

1979

# Multiple criteria optimization and statistical design for electronic circuits

M Lightner  
*Carnegie Mellon University*

Stephen W. Director

Follow this and additional works at: <http://repository.cmu.edu/ece>

---

This Technical Report is brought to you for free and open access by the Carnegie Institute of Technology at Research Showcase @ CMU. It has been accepted for inclusion in Department of Electrical and Computer Engineering by an authorized administrator of Research Showcase @ CMU. For more information, please contact [research-showcase@andrew.cmu.edu](mailto:research-showcase@andrew.cmu.edu).

**NOTICE WARNING CONCERNING COPYRIGHT RESTRICTIONS:**

The copyright law of the United States (title 17, U.S. Code) governs the making of photocopies or other reproductions of copyrighted material. Any copying of this document without permission of its author may be prohibited by law.

MULTIPLE CRITERIA OPTIMIZATION AND STATISTICAL DESIGN  
FOR ELECTRONIC CIRCUITS

by

M.R. Lightner & S.W. Director

DRC-18-4-79

January 1979

Department of Electrical Engineering  
Carnegie-Mellon University  
Pittsburgh, PA 15213

This work was supported in part by NSF Grant ENG-77-20895.

---

620.0042  
C282  
DRC-18-4-79

ABSTRACT

In this paper we examine the problem of designing electronic circuits using Multiple Criteria Optimization where one of the competing criteria is the circuit yield. A new technique for generating solutions to the MCO problem based upon a family of weighted p-norms is presented. We concentrate on the max norm member of this family (this gives a minimax problem) and propose a method of solution based upon a new constrained optimization method due to Powell. The yield and gradient of yield are estimated using a method based upon Simplicial Approximation which is used to form a piecewise linear approximation to the probability density function of the designable parameters. An example illustrates that it may be possible to significantly alter the values of various circuit criteria, over their value at the maximum yield point, with very little change in the circuit yield.

f. Introduction

Historically circuit design can be viewed as consisting of two broad methodologies: performance design and statistical design. In performance design the circuit designer chooses a circuit configuration, adjusts parameters to attain a desired performance and then tests the circuit yield. If the yield is too small the parameters are re-adjusted. Statistical design arose mainly in response to integrated circuit design problems. In statistical design a circuit configuration is chosen and then the parameters are adjusted to achieve maximum circuit yield (worst case design being the extreme of 100% yield).

These two methodologies can be unified by considering circuit design as a Multiple Criteria Optimization (MCO) problem with yield as one of the

competing objectives. Specifically, after a circuit configuration is chosen, we consider the following MCO problem:

$$\begin{array}{l} \text{Min} \\ x_0 \end{array} \left[ \begin{array}{c} f_1(x_0) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_m(x_0) \\ 1-Y(x_0) \end{array} \right] \quad (1)$$

$$\text{subject to } g_i(x_0) \leq p \quad i=1, \dots, I$$

where  $x_0$  are the designable parameters, such as length, width, and various component values;  $f_i(x_0)$  are various objectives to be minimized such as power, area, propagation delay, insertion loss at a given frequency;  $Y(X_0)$  is the circuit yield at  $X_0$ ; and the  $g^i x^i$ 's are circuit constraints such as limits on parameter values, upper limits on propagation delay and requirements on various voltage levels.

In this paper we intend to examine computationally effective means of solving (1). To do this we will discuss the MCO problem and the yield calculation problem separately. In the next section we present a new method for generating solutions to the MCO problem and a suggestion for effective implementation of this method. In Section III we present a method based upon the Simplicial Approximation [1] for efficiently estimating the circuit yield and the gradient of the yield with respect to the nominal point. In Section IV we present a geometric example of a MCO problem with yield as a criteria. Finally, Section V presents a summary and conclusion.

II. Multiple Criteria Optimization

In this section we will consider the MCO problem in the following form:

$$\begin{array}{l}
 \text{Min} \\
 \mathbf{x} \\
 \text{subject to}
 \end{array}
 \begin{array}{l}
 \left[ \begin{array}{l}
 f_1(x) \\
 \cdot \\
 \cdot \\
 \cdot \\
 \cdot \\
 f_m(x)
 \end{array} \right]
 \end{array}
 \tag{2}$$

$$g_i(x) \leq 0 \quad i = 1, \dots, I.$$

where  $x$  is in  $R^n$ . The set

$$A = \{x \mid g_i(x) \leq 0, i = 1, \dots, I\}$$

will be called the feasible region in input space. The image of  $A$

by  $f = (f_1(x), \dots, f_m(x))$ , i.e.,

$$A_f = \{f \mid f \in f(A)\},$$

will be called the feasible region in output space. The solution concept

for the MCO problem that we will use is that of noninferiority [2].

Definition. A point  $x \in A_f$  is called a noninferior point if and only if there does not exist another point  $x^1 \in A_f, x \neq x^1$ , such that

$$f_i(x^1) \leq f_i(x) \text{ for all } i$$

$$f_j(x^1) < f_j(x) \text{ for some } j.$$

The image  $f(x)$  of a noninferior point  $x$  will be called a noninferior solution.

Alternate terminology for noninferiority is Pareto optimality [3], efficiency [4], admissibility and a form of nondomination [5].

Many methods have been developed for generating noninferior solutions [2-15], reviews are found in [9] and [10]. We will present two standard methods in order to illustrate the problems and ideas of MCO, then a new family of solutions will be described.

Historically the first method used to generate a noninferior solution was to form a weighted sum of the objectives [11-12], i.e.,

$$F(x,w) = \sum_{i=1}^m w_i f_i(x) \quad w_i \geq 0. \quad (3)$$

Then minimizing (3) subject to the given constraints would generate a noninferior solution. In output space this method can be viewed as finding a supporting hyperplane, with normal  $w$ , to the feasible region  $A$ . This is illustrated in Figure 1. (Throughout this paper we will assume that the feasible region in output space,  $A$ , is compact, and so without loss of generality we can always consider  $A$  in the positive orthant, i.e.,  $A \geq 0$ .) Figure 2 indicates the problem that arises in using the weighted sum method when some noninferior solutions are on a non-convex portion of the surface of  $A$ .

One method (of many) that overcomes the convexity limitation of the weighted sum method, is the constraint method [13-15]. In this method one of the objectives is minimized subject to the remaining objectives being less than or equal to certain constraints, i.e.,

---

$$\begin{aligned}
& \text{Min} && f_1(x) \\
& x \in O && \\
& \text{subject to} && f_i(x) \leq a_i \quad i = 2, \dots, m
\end{aligned} \tag{4}$$

where the  $a_i$ 's are chosen by the user. This method is illustrated in Figure 3. The problem with the constraint method is that not every choice of  $a$ 's yield a noninferior solution and thus each solution must be checked for noninferiority.

An alternate to either of these methods is to consider the family of weighted  $L_p$  norms,

$$\|Wf\|_p = \left[ \sum_{i=1}^m (w_{ii} f_i)^p \right]^{1/p} \quad 1 \leq p \leq \infty \tag{5}$$

and

$$\|Wf\|_\infty = \max_{i=1, \dots, m} (w_{ii} f_i)$$

where  $W$  is a weight matrix with non-negative diagonal entries and all other entries zero. Notice that in (5) we implicitly use the assumption that  $A \geq 0$ , hence  $f_i \geq 0$ . The least  $p^{\text{th}}$  method for generating a noninferior solution is to

$$\min_{x \in O} \|Wf\|_p \tag{6}$$



For the  $I_1$  norm (6) is simply the weighted sum method. A geometric interpretation of (6) in output space can be found by considering the level set associated with the  $p$  norm i.e.,

$$L_p(a) = \{f \mid \|wf\|_p \leq a\} .$$

The geometric interpretation of the  $\infty$  norm method (minimax) is shown in Figure 4, for two different weights. We now present two theorems concerning the  $I_p$  method where we assume that the minimum of each function, considered separately, is unique. The proofs of the theorems are straightforward and will not be presented to conserve space, they can be found in [10].

Theorem 1 A solution  $f^* \in A$  is noninferior if and only if there exists a diagonal weight matrix,  $W \succeq 0$ , and a  $p$ ,  $1 < p < \infty$ , such that  $f^*$  is the unique minimizer of  $\|wf\|_p$  over  $A$ .

We know that if  $f^*$  is a noninferior solution found by minimizing an  $I_p$  norm, that the normal to the level set of the  $I_p$  norm at  $f^*$  must be collinear with the inward pointing normal,  $n^*$ , of  $A$  at  $f^*$ . Let us normalize  $n^*$  so that

$$n^{*T} f^* = 1$$

A canonical weight,  $W(p)$ , can now be defined such that if  $f^*$  is found using a  $I_p$  norm then  $\|w(p)f^*\|_p = 1$ . This weight is

$$w_{ii}(p) = \frac{n_i^*}{\|f_i^*\|_p} \quad (?)$$

Note that for  $p = 1$ ,  $w_{ii}(1) = n_i^*$  as required, for  $p = \infty$ ,  $w_{ii}(\infty) = \frac{1}{\|f_i^*\|_\infty}$ .

Definition A noninferior solution  $f^*$  will be called a solution of degree  $p$  when  $I_p$  is the smallest canonically weighted norm (i.e., smallest  $p$ ) that can be used to find  $f^*$  uniquely.

With the canonic weight and degree of solution defined we now have

Theorem 2 If  $f^*$  is a noninferior solution of degree  $p$ , then  $f^*$  can be found by minimizing  $\|w(q)f\|_q$  over  $A$ , for all  $q \geq p$ , where  $W(q)$  is the canonic weight for  $f^*$ .

We will now discuss the computational implementation of the  $l^1$  method for generating noninferior solutions. The  $l^1$  problem is the minimax problem and is stated as

$$\min_{x \in \{f_1, \dots, f_m\}} \max_i w_i |f_i|$$

There are many methods available for solving the minimax problem [16-19]. We will use the simplest method which is to introduce an extra parameter  $Y$  and rewrite (8) as

$$\begin{aligned} & \min_{(x \in Q), Y} Y \\ & \text{subject to} \\ & w_i |f_i| \leq Y \quad i = 1, \dots, m. \end{aligned} \tag{9}$$

In order to solve (9) it is imperative that a powerful constrained optimization method be available. We use the new method proposed by Powell [20-22] which is similar to methods discussed by Han [23] and Tapia [24]. Powell's method consists of solving a sequence of quadratic programs of the form

$$\min_d \quad F(x^k) + d^T G + \frac{1}{2} d^T B d$$

subject to

(10)

$$d^T \nabla g_i(x^k) + g_i(x^k) \leq 0 \quad i = 1, \dots, \ell$$

where  $F(x^k)$  is the value of the function being minimized at  $x^k$ ,  $G$  is the gradient of  $F$  at  $x^k$ ,  $B$  is an estimate of the Hessian of the Lagrangian with respect to  $x$ . The solution of (10) yields a search direction that tries to simultaneously reduce  $F(x)$  and satisfy the constraints. The multipliers from the quadratic program are used in the symmetric rank two update of  $B$ . By linearizing all the constraints the active set strategy is completely handled by the quadratic program. Further details of implementing this method as well as a proof of  $R$  superlinear convergence can be found in the references. In Powell's work he finds his method a factor of 3-5 times better than any technique he compared it with. In experiments of our own [10] Powell's method was found to be at least 3 times faster than the Augmented Lagrangian methods [25-26].

The final problem associated with the generation of noninferior solutions is the choice of the weights  $w_{ii}$ . Clearly interactive heuristic methods for choosing weights should be used in order to minimize the number of constrained minimizations required to find a satisfactory noninferior design. Much work needs to be done in developing these methods and we will only give some suggestions to aid in developing specific heuristics.

---

In order to know the possible range of noninferior solutions we suggest that the first step in a heuristic be the minimization of each objective function separately. Using this information the designer might then select a set of weights  $g_i$  indicating his preference for each of these solutions. Upon normalizing  $g_i$  so that  $\sum g_i = 1$  we generate the point  $\hat{f}$

$$\hat{f} = \sum \beta_i f_i^* \quad (11)$$

where  $f_i^*$  is the point found by minimizing the  $i$ th objective. Now based upon the ideas embodied in the canonic weight we choose the weight

$$w_{ii} = \frac{1}{K} \quad i = 1, \dots, m \quad (12)$$

for the next  $I_{\infty}$  minimization. In succeeding iterations the designer will pick known noninferior solutions among which he wishes to trade-off and gives preferences  $g_i$ . Then the weights are generated using (11) and (12). This procedure continues until the designer finds a noninferior design with which he is satisfied.

This  $I_{\infty}$  norm method of generating noninferior solutions has proved completely satisfactory on a number of examples. Furthermore, by using Powell's method we have found that the minimax approach is no more computationally expensive than the constraint method while being much more appealing. Methods similar to the  $I_p$  norm family have been proposed in [27-29]. In the next section we will discuss a method for efficiently estimating yield and its gradient so that it can be included in an MCO statement of the circuit design problem.

### III. Yield Maximization

In choosing a method to calculate yield and its gradient it is important to realize that in an MCO problem the yield and gradient of yield will have to be estimated many times. Therefore, yield estimation must be efficient. This

efficiency may be obtained at the cost of a relatively large amount of preprocessing and still be computationally viable over the entire MCO solution procedure. The basis of our method will be the Simplicial Approximation method [1].

The key feature of the Simplicial Approximation which we use is the generation of the piecewise-linear inner approximation to the feasible region  $\mathcal{Q}$  in input space. Although a relatively costly preprocessing step this approximation has a two-fold beneficial effect in the MCO problem: first the nonlinear constraints on circuit behavior will be replaced by piecewise linear constraints making the constrained optimization easier. Secondly, we will be able to estimate yield inexpensively using the Simplicial Approximation.

Let  $F(x, x_0)$  be the p.d.f. of the parameters with nominal parameter vector  $x_0$ . With  $\mathcal{Q}$  as the feasible region in input space we have

$$Y(x_0) = \int_{\mathcal{Q}} F(x, x_0) dx. \quad (13)$$

By replacing  $\mathcal{Q}$  with the simplicial approximation, SA, we have

$$Y(x_0) \approx \int_{SA} F(x, x_0) dx. \quad (14)$$

The key to the method is to realize that, by construction, each face of the simplicial approximation is a simplex. Thus, as shown in Figure 5, each nominal point  $x_0$  induces a unique interior simplicial decomposition of the approximation SA. We will consider various methods of approximating (14) over each of these interior simplices. Generically this approximation will take the form

$$A_i(x_0) = \frac{1}{k} \sum_{k=1}^l F(x_{ik}, x_0) \quad (15)$$

where  $A_i(x_0)$  is the hypervolume of the  $i^{\text{th}}$  interior simplex,  $F(x, \dots, x)$  is the

value of the p.d.f. at various points in the  $i^{\text{th}}$  region and  $d_k$  a weight. We will approximate the gradient of the yield over  $i^{\text{th}}$  region as the gradient of the approximation over that region - which we can calculate exactly.

Generically the gradient will have the form

$$A_i(x_0) \left\{ \sum_{k=1}^l d_k \nabla_{x_0} F(x_{ik}, x_0) \right\} + \nabla_{x_0} A_i(x_0) \left\{ \sum_{k=1}^l d_k F(x_{ik}, x_0) \right\} \quad (16)$$

By examining (15) and (16) we see that to be able to efficiently estimate yield and its gradient over the  $i^{\text{th}}$  region we must efficiently evaluate  $A_i(x_0)$  and  $\nabla_{x_0} A_i(x_0)$ . Let  $X_i$  be a matrix whose columns are the coordinates of the vertices of the  $i^{\text{th}}$  interior simplex exclusive of  $x_0$ , the nominal. This matrix is independent of the nominal. It can be shown [10] that

$$A_i(x_0) = \frac{1}{n!} \det(X_i) \{1 - x_0^T X_i^{-1} e\} \quad (17)$$

and

$$\nabla_{x_0} A_i(x_0) = \frac{1}{n!} \det(X_i) \text{sgn} \{1 - x_0^T X_i^{-1} e\} (-X_i^{-1} e) \quad (18)$$

where  $n$  is the number of designable parameters  $\det(X_i)$  is the determinant of  $X_i$ ,  $e$  is a column vector of all ones, and  $\text{sgn}$  is the sign function. Thus we need to find  $X_i^{-1} e$  and  $\det(X_i)$ . This can be easily done for each  $X_i$  by solving

$$X_i y = e$$

or

$$L_i U_i y = e$$

$$y = U_i^{-1} L_i^{-1} e = X_i^{-1} e$$

$$\det(X_i) = \prod_{i=1}^n u_{ii}$$

where  $L_i$  and  $U_i$  are determined by triangular factorization of  $X_i$ .

**Note that** this is a preprocessing step that only needs to be done once for each  $X_i$ . Also as we progress from the  $i^{\text{th}}$  face to its neighbor only one column of  $X_i$  changes and so further economies are possible in these calculations.

For this paper we will consider breaking each interior simplex into a number of segments by cutting the  $i^{\text{th}}$  simplex with hyperplanes parallel to the face of the  $i^{\text{th}}$  simplex on the boundary of the SA. The integral of the yield over each segment will be found by using a piecewise linear estimate of the p.d.f. over the segment. This division is shown in Figure 6. Using Figure 6 with

$$x_{Hmk} = x_0 + \frac{k}{I} (x_{im} - x_0) \quad \begin{matrix} m = 1, \dots, n \\ k = 1, \dots, l \end{matrix}$$

where  $I$  is the number of segments desired,  $x_m$  the vertices of the SA, we estimate the yield over the  $i^{\text{th}}$  simplex as

$$y_i(x_0) = \frac{1}{l^n} A_i(x_0) \frac{1}{n+1} \left\{ f(x_0, x_0) + \sum_{k=1}^n f(\xi_{ikl}, x_0) \right\} + \sum_{j=2}^l \frac{j^n - (j-1)^n}{l^n} A_i(x_0) \frac{1}{2^n} \left\{ \sum_{k=j-1}^j \left[ \sum_{m=1}^n f(\xi_{imk}, x_0) \right] \right\}. \quad (19)$$

The gradient of the yield over the  $i^{\text{th}}$  simplex is estimated by taking the gradient of (19), i.e.,

$$\begin{aligned}
\nabla_{x_0} Y_i(x_0) &= \frac{1}{2^n} A_i(x_0) \frac{1}{(n+1)} \left\{ \nabla_{x_0} f(x_0, x_0) + 3 \frac{\partial f(\xi_{ik1}, x_0)}{\partial x_0} \right\} \\
&+ \frac{1}{2^n} \frac{\partial A_i(x_0)}{\partial x_0} \frac{1}{(n+1)} \left\{ f(V^*_0) + \sum_{k=1}^n f(\xi_{ik1}, x_0) \right\} \\
&+ \sum_{j=2}^n \frac{1}{2^n} \frac{\partial A_i(x_0)}{\partial x_0} \frac{1}{(n+1)} \left\{ \sum_{k=j-1}^n \sum_{m=1}^n \frac{\partial f(\xi_{ikm}, x_0)}{\partial x_0} \right\} \\
&+ \sum_{j=2}^n \frac{1}{2^n} \frac{\partial A_i(x_0)}{\partial x_0} \frac{1}{(n+1)} \left\{ \sum_{k=j-1}^n \sum_{m=1}^n \frac{\partial f(\xi_{ikm}, x_0)}{\partial x_0} \right\}
\end{aligned} \tag{20}$$

Notice that (20) is the exact gradient of (19). Further since we have reduced the work of calculating  $A_i(x_0)$  to an inner product, and  $\nabla_{x_0} A_i(x_0)$  to checking a sign, the major work in evaluating (19) and (20) is the evaluation of  $f(x, x_0)$  and  $\nabla_{x_0} f(x, x_0)$  at various points over the simplex. Summing (19) and (20) over all interior simplices gives the estimates of yield and gradient of yield for the simplicial approximation.

For a particular simplicial approximation- (Fig. 7) we compared a 1000 sample Monte Carlo estimate of yield to the method described above using 10 segments per interior simplex. For a gaussian p.d.f. with equal variance and no correlation the results are shown in Table 1. Comparisons are made for two different standard deviations and for several nominal points throughout the approximation.

In this section we have presented a new method based upon the Simplicial Approximation for estimating yield and its gradient. Because much of the work involved is preprocessing and because the Simplicial Approximation reduces the work involved in solving a constrained optimization problem we



feel that this is a computationally viable approach for use when adding yield as an objective function in a Multiple Criteria Optimization circuit design problem.

#### IV. Numerical Example

In order to illustrate an MCO problem including yield we will consider the following problem

$$\begin{aligned}
 \text{Min} \quad & f_1 = (x_1^{1.5})^2 + (x_2 - 3)^2 \\
 & x_1 > x_2 \\
 & f_2 = (x_1 - 7)^2 + (x_2 - 3.5)^2 \\
 & 1 - Y(x)
 \end{aligned} \tag{21}$$

subject to

$$\frac{(x_1 - 6)^2}{(5.5)^2} + \frac{(x_2 - 6)^2}{(2)^2} < 1 \tag{22}$$

where  $Y(x)$  is the yield and  $x_1$  and  $x_2$  are independent gaussian with equal variance. The first step is to generate a simplicial approximation to the feasible region in input space defined by (22). This is shown in Figure 7. Next we find the minimum of  $f_1$ ,  $f_2$  and  $1 - Y(x)$  separately, these points are indicated in Figure 7. We note in passing that, because of the symmetries involved, no matter what standard deviation is chosen the maximum yield point remains the same.

As an analog of the MCO circuit design without yield we indicate in Figure 8 the set of noninferior points associated with minimizing  $f_1$  and  $f_2$  simultaneously. All of these points are quite far from the maximum yield point. This indicates that if we consider only performance criteria and not yield in an MCO design problem, no matter how many noninferior points for  $f_1$  and  $f_2$  we find, the yield could still be unacceptable.

Next we consider the full MCO problem (21). Figure 9 indicates two noninferior points generated using the minimax method. Yield was evaluated as described in section III using 10 segments per interior simplex. A standard deviation of 1 was used for the example in Figure 9. Examining the values of the objective functions shown in Figure 9 we see the most important feature of MCO including yield: it is possible to significantly alter the values of performance criteria at the cost of a minor change in yield.

Figure 10 shows the same example as Figure 9 except the standard deviation is now 2 instead of 1. Again, two noninferior solutions were generated using the weighted minimax method. Examining the objective function values we again see that significant changes in  $f_1$  and  $f_2$ , over their values at the maximum yield point can be found without significant reduction in yield.

The phenomenon noted above has been demonstrated in examples using small filter networks, in our talk we will present an example of MCO circuit design, including yield, on an MOSFET logic gate.

---

## V. Summary and Conclusions

In this paper we have briefly reviewed multiple criteria optimization and presented a new method of generating noninferior solutions. The weighted minimax generation method, together with Powells' new algorithm, is a powerful technique, with appealing heuristics, for interactively generating noninferior solutions to the MCO circuit design problem.

Further, a method for estimating yield and its gradient, based upon Simplicial Approximation, has been discussed. This method provides a computationally viable method for including yield as an objective function in an MCO problem.

Finally, a numerical example embodying the above ideas has been presented which illustrates the MCO problem including yield. This example illustrates the most important feature of using MCO to trade-off between performance and yield: that it is possible to achieve significant variation of performance criteria, over their values at the maximum yield point, without significantly effecting the yield of a particular circuit.

---

## REFERENCES

1. S. W. Director and C. D. Hachtel, "The Simplicial Approximation Approach to Design Centering," *IEEE Trans. Cir. and Sys.*, Vol. CAS-24, No. 7, July 1977, pp. 363-372.
  2. R. W. Reid and S. J. Citron, "On Noninferior Index Vectors," *J. Optim. Th. and Appl.*, Vol. 7, No. 1, 1971.
  3. N. O. Da Cunha and E. Polak, "Constrained Minimization Under Vector-Valued Criteria in Finite Dimensional Space," *J. Math. Anal, and Appl.*, Vol. 19, 1967, pp. 103-124.
  4. A. M. Geoffrion, "Proper Efficiency and the Theory of Vector Maximization," *J. Math. Anal, and Appl.*, Vol. 22, 1968, pp. 618-630.
  5. P. L. Yu, "Cone Convexity, Cone Extreme Points and Nondominated Solutions in Decision Problems with Multiobjective," *J. Optim. Th. and Appl.*, Vol. 14, No. 3, 1974, pp. 319-377.
  6. Conference on Multiple Criteria Decision Making, eds. Cochrane and Zeleney, 1972, Univ. of South Carolina Press.
  7. Multiple Criteria Decision Making, Kyoto 1975, ed. M. Zeleney, Vol. 123 Lecture Notes in Economics and Mathematical Systems, Springer-Verlag.
  8. Multiple Criteria Decision Making, France 1975, ed. H. Thiriez and S. Zions, Vol. 130, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag.
  9. J. L. Cohon and D. H. Marks, "A Review and Evaluation of Multiobjective Programming Techniques," *Water Resources Research*, Vol. 11, No. 2, April 1975, pp. 208-220.
  10. M. R. Lightner, "Multiple Criteria Optimization and Statistical Design for Electronic Circuits," Ph.D. Dissertation, Carnegie-Mellon Univ., Pittsburgh, Pa., Dec. 1978.
  11. H. W. Kuhn and A. W. Tucker, "Nonlinear Programming," in *Proc. of Second Berkeley Symposium on Math. Stat. and Prob.*, pp. 481-492, Univ. of Calif. Press, Berkeley, Calif. 1951.
  12. L. A. Zadeh, "Optimality and Non-Scalar-Valued Performance Criteria," *IEEE Trans. Auto. Control*, Vol. AC-8, No. 1, Jan., 1963.
  13. J. G. Lin, "Maximal Vectors and Multi-Objective Optimization," *J. Optim. Th. and Appl.*, Vol. 18, No. 1, Jan. 1976.
  14. J. G. Lin, "Multi-Objective Problems: Pareto-Optimal Solutions by Method of Proper Equality Constraints," *IEEE Trans. Auto. Control*, Vol. AC-21, No. 5, Oct. 1976.
  15. E. Polak, "On the Approximation of Solutions to Multiple Criteria Decision Making Problems," in Reference #7 this paper.
-

16. W. I. Zangwill, "An Algorithm for the Chebyshev Problem - With an Application to Concave Programming," *Management Science*, Vol. 14, No. 1, Sept. 1967.
  17. V. F. Demyanov and V. N. Malozemov, "Introduction to Minimax," A Halsted Press Book - John Wiley & Sons, New York, N. Y., 1974.
  18. J. W. Bandler and C. Charalambous, "Nonlinear Programming Using Minimax Techniques," *J. Optim. Th. and Appl.*, Vol. 13, No. 6, 1974.
  19. K. Madsen and H. Schjaer-Jacobsen, "Linearly Constrained Minimax Optimization," *Math. Prog.*, Vol. 14, 1978 pp. 208-223.
  20. M. J. D. Powell, "A Fast Algorithm for Nonlinearly Constrained Optimization Calculations," in *Numerical Analysis, Dundee 1977*, eds. A. Dodd and B. Eckmann, #630, *Lecture Notes in Math.*, Springer-Verlag.
  21. M. J. D. Powell, "The Convergence of Variable Metric Methods for Nonlinearly Constrained Optimization Calculations," presented at "Nonlinear Programming 3," Aug. 15, 1977, Madison Wisconsin.
  22. M. J. D. Powell, "Algorithms for Nonlinearly Constraints that Use Lagrangian Functions," *Math. Prog.*, Vol. 14, 1978, pp. 224-248.
  23. S. P. Han, "Superlinearly Convergent Variable Metric Algorithms for General Nonlinear Programming Problems," *Math. Prog.*, Vol. 11, 1976, pp. 263-282.
  24. R. A. Tapia, "Diagonalized Multiplier Methods and Quasi-Newton Methods for Constrained Optimization," *J. Optim. Th. and Appl.*, Vol. 22, No. 2, June 1977, pp. 135-194.
  25. R. T. Rockafellar, "Augmented Lagrange Multiplier Functions and Duality in Nonconvex Programming," *SIAM J. Control*, Vol. 12, No. 2, May 1974, pp. 268-285.
  26. D. P. Bertsekas, "Multiplier Methods: A Survey," *Automatica*, Vol. 12, pp. 133-145, 1976.
  27. P. L. Yu, "A Class of Solutions for Group Decision Problems," *Manag. Science*, Vol. 19, No. 8, April 1973.
  28. P. L. Yu and G. Leitmann, "Compromise Solutions, Domination Structures, and Salukvadze's Solution," *J. Optira. Th. and Appl.*, Vol. 13, No. 3, pp. 362-378, 1974.
  29. V. J. Bowman, Jr., "On the Relationship of the Tchebycheff Norm and the Efficient Frontier of Multiple-Criteria Objectives," in Reference #8, in this paper.
-

FIGURE CAPTIONS

- Fig. 1 Geometric Interpretation in Output Space of Weighted Sum Minimization.
- Fig. 2 Noninferior Solutions Not Attainable by Weighted Sum Minimization Due to Nonconvexity.
- Fig. 3 Constraint Method for Finding all Noninferior Solutions.
- Fig. 4 Portions of the Level Sets of the Weighted  $L_\infty$  Norm Associated with Two Different Noninferior Solutions.
- Fig. 5 Interior Simplicial Decomposition of the Simplicial Approximation to the Feasible Region in Input Space.
- Fig. 6 Division of the  $i^{\text{th}}$  Interior Simplex into Regions for Piecewise Linear Approximation of the Yield.
- Fig. 7 Simplicial Approximation of Eqn. (22) Showing Extremes of  $f_1$ ,  $f_2$ , and Yield.
- Fig. 8 Noninferior Points for  $f_1$  and  $f_2$  exclusive of Yield.
- Fig. 9 Noninferior Points for a Standard Deviation of 1, for the MCO Problem Including Yield.
- Fig. 10 Noninferior Points for a Standard Deviation of 2, for the MCO Problem Including Yield.
-

| PL Approximation       | Monte Carlo (1000 Samples) |
|------------------------|----------------------------|
| Standard Deviation = 1 |                            |
| 37.68                  | 42.2                       |
| 46.78                  | 48.8                       |
| 96.44                  | 94.9                       |
| 93.22                  | 93.5                       |
| 90.24                  | 91.8                       |
| Standard Deviation = 2 |                            |
| 26.97                  | 29.1                       |
| 42.9                   | 44.9                       |
| 62.47                  | 63.7                       |
| 56.61                  | 56.2                       |
| 60.62                  | 60.8                       |

Table 1 Comparison of Piecewise Linear Approximation of Yield Using 10 Divisions of Each Interior Simplex with a 1000 Sample Monte Carlo Over the Simplicial Approximation.

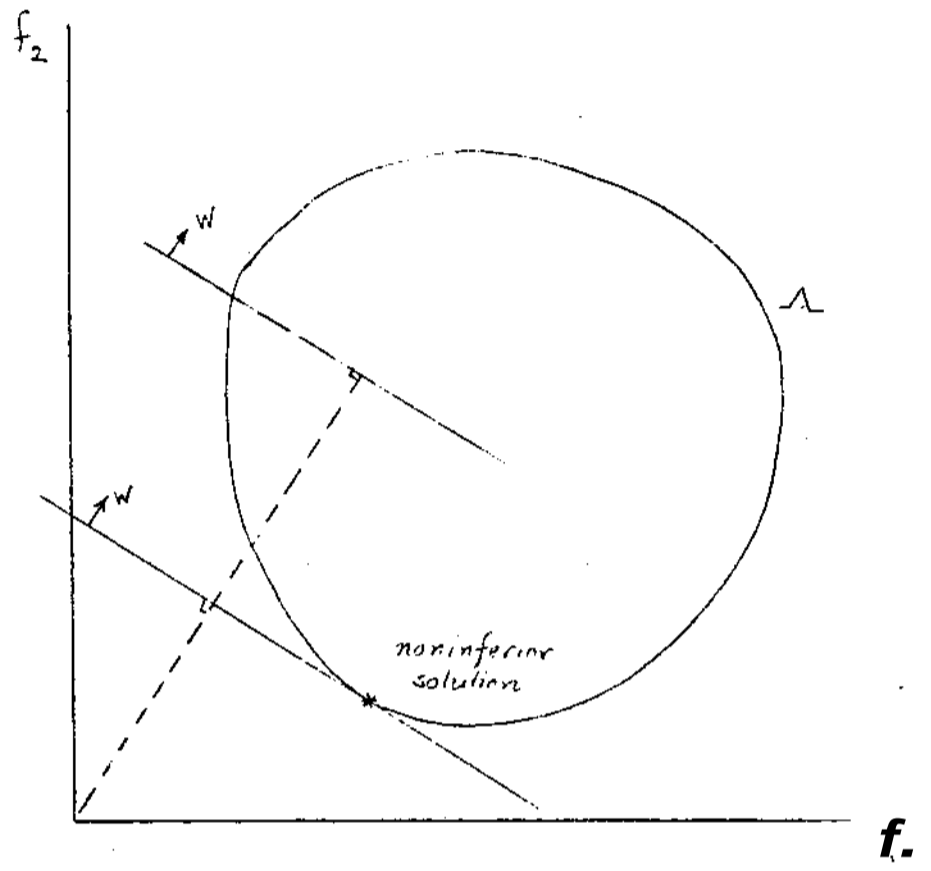
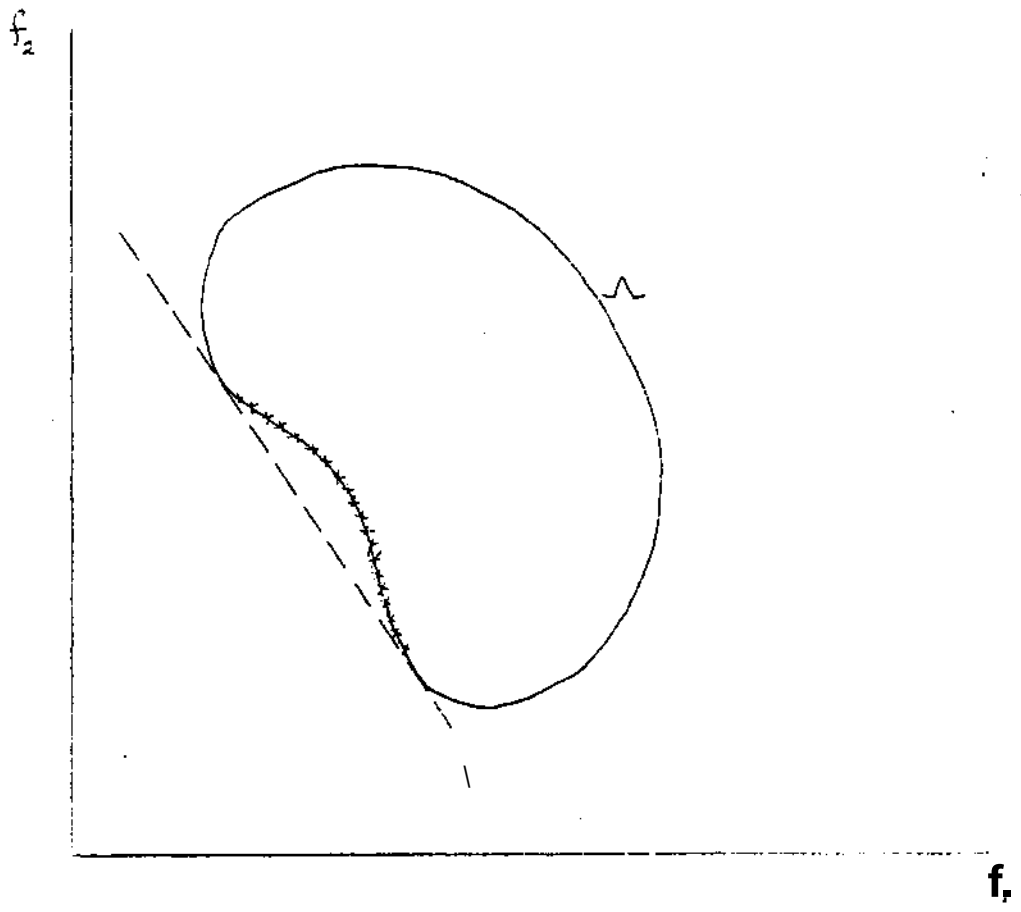


Fig. 1





$z$

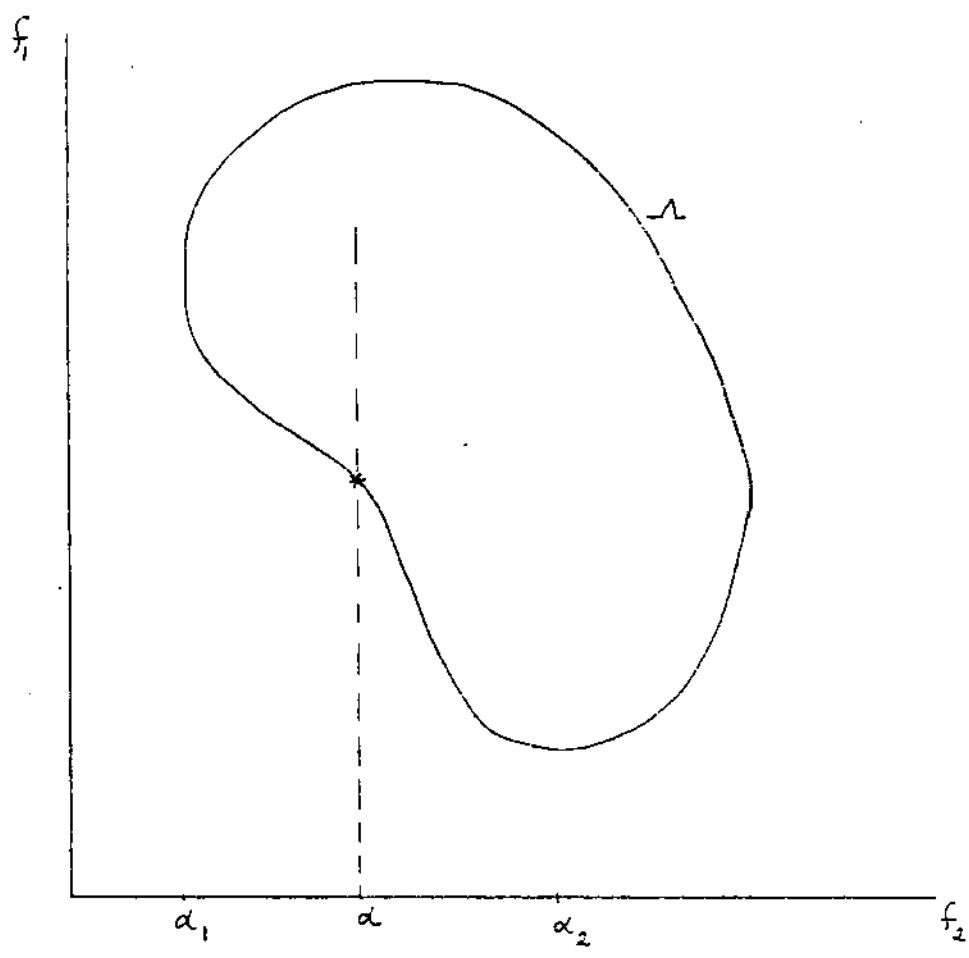


Fig 2

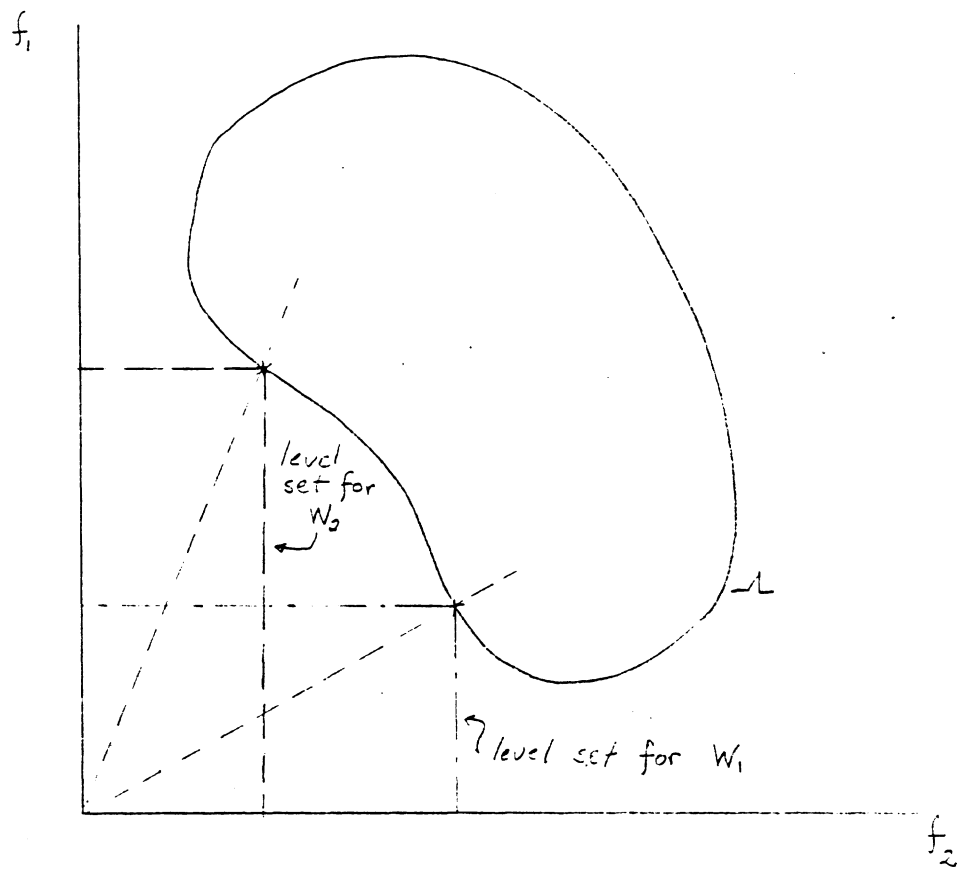


Fig. 41

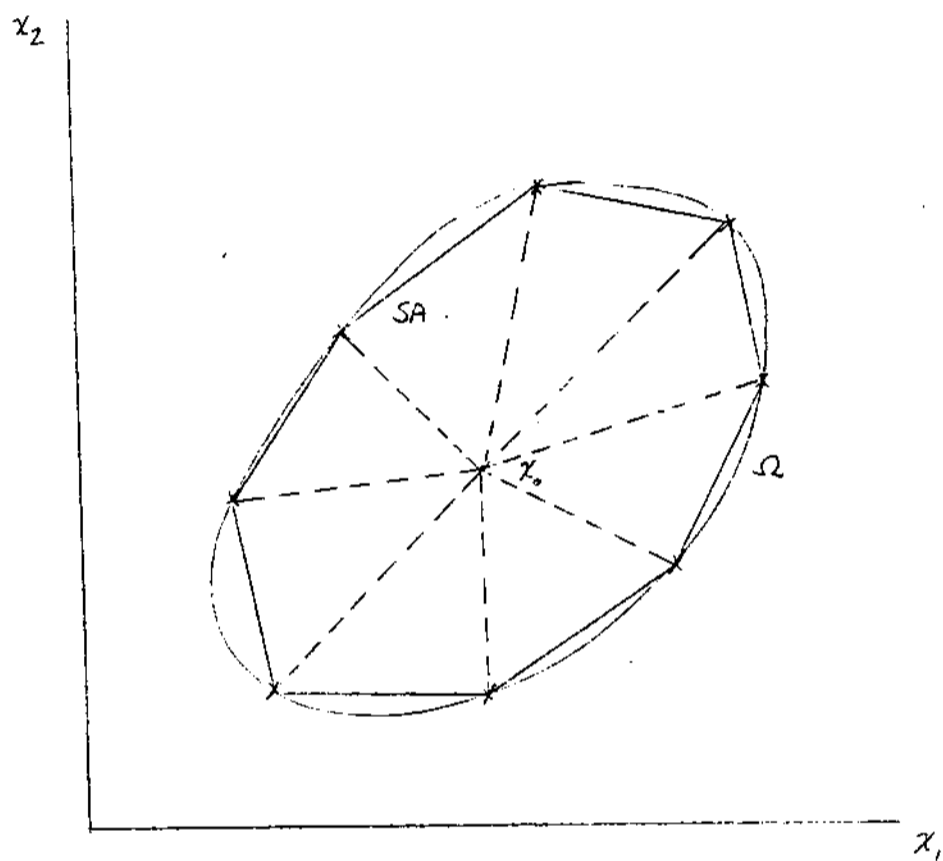


Fig 5

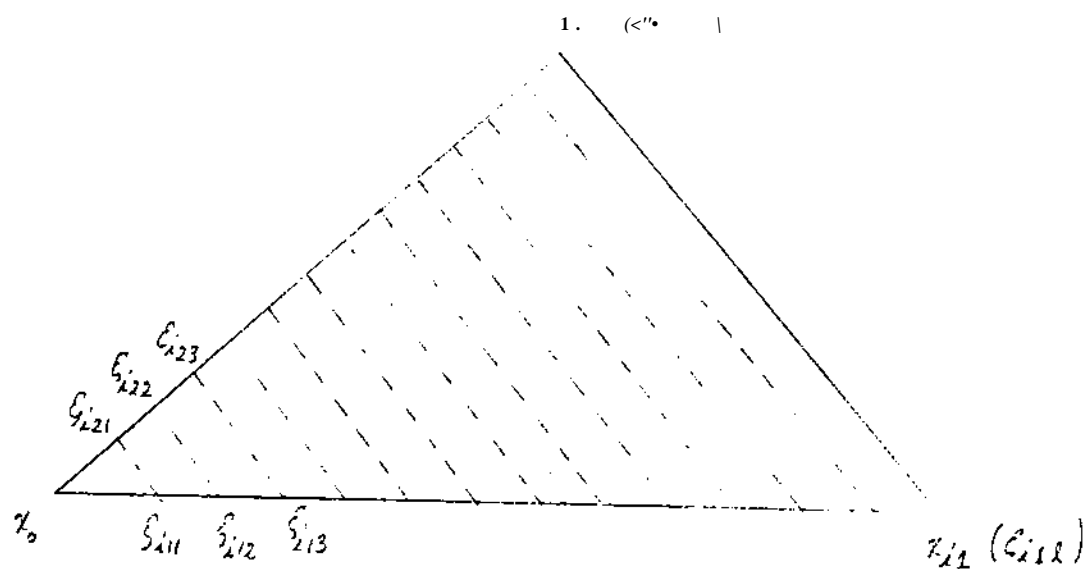


Fig 6

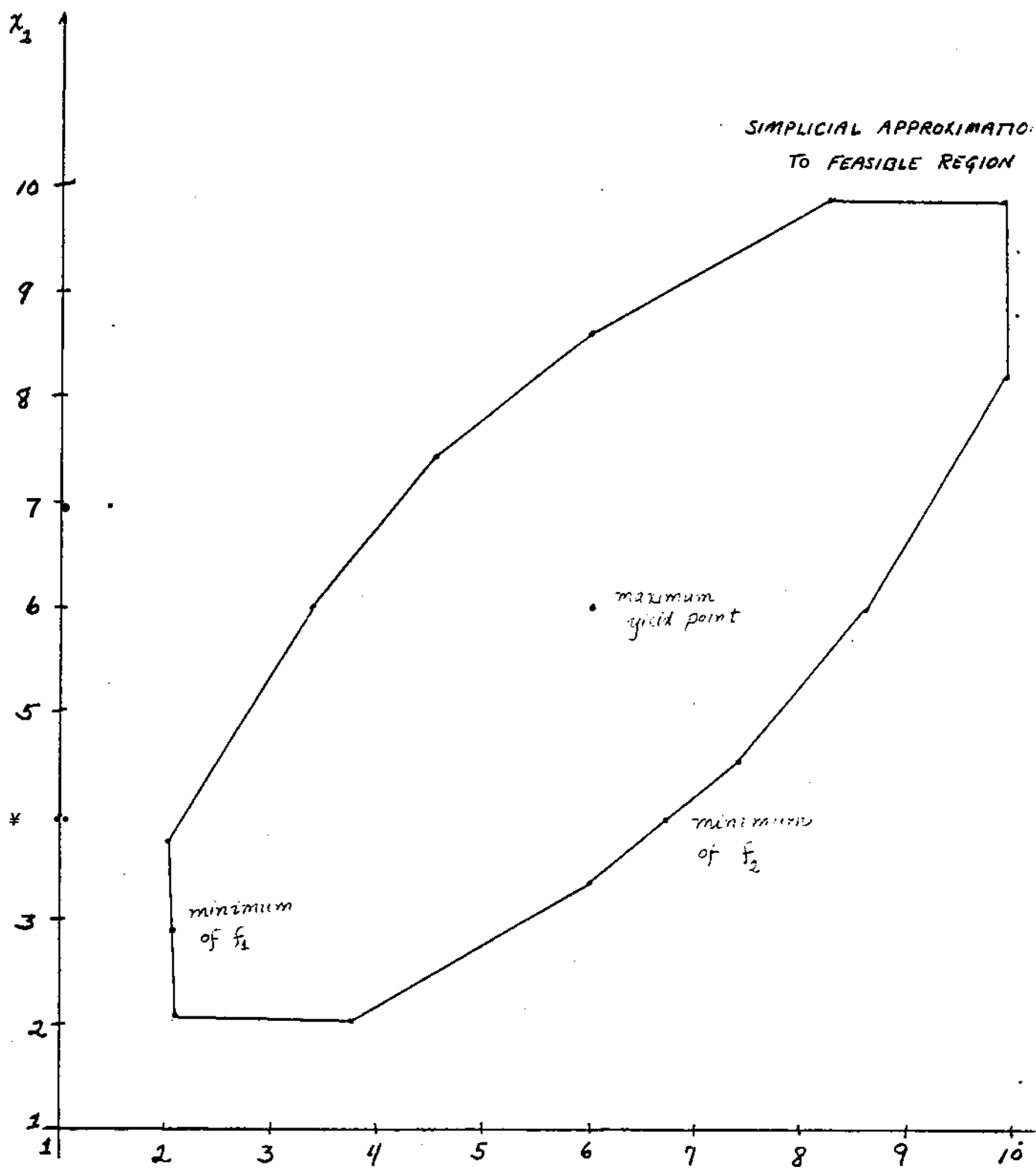
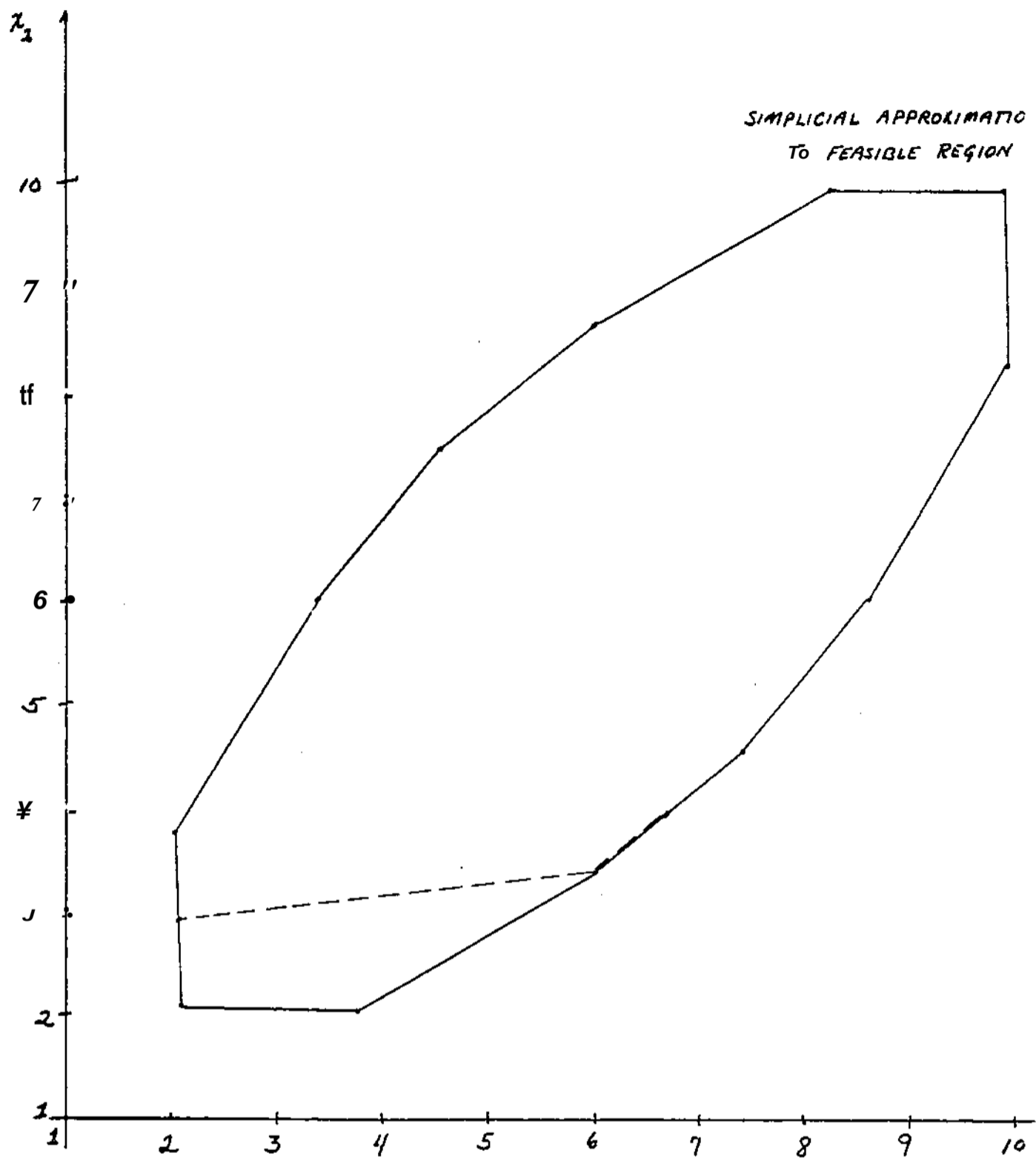


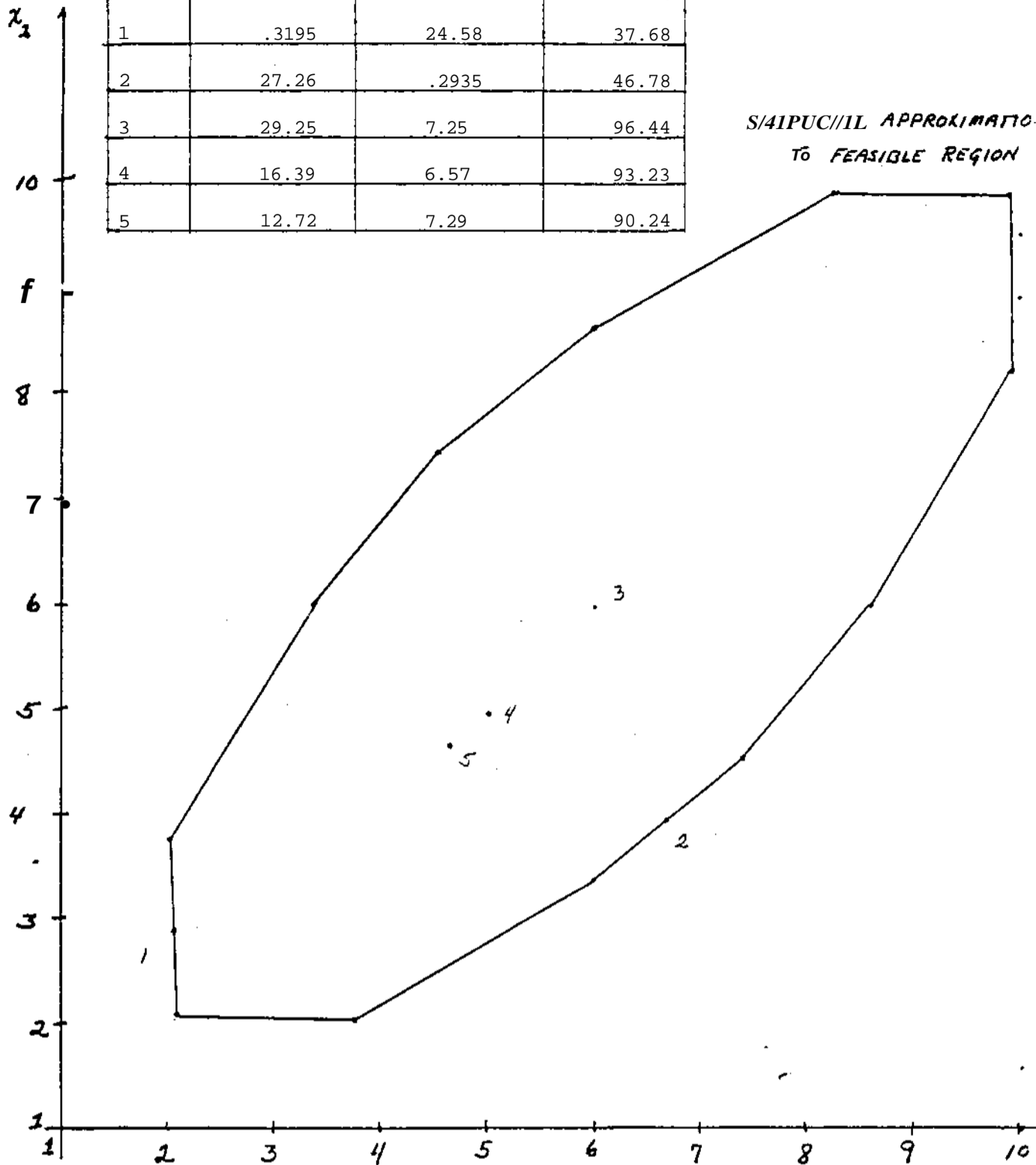
Fig 1



NONINFERRIOR POINTS

| PT | F1    | F2    | YIELD |
|----|-------|-------|-------|
| 1  | .3195 | 24.58 | 37.68 |
| 2  | 27.26 | .2935 | 46.78 |
| 3  | 29.25 | 7.25  | 96.44 |
| 4  | 16.39 | 6.57  | 93.23 |
| 5  | 12.72 | 7.29  | 90.24 |

S/4IPUC//IL APPROXIMATION  
TO FEASIBLE REGION





MINIMIZING VLANTS

| PT | F1    | F2    | YIELD |
|----|-------|-------|-------|
| 1  | .3195 | 24.58 | 26.97 |
| 2  | 27.26 | .2935 | 42.9  |
| 3  | 29.25 | 7.25  | 62.47 |
| 4  | 25.28 | 2.41  | 56.61 |
| 5  | 18.86 | 6.53  | 60.62 |

SIMPLICIAL APPROXIMATION  
TO FEASIBLE REGION

