Minimal inequalities for an infinite relaxation of integer programs

Amitabh Basu
Carnegie Mellon University, abasu1@andrew.cmu.edu

Michele Conforti
Università di Padova, conforti@math.unipd.it

Gérard Cornuéjols *
Carnegie Mellon University and Université d'Aix-Marseille
gc0v@andrew.cmu.edu

Giacomo Zambelli
Università di Padova, giacomo@math.unipd.it

April 17, 2009

Abstract

We show that maximal S-free convex sets are polyhedra when S is the set of integral points in some rational polyhedron of $\mathbb{R}^n$. This result extends a theorem of Lovász characterizing maximal lattice-free convex sets. We then consider a model that arises in integer programming, and show that all irredundant inequalities are obtained from maximal S-free convex sets.

1 Maximal S-free convex sets

Let $S \subseteq \mathbb{Z}^n$ be the set of integral points in some rational polyhedron of $\mathbb{R}^n$. We say that $B \subset \mathbb{R}^n$ is an *S-free convex set* if B is convex and does not contain any point of S in its interior. We say that B is a *maximal S-free convex set* if it is convex, and it is not properly contained in any S-free convex set. Our interest in this notion arose from a recent paper of Dey and Wolsey [5] showing the relevance of maximal S-free convex sets in integer programming. When $S = \mathbb{Z}^n$, an S-free set is called a *lattice-free set*. The following theorem of Lovász characterizes maximal lattice-free convex sets.

Theorem 1. (Lovász [6]) *A set $S \subset \mathbb{R}^n$ is a maximal lattice-free convex set if and only if one of the following holds:*

*Supported by NSF grant CMMI0653419, ONR grant N00014-03-1-0188 and ANR grant BLAN06-1-138894.

(i) *S is a polyhedron of the form $S = P + L$ where P is a polytope, L is a rational linear space, $\dim(S) = \dim(P) + \dim(L) = n$, S does not contain any integral point in its interior and there is an integral point in the relative interior of each facet of S;*

(ii) *S is a hyperplane of $\mathbb{R}^n$ that is not rational.*

Lovász only gives a sketch of the proof. A complete proof can be found in [2]. It follows from the proof of Theorem 1 that every lattice-free convex set is contained in a maximal lattice-free convex set. The next theorem is an extension of Lovász' theorem to maximal S-free convex sets. It was proved by Dey and Wolsey [5] in a special case.

Given a convex set $K \subset \mathbb{R}^n$, we denote by $\mathrm{rec}(K)$ its recession cone and by $\mathrm{lin}(K)$ its lineality space. Given a set $X \subset \mathbb{R}^n$, we denote by $\langle X \rangle$ the linear space generated by X.

Given a k-dimensional linear space V and a subset Λ of V, we say that Λ is a *lattice of* V if there exists a linear bijection $f : \mathbb{R}^k \to V$ such that $\Lambda = f(\mathbb{Z}^k)$.

Theorem 2. *Let S be the set of integral points in some rational polyhedron of $\mathbb{R}^n$ such that $\dim(S) = n$. A set $B \subset \mathbb{R}^n$ is a maximal S-free convex set if and only if one of the following holds:*

(i) *B is a polyhedron such that $B \cap \mathrm{conv}(S)$ has nonempty interior, B does not contain any point of S in its interior and there is a point of S in the relative interior of each of its facets. The recession cone of $B \cap \mathrm{conv}(S)$ is rational and it is contained in the lineality space of B.*

(ii) *B is a half-space of $\mathbb{R}^n$ such that $B \cap \mathrm{conv}(S)$ has empty interior and the boundary of B is a supporting hyperplane of $\mathrm{conv}(S)$.*

(iii) *B is a hyperplane of $\mathbb{R}^n$ such that $\mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$ is not rational.*

We will need the following lemmas. The first one is proved in [2] and is an easy consequence of Dirichlet's theorem.

Lemma 3. *Let $y \in \mathbb{Z}^n$ and $r \in \mathbb{R}^n$. For every $\varepsilon > 0$ and $\bar{\lambda} \geq 0$, there exists an integral point at distance less than ε from the half line $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$.*

Lemma 4. *Let B be an S-free convex set such that $B \cap \mathrm{conv}(S)$ has nonempty interior. For every $r \in \mathrm{rec}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$, $B + \langle r \rangle$ is S-free.*

Proof. Let $C = \mathrm{rec}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$ and $r \in C \setminus \{0\}$. Suppose by contradiction that there exists $y \in S \cap \mathbf{int}(B + \langle r \rangle)$. We show that $y \in \mathbf{int}(B) + \langle r \rangle$. If not, $(y + \langle r \rangle) \cap \mathbf{int}(B) = \emptyset$, which implies that there is a hyperplane H separating the line $y + \langle r \rangle$ and $B + \langle r \rangle$, a contradiction. Thus there exists $\bar{\lambda}$ such that $\bar{y} = y + \bar{\lambda} r \in \mathbf{int}(B)$, i.e. there exists $\varepsilon > 0$ such that B contains the open ball $B_\varepsilon(\bar{y})$ of radius ε centered at $\bar{y}$. Since $r \in C \subseteq \mathrm{rec}(B)$, it follows that $B_\varepsilon(\bar{y}) + \{\lambda r \mid \lambda \geq 0\} \subset B$. Since $y \in \mathbb{Z}^n$, by Lemma 3 there exists $z \in \mathbb{Z}^n$ at distance less than ε from the half line $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$. Thus $z \in B_\varepsilon(\bar{y}) + \{\lambda r \mid \lambda \geq 0\}$, hence $z \in \mathbf{int}(B)$. Note that the half-line $\{y + \lambda r \mid \lambda \geq \bar{\lambda}\}$ is in $\mathrm{conv}(S)$, since $y \in S$ and $r \in \mathrm{rec}(\mathrm{conv}(S))$. Since $\mathrm{conv}(S)$ is a rational polyhedron, for $\varepsilon > 0$ sufficiently small every integral point at distance at most ε from $\mathrm{conv}(S)$ is in $\mathrm{conv}(S)$. Therefore $z \in S$, a contradiction. □

Proof of Theorem 2. The the proof of the "if" part is standard, and it is similar to the proof for the lattice-free case (see [2]). We show the "only if" part. Let B be a maximal S-free convex set. If $\dim(B) < n$, then B is contained in some affine hyperplane K. Since K has empty interior, K is S-free, thus $B = K$ by maximality of B. Next we show that $\text{lin}(B) \cap \text{rec}(\text{conv}(S))$ is not rational. Suppose not. Then the linear subspace $L = \langle \text{lin}(B) \cap \text{rec}(\text{conv}(S)) \rangle$ is rational. Therefore the projection Λ of $\mathbb{Z}^n$ onto $L^\perp$ is a lattice of $L^\perp$ (see, for example, Barvinok [1] p 284 problem 3). The projection S' of S onto $L^\perp$ is a subset of Λ. Let B' be the projection of B onto $L^\perp$. Then $B' \cap \text{conv}(S')$ is the projection of $B \cap \text{conv}(S)$ onto $L^\perp$. By construction, $B' \cap \text{conv}(S')$ is bounded. Fix $\delta > 0$. Since Λ is a lattice and $S' \subseteq \Lambda$, there is a finite number of points at distance less than δ from the bounded set $B' \cap \text{conv}(S')$ in $L^\perp$. It follows that there exists $\varepsilon > 0$ such that every point of S' has distance at least ε from $B' \cap \text{conv}(S')$. Let $B'' = \{v + w \mid v \in B,\, w \in L^\perp, \|w\| \le \varepsilon\}$. The set B'' is S-free by the choice of ε, but B'' strictly contains B, contradicting the maximality of B. Therefore (iii) holds when $\dim(B) < n$. Hence we may assume $\dim(B) = n$. If $B \cap \text{conv}(S)$ has empty interior, then there exists a hyperplane separating B and $\text{conv}(S)$ which is supporting for $\text{conv}(S)$. By maximality of B case (ii) follows.

Therefore we may assume that $B \cap \text{conv}(S)$ has nonempty interior. We show that B satisfies (i).

Claim 1. *There exists a rational polyhedron P such that:*
i) $\text{conv}(S) \subset \textbf{int}(P)$*,*
ii) The set $K = B \cap P$ is lattice-free,
iii) For every facet F of P, $F \cap K$ is a facet of K,
iv) For every facet F of P, $F \cap K$ contains an integral point in its relative interior.

Since $\text{conv}(S)$ is a rational polyhedron, there exist integral A and b such that $\text{conv}(S) = \{x \in \mathbb{R}^n \mid Ax \le b\}$. The set $P' = \{x \in \mathbb{R}^n \mid Ax \le b + \frac{1}{2}\mathbf{1}\}$ satisfies $i)$. The set $B \cap P'$ is lattice-free since B is S-free and P' does not contain any point in $\mathbb{Z}^n \setminus S$, thus P' also satisfies $ii)$. Let $\bar{A}x \le \bar{b}$ be the system containing all inequalities of $Ax \le b + \frac{1}{2}\mathbf{1}$ that define facets of $B \cap P'$. Let $P_0 = \{x \in \mathbb{R}^n \mid \bar{A}x \le \bar{b}\}$. Then P_0 satisfies $i), ii), iii)$.

It will be more convenient to write P_0 as intersection of the half-spaces defining the facets of P_0, $P_0 = \cap_{H \in \mathcal{F}_0} H$. We construct a sequence of rational polyhedra $P_0 \subset P_1 \subset \ldots \subset P_t$ such that P_i satisfies $i), ii), iii)$, $i = 1, \ldots, t$, and such that P_t satisfies $iv)$. Given P_i, we construct P_{i+1} as follows. Let $P_i = \cap_{H \in \mathcal{F}_i} H$, where $\mathcal{F}_i$ is the set of half spaces defining facets of P_i. Let $\bar{H}$ be a half-space in $\mathcal{F}_i$ defining a facet of $B \cap P_i$ that does not contain an integral point in its relative interior; if no such $\bar{H}$ exists, then P_i satisfies $iv)$ and we are done. If $B \cap \bigcap_{H \in \mathcal{F}_i \setminus \{\bar{H}\}} H$ does not contain any integral point in its interior, let $\mathcal{F}_{i+1} = \mathcal{F}_i \setminus \{\bar{H}\}$. Otherwise, since P_i is rational, among all integral points in the interior of $B \cap \bigcap_{H \in \mathcal{F}_i \setminus \{\bar{H}\}} H$ there exists one, say $\bar{x}$, at minimum distance from $\bar{H}$. Let H' be the half-space containing $\bar{H}$ with $\bar{x}$ on its boundary. Let $\mathcal{F}_{i+1} = \mathcal{F}_i \setminus \{\bar{H}\} \cap \{H'\}$. Observe that H' defines a facet of P_{i+1} since $\bar{x}$ is in the interior of $B \cap \bigcap_{H \in \mathcal{F}_{i+1} \setminus \{H'\}} H$ and it is on the boundary of H'. So $i), ii), iii)$ are satisfied and P_{i+1} has fewer facets that violate $iv)$ than P_i. ◇

Let T be a maximal lattice-free convex set containing the set K defined in Claim 1. By Theorem 1, T is a polyhedron with an integral point in the relative interior of each of its facets. Let H be a hyperplane that defines a facet of P. Since $K \cap H$ is a facet of K with an integral point in its relative interior, it follows that H defines a facet of T. This implies that $T \subset P$. Therefore we can write T as

$$T = P \cap \bigcap_{i=1}^{k} H_i, \tag{1}$$

where H_i are halfspaces. Let $\bar{H}_i = \mathbb{R}^n \setminus \mathbf{int}(H_i)$, $i = 1, \dots, k$.

Claim 2. *B is a polyhedron.*

We first show that, for $i = 1, \dots, k$, $\mathbf{int}(B) \cap (\bar{H}_i \cap \operatorname{conv}(S)) = \emptyset$. Consider $y \in \mathbf{int}(B) \cap \bar{H}_i$. Since $y \in \bar{H}_i$ and K is contained in T, $y \notin \mathbf{int}(K)$. Since $K = B \cap P$ and $y \in \mathbf{int}(B) \setminus \mathbf{int}(K)$, it follows that $y \notin \mathbf{int}(P)$. Hence $y \notin \operatorname{conv}(S)$ because $\operatorname{conv}(S) \subseteq \mathbf{int}(P)$.

Thus, for $i = 1, \dots, k$, there exists a hyperplane separating B and $\bar{H}_i \cap \operatorname{conv}(S)$. Hence there exists a halfspace K_i such that $B \subset K_i$ and $\bar{H}_i \cap \operatorname{conv}(S)$ is disjoint from the interior of K_i. We claim that the set $B' = \cap_{i=1}^{k} K_i$ is S-free. Indeed, let $y \in S$. Then y is not interior of T. Since $y \in \operatorname{conv}(S)$ and $\operatorname{conv}(S) \subseteq \mathbf{int}(P)$, y is in the interior of P. Hence, by (1), there exists $i \in \{1, \dots, k\}$ such that y is not in the interior of H_i. Thus $y \in \bar{H}_i \cap \operatorname{conv}(S)$. By construction, y is not in the interior of K_i, hence y is not in the interior of B'. Thus B' is an S-free convex set containing B. Since B is maximal, $B' = B$. $\diamond$

Claim 3. $\operatorname{lin}(K) = \operatorname{rec}(K)$.

Let $r \in \operatorname{rec}(K)$. We show $-r \in \operatorname{rec}(K)$. By Lemma 4 applied to $\mathbb{Z}^n$, $K + \langle r \rangle$ is lattice-free. We observe that $B + \langle r \rangle$ is S-free. If not, let $y \in S \cap \mathbf{int}(B + \langle r \rangle)$. Since $S \subseteq \mathbf{int}(P)$, $y \in \mathbf{int}(P + \langle r \rangle)$, hence $y \in \mathbf{int}(K + \langle r \rangle)$, a contradiction. Hence, by maximality of B, $B = B + \langle r \rangle$. Thus $-r \in \operatorname{rec}(B)$. Suppose that $-r \notin \operatorname{rec}(P)$. Then there exists a facet F of P that is not parallel to r. By construction, $F \cap K$ is a facet of K containing an integral point $\bar{x}$ in its relative interior. The point $\bar{x}$ is then in the interior of $K + \langle r \rangle$, a contradiction. $\diamond$

Claim 4. $\operatorname{lin}(K)$ *is rational.*

Consider the maximal lattice-free convex set T containing K considered earlier. By Theorem 1, $\operatorname{lin}(T) = \operatorname{rec}(T)$, and $\operatorname{lin}(T)$ is rational. Clearly $\operatorname{lin}(T) \supseteq \operatorname{lin}(K)$. Hence, if the claim does not hold, there exists a rational vector $r \in \operatorname{lin}(T) \setminus \operatorname{lin}(K)$. By (1), $r \in \operatorname{lin}(P)$. Since $K = B \cap P$, $r \notin \operatorname{lin}(B)$. Hence $B \subset B + \langle r \rangle$. We will show that $B + \langle r \rangle$ is S-free, contradicting the maximality of B. Suppose there exists $y \in S \cap \mathbf{int}(B + \langle r \rangle)$. Since $\operatorname{conv}(S) \subseteq \mathbf{int}(P)$, $y \in \mathbf{int}(P) \subseteq \mathbf{int}(P) + \langle r \rangle$. Therefore $y \in \mathbf{int}(B \cap P) + \langle r \rangle$. Since $B \cap P \subseteq T$, then $y \in \mathbf{int}(T) + \langle r \rangle = \mathbf{int}(T)$ where the last equality follows from $r \in \operatorname{lin}(T)$. This contradicts the fact that T is lattice-free. $\diamond$

By Lemma 4 and by the maximality of B, $\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S)) = \operatorname{rec}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$.

Claim 5. $\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S))$ *is rational.*

Since $\operatorname{lin}(K)$ and $\operatorname{rec}(\operatorname{conv}(S))$ are both rational, we only need to show $\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S)) = \operatorname{lin}(K) \cap \operatorname{rec}(\operatorname{conv}(S))$. The "$\supseteq$" direction follows from $B \supseteq K$. For the other direction, note that, since $\operatorname{conv}(S) \subseteq P$, we have $\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S)) \subseteq \operatorname{lin}(B) \cap \operatorname{rec}(P) = \operatorname{lin}(B \cap P) = \operatorname{lin}(K)$, hence $\operatorname{lin}(B) \cap \operatorname{rec}(\operatorname{conv}(S)) \subseteq \operatorname{lin}(K) \cap \operatorname{rec}(\operatorname{conv}(S))$. $\diamond$

Claim 6. *Every facet of B contains a point of S in its relative interior.*

Let L be the linear space generated by $\mathrm{lin}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$. By Claim 5, L is rational. Let B', S', Λ be the projections of B, S, $\mathbb{Z}^n$, respectively, onto $L^\perp$. Since L is rational, Λ is a lattice of $L^\perp$ and $S' = \mathrm{conv}(S') \cap \Lambda$. Also, B' is a maximal S'-free convex set of $L^\perp$, since for any S'-free set D of $L^\perp$, $D + L$ is S-free. By construction, $B' \cap \mathrm{conv}(S')$ is bounded. Let $B' = \{x \in L^\perp \,|\, \alpha_i x \le \beta_i,\, i = 1, \dots, t\}$. Given $\varepsilon > 0$, let $\bar{B} = \{x \in L^\perp \,|\, \alpha_i x \le \beta_i,\, i = 1, \dots, t-1,\, \alpha_t x \le \beta_t + \varepsilon\}$. The polyhedron $\mathrm{conv}(S') \cap \bar{B}$ is a polytope since it has the same recession cone as $\mathrm{conv}(S') \cap B'$. The polytope $\mathrm{conv}(S') \cap \bar{B}$ contains points of S' in its interior by the maximality of B'. Since Λ is a lattice of $L^\perp$, $\mathbf{int}(\mathrm{conv}(S') \cap \bar{B})$ has a finite number of points in S', hence there exists one minimizing $\alpha_t x$, say z. By construction, the polyhedron $B'' = \{x \in L^\perp \,|\, \alpha_i x \le \beta_i,\, i = 1, \dots, t-1,\, \alpha_t x \le \alpha_t z\}$ does not contain any point of S' in its interior and contains B'. By the maximality of B', $B' = B''$ hence B' contains z in its relative interior, and B contains a point of S in its relative interior. □

Corollary 5. *For every maximal S-free convex set B there exists a maximal lattice-free convex set K such that, for every facet F of B, $F \cap K$ is a facet of K.*

Proof. Let K be defined as in Claim 1 in the proof of Theorem 2. It follows from the proof that K is a maximal lattice-free convex set with the desired properties. □

Remark 6. *It follows from the proof of Theorem 2 that every S-free convex set is contained in a maximal S-free convex set.*

2 Minimal inequalities

Let S be the set of integral points in some rational polyhedron in $\mathbb{R}^n$ such that $\dim(S) = n$. We consider the following model introduced by Dey and Wolsey [5]. Let $R(f, S)$ be the set of all $s = (s_r)_{r \in \mathbb{R}^n}$ such that

$$\begin{aligned} & f + \sum_{r \in \mathbb{R}^n} r s_r \in S \\ & s_r \ge 0, \quad r \in \mathbb{R}^n \\ & s \text{ has finite support.} \end{aligned} \tag{2}$$

Given a function $\psi : \mathbb{R}^n \to \mathbb{R}$, the linear inequality

$$\sum_{r \in \mathbb{R}^n} \psi(r) s_r \ge 1 \tag{3}$$

is *valid for* $R(f, S)$ if it is satisfied by every $s \in R(f, S)$.

Next we observe how maximal lattice-free convex sets in $\mathbb{R}^n$ give valid linear inequalities for $R(f, S)$. Let B be a maximal S-free convex set in $\mathbb{R}^n$ containing f in its interior. Since, by Theorem 2, B is a polyhedron and since f is in its interior, there exist $a_1, \dots, a_t \in \mathbb{R}^q$ such that $B = \{x \in \mathbb{R}^n \,|\, a_i(x - f) \le 1,\, i = 1 \dots, t\}$. We define the function $\psi_B : \mathbb{R}^n \to \mathbb{R}$ by

$$\psi_B(r) = \max_{i=1,\dots,t} a_i r.$$

Note that the function ψ_B is *subadditive*, i.e. $\psi_B(r) + \psi_B(r') \geq \psi_B(r + r')$, and *positively homogeneous*, i.e. $\psi_B(\lambda r) = \lambda \psi_B(r)$ for every $\lambda \geq 0$. We claim that

$$\sum_{r \in \mathbb{R}^n} \psi_B(r) s_r \geq 1$$

is valid for $R(f, S)$.

Indeed, let $s \in R(f, S)$, and $x = f + \sum_{r \in \mathbb{R}^n} r s_r$. Note that $x \in S$, thus $x \notin \mathbf{int}(B)$. Then

$$\sum_{r \in \mathbb{R}^n} \psi_B(r) s_r = \sum_{r \in \mathbb{R}^n} \psi_B(r s_r) \geq \psi_B(\sum_{r \in \mathbb{R}^n} r s_r) = \psi_B(x - f) \geq 1,$$

where the first equation follows from positive homogeneity, the first inequality follows from subadditivity of ψ_B and the last one follows from the fact that $x \notin \mathbf{int}(B)$.

We will show that all nontrivial irredundant valid linear inequalities for $R(f, S)$ are indeed of the type described above. We make this more precise in the next theorem.

A linear inequality $\sum_{r \in \mathbb{R}^n} \psi(r) s_r \geq 1$ that is satisfied by every nonnegative element s with finite support such that $f + \sum_{r \in \mathbb{R}^n} r s_r \in \text{conv}(S)$ is said to be *trivial*.

We say that inequality $\sum_{r \in \mathbb{R}^n} \psi(r) s_r \geq 1$ *dominates* inequality $\sum_{r \in \mathbb{R}^n} \psi'(r) s_r \geq 1$ if $\psi(r) \leq \psi'(r)$ for all $r \in \mathbb{R}^n$. Note that, for any $\bar{s} \in \mathbb{R}^n$ such that $\bar{s}_r \geq 0$ for all $r \in \mathbb{R}^n$, if $\bar{s}$ satisfies the first inequality, then $\bar{s}$ also satisfies the second. A valid inequality $\sum_{r \in \mathbb{R}^n} \psi(r) s_r \geq 1$ for $R(f, S)$ is *minimal* if it is not dominated by any valid linear inequality $\sum_{r \in \mathbb{R}^n} \psi'(r) s_r \geq 1$ for $R(f, S)$ such that $\psi' \neq \psi$.

Theorem 7.
Every nontrivial valid linear inequality for $R(f, S)$ of the form $\sum_{r \in \mathbb{R}^n} \psi(r) s_r \geq 1$ is dominated by a nontrivial minimal valid linear inequality for $R(f, S)$.
Every nontrivial minimal valid linear inequality for $R(f, S)$ of the form $\sum_{r \in \mathbb{R}^n} \psi(r) s_r \geq 1$ is an inequality

$$\sum_{r \in \mathbb{R}^n} \psi_B(r) s_r \geq 1,$$

where B is a maximal S-free convex set in $\mathbb{R}^n$ with f in its interior such that $\mathbf{int}(B \cap \text{conv}(S)) \neq \emptyset$.

A linear function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is of the form

$$\Psi(s) = \sum_{r \in \mathbb{R}^n} \psi(r) s_r \tag{4}$$

for some $\psi : \mathbb{R}^n \to \mathbb{R}$. Throughout the rest of the paper, capitalized Greek letters indicate linear functions from $\mathbb{R}^n$ to $\mathbb{R}$, while the corresponding lowercase letters indicate functions from $\mathbb{R}^n$ to $\mathbb{R}$ as defined in (4).

We will need the following lemma, whose proof is identical to the proof of Lemma 18 in [2]. A function is *sublinear* if it is positively homogeneous and subadditive.

Lemma 8. *Let $\Psi(s) \geq 1$ be a valid linear inequality for $R(f, S)$. Then $\Psi(s) \geq 1$ is dominated by a valid linear inequality $\Psi'(s) \geq 1$ for $R(f, S)$ such that ψ' is sublinear.*

Given a nontrivial valid linear inequality $\Psi(s) \geq 1$ for $R(f, S)$ such that ψ is sublinear, we consider the set

$$B_\psi = \{x \in \mathbb{R}^n \,|\, \psi(x - f) \leq 1\}.$$

Since ψ is continuous, B_ψ is closed. Since ψ is convex, B_ψ is convex. Since ψ defines a valid inequality, B_ψ is S-free. Indeed the interior of B_ψ is $\mathbf{int}(B_\psi) = \{x \in \mathbb{R}^n : \psi(x-f) < 1\}$. Its boundary is $\mathbf{bd}(B_\psi) = \{x \in \mathbb{R}^n : \psi(x - f) = 1\}$, and its recession cone is $\text{rec}(B_\psi) = \{x \in \mathbb{R}^n : \psi(x - f) \leq 0\}$. Note that f is in the interior of B_ψ.

Before proving Theorem 7, we need the following general theorem about sublinear functions. Let K be a closed, convex set in $\mathbb{R}^n$ with the origin in its interior. The *polar* of K is the set $K^* = \{y \in \mathbb{R}^n \,|\, ry \leq 1 \text{ for all } r \in K\}$. Clearly K^* is closed and convex, and since $0 \in \mathbf{int}(K)$, it is well known that K^* is bounded. In particular, K^* is a compact set. Also, since $0 \in K$, $K^{**} = K$. Let

$$\hat{K} = \{y \in K^* \,|\, \exists x \in K \text{ such that } xy = 1\}. \tag{5}$$

Note that $\hat{K}$ is contained in the relative boundary of K^*. Let $\rho_K : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\rho_K(r) = \sup_{y \in \hat{K}} ry, \quad \text{for all } r \in \mathbb{R}^n. \tag{6}$$

It is easy to show that ρ_K is sublinear.

Theorem 9 (Basu et al. [3]). *Let $K \subset \mathbb{R}^n$ be a closed convex set containing the origin in its interior. Then $K = \{r \in \mathbb{R}^n \,|\, \rho_K(r) \leq 1\}$. Furthermore, for every sublinear function σ such that $K = \{r \,|\, \sigma(r) \leq 1\}$, we have $\rho_K(r) \leq \sigma(r)$ for every $r \in \mathbb{R}^n$.*

Remark 10. *Let $K \subset \mathbb{R}^n$ be a polyhedron containing the origin in its interior. Let $a_1, \ldots, a_t \in \mathbb{R}^n$ such that $K = \{r \in \mathbb{R}^n \,|\, a_i r \leq 1,\, i = 1, \ldots, t\}$. Then $\rho_K(r) = \max_{i=1,\ldots,t} a_i r$.*

Proof. The polar of K is $K^* = \text{conv}\{0, a_1, \ldots, a_t\}$ (see Theorem 9.1 in Schrijver [7]). Furthermore, $\hat{K}$ is the union of all the facets of K^* that do not contain the origin, therefore

$$\rho_K(r) = \sup_{y \in \hat{K}} yr = \max_{i=1,\ldots,t} a_i r$$

for all $r \in \mathbb{R}^n$. □

Remark 11. *Let B be a closed S-free convex set in $\mathbb{R}^n$ with f in its interior, and let $K = B - f$. Then the inequality $\sum_{r \in \mathbb{R}^n} \rho_K(r) s_r \geq 1$ is valid for $R(f, S)$.*

Proof: Let $s \in R(f, S)$. Then $x = f + \sum_{r \in \mathbb{R}^n} r s_r$ is in S, therefore $x \notin \mathbf{int}(B)$ because B is S-free. By Theorem 9, $\rho_K(x - f) \geq 1$. Thus

$$1 \leq \rho_K(\sum_{r \in \mathbb{R}^n} r s_r) \leq \sum_{r \in \mathbb{R}^n} \rho_K(r) s_r,$$

where the second inequality follows from the sublinearity of ρ_K. □

Lemma 12. *Let C be a closed S-free convex set such that* $\mathbf{int}(C \cap \text{conv}(S)) \neq \emptyset$*. Given a point f in the interior of C, let $K = C - f$. There exists a maximal S-free convex set* $B = \{x \in \mathbb{R}^n \mid a_i(x - f) \leq 1,\ i = 1, \ldots, k\}$ *such that* $a_i \in \mathbf{cl}(\text{conv}(\hat{K}))$ *for* $i = 1, \ldots, k$.

Proof. Since C is an S-free convex set, it is contained in some maximal S-free convex set T. The set T satisfies one of the statements (i)-(iii) of Theorem 2. Since $\mathbf{int}(C \cap \text{conv}(S)) \neq \emptyset$, $\mathbf{int}(T \cap \text{conv}(S)) \neq \emptyset$, hence case (i) applies. Thus T is a polyhedron and $\text{rec}(T \cap \text{conv}(S)) = \text{lin}(T) \cap \text{rec}(\text{conv}(S))$ is rational. Let us choose T such that the dimension of $\text{lin}(T)$ is largest possible.

Since T is a polyhedron containing f in its interior, there exists $D \in \mathbb{R}^{t \times q}$ and $b \in \mathbb{R}^t$ such that $b_i > 0$, $i = 1, \ldots, t$, and $T = \{x \in \mathbb{R}^n \mid D(x - f) \leq b\}$. Without loss of generality, we may assume that $\sup_{x \in C} d_i(x - f) = 1$ where d_i denotes the ith row of D, $i = 1, \ldots, t$. By our assumption, $\sup_{r \in K} d_i r = 1$. Therefore $d_i \in K^*$, since $d_i r \leq 1$ for all $r \in K$. Furthermore $d_i \in \mathbf{cl}(\hat{K})$, since $\sup_{r \in K} d_i r = 1$.

Let $P = \{x \in \mathbb{R}^n \mid D(x - f) \leq e\}$. Note that $\text{lin}(P) = \text{lin}(T)$. By our choice of T, $P + \langle r \rangle$ is not S-free for any $r \in \text{rec}(\text{conv}(S)) \setminus \text{lin}(P)$, otherwise P would be contained in a maximal S-free convex set whose lineality space contains $\text{lin}(T) + \langle r \rangle$, a contradiction.

Let $L = \langle \text{rec}(P \cap \text{conv}(S)) \rangle$. Since $\text{lin}(P) = \text{lin}(T)$, L is a rational space. Note that $L \subseteq \text{lin}(P)$, implying that $d_i \in L^\perp$ for $i = 1, \ldots, t$.

We observe next that we may assume that $P \cap \text{conv}(S)$ is bounded. Indeed, let $\bar{P}$, $\bar{S}$, Λ be the projections onto $L^\perp$ of P, S, and $\mathbb{Z}^n$, respectively. Since L is a rational space, Λ is a lattice of $L^\perp$ and $\bar{S} = \text{conv}(\bar{S}) \cap \Lambda$. Note that $\bar{P} \cap \text{conv}(\bar{S})$ is bounded, since $L \supseteq \text{rec}(P \cap \text{conv}(S))$. If we are given a maximal $\bar{S}$-free convex set $\bar{B}$ in $L^\perp$ such that $\bar{B} = \{x \in L^\perp \mid a_i(x - f) \leq 1, i = 1, \ldots, h\}$ and $a_i \in \text{conv}\{d_1, \ldots, d_t\}$ for $i = 1, \ldots, h$, then $B = \bar{B} + L$ is the set $B = \{x \in \mathbb{R}^n \mid a_i(x - f) \leq 1, i = 1, \ldots, h\}$. Since $\bar{B}$ contains a point of $\bar{S}$ in the relative interior of each of its facets, B contains a point of S in the relative interior of each of its facets, thus B is a maximal S-free convex set.

Thus we assume that $P \cap \text{conv}(S)$ is bounded, so $\dim(L) = 0$. If all facets of P contain a point of S in their relative interior, then P is a maximal S-free convex set, thus the statement of the lemma holds. Otherwise we describe a procedure that replaces one of the inequalities defining a facet of P without any point of S in its relative interior with an inequality which is a convex combination of the inequalities of $D(x - f) \leq e$, such that the new polyhedron thus obtained is S-free and has one fewer facet without points of S in its relative interior. More formally, suppose the facet of P defined by $d_1(x - f) \leq 1$ does not contain any point of S in its relative interior. Given $\lambda \in [0, 1]$, let

$$P(\lambda) = \{x \in \mathbb{R}^n \mid [\lambda d_1 + (1 - \lambda) d_2](x - f) \leq 1, \quad d_i(x - f) \leq 1 \;\; i = 2, \ldots, t\}.$$

Note that $P(1) = P$ and $P(0)$ is obtained from P by removing the inequality $d_1(x - f) \leq 1$. Furthermore, given $0 \leq \lambda' \leq \lambda'' \leq 1$, we have $P(\lambda') \supseteq P(\lambda'')$.

Let $r_1, \ldots, r_m$ be generators of $\text{rec}(\text{conv}(S))$. Note that, since $P \cap \text{conv}(S)$ is bounded, for every $j = 1, \ldots, m$ there exists $i \in \{1, \ldots, t\}$ such that $d_i r_j > 0$. Let $r_1, \ldots, r_h$ be the generators of $\text{rec}(\text{conv}(S))$ satisfying

$$\begin{aligned} &d_1 r_j > 0 \\ &d_i r_j \leq 0 \qquad i = 2, \ldots, t. \end{aligned}$$

Note that, if no such generators exist, then $P(0) \cap \text{conv}(S)$ is bounded. Otherwise $P(\lambda) \cap \text{conv}(S)$ is bounded if and only if, for $j = 1, \ldots, h$

$$[\lambda d_1 + (1 - \lambda) d_2] r_j > 0.$$

This is the case if and only if $\lambda > \lambda^*$, where

$$\lambda^* = \max_{j=1,\ldots,h} \frac{-d_2 r_j}{(d_1 - d_2) r_j}.$$

Let r^* be one of the vectors $r_1, \ldots, r_h$ attaining the maximum in the previous equation. Then $r^* \in \text{rec}(P(\lambda^* \cap \text{conv}(S)$.

Note that $P(\lambda^*)$ is not S-free otherwise $P(\lambda^*) + \langle r^* \rangle$ is S-free by Lemma 4, and so is $P + \langle r^* \rangle$, a contradiction.

Thus $P(\lambda^*)$ contains a point of S in its interior. That is, there exists a point $\bar{x} \in S$ such that $[\lambda^* d_1 + (1 - \lambda^*) d_2](\bar{x} - f) < 1$ and $d_i(\bar{x} - f) < 1$ for $i = 2, \ldots, t$. Since P is S-free, $d_1(\bar{x} - f) > 1$. Thus there exists $\bar{\lambda} > \lambda^*$ such that $[\bar{\lambda} d_1 + (1 - \bar{\lambda}) d_2](\bar{x} - f) = 1$. Note that, since $P(\bar{\lambda}) \cap \text{conv}(S)$ is bounded, there is a finite number of points of S in the interior of $P(\bar{\lambda})$. So we may choose $\bar{x}$ such that $\bar{\lambda}$ is maximum. Thus $P(\bar{\lambda})$ is S-free and $\bar{x}$ is in the relative interior of the facet of $P(\bar{\lambda})$ defined by $[\bar{\lambda} d_1 + (1 - \bar{\lambda}) d_2](x - f) \leq 1$.

Note that, for $i = 2, \ldots, t$, if $d_i(x - f) \leq 1$ defines a facet of P with a point of S in its relative interior, then it also defines a facet of $P(\bar{\lambda})$ with a point of S in its relative interior, because $P \subset P(\bar{\lambda})$. Thus repeating the above construction at most t times, we obtain a set B satisfying the lemma. □

Remark 13. *Let C and K be as in Lemma 12. Given any maximal S-free convex set $B = \{x \in \mathbb{R}^n \mid a_i(x - f) \leq 1,\ i = 1, \ldots, k\}$ containing C, then $a_1, \ldots, a_k \in K^*$. If $\text{rec}(C)$ is not full dimensional, then the origin is not an extreme point of K^*. Since all extreme points of K^* are contained in $\{0\} \cup \hat{K}$, in this case $\mathbf{cl}(\text{conv}(\hat{K})) = K^*$. Therefore, when $\text{rec}(C)$ is not full dimensional, every maximal S-free convex set containing C satisfies the statement of Lemma 12.*

Lemma 14. *Given a maximal S-free convex set B of $\mathbb{R}^n$ containing f in its interior, $\Psi_B(s) \geq 1$ is a minimal valid inequality for $R(f, S)$.*

Proof. Let $\Psi(s) \geq 1$ be a valid linear inequality for $R(f, S)$ such that $\psi(r) \leq \psi_B(r)$ for all $r \in \mathbb{R}^n$. Then $B_\psi \supset B$ and B_ψ is S-free. By maximality of B, $B = B_\psi$. By Theorem 9 and Remark 10, $\psi_B(r) \leq \psi(r)$ for all $r \in \mathbb{R}^n$, proving $\psi = \psi_B$. □

Proof of Theorem 7.
Let $\Psi(s) \geq 1$ be a nontrivial valid linear inequality for $R(f, S)$. By Lemma 8, we may assume that ψ is sublinear.

We first show that, if $\mathbf{int}(B_\psi \cap \text{conv}(S)) = \emptyset$, then $\Psi(s) \geq 1$ is trivial. Indeed, suppose $\mathbf{int}(B_\psi \cap \text{conv}(S)) = \emptyset$ and let s nonnegative with finite support such that $x = f + \sum_{r \in \mathbb{R}^n} r s_r \in \text{conv}(S)$. This implies $x \notin \mathbf{int}(B_\psi)$. It follows

$$1 \leq \psi(x - f) = \psi(\sum_{r \in \mathbb{R}^n} r s_r) \leq \sum_{r \in \mathbb{R}^n} \psi(r) s_r = \Psi(s),$$

where the last inequality follows from the sublinearity of ψ.

Let $K = \{r \in \mathbb{R}^n \,|\, \psi(r) \leq 1\}$, and let $\hat{K}$ be defined as in (5). Note that $K = B_\psi - f$. Thus, by Remark 11, $\sum_{r\in\mathbb{R}^n} \rho_K(r)s_r \geq 1$ is valid for $R(f,S)$. Since ψ is sublinear, it follows from Theorem 9 that $\rho_K(r) \leq \psi(r)$ for every $r \in \mathbb{R}^n$.

By Lemma 12, there exists a maximal S-free convex set $B = \{x \in \mathbb{R}^n \,|\, a_i(x-f) \leq 1,\ i = 1,\ldots,k\}$ such that $a_i \in \mathbf{cl}(\mathrm{conv}(\hat{K}))$ for $i = 1,\ldots,k$.

Then

$$\psi(r) \geq \rho_K(r) = \sup_{y\in\hat{K}} yr = \max_{y\in\mathbf{cl}(\mathrm{conv}(\hat{K}))} yr \geq \max_{i=1,\ldots,k} a_i r = \psi_B(r).$$

This shows that $\psi_B(r) \leq \psi(r)$ for all $r \in \mathbb{R}^n$, which concludes the proof of the theorem. □

Remark 15. *Note that ψ is nonnegative if and only if $\mathrm{rec}(B_\psi)$ is not full-dimensional. It follows from Remark 13 that, for every maximal S-free convex set B containing B_ψ, we have $\psi_B(r) \leq \psi(r)$ for every $r \in \mathbb{R}^n$ when ψ is nonnegative.*

A statement similar to the one of Theorem 7 was shown by Borozan-Cornuéjols [4] for a model similar to (2) when $S = \mathbb{Z}^n$ and the vectors s are elements of $\mathbb{R}^{\mathbb{Q}^n}$. In this case, it is easy to show that, for every valid inequality $\sum_{r\in\mathbb{Q}^n} \psi(r)s_r \geq 1$, the function $\psi : \mathbb{Q}^n \to \mathbb{R}$ is nonnegative. Remark 15 explains why in this context it is much easier to prove that minimal inequalities arise from maximal lattice-free convex sets.

References

[1] A. Barvinok, *A Course in Convexity*, Graduate Studies in Mathematics, vol. 54, American Mathematical Society, Providence, Rhode Island, 2002.

[2] A. Basu, M. Conforti, G. Cornuéjols, G. Zambelli, Maximal lattice-free convex sets in linear subspaces, manuscript (March 2009).

[3] A. Basu, G. Cornuéjols, G. Zambelli, Convex Sets and Minimal Sublinear Functions, manuscript (March 2009).

[4] V. Borozan, G. Cornuéjols, Minimal Valid Inequalities for Integer Constraints, technical report (July 2007), to appear in *Mathematics of Operations Research*.

[5] S.S. Dey, L.A. Wolsey, Constrained Infinite Group Relaxations of MIPs, manuscript (March 2009).

[6] L. Lovász, Geometry of Numbers and Integer Programming, *Mathematical Programming: Recent Developements and Applications*, M. Iri and K. Tanabe eds., Kluwer (1989), 177-210.

[7] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, New York (1986).