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Convex Sets and Minimal Sublinear Functions

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Abstract

We show that, given a closed convex set K with the origin in its interior, the support function of the set $\{y \in K^* \mid \exists x \in K \text{ such that } xy = 1\}$ is the pointwise smallest sublinear function σ such that $K = \{x \mid \sigma(x) \leq 1\}$.

1 Introduction

Let K be a closed, convex set with the origin in its interior. A standard concept in convex analysis [1, 2] is that of *gauge* (sometimes called Minkowski function), which is the function γ_K defined by

$$\gamma_K(x) = \inf\{t > 0 \mid t^{-1}x \in K\}, \quad \text{for all } x \in \mathbb{R}^n.$$

By definition γ_K is nonnegative. It is also sublinear, another classical concept that we define next. A function $\sigma : \mathbb{R}^n \to \mathbb{R}$ is *positively homogeneous* if $\sigma(tx) = t\sigma(x)$ for every $x \in \mathbb{R}^n$ and $t > 0$, and it is *sublinear* if it is convex and positively homogeneous. One can readily verify that $K = \{x \mid \gamma_K(x) \leq 1\}$.

Given any sublinear function σ such that $K = \{x \mid \sigma(x) \leq 1\}$, it follows from positive homogeneity that $\sigma(x) = \gamma_K(x)$ for every x where $\sigma(x) > 0$. Hence $\sigma(x) \leq \gamma_K(x)$ for all $x \in \mathbb{R}^n$. In this paper we introduce a sublinear function ρ_K such that $K = \{x \mid \rho_K(x) \leq 1\}$ and $\rho_K(x) \leq \sigma(x)$ for all $x \in \mathbb{R}^n$.

The *polar* of K is the set $K^* = \{y \in \mathbb{R}^n \mid xy \leq 1 \text{ for all } x \in K\}$. Clearly K^* is closed and convex, and since $0 \in \mathbf{int}(K)$, it is well known that K^* is bounded. In particular, K^* is a compact set. Also, since $0 \in K$, $K^{**} = K$.

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Given any $T \subset \mathbb{R}^n$, the *support function of* T is defined by

$$\sigma_T(x) = \sup_{y \in T} xy, \quad \text{for all } x \in \mathbb{R}^n.$$

It is straightforward to show that support functions are sublinear [1]. It is well known that γ_K is the support function of K^* (see [1] Proposition 3.2.4).

We define our function ρ_K as the support function of the set

$$\hat{K} = \{y \in K^* \mid \exists x \in K \text{ such that } xy = 1\}.$$

Note that $\hat{K}$ is contained in the relative boundary of K^*. By definition

$$\rho_K(x) = \sup_{y \in \hat{K}} xy, \quad \text{for all } x \in \mathbb{R}^n.$$

Note that ρ_K is sublinear. Furthermore we will show that $K = \{x \mid \rho_K(x) \leq 1\}$. The next theorem shows that ρ_K is the smallest function with these two properties.

Theorem 1 *Let $K \subset \mathbb{R}^n$ be a closed convex set containing the origin in its interior. For every sublinear function σ such that $K = \{x \mid \sigma(x) \leq 1\}$, we have $\rho_K(x) \leq \sigma(x)$ for every $x \in \mathbb{R}^n$.*

Note that the recession cone of K, which is the set $\text{rec}(K) = \{x \in K \mid tx \in K \text{ for all } t > 0\}$, coincides with $\{x \in K \mid \sigma(x) \leq 0\}$ for every sublinear function σ such that $K = \{x \mid \sigma(x) \leq 1\}$. In particular $\rho_K(x)$ can be negative for $x \in \text{rec}(K)$, so in general it is different from the gauge.

For example, let $K = \{x \in \mathbb{R}^2 \mid x_1 \leq 1, x_2 \leq 1\}$. Then $K^* = \text{conv}\{(0,0), (1,0), (0,1)\}$ and $\hat{K} = \text{conv}\{(1,0), (0,1)\}$. Therefore, for every $x \in \mathbb{R}^2$, $\gamma_K(x) = \max\{0, x_1, x_2\}$ and $\rho_K(x) = \max\{x_1, x_2\}$. In particular, $\rho_K(x) < 0$ for every x such that $x_1 < 0$, $x_2 < 0$.

2 Proof of Theorem 1

We will need Straszewicz's theorem [3] (see [2] Theorem 18.6). Given a closed convex set C, a point $x \in C$ is *extreme* if it cannot be written as a proper convex combination of two distinct points in C. A point $x \in C$ is *exposed* if there exists a supporting hyperplane H for C such that $H \cap C = \{x\}$. Clearly exposed points are extreme. We will denote by $\text{ext}(C)$ the set of extreme points and $\exp(C)$ the set of exposed points of C.

Theorem 2 *Given a closed convex set C, the set of exposed points of C is a dense subset of the set of extreme points of C.*

Let K be a closed convex set with the origin in its interior. Let σ be a sublinear function such that $K = \{x \,|\, \sigma(x) \le 1\}$. The boundary of K, denoted by $\mathbf{bd}(K)$, is the set $\{x \in K \,|\, \sigma(x) = 1\}$.

Lemma 3 *For every $x \notin \mathrm{rec}(K)$, $\sigma(x) = \rho_K(x) = \sup_{y \in K^*} xy$. In particular, $K = \{x \,|\, \rho_K(x) \le 1\}$.*

Proof. Let $x \notin \mathrm{rec}(K)$. Then $t = \sigma(x) > 0$. By positive homogeneity, $\sigma(t^{-1}x) = 1$, hence $t^{-1}x \in \mathbf{bd}(K)$. Since K is closed and convex, there exists a supporting hyperplane for K containing $t^{-1}x$. Since $0 \in \mathbf{int}(K)$, this implies that there exists $\bar{y} \in K^*$ such that $(t^{-1}x)\bar{y} = 1$. In particular $\bar{y} \in \hat{K}$, hence by definition $\rho_K(x) \ge x\bar{y} = t$.

Furthermore, for any $y \in K^*$, $(t^{-1}x)y \le 1$, hence $xy \le t$, which implies $t \ge \sup_{y \in K^*} xy$. Thus

$$\rho_K(x) \ge t \ge \sup_{y \in K^*} xy \ge \sup_{y \in \hat{K}} xy = \rho_K(x),$$

where the last inequality holds since $\hat{K} \subset K^*$, hence equality holds throughout. □

Lemma 4 *Given an exposed point $\bar{y}$ of K^* different from the origin, there exists $x \in K$ such that $x\bar{y} = 1$ and $xy < 1$ for all $y \in K^*$ distinct from $\bar{y}$.*

Proof. If $\bar{y} \neq 0$ is an exposed point of K^*, then there exists a supporting hyperplane $H = \{y \,|\, ay = \beta\}$ such that $a\bar{y} = \beta$ and $ay < \beta$ for every $y \in K^* \setminus \{\bar{y}\}$. Since $0 \in K^*$ and $\bar{y} \neq 0$, $\beta > 0$. Thus the point $x = \beta^{-1}a \in K^{**} = K$ satisfies the statement. □

Lemma 5 *For every $x \in \mathbb{R}^n$, $\rho_K(x) = \sup_{y \in \hat{K} \cap \exp(K^*)} xy$.*

Proof. We first show that $\rho_K(x) = \sup_{y \in \hat{K} \cap \mathrm{ext}(K^*)} xy$. Given $y \in \hat{K}$ we show that there exists an extreme point y' of K^* in $\hat{K}$ such that $xy \le xy'$. Since $y \in \hat{K}$, there exists $\bar{x} \in K$ such that $\bar{x}y = 1$. The point y is a convex combination of extreme points $y_1, \ldots, y_k$ of K^*, and each y_i satisfies $\bar{x}y^i = 1$. Thus $y^1, \ldots, y^k \in \hat{K}$, and $xy^i \ge xy$ for at least one i.

By Straszewicz's theorem (Theorem 2) the set of exposed points in K^* is a dense subset of the extreme points of K^*. By Lemma 4, all exposed points of K^* except the origin are in $\hat{K}$, hence $\exp(K^*) \cap \hat{K}$ is dense in $\mathrm{ext}(K^*) \cap \hat{K}$. Therefore $\rho_K(x) = \sup_{y \in \hat{K} \cap \exp(K^*)} xy$. □

A function σ is *subadditive* if $\sigma(x_1 + x_2) \le \sigma(x_1) + \sigma(x_2)$ for every $x_1, x_2 \in \mathbb{R}^n$. It is easy to show that σ is sublinear if and only if it is subadditive and positively homogeneous.

Proof of Theorem 1. By Lemma 3, we only need to show $\sigma(x) \ge \rho_K(x)$ for points $x \in \mathrm{rec}(K)$. By Lemma 5 it is sufficient to show that, for every exposed point $\bar{y}$ of K^* contained in $\hat{K}$, $\sigma(x) \ge x\bar{y}$.

Let $\bar{y}$ be an exposed point of K^* in $\hat{K}$. By Lemma 4 there exists $\bar{x} \in K$ such that $\bar{x}\bar{y} = 1$ and $\bar{x}y < 1$ for all $y \in K^*$ distinct from $\bar{y}$. Note that $\bar{x} \in \mathbf{bd}(K)$.

We observe that for all $\delta > 0$, $\bar{x} - \delta^{-1}x \notin \text{rec}(K)$. Indeed, since $x \in \text{rec}(K)$, $\bar{x} + \delta^{-1}x \in K$. Hence $\bar{x} - \delta^{-1}x \notin \mathbf{int}(K)$ because $\bar{x} \in \mathbf{bd}(K)$. Since $0 \in \mathbf{int}(K)$ and $\bar{x} - \delta^{-1}x \notin \mathbf{int}(K)$, then $\bar{x} - \delta^{-1}x \notin \text{rec}(K)$. Thus by Lemma 3

$$\sigma(\bar{x} - \delta^{-1}x) = \sup_{y \in K^*} (\bar{x} - \delta^{-1}x)y. \tag{1}$$

Since $\bar{x} \in \mathbf{bd}(K)$, $\sigma(\bar{x}) = 1$. By subadditivity, $1 = \sigma(\bar{x}) \leq \sigma(\bar{x} - \delta^{-1}x) + \sigma(\delta^{-1}x)$. By positive homogeneity, the latter implies that $\sigma(x) \geq \delta - \delta\sigma(\bar{x} - \delta^{-1}x)$ for all $\delta > 0$. By (1),

$$\sigma(x) \geq \inf_{y \in K^*} [\delta(1 - \bar{x}y) + xy],$$

hence

$$\sigma(x) \geq \sup_{\delta > 0} \inf_{y \in K^*} [\delta(1 - \bar{x}y) + xy].$$

Let $g(\delta) = \inf_{y \in K^*} \delta(1 - \bar{x}y) + xy$. Since $\bar{x} \in K$, $1 - \bar{x}y \geq 0$ for every $y \in K^*$. Hence $\delta(1 - \bar{x}y) + xy$ defines an increasing affine function of δ for each $y \in K^*$, therefore $g(\delta)$ is increasing and concave. Thus $\sup_{\delta > 0} g(\delta) = \lim_{\delta \to +\infty} g(\delta)$.

Since K^* is compact, for every $\delta > 0$ there exists $y(\delta) \in K^*$ such that $g(\delta) = \delta(1 - \bar{x}y(\delta)) + xy(\delta)$. Furthermore there exists a sequence $(\delta_i)_{i \in \mathbb{N}}$ such that $\lim_{i \to +\infty} \delta_i = +\infty$ and the sequence $(y_i)_{i \in \mathbb{N}}$ defined by $y_i = y(\delta_i)$ converges, because in a compact set every sequence has a convergent subsequence. Let $y^* = \lim_{i \to +\infty} y_i$.

We conclude the proof by showing that $\sigma(x) \geq xy^*$ and $y^* = \bar{y}$.

$$\begin{aligned}
\sigma(x) \geq \sup_{\delta > 0} g(\delta) &= \lim_{i \to +\infty} g(\delta_i) \\
&= \lim_{i \to +\infty} [\delta_i(1 - \bar{x}y_i) + xy_i] \\
&= \lim_{i \to +\infty} \delta_i(1 - \bar{x}y_i) + xy^* \\
&\geq xy^*
\end{aligned}$$

where the last inequality follows from the fact that $\delta_i(1 - \bar{x}y_i) \geq 0$ for all $i \in \mathbb{N}$. Finally, since $\lim_{i \to +\infty} \delta_i(1 - \bar{x}y_i)$ is bounded and $\lim_{i \to +\infty} \delta_i = +\infty$, it follows that $\lim_{i \to +\infty}(1 - \bar{x}y_i) = 0$, hence $\bar{x}y^* = 1$. By our choice of $\bar{x}$, $\bar{x}y < 1$ for every $y \in K^*$ distinct from $\bar{y}$. Hence $y^* = \bar{y}$. □

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