomain \mathbf{rGSet} in the diagram) create filtered colimits. \square

Corollary 6.23. *The adjunction $\Psi \dashv \Phi$ between \mathbf{Gpd} and \mathbf{Th}_1 is a Quillen equivalence with respect to the standard model structure on \mathbf{Gpd} and the model structure on \mathbf{Th}_1 from Theorem 6.22.*

Proof. By Lemma 6.20 it suffices to prove that if \mathbb{T}' is an object of \mathbf{Th}_1 , then the counit $\varepsilon_{\mathbb{T}'} : \Psi \circ \Phi(\mathbb{T}') \rightarrow \mathbb{T}'$ is a weak equivalence. However, in general given categories \mathcal{C} and \mathcal{D} satisfying the conditions of Theorem 6.19, if the components of the unit of the adjunction are weak equivalences, then it follows from the three-for-two property in \mathcal{C} and the triangle law for adjunctions that the components of the counit are also weak equivalences. \square

Corollary 6.24. *The 1-truncated Martin-Löf complexes are a model of homotopy 1-types.*

Proof. By Corollary 6.23 and the corresponding fact for groupoids we have the following equivalences of categories:

$$\mathbf{Ho}(\mathbf{1-Type}) \simeq \mathbf{Ho}(\mathbf{Gpd}) \simeq \mathbf{Ho}(\mathbf{Th}_1) \simeq \mathbf{Ho}(\mathbf{MLCx}_1).$$

\square

APPENDIX A. RULES OF TYPE THEORY

In this appendix we describe the syntax of the system \mathbb{T}_ω . All rules below are stated in an ambient context which is omitted for ease of presentation.

A.1. Structural rules.

$$\frac{\Gamma \vdash \mathcal{J}}{\Delta, \Gamma \vdash \mathcal{J}} \text{ Weakening}$$

where \mathcal{J} ranges over judgements and we assume without loss of generality that the variables declared in Δ and Γ are disjoint.

$$\frac{a : A \quad x : A, \Delta \vdash B(x) : \mathbf{type}}{\Delta[a/x] \vdash B(a) : \mathbf{type}} \text{ Type substitution}$$

$$\frac{a : A \quad x : A, \Delta \vdash b(x) : B(x)}{\Delta[a/x] \vdash b(a) : B(a)} \text{ Term substitution}$$

$$\frac{A : \mathbf{type}}{x : A, \Delta \vdash x : A} \text{ Variable declaration}$$

A.2. Rules governing definitional equality.

$$\frac{A : \mathbf{type}}{A = A : \mathbf{type}} \qquad \frac{A = B : \mathbf{type}}{B = A : \mathbf{type}}$$

$$\frac{A = B : \mathbf{type} \quad B = C : \mathbf{type}}{A = C : \mathbf{type}}$$

$$\frac{a : A}{a = a : A} \qquad \frac{a = b : A}{b = a : A}$$

$$\frac{a = b : A \quad b = c : A}{a = c : A}$$

$$\frac{a = b : A \quad x : A \vdash B(x) : \mathbf{type}}{B(a) = B(b) : \mathbf{type}}$$

$$\frac{a = b : A \quad x : A \vdash f(x) : B(x)}{f(a) = f(b) : B(a)}$$

$$\frac{A = B : \mathbf{type} \quad a : A}{a : B}$$

A.3. Formation rules.

$$\frac{x : A \vdash B(x) : \mathbf{type}}{\prod_{x:A} B(x) : \mathbf{type}} \quad \prod \text{ formation}$$

$$\frac{x : A \vdash B(x) : \mathbf{type}}{\sum_{x:A} B(x) : \mathbf{type}} \quad \sum \text{ formation}$$

$$\frac{a, b : A}{\vdash \underline{A}(a, b) : \mathbf{type}} \quad \text{Id formation}$$

$$\frac{}{\vdash \mathbf{N} : \mathbf{type}} \quad \mathbf{N} \text{ formation}$$

A.4. Introduction and elimination rules for dependent products.

$$\frac{x : A \vdash f(x) : B(x)}{\lambda_{x:A} f(x) : \prod_{x:A} B(x)} \quad \prod \text{ introduction}$$

$$\frac{f : \prod_{x:A} B(x) \quad a : A}{\text{app}(f, a) : B(a).} \quad \prod \text{ elimination}$$

A.5. Introduction and elimination rules for dependent sums.

$$\frac{a : A \quad b : B(a)}{\mathbf{pair}(a, b) : \sum_{x:A} B(x)} \Sigma \text{ introduction}$$

$$\frac{\vdash p : \sum_{x:A} B(x) \quad x : A, y : B(x) \vdash \psi(x, y) : C(\mathbf{pair}(x, y))}{\mathbf{R}([x : A, y : B(x)]\psi(x, y), p) : C(p)} \Sigma \text{ elimination}$$

A.6. Introduction and elimination rules for identity types.

$$\frac{a : A}{r(a) : \underline{A}(a, a)} \text{Id introduction}$$

$$\frac{\begin{array}{l} x : A, y : A, z : \underline{A}(x, y) \vdash B(x, y, z) : \text{type} \\ x : A \vdash \varphi(x) : B(x, x, r(x)) \\ f : \underline{A}(a, b) \end{array}}{\mathbf{J}_{[x,y:A,z:\underline{A}(x,y)]B(x,y,z)}([x : A]\varphi(x), a, b, f) : B(a, b, f)} \text{Id elimination}$$

A.7. Introduction and elimination rules for natural numbers.

$$\frac{}{0 : \mathbf{N}} \mathbf{N} \text{ introduction (i)}$$

$$\frac{n : \mathbf{N}}{\mathbf{S}(n) : \mathbf{N}} \mathbf{N} \text{ introduction (ii)}$$

$$\frac{n : \mathbf{N} \quad c : C(0) \quad x : \mathbf{N}, y : C(x) \vdash \gamma(x, y) : C(\mathbf{S}(x))}{\mathbf{rec}(n, c, [x : \mathbf{N}, y : C(x)]\gamma(x, y)) : C(n)} \mathbf{N} \text{ elimination}$$

A.8. Conversion rules.

$$\frac{\lambda_{x:A} f(x) : \prod_{x:A} B(x) \quad a : A}{\mathbf{app}(\lambda_{x:A} f(x), a) = f(a) : B(a)} \prod \text{ conversion}$$

$$\frac{a : A \quad b : B(a) \quad x : A, y : B(x) \vdash \psi(x, y) : C(\mathbf{pair}(x, y))}{\mathbf{R}([x : A, y : B(x)]\psi(x, y), \mathbf{pair}(a, b)) = \psi(a, b) : C(\mathbf{pair}(a, b))} \Sigma \text{ conversion}$$

$$\frac{a : A}{\mathbf{J}_{[x,y:A,z:\underline{A}(x,y)]B(x,y,z)}([x : A]\varphi(x), a, a, r(a)) = \varphi(a) : B(a, a, r(a))} \text{Id conversion}$$

$$\frac{}{\mathbf{rec}(0, c, [x : \mathbf{N}, y : C(x)]\gamma(x, y)) = c : C(0)} \mathbf{N} \text{ conversion (i)}$$

$$\frac{n : \mathbf{N}}{\mathbf{rec}(\mathbf{S}(n), c, [x : \mathbf{N}, y : C(x)]\gamma(x, y)) = \gamma(n, \mathbf{rec}(c, [x : \mathbf{N}, y : C(x)]\gamma(x, y), n)) : C(\mathbf{S}(n))} \mathbf{N} \text{ conversion (ii)}$$

APPENDIX B. ALL TERMS OF TYPE N ARE PROPOSITIONALLY EQUAL TO
NUMERALS

In order to show that all closed terms of type N are propositionally equal to numerals we will employ, as in the results on relevant types and terms in Section 5, a logical relations style argument. Recall that a term τ of type N is a **numeral** if it is either 0 or is of the form $\mathbf{S}^n(0)$ for a natural number n .

Definition B.1. A type T is N-relevant if it is of one of the following forms:

- T is N.
- T is of the form $\Pi_{x:A}.S(x)$ and $x : A \vdash S(x)$ is N-relevant.
- T is of the form $\Sigma_{x:A}.S(x)$ and either A or $x : A \vdash S(x)$ is relevant.

A term τ of N-relevant type T in the empty context is **N-relevant** if

- T is N and τ is propositionally equal to a numeral.
- T is of the form $\Pi_{x:A}.S(x)$ and τ is propositionally equal to a term τ' such that $\mathbf{app}(\tau', a)$ is N-quasirelevant, for all N-quasirelevant terms $a : A$, where N-quasirelevant is defined by analogy with the definition of quasirelevant terms in Definition 5.5.
- T is of the form $\Pi_{x:A}.S(x)$ and τ is propositionally equal to a term τ' such that $\pi_0(\tau')$ and $\pi_1(\tau')$ are N-quasirelevant.

A term $\tau(x_1, \dots, x_n)$ of N-relevant type $T(x_1, \dots, x_n)$ in the non-empty context $(x_1 : A_1, \dots, x_n : A_n(x_1, \dots, x_{n-1}))$ is N-relevant if, for any sequence of closed N-quasirelevant terms $a_1 : A_1, \dots, a_n : A_n(a_1, \dots, a_{n-1})$, $\tau(a_1, \dots, a_n)$ is N-quasirelevant.

It is an immediate consequence of Definition B.1 that the N-relevant terms possess the same general closure properties as the relevant terms from Section !!! . In order to show that all terms of N-relevant type are N-relevant it is necessary to show first that the N-relevant terms are closed under the formation of elimination terms for identity types and the proof of this fact is analogous to that given for the relevant terms.

Proposition B.2. *If Δ is a context*

$$\Delta = (x_0, x_1 : A, z : A(x_0, x_1), v_1 : B_1(x_0, x_1, z), \dots, v_n : B_n(x_0, x_1, z, v_1, \dots, v_{n-1})),$$

then all of the terms

$$\begin{aligned} \Delta &\vdash z \triangleleft v_n, \\ \Delta_\sigma &\vdash z \triangleright w_n, \\ x_0, x_1 : A, z : A(x_0, x_1) &\vdash \kappa_n(z), \text{ and} \\ x_0, x_1 : A, z : A(x_0, x_1) &\vdash \nu_n(z) \end{aligned}$$

are N-quasirelevant, for $i = 0, 1$.

Proof. Let N-quasirelevant terms $a_0, a_1 : A, f : A(a_0, a_1), b_1 : B_1(a_0, a_1, f), \dots, b_n : B_n(a_0, \dots, b_{n-1})$ and $d_1 : B_1(a_i, a_i, r(a_i)), \dots, d_n : B_n(a_i, \dots, d_{n-1})$ be given to show that the terms $f \triangleleft b_n, f \triangleright d_n, \kappa_n(f)$ and $\nu_n(f)$ are N-quasirelevant. The proof is by induction on the structure of $B_n(x_0, x_1, z, v_1, \dots, v_{n-1})$ as a N-relevant type. (When it is not N-relevant, the claim is trivial.)

First, when $B_n(x_0, \dots, v_{n-1})$ is just N itself, we have that $f \triangleleft b_n \simeq b_n$ and $f \triangleright d_n \simeq d_n$. Thus, these terms are N-quasirelevant. Likewise, $\kappa_n(f) \simeq$

$\lambda_{v_1} \cdots \lambda_{v_{n-1}} \cdot \lambda_{y:\mathbb{N}} \cdot y$ and $\nu_n(f) \simeq \lambda_{w_1} \cdots \lambda_{w_{n-1}} \cdot \lambda_{y:\mathbb{N}} \cdot y$ and these terms are therefore N-relevant.

The remaining cases are proved in precisely the same way as in the proof of Proposition 5.11. \square

Proposition B.3. *All terms of N-relevant type are N-relevant.*

Proof. The proof is essentially the same as the proof of Proposition 5.13 using Proposition B.2. \square

Corollary B.4. *Every closed term of type N is propositionally equal to a numeral.*

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DEPARTMENT OF PHILOSOPHY, CARNEGIE MELLON UNIVERSITY, 5000 FORBES AVE., PITTSBURGH, PA 15213 USA

E-mail address: `awodey@cmu.edu`

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF OTTAWA, 585 KING EDWARD AVE., OTTAWA, ON K1N 6N5 CANADA

E-mail address: `phofstra@uottawa.ca`

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF OTTAWA, 585 KING EDWARD AVE., OTTAWA, ON K1N 6N5 CANADA

E-mail address: `mwarren@uottawa.ca`