A geometric perspective on lifting

Michele Conforti
Università di Padova, conforti@math.unipd.it

Gérard Cornuéjols *
Carnegie Mellon University and Université d'Aix-Marseille, gc0v@andrew.cmu.edu

Giacomo Zambelli
Università di Padova, giacomo@math.unipd.it

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Abstract

Recently, it has been shown that minimal inequalities for a continuous relaxation of mixed integer linear programs are associated with maximal lattice-free convex sets. In this paper we show how to lift these inequalities for integral nonbasic variables by considering maximal lattice-free convex sets in a higher-dimensional space. We apply this approach to several examples. In particular we identify cases where the lifting is sequence-independent, and therefore unique.

1 Introduction

Let S be the set of integral points in some rational polyhedron in $\mathbb{R}^n$ such that $\dim(S) = n$. We consider the following semi-infinite relaxation to a general MILP

$$\begin{aligned} x = \quad & f + \sum_{r\in\mathbb{R}^n} r s_r + \sum_{r\in\mathbb{R}^n} r y_r \\ & x \in S \\ & s_r \geq 0, \qquad\qquad r \in \mathbb{R}^n \\ & y_r \geq 0,\ y_r \in \mathbb{Z}, \quad r \in \mathbb{R}^n \\ & s, y \text{ have finite support.} \end{aligned} \tag{1}$$

Given two functions ψ and π from $\mathbb{R}^n$ to $\mathbb{R}$, the inequality

$$\sum_{r\in\mathbb{R}^n} \psi(r) s_r + \sum_{r\in\mathbb{R}^n} \pi(r) y_r \geq 1 \tag{2}$$

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is *valid* for (1) if it holds for every (x, s, y) satisfying (1). If (2) is valid, we say that the function (ψ, π) is *valid* for (1). A valid function (ψ, π) is *minimal* if there is no valid function (ψ', π') distinct from (ψ, π) such that $\psi'(r) \leq \psi(r)$, $\pi'(r) \leq \pi(r)$ for all $r \in \mathbb{R}^n$.

The following simpler model has been studied recently [9]

$$\begin{aligned} x = \quad & f + \sum_{r \in \mathbb{R}^n} r s_r \\ & x \in S \\ & s_r \geq 0, \quad r \in \mathbb{R}^n \\ & s \text{ has finite support.} \end{aligned} \tag{3}$$

We refer to this model as the *continuous semi-infinite relaxation relative to f*. Given a valid function ψ for (3), the function π is a *lifting* of ψ if (ψ, π) is valid for (1).

Minimal valid inequalities for (3) are well understood in terms of maximal S-free convex sets. We are interested in characterizing liftings of minimal valid inequalities for (3).

If ψ is a minimal valid function for (3) and π is a lifting of ψ such that (ψ, π) is minimal, we say that π is a *minimal lifting* of ψ.

We remark that, given any valid function ψ for (3) and a lifting π of ψ, the function π' defined by $\pi'(r) = \min\{\psi(r), \pi(r)\}$ is also a lifting for ψ. Indeed, given $(\bar{s}, \bar{y})$ satisfying (1), we show that $\sum_{r \in \mathbb{R}^n} \psi(r)\bar{s}_r + \sum_{r \in \mathbb{R}^n} \pi'(r)\bar{y}_r \geq 1$. Let $(\tilde{s}, \tilde{y})$ be defined by $\tilde{s}_r = \bar{s}_r$, $\tilde{y}_r = \bar{y}_r$ for every $r \in \mathbb{R}^n$ such that $\pi(r) \leq \psi(r)$, and $\tilde{s}_r = \bar{s}_r + \bar{y}_r$, $\tilde{y}_r = 0$ for every $r \in \mathbb{R}^n$ such that $\psi(r) < \pi(r)$. One can readily verify that $(\tilde{s}, \tilde{y})$ satisfies (1), hence $\sum_{r \in \mathbb{R}^n} \psi(r)\tilde{s}_r + \sum_{r \in \mathbb{R}^n} \pi(r)\tilde{y}_r \geq 1$. Furthermore, $\sum_{r \in \mathbb{R}^n} \psi(r)\bar{s}_r + \sum_{r \in \mathbb{R}^n} \pi'(r)\bar{y}_r = \sum_{r \in \mathbb{R}^n} \psi(r)\tilde{s}_r + \sum_{r \in \mathbb{R}^n} \pi(r)\tilde{y}_r \geq 1$

In particular, if ψ is a minimal valid function for (3) and π is a minimal lifting of ψ, then $\pi \leq \psi$.

We first concentrate on deriving the best possible lifting coefficient of one single integer variable. Namely, given $d \in \mathbb{R}^n$, we consider the model

$$\begin{aligned} x = \quad & f + \sum_{r \in \mathbb{R}^n} r s_r + dz \\ & x \in S \\ & s_r \geq 0, \qquad\qquad r \in \mathbb{R}^n \\ & z \geq 0,\ z \in \mathbb{Z}, \\ & s \text{ has finite support.} \end{aligned} \tag{4}$$

Given a minimal valid function ψ for (3), we want to determine the minimum scalar λ such that the inequality

$$\sum_{r \in \mathbb{R}^n} \psi(r) s_r + \lambda z \geq 1$$

is valid for (4). Given $d \in \mathbb{R}^n$, let $\pi_\ell(d)$ be such minimum λ. By definition, $\pi_\ell \leq \pi$ for every lifting π of ψ. In general, the function (ψ, π_ℓ) is not valid for (1). However, when (ψ, π_ℓ) is valid, π_ℓ is the unique minimal lifting for ψ.

In this paper we give a geometric characterization of the function π_ℓ, and use this characterization to analyze specific functions ψ in which π_ℓ is the unique minimal lifting.

A valid function (ψ, π) is *extreme* for (1) if there do not exist distinct valid functions (ψ^1, π^1), (ψ^2, π^2) such that $(\psi, \pi) = \frac{1}{2}(\psi^1, \pi^1) + \frac{1}{2}(\psi^2, \pi^2)$. Note that if ψ is extreme for (3), then ψ is minimal.

Remark 1. *If ψ is extreme for* (3) *and π_ℓ is a lifting for ψ, then (ψ, π_ℓ) is extreme for* (1).

Indeed, given valid functions (ψ^1, π^1), (ψ^2, π^2) such that $(\psi, \pi) = \frac{1}{2}(\psi^1, \pi^1) + \frac{1}{2}(\psi^2, \pi^2)$, then $\psi_1 = \psi_2 = \psi$, since ψ is extreme for (3), and $\pi_1 = \pi_2 = \pi_\ell$ since $\pi_1 \geq \pi_\ell$ and $\pi_2 \geq \pi_\ell$.

2 Lifting and S-free convex sets

We observe that (4) is equivalent to the following

$$\begin{aligned} \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} = \; & \begin{pmatrix} f \\ 0 \end{pmatrix} + \sum_{r \in \mathbb{R}^n} \begin{pmatrix} r \\ 0 \end{pmatrix} s_r + \begin{pmatrix} d \\ 1 \end{pmatrix} z \\ & (x, x_{n+1}) \in S \times \mathbb{Z}_+ \\ & s_r \geq 0, \quad r \in \mathbb{R}^n \\ & z \geq 0, \\ & s \text{ has finite support.} \end{aligned} \tag{5}$$

Indeed (x, s, z) is a solution for (4) if and only if (x, x_{n+1}, s, z) is a solution to (5) by setting $x_{n+1} = z$. Note that the above is obtained from the continuous semi-infinite relaxation relative to $\binom{f}{0}$ by setting to 0 all variables relative to rays with nonzero $(n+1)$-th component, except for $\binom{d}{1}$. Therefore, given any valid function $\bar{\psi}$ for the continuous semi-infinite relaxation relative to $\binom{f}{0}$, then if we let $\psi(r) = \bar{\psi}\binom{r}{0}$ for $r \in \mathbb{R}^n$ and $\lambda = \bar{\psi}\binom{d}{1}$, the inequality $\sum_{r \in \mathbb{R}^n} \psi(r) s_r + \lambda z \geq 1$ is valid for (5) and for (4).

A convex set is *S-free* if it does not contain any point of S in its interior. Maximal S-free convex sets were characterized in [4], where it was also shown that there is a one-to-one correspondence between minimal valid functions for (3) and maximal S-free convex sets with f in their interior.

Theorem 2. *A full-dimensional convex set B is a maximal S-free convex set if and only if it is a polyhedron such that B does not contain any point of S in its interior and each facet of B contains a point of S in its relative interior. Furthermore if $B \cap \operatorname{conv}(S)$ has nonempty interior,* $\operatorname{lin}(B)$ *contains* $\operatorname{rec}(B \cap \operatorname{conv}(S))$.

We explain how minimal valid inequalities for (3) arise from maximal S-free convex sets. Let B a polyhedron with f in its interior, and let $a_1, \ldots, a_t \in \mathbb{R}^q$ such that $B = \{x \in \mathbb{R}^n \mid a_i(x - f) \leq 1,\, i = 1 \ldots, t\}$. We define the function $\psi_B : \mathbb{R}^n \to \mathbb{R}$ by

$$\psi_B(r) = \max_{i=1,\ldots,t} a_i r.$$

Note that the function ψ_B is *subadditive*, i.e. $\psi_B(r) + \psi_B(r') \geq \psi_B(r + r')$, and *positively homogeneous*, i.e. $\psi_B(\lambda r) = \lambda \psi_B(r)$ for every $\lambda \geq 0$.

We claim that, if B is a maximal S-free convex set,

$$\sum_{r\in\mathbb{R}^n} \psi_B(r)s_r \geq 1 \quad \text{is valid for (3).} \tag{6}$$

Indeed, let (x, s) be a solution of (3). Note that $x \in S$, thus $x \notin \mathbf{int}(B)$. Then

$$\sum_{r\in\mathbb{R}^n} \psi_B(r)s_r = \sum_{r\in\mathbb{R}^n} \psi_B(rs_r) \geq \psi_B(\sum_{r\in\mathbb{R}^n} rs_r) = \psi_B(x-f) \geq 1,$$

where the first equation follows from positive homogeneity, the first inequality follows from subadditivity of ψ_B and the last one follows from the fact that $x \notin \mathbf{int}(B)$.

The above functions are minimal [4],[8]. It was proved in [4] that the converse is also true, namely that every minimal function valid for (3) is of the form ψ_B where B is a maximal S-free convex set with f in its interior.

Example. We consider problem (1) when $n = 1$, $0 < f < 1$ and $S = \mathbb{Z}$. In this case the only maximal S-free convex set containing f is the interval $B = [0, 1]$. Thus $B = \{x \in \mathbb{R} \,|\, -f^{-1}(x-f) \leq 1,\, (1-f)^{-1}(x-f) \leq 1\}$ and $\psi_B(r) = \max\{-f^{-1}r, (1-f)^{-1}r\}$.

Let ψ be a minimal valid function for (3), and let $B = \{x \in \mathbb{R}^n \,|\, a_i(x-f) \leq 1,\ i = 1, \ldots, t\}$ be a maximal S-free convex set with f in its interior such that $\psi = \psi_B$. We define the set $B(\lambda) \subset \mathbb{R}^{n+1}$ as follows

$$B(\lambda) = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, a_i(x-f) + (\lambda - a_i d)x_{n+1} \leq 1,\ i = 1, \ldots, t\}. \tag{7}$$

Theorem 3. *The inequality* $\sum_{r\in\mathbb{R}^n} \psi(r)s_r + \lambda z \geq 1$ *is valid for* (4) *if and only if* $B(\lambda)$ *is* $(S \times \mathbb{Z}_+)$*-free.*

Proof. Let $\bar{\psi} = \psi_{B(\lambda)}$. By construction, $\bar{\psi}\binom{r}{0} = \psi(r)$ for all $r \in \mathbb{R}^n$, while $\bar{\psi}\binom{d}{1} = \lambda$.

We show the "if" part of the statement. Given λ such that $B(\lambda)$ is $(S \times \mathbb{Z}_+)$-free, it follows by claim (6) that the function $\bar{\psi}$ is valid for the continuous semi-infinite relaxation relative to $\binom{f}{0}$. This implies that $\sum_{r\in\mathbb{R}^n} \psi(r)s_r + \lambda z \geq 1$ is valid for (4).

We now prove the "only if" part. Let λ be such that $\sum_{r\in\mathbb{R}^n} \psi(r)s_r + \lambda z \geq 1$ is valid for (4). Given a point $\binom{\bar{x}}{\bar{x}_{n+1}} \in S \times \mathbb{Z}_+$, we show that such point is not in the interior of $B(\lambda)$. Indeed, let $\bar{r} = \bar{x} - \bar{x}_{n+1}d - f$, $\bar{z} = \bar{x}_{n+1}$, and $(\bar{s}_r)_{r\in\mathbb{R}^n}$ be defined by

$$\bar{s}_r = \begin{cases} 1 & \text{if } r = \bar{r}, \\ 0 & \text{otherwise .} \end{cases}$$

Note that $f + \sum_{r\in\mathbb{R}^n} r\bar{s}_r + d\bar{z} = f + \bar{r} + \bar{x}_{n+1}d = \bar{x}$. Since $\bar{x} \in S$ and $\sum_{r\in\mathbb{R}^n} \psi(r)s_r + \lambda z \geq 1$ is valid for (4), we have

$$\begin{aligned} 1 \ &\leq \ \sum_{r\in\mathbb{R}^n} \psi(r)\bar{s}_r + \lambda\bar{z} = \psi(\bar{r}) + \lambda\bar{x}_{n+1} = \max_{i=1,\ldots,t} a_i\bar{r} + \lambda\bar{x}_{n+1} \\ &= \ \max_{i=1,\ldots,t} [a_i(\bar{x}-f) + (\lambda - a_i d)\bar{x}_{n+1}]. \end{aligned}$$

Thus there exists $i \in \{1, \ldots, t\}$ such that $a_i(\bar{x} - f) + (\lambda - a_i d)\bar{x}_{n+1} \geq 1$. This shows that $\binom{\bar{x}}{\bar{x}_{n+1}}$ is not in the interior of $B(\lambda)$. □

Example (continued). In the previous example, let $d \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. If $\lambda \neq 0$, then the set $B(\lambda)$ is the 2-dimensional polyhedron with two facets, containing the points $\binom{0}{0}$ and $\binom{1}{0}$ respectively and with one vertex, namely $\binom{f}{0} + \lambda^{-1}\binom{d}{1}$. If $\lambda = 0$, then $B(\lambda)$ is the split $[0,1] + \langle\binom{d}{1}\rangle$. It is immediate to verify that, for $\lambda < 0$, the interior of $B(\lambda)$ contains one of the integral points $\binom{\lfloor d \rfloor}{1}$ or $\binom{\lceil d \rceil}{1}$.
For example, let $f = \frac{1}{4}$. For $d = \frac{3}{2}$, $\psi_B(d) = 2$. One can readily verify that $B(\lambda)$ is $\mathbb{Z}\times\mathbb{Z}_+$-free if and only if $\lambda \geq \frac{2}{3}$, otherwise it contains the point $\binom{2}{1}$. Hence $\pi_\ell(d) = \frac{2}{3}$.
For $d = 1$, $\psi_B(d) = \frac{4}{3}$. It is immediate that $B(\lambda)$ is $\mathbb{Z} \times \mathbb{Z}_+$-free if and only if $\lambda \geq 0$, hence $\pi_\ell(d) = 0$.

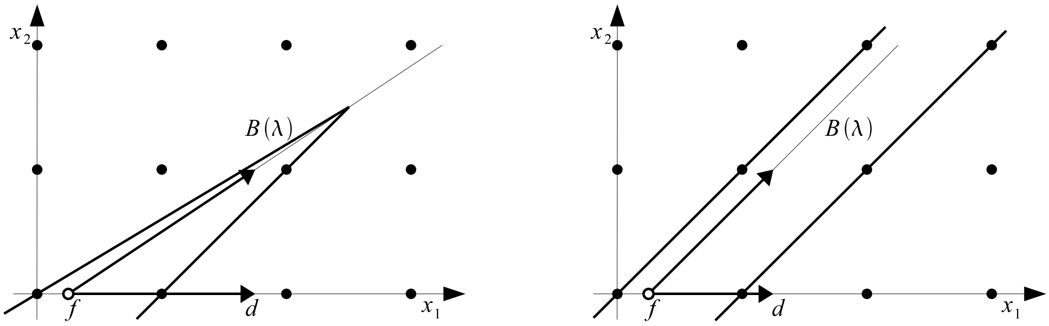

Figure 1: Example: $f = \frac{1}{4}$. Left: $d = \frac{3}{2}$. Right: $d = 1$.

Theorem 4. *Let ψ be a minimal valid function for (3) and π be a minimal lifting of ψ. Then there exists $\varepsilon > 0$ such that ψ, π and π_ℓ coincide on the ball of radius ε centered at the origin.*

Proof. Since ψ is a minimal valid function for (3), there exists a maximal S-free convex set $B = \{x \in \mathbb{R}^n \mid a_i(x - f) \leq 1,\ i = 1, \dots, t\}$ such that $\psi = \psi_B$.

Let

$$\alpha = \max_{1 \leq i,j \leq t} \max_{\|r\|=1} (a_i - a_j)r$$

Since B is a maximal S-free convex set, every facet of B contains a point of S in its relative interior. Hence, for $i = 1, \dots, t$, there exists $x^i \in S$ such that $a_i(x^i - f) = 1$ and $a_j(x^i - f) \leq 1 - \gamma_i$, $j \neq i$, for some positive γ_i. Let $\varepsilon > 0$ such that $\varepsilon\alpha \leq \gamma_i$ for $i = 1, \dots, t$.

Let $d \in \mathbb{R}^n$ such that $\|d\| \leq \varepsilon$. We will show that, for every $\lambda < \psi(d)$, $B(\lambda)$ contains a point of $S \times \mathbb{Z}_+$ in its interior. By Theorem 3, this implies that $\pi_\ell(d) \geq \psi(d)$. Since $\pi_\ell \leq \pi \leq \psi$, this implies $\pi_\ell(d) = \pi(d) = \psi(d)$.

Let i, $1 \leq i \leq t$, such that $\psi(d) = a_i d$. Let $\lambda = \psi(d) - \delta$ for some $\delta > 0$. We show that $B(\lambda)$ contains the point $\binom{x^i}{1}$ in its interior. Indeed, by (7), $B(\lambda)$ is the set of points in $\mathbb{R}^{n+1}$ satisfying the inequalities

$$a_j(x - f) + [(a_i - a_j)d - \delta]x_{n+1} \leq 1, \quad j = 1, \dots, t.$$

Substituting $\binom{x^i}{1}$ we obtain

$$\begin{array}{rcl} a_i(x^i - f) \qquad\qquad - \delta & < & 1, \\ a_j(x^i - f) + (a_i - a_j)d - \delta & < & 1, \quad j = 1, \dots, t,\, j \neq i, \end{array}$$

where the first inequality follows from $a_i(x^i - f) = 1$, while the second follows from $a_j(x^i - f) \leq 1 - \gamma_i$, $\|d\| \leq \varepsilon$, and $(a_i - a_j)(d/\|d\|) \leq \alpha$ by our choice of α.

Thus $\binom{x^i}{1}$ is in the interior of $B(\lambda)$. □

Example (continued). From the previous example where $n = 1$, $0 < f < 1$ and $S = \mathbb{Z}$, note that $\pi_\ell(d) = \psi_B(d)$ for every $d \in [-f, 1 - f]$. Indeed, if $d < 0$, then $B(\lambda)$ contains $\binom{0}{1}$ for all $\lambda < \psi_B(d)$, while if $d \geq 0$ then $B(\lambda)$ contains $\binom{1}{1}$ for all $\lambda < \psi_B(d)$. Furthermore, for $\lambda = \psi_B(d)$, if $d < 0$ the facet of $B(\lambda)$ containing $\binom{0}{0}$ is vertical and contains the point $\binom{0}{1}$, if $d \geq 0$ then the facet of $B(\lambda)$ containing $\binom{1}{0}$ is vertical and contains the point $\binom{1}{1}$.

Theorem 4 implies that, for every minimal valid function ψ for (3), there exists a region $R_\psi \subseteq \mathbb{R}^n$ containing the origin in its interior such that ψ and π coincide in R_ψ for every minimal lifting π of ψ for (1).

Lemma 5. *Let ψ be a minimal valid function, and π be a minimal lifting of ψ. Then*
i) For every $r \in \mathbb{R}^n$ and $w \in \mathbb{Z}^n \cap \mathrm{lin}(\mathrm{conv}(S))$, $\pi(r) = \pi(r + w)$.
ii) For every $r \in \mathbb{R}^n$ such that $r + w \in R_\psi$ for some $w \in \mathbb{Z}^n \cap \mathrm{lin}(\mathrm{conv}(S))$, $\pi(r) = \psi(r + w)$.

Proof. i) Let $\bar{r} \in \mathbb{R}^n$ and $w \in \mathbb{Z}^n \cap \mathrm{lin}(\mathrm{conv}(S))$. Suppose $\pi(\bar{r}) \neq \pi(\bar{r} + w)$. Since $-w \in \mathbb{Z}^n \cap \mathrm{lin}(\mathrm{conv}(S))$, we may assume $\pi(\bar{r}) > \pi(\bar{r} + w)$. Since $w \in \mathbb{Z}^n \cap \mathrm{lin}(\mathrm{conv}(S))$, then a point $x \in \mathbb{R}^n$ is in S if and only if $x + w \in S$. Thus a point $(\bar{x}, \bar{s}, \bar{y})$ satisfies (1) if and only if $(\bar{x} + w\bar{y}_{\bar{r}}, \bar{s}, \tilde{y})$ satisfies (1), where $\tilde{y}_{\bar{r}} = 0$, $\tilde{y}_{\bar{r}+w} = \bar{y}_{\bar{r}+w} + \bar{y}_{\bar{r}}$, and $\tilde{y}_r = \bar{y}_r$ for every $r \in \mathbb{R}^n \setminus \{\bar{r}, \bar{r} + w\}$. This shows that the function π' defined by $\pi'(\bar{r}) = \pi(\bar{r} + w)$, $\pi'(r) = \pi(r)$ for every $r \in \mathbb{R}^n \setminus \{\bar{r}\}$ is a lifting of ψ, contradicting the minimality of π.

ii) It follows from *i)* that $\pi(r) = \pi(r + w)$. By definition of R_ψ, $\pi(r + w) = \psi(r + w)$. □

The above lemma implies the following result.

Theorem 6. *If for every $r \in \mathbb{R}^n$ there exists $w^r \in \mathbb{Z}^n \cap \mathrm{lin}(\mathrm{conv}(S))$ such that $r + w \in R_\psi$, then there exists a unique minimal lifting for ψ, namely the function π defined by $\pi(r) = \psi(r + w^r)$. Furthermore $\pi = \pi_\ell$.*

Note that, if for some $r \in R_\psi$ there exists $w \in \mathbb{Z}^n \cap \mathrm{lin}(\mathrm{conv}(S))$ such that $r + w \in R_\psi$, then $\psi(r + w) = \psi(r)$.

Example (continued). From the previous example where $n = 1$, $0 < f < 1$ and $S = \mathbb{Z}$, we have shown that $\psi(r) = \pi_\ell(r)$ for every $r \in [-f, 1 - f]$. Note that, for every $r \in \mathbb{R}$, $r - \lfloor r + f \rfloor \in [-f, 1 - f]$. Thus $\pi_\ell(r) = \psi(r - \lfloor r + f \rfloor)$ for all $r \in \mathbb{R}$, and π_ℓ is the unique minimal lifting for ψ. Thus $\pi_\ell(r) = \max\{-f^{-1}(r - \lfloor r + f \rfloor), (1 - f)^{-1}(r - \lfloor r + f \rfloor)\}$. More explicitly, if $r - \lfloor r \rfloor < 1 - f$, then $\pi_\ell(r) = \frac{r - \lfloor r \rfloor}{1 - f}$, while if $r - \lfloor r \rfloor \geq 1 - f$, $\pi_\ell(r) = \frac{\lceil r \rceil - r}{f}$.

Given a tableau row $x = f + \sum_{i=1}^{h} p^i s_i + \sum_{j=1}^{k} q^j y_j$, where $s_i \geq 0$, $i = 1, \dots, h$, and $y_j \geq 0$ and integer, $j = 1, \dots, h$, the inequality $\sum_{i=1}^{h} \psi(p^i) s_i + \sum_{j=1}^{k} \pi_\ell(q^j) y_j \geq 1$ is

$$\sum_{\substack{i=1 \\ p^i \geq 0}}^{h} \frac{p^i}{1-f} s_i + \sum_{\substack{i=1 \\ p^i < 0}}^{h} -\frac{p^i}{f} s_i + \sum_{\substack{j=1 \\ q^j - \lfloor q^j \rfloor < 1-f}}^{k} \frac{q^j - \lfloor q^j \rfloor}{1-f} y_j + \sum_{\substack{j=1 \\ q^j - \lfloor q^j \rfloor \geq 1-f}}^{k} \frac{\lceil q^j \rceil - q^j}{f} y_j \geq 1,$$

which is the Gomory Mixed Integer Cut associated with the tableau row.

3 Applications

3.1 Wedge inequalities

We consider the problem (1) where $n = 2$ and $S = \mathbb{Z} \times \mathbb{Z}_+$. We focus on inequalities arising from maximal S-free convex sets with 2 sides and one vertex. We call such sets *wedges*.

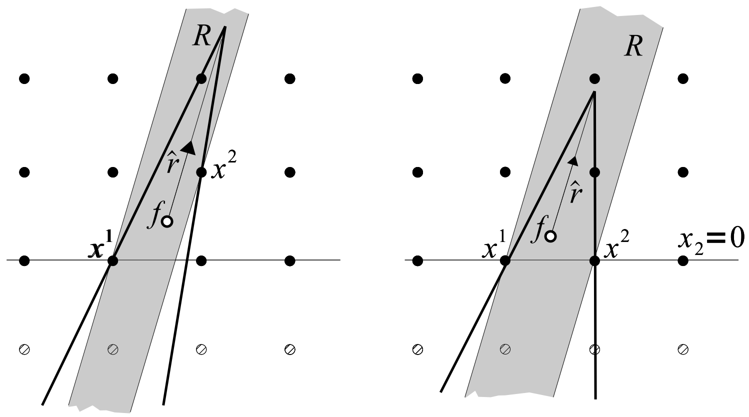

Figure 2: Wedges and corresponding region R shaded in gray. The inequality corresponding to the wedge on the right has a unique minimal lifting.

Let $B = \{x \in \mathbb{R}^2 \mid a_i(x - f) \leq 1,\ i = 1, 2\}$ be such a maximal S-free convex set. Since B is S-free, its only vertex must be in the interior of $\mathrm{conv}(S)$, $\mathrm{rec}(B)$ has dimension 2 and for every nonzero element $r \in \mathrm{rec}(B)$, $r_2 < 0$.

Note that $\mathrm{rec}(\mathrm{conv}(S)) = \mathbb{R} \times \mathbb{R}_+$ and B has empty lineality space. By Theorem 2, $\mathrm{lin}(B) \supseteq \mathrm{rec}(B \cap \mathrm{conv}(S))$, hence $\mathrm{rec}(B) \cap \mathrm{conv}(S) = \emptyset$. In particular, $(\mathbb{R} \times \{0\}) \cap \mathrm{rec}(B) = \emptyset$, thus by symmetry we may assume $a_1 \binom{1}{0} < 0$ and $a_2 \binom{1}{0} > 0$, that is $a_{11} < 0$ and $a_{21} > 0$.

Let $\hat{r}$ be a nonzero vector such that $a_1 \hat{r} = a_2 \hat{r}$. Note that any point $x \in \mathbb{R}^2$ can be uniquely written as $x = f + \alpha^x \hat{r} + \beta^x \binom{1}{0}$ where $\alpha^x, \beta^x \in \mathbb{R}$. Let $\bar{x} \in S$ be a point in the relative interior of one of the two facets of B, say $a_h(\bar{x} - f) = 1$, $a_k(\bar{x} - f) < 1$. Note that $0 > (a_k - a_h)(\bar{x} - f) = \beta^{\bar{x}}(a_{k1} - a_{h1})$, hence $\beta^{\bar{x}} < 0$ if $h = 1$ and $\beta^{\bar{x}} > 0$ if $h = 2$. Let x^1 be a point of S in the relative interior of the facet defined by $a_1(x - f) \leq 1$ such that β^{x^1} is largest possible, and x^2 be a point of S in the relative interior of the facet defined by $a_2(x - f) \leq 1$ such that β^{x^2} is smallest possible. Let $\beta_i = \beta^{x^i}$. Note that $\beta_1 < 0 < \beta_2$. We define the region $R = [\beta_1, \beta_2] + \langle \hat{r} \rangle$. (See Figure 2.)

Lemma 7. *For every $d \in R$, $\pi_\ell(d) = \psi_B(d)$.*

Proof. Let $d \in R$, that is $d = \alpha\hat{r} + \beta\binom{1}{0}$, for some $\alpha \in \mathbb{R}$ and $\beta \in [\beta_1, \beta_2]$. We consider the case $\beta \leq 0$. The case $\beta \geq 0$ is similar.

Note that $(a_1 - a_2)d = \alpha(a_1 - a_2)\hat{r} + \beta(a_{11} - a_{21}) \geq 0$ since $(a_1 - a_2)\hat{r} = 0$, $\beta \leq 0$, $a_{11} < 0$ and $a_{21} > 0$. Hence $\psi_B(d) = \max\{a_1 d, a_2 d\} = a_1 d$.

We will show that, for every $\lambda < \psi_B(d)$, the set $B(\lambda)$ defined in (7) contains the point $\binom{x^1}{1}$ in its interior. By Theorem 3, this will imply $\pi_\ell(d) \geq \psi_B(d)$, and thus $\pi_\ell(d) = \psi_B(d)$.

Let $\lambda = \psi_B(d) - \delta$ for some $\delta > 0$. Then $B(\lambda)$ is the set of $x \in \mathbb{R}^3$ satisfying

$$\begin{array}{rcl} a_1(x - f) \qquad\qquad\qquad - \delta x_3 & \leq & 1, \\ a_2(x - f) + (a_1 - a_2)dx_3 - \delta x_3 & \leq & 1. \end{array}$$

Substituting $\binom{x^1}{1}$ in the first inequality, we obtain $a_1(x^1 - f) - \delta = 1 - \delta < 1$. Substituting in the second inequality, we obtain

$$\begin{array}{rcl} a_2(x^1 - f) + (a_1 - a_2)d - \delta & = & \alpha^{x^1} a_2\hat{r} + \beta_1 a_{21} + \alpha(a_1 - a_2)\hat{r} + \beta(a_{11} - a_{21}) - \delta \\ & = & \alpha^{x^1} a_1\hat{r} + \beta_1 a_{11} + (\beta - \beta_1)(a_{11} - a_{21}) - \delta \\ & \leq & a_1(x^1 - f) - \delta = 1 - \delta < 1 \end{array}$$

where the first inequality in the last row follows from $\beta_1 \leq \beta$, $a_{11} < 0$, $a_{21} > 0$. Thus $\binom{x^1}{1}$ is in the interior of $B(\lambda)$. □

Let y^1 and y^2 be the intersection of the facets defined by $a_1(x - f) \leq 1$ and $a_2(x - f) \leq 1$, respectively, with the axis $x_2 = 0$. That is $a_1(y^1 - f) = 1$, $y^1_2 = 0$, and $a_2(y^2 - f) = 1$, $y^2_2 = 0$. Since B is S-free, $y^2_1 - y^1_1 \leq 1$, where equality holds if and only if y^1, y^2 are integral. Furthermore, it is not difficult to show that $\beta_2 - \beta_1 \leq y^2_1 - y^1_1$. Thus $\beta_2 - \beta_1 = 1$ if and only if y^1, y^2 are integral vectors. In this case, for every $r \in \mathbb{R}^2$ there exists $w^r \in \mathbb{Z} \times \{0\}$ such that $r + w^r \in R$. Since $\text{lin}(\text{conv}(S)) = \mathbb{R} \times \{0\}$, by Theorem 6, $\pi_\ell(r)$ is the unique minimal lifting of ψ_B, and $\pi_\ell(r) = \psi_B(r + w^r)$ for every $r \in \mathbb{R}^2$.

Dey and Wolsey [9] show that ψ_B is extreme for (3) if and only if B contains at least three points of S. Thus Remark 1 implies the following:

Theorem 8. *If B contains at least three points of S and $B \cap (\mathbb{R} \times \{0\})$ is an interval of length one, then (ψ_B, π_ℓ) is a valid extreme inequality for* (1).

3.2 Simplicial polytopes

In this section we focus on valid inequalities for (3) arising from maximal lattice-free simplicial polytopes, in the case where $S = \mathbb{Z}^n$. Recall that a polytope is *simplicial* if each of its facets is a simplex.

Let $B = \{x \in \mathbb{R}^n \,|\, a_i(x - f) \leq 1,\ i = 1, \ldots, t\}$ be an n-dimensional maximal lattice-free simplicial polytope and let $v^1, \ldots, v^p$ be its vertices. For $i = 1, \ldots, t$, let $V_i \subset \{1, \ldots, p\}$ be the set of indices of vertices of the facet defined by $a_i(x - f) \leq 1$, that is $a_i v^j = 1$ for all $j \in V_i$. Let $r^i = v^i - f$, $i = 1, \ldots, p$. Note that, since B is simplicial, $\{r^j \,|\, j \in F_i\}$ consists of

n linearly independent vectors, for $i = 1, \ldots, t$, and $a_i r^j = 1$ for all $j \in F_i$, while $a_i r^j < 1$ for all $j \notin F_i$.

Let $\bar{x}$ be an integral point in the interior of the facet defined by $a_i(x - f) \leq 1$, that is $a_i(\bar{x} - f) = 1$, $a_j(\bar{x} - f) < 1$, $j \neq i$. Then $\bar{x}$ can be uniquely written as $\bar{x} = f + \sum_{j \in F_i} \bar{\alpha}_j r^j$, where $\sum_{j \in F_i} \bar{\alpha}_j = 1$, $\bar{\alpha}_j \geq 0$, $j \in F_i$. Let $R(\bar{x}) = \{\sum_{j \in F_i} \alpha_j r^j \,|\, 0 \leq \alpha_j \leq \bar{\alpha}_j,\ j \in F_i\}$.

Let us denote by $\mathcal{I}$ the set of all points $\bar{x}$ in $\mathbb{Z}^n$ such that $\bar{x}$ is contained in the relative interior of some facet of B. Let $R = \cup_{\bar{x} \in \mathcal{I}} R(\bar{x})$.

Lemma 9. *For every $d \in R$, $\pi_\ell(d) = \psi_B(d)$.*

Proof. We only need to show that, given $\bar{x} \in \mathcal{I}$ and $d \in R(\bar{x})$, $\pi_\ell(d) = \psi_B(d)$. By symmetry we may assume that $\bar{x}$ is in the relative interior of the facet defined by $a_1(\bar{x} - f) \leq 1$, and that $F_1 = \{1, \ldots, n\}$. Let $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ nonnegative such that $\sum_{j=1}^n \bar{\alpha}_j = 1$ and $\bar{x} = f + \sum_{j=1}^n \bar{\alpha}_j r^j$. Since $d \in R(\bar{x})$, there exist $\alpha_1, \ldots, \alpha_n$ such that $d = \sum_{j=1}^n \alpha_j r^j$ and $0 \leq \alpha_j \leq \bar{\alpha}_j$, $j = 1, \ldots, n$. Note that, for $i = 1, \ldots, t$, $(a_1 - a_i)d = \sum_{j=1}^n \alpha_j (a_1 - a_i) r^j \geq 0$. Thus $\psi_B(d) = a_1 d$.

We will show that, for every $\lambda < \psi_B(d)$, the set $B(\lambda)$ defined as in (7) contains the point $\binom{\bar{x}}{1}$ in its interior. By Theorem 3, this will imply $\pi_\ell(d) \geq \psi_B(d)$, and thus $\pi_\ell(d) = \psi_B(d)$.

Let $\lambda = \psi_B(d) - \delta$ for some $\delta > 0$. Then $B(\lambda)$ is the set of $x \in \mathbb{R}^{n+1}$ satisfying

$$\begin{array}{rcll} a_1(x - f) \qquad\qquad\qquad\quad - \delta x_{n+1} & \leq & 1, & \\ a_i(x - f) + (a_1 - a_i) d x_{n+1} - \delta x_{n+1} & \leq & 1, & i = 2, \ldots, t. \end{array}$$

Substituting $\binom{\bar{x}}{1}$ in the first inequality, we obtain $a_1(\bar{x} - f) - \delta = 1 - \delta < 1$. Substituting in the ith inequality, $i = 2, \ldots, n+1$, we obtain

$$\begin{aligned} a_i(\bar{x} - f) + (a_1 - a_i)d - \delta &= \sum_{j=1}^n \bar{\alpha}_j a_i r^j + \sum_{j=1}^n \alpha_j (a_1 - a_i) r^j - \delta \\ &= \sum_{j=1}^n \bar{\alpha}_j - \sum_{j=1}^n \bar{\alpha}_j (1 - a_i r^j) + \sum_{j=1}^n \alpha_j (1 - a_i r^j) - \delta \\ &= 1 - \sum_{j=1}^n (\bar{\alpha}_j - \alpha_j)(1 - a_i r^j) - \delta \\ &\leq 1 - \delta < 1 \end{aligned}$$

where the equality in the second line follows from $a_i r^j = 1$ for $j = 1, \ldots, n$, the equality on the third line follows from $\sum_{j=1}^n \bar{\alpha}_j = 1$, while the first inequality on the last line follows from $\alpha_j \leq \bar{\alpha}_j$ and $a_i r^j \leq 1$. □

In light of Theorem 6, we are interested in cases where for every $r \in \mathbb{R}^n$ there exists $w^r \in \mathbb{Z}^n$ such that $r + w^r \in R$, since in this case π_ℓ is the unique minimal lifting.

Dey and Wolsey [8] studied the case $n = 2$. In this case maximal lattice free polytopes are either triangles or quadrilaterals [10]. Dey and Wolsey show that the above property holds if and only if B is a triangle containing at least four integral points (see Figure 3), while it does not hold if B is a triangle containing exactly three integral points or if B is a quadrilateral. They also show that, when B is a triangle with at least four integral points,

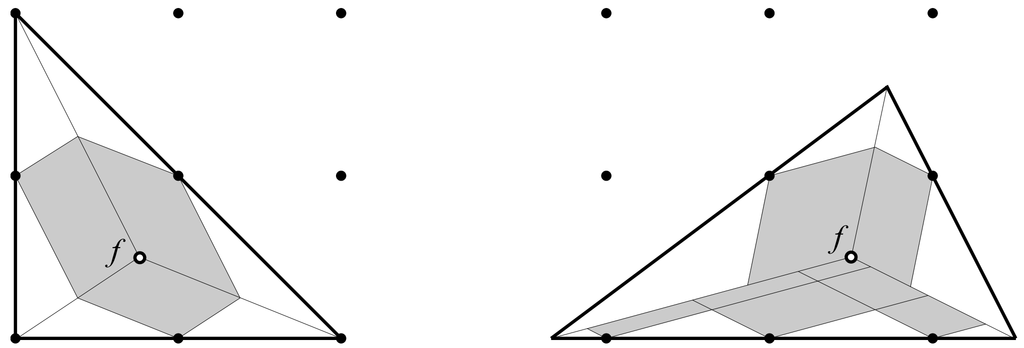

Figure 3: Lattice free triangles giving inequalities with a unique minimal lifting. Region R is shaded.

(ψ_B, π_ℓ) is extreme for (1). This fact also follows from Remark 1 and from the fact that ψ_B is extreme for (3) whenever B is a maximal lattice-free triangle [7].

We next show that the above property holds when B is the n-dimensional simplex $\mathrm{conv}\{0, ne_1, \ldots, ne_n\}$, where e_i denotes the ith unit vector. We assume that f is in the interior of B. The picture on the left in Figure 3 shows the case $n = 2$.
Note that $B = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i \leq n,\, x_i \geq 0,\, i = 1, \ldots, n\}$. The point $e - e_i$, where e denotes the vector of all ones, is the unique integral point in the relative interior of the facet of B defined by $x_i \geq 0$ and e is the unique integral point in the relative interior of the facet of B defined by $\sum_{i=1}^n x_i \leq n$. Thus $\mathcal{I} = \{e, e - e_1, \ldots, e - e_n\}$ where e denotes the vector of all ones.

Let $d^1, \ldots, d^{n+1}$ be defined as follows: $d^i = e_i - \frac{1}{n} f$, $i = 1, \ldots, n$ and $d^{n+1} = -\frac{1}{n} f$. Then $R(e) = \{\sum_{j=1}^n \alpha_j d^j \mid 0 \leq \alpha_i \leq 1,\, i = 1, \ldots, n\}$ and $R(e - e_i) = \{\sum_{j=1}^{n+1} \alpha_j d^j \mid 0 \leq \alpha_k \leq 1,\, k = 1, \ldots, n+1,\, \alpha_i = 0\}$. Therefore $R = \{\sum_{j=1}^{n+1} \alpha_j d^j \mid 0 \leq \alpha_i \leq 1,\, i = 1, \ldots, n+1,\, \alpha_i = 0 \text{ for some } i,\, 1 \leq i \leq n+1\}$

Lemma 10. *Let $B = \mathrm{conv}\{0, ne_1, \ldots, ne_n\}$. For every $r \in \mathbb{R}^n$, there exists $w \in \mathbb{Z}^n$ such that $r + w \in R$.*

Proof. Note that, for $1 \leq i, j \leq n+1$, $d^i - d^j \in \mathbb{Z}^n$.

Let $C_i = \mathrm{cone}\{d^j \mid j \neq i, 1 \leq j \leq n+1\}$, $i = 1, \ldots, n+1$. Note that $\cup_{i=1}^{n+1} C_i = \mathbb{R}^n$ and $C_i \cap C_k = \mathrm{cone}\{d^j \mid j \neq i, k,\, 1 \leq j \leq n+1\}$. Furthermore, $-d^i \in C_i$ for $i = 1, \ldots, n+1$.

Claim: Let $r \in \mathbb{R}^n$ and let i such that $r \in C_i$. There exists a unique $\alpha \in \mathbb{R}^{n+1}$ such that $r = \sum_{j=1}^{n+1} \alpha_j d^j$ and $\alpha_i = 0$. Furthermore, α is nonnegative and $\alpha_j \leq \alpha'_j$ for every nonnegative $\alpha' \in \mathbb{R}^{n+1}$ such that $r = \sum_{j=1}^{n+1} \alpha'_j d^j$.

We prove the claim. Since C_i is generated by n linearly independent vectors, r can be uniquely written as $r = \sum_{j=1}^{n+1} \alpha_j d^j$ such that $\alpha_i = 0$, and α must be nonnegative since $r \in C_i$.

Given a nonnegative $\alpha' \in \mathbb{R}^{n+1}$ such that $r = \sum_{j=1}^{n+1} \alpha'_j d^j$ distinct from α, then $\alpha'_i > 0$. Hence

$$-d^i = (\alpha'_i)^{-1} \sum_{\substack{j=1 \\ j \neq i}}^{n+1} (\alpha'_j - \alpha_j) d^j$$

thus $\alpha'_j - \alpha_j \geq 0$ since $-d^i \in C_i$, hence by the above argument $-d^i$ can be uniquely written as a linear combination of the extreme rays of C_i, and such combination is nonnegative. This proves the claim.

Let us now consider $r \in \mathbb{R}^n$. Let i be such that $r \in C_i$, $1 \leq i \leq n+1$. Let $\alpha \in \mathbb{R}^{n+1}$ such that $r = \sum_{j=1}^{n+1} \alpha_j d^j$ and $\alpha_i = 0$. By the above claim α is nonnegative. Let $\bar{\alpha} = \max_{j=1,\ldots,n+1} \alpha_j$. If $\bar{\alpha} \leq 1$, then $r \in R$. If not, $\alpha_k = \bar{\alpha} > 1$ for some $1 \leq k \leq n+1$.

Let $r' = r + (e_i - e_k) = r + (d^i - d^k)$. Then $r' = \sum_{j \neq i,k} \alpha_j d^j + d^i + (\alpha_k - 1) d^k$. Let h be such that $r' \in C_h$, $1 \leq h \leq n+1$ and let $\alpha' \in \mathbb{R}^{n+1}$ be the unique vector such that $r' = \sum_{j=1}^{n+1} \alpha'_j d^j$ and $\alpha'_h = 0$. By the previous claim, α' satisfies the following properties

- $r' - r \in \mathbb{Z}^n$ and $\alpha'_h = 0$,
- $0 \leq \alpha'_j \leq \alpha_j$, $j \neq i$, $1 \leq j \leq n+1$,
- $0 \leq \alpha'_i \leq 1$, $0 \leq \alpha'_k \leq \alpha_k - 1$.

Thus, either $\max_{j=1,\ldots,n+1} \alpha'_j \leq \bar{\alpha} - 1$, or the number of indices j such that $\alpha'_j = \bar{\alpha}$ is smaller than the number of indices j such that $\alpha_j = \bar{\alpha}$. This implies the statement of the lemma. □

It can be shown that, in this case, R is a polytope with $\binom{n+1}{2}$ pairs of parallel facets, and that R has volume 1. Thus, by Lemma 10, all possible translations of R by integral vectors form a tiling of $\mathbb{R}^n$. Therefore for every $d \in \mathbb{R}^n$, there exists $w^d \in \mathbb{Z}^n$ such that $d + w^d \in R$. By Theorem 6, the function π_ℓ defined by $\pi_\ell(d) = \psi_B(d + w^d)$ is the unique minimal lifting of ψ_B.

Whenever B is a maximal lattice-free simplex, ψ_B is extreme for (3). Indeed, if $v^1, \ldots, v^{n+1}$ are the vertices of B and we define $r^j = v^j - f$, $j = 1, \ldots, n+1$, ψ_B is extreme for (3) if and only if $\sum_{j=1}^{n+1} \psi_B(r^j) s_j \geq 1$ is extreme for the convex hull of the set $R_f(r^1, \ldots, r^{n+1})$ defined as the set of all $s \in \mathbb{R}^{n+1}$ such that $f + \sum_{j=1}^{n+1} r^j s_j \in \mathbb{Z}^n$ and $s \geq 0$ (see [9]). In this case, since each facet of B contains an integral point, for $i = 1, \ldots, n+1$ there exists $s^i \in \mathbb{R}^{n+1}$ such that $s^i_j > 0$ for all $j \neq i$, $1 \leq j \leq n+1$, $s^i_i = 0$ and $\sum_{j=1}^{n+1} s^i_j r^j \in \mathbb{Z}^n$. Hence $s^1, \ldots, s^{n+1}$ are linearly independent points of $R_f(r^1, \ldots, r^{n+1})$, and $\sum_{j=1}^{n+1} \psi_B(r^j) s^i_j = 1$ for $i = 1, \ldots, n+1$. This shows that $\sum_{j=1}^{n+1} \psi_B(r^j) s_j \geq 1$ defines a facet of $\operatorname{conv}(R_f(r^1, \ldots, r^{n+1}))$, and thus it is extreme for $\operatorname{conv}(R_f(r^1, \ldots, r^{n+1}))$. Therefore ψ_B is extreme for (3).

The above statement and Remark 1 imply the following.

Theorem 11. *If $B = \operatorname{conv}(0, ne_1, \ldots, ne_n)$, (ψ_B, π_ℓ) is extreme for (1) with $S = \mathbb{Z}^n$.*

3.3 Simple cones

We consider the case were $S = \mathbb{Z}^{n-1} \times \mathbb{Z}_+$ and the maximal S-free convex set B is the translation of a simple cone. That is, B has a unique vertex v, and $B - v$ is a simple cone. Recall that a polyhedral cone in $\mathbb{R}^n$ is *simple* if it is generated by n linearly independent vectors, and therefore it has n facets. This case extends the wedge inequalities of Section 3.1.

Let $B = \{x \in \mathbb{R}^n \,|\, a_i(x - f) \leq 1,\ i = 1 \ldots, n\}$. By Theorem 2, $\mathrm{rec}(B) \cap \mathrm{rec}(\mathrm{conv}(S))$ is contained in the lineality space of B, which is empty. Therefore $B \cap \mathrm{conv}(S)$ is bounded. Therefore the polytope $B \cap (\mathbb{R}^{n-1} \times \{0\})$ is an an $(n-1)$-dimensional simplex P. Let $v^1, \ldots, v^n$ be the vertices of P, and let $r^j = v^j - f$, $j = 1, \ldots, n$. By symmetry, we may assume that $a_i r^j = 1$ for $1 \leq i, j \leq n$, $i \neq j$, and $a_i r^i < 1$. Let $\hat{r} = v - f$. Note that, for $i = 1, \ldots, n$, $a_i \hat{r} = 1$.

Let $\bar{x}$ be a point of S in the relative interior of one of the facets of B, say the facet defined by $a_h(x - f) \leq 1$. Then $\bar{x}$ can be uniquely written as $\bar{x} = f + \bar{\alpha}\hat{r} + \sum_{j=1}^n \bar{\alpha}_j r^j$ such that $0 \leq \bar{\alpha}_j$, $j = 1, \ldots, n$, and $\bar{\alpha}_h = 0$. Let $R(\bar{x}) = \{\sum_{j=1}^n \alpha_j r^j \,|\, 0 \leq \alpha_j \leq \bar{\alpha}_j,\ j = 1, \ldots, n\} + \langle \hat{r} \rangle$. Let us denote by $\mathcal{I}$ the set of all points $\bar{x}$ in S such that $\bar{x}$ is contained in the relative interior of some facet of B. Let $R = \cup_{\bar{x} \in \mathcal{I}} R(\bar{x})$.

Lemma 12. *For every $d \in R$, $\pi_\ell(d) = \psi_B(d)$.*

Proof. We only need to show that, given $\bar{x} \in \mathcal{I}$ and $d \in R(\bar{x})$, $\pi_\ell(d) = \psi_B(d)$. By symmetry we may assume that $\bar{x}$ is in the relative interior of the facet defined by $a_1(x - f) \leq 1$. Let $\bar{\alpha} \in \mathbb{R}$ and $\bar{\alpha}_2, \ldots, \bar{\alpha}_n$ nonnegative such that $\bar{x} = f + \bar{\alpha}\hat{r} + \sum_{j=2}^n \bar{\alpha}_j r^j$. Since $d \in R(\bar{x})$, there exist $\alpha \in \mathbb{R}$ and $\alpha_1, \ldots, \alpha_n$ such that $d = \alpha\hat{r} + \sum_{j=2}^n \alpha_j r^j$ and $0 \leq \alpha_j \leq \bar{\alpha}_j$, $j = 2, \ldots, n$.

Note that, for $i = 2, \ldots, t$, $(a_1 - a_i)d = \alpha(a_1 - a_i)\hat{r} + \sum_{j=2}^n \alpha_j (a_1 - a_i) r^j \geq 0$, since $(a_1 - a_i)\hat{r} = 0$ and $(a_1 - a_i)r^j \geq 0$. Thus $\psi_B(d) = a_1 d$.

We will show that, for every $\lambda < \psi_B(d)$, the set $B(\lambda)$ defined in (7) contains the point $\binom{\bar{x}}{1}$ in its interior. By Theorem 3, this will imply $\pi_\ell(d) \geq \psi_B(d)$, and thus $\pi_\ell(d) = \psi_B(d)$.

Let $\lambda = \psi_B(d) - \delta$ for some $\delta > 0$. Then $B(\lambda)$ is the set of $x \in \mathbb{R}^{n+1}$ satisfying

$$\begin{array}{lcll} a_1(x - f) \qquad\qquad\qquad\qquad - \delta x_{n+1} & \leq & 1, & \\ a_i(x - f) + (a_1 - a_i) d x_{n+1} - \delta x_{n+1} & \leq & 1 & i = 2, \ldots, t. \end{array}$$

Substituting $\binom{\bar{x}}{1}$ in the first inequality, we obtain $a_1(\bar{x} - f) - \delta = 1 - \delta < 1$. Substituting in the ith inequality, $i = 2, \ldots, n+1$, we obtain

$$\begin{aligned} a_i(\bar{x} - f) + (a_1 - a_i)d - \delta &= \bar{\alpha} a_i \hat{r} + \sum_{j=2}^n \bar{\alpha}_j a_i r^j + \alpha(a_1 - a_i)\hat{r} + \sum_{j=2}^n \alpha_j (a_1 - a_i) r^j - \delta \\ &= \bar{\alpha} a_1 \hat{r} + \sum_{j=2}^n \bar{\alpha}_j a_1 r^j - \bar{\alpha}_i (a_1 - a_i) r^i + \alpha_i (a_1 - a_i) r^i - \delta \\ &= a_1(\bar{x} - f) - (\bar{\alpha}_i - \alpha_i)(a_1 - a_i) r^i - \delta \\ &\leq 1 - \delta < 1 \end{aligned}$$

where the equality in the second line follows from $a_i \hat{r} = a_1 \hat{r}$ and $a_1 r^j = a_i r^j$ for all $2 \leq j \leq n$ such that $i \neq j$, while the first inequality on the last line follows from $\alpha_i \leq \bar{\alpha}_i$ and $a_i r^i < 1 = a_1 r^i$. □

Note that P is an $n-1$-dimensional simplex in $\mathbb{R}^{n-1} \times \{0\}$ and P does not contain any point of $\mathbb{Z}^{n-1} \times \{0\}$ in its interior. Suppose that P is maximal lattice free in $\mathbb{R}^{n-1} \times \{0\}$. In this case we can apply the results of Section 3.2 to identify cases where π_ℓ is a lifting of ψ_B.

Let $\bar{f}$ be the intersection of the line $f + \langle \hat{r} \rangle$ with $\mathbb{R}^{n-1} \times \{0\}$, and let $\bar{r}^j = v^j - \bar{f}$. For every point $\bar{x} \in \mathbb{Z}^{n-1} \times \{0\}$ in the relative interior of one of the facets of P, say the facet defined by $a_h(x-f) \leq 1$, $\bar{x}$ can be uniquely written as $\bar{x} = \bar{f} + \sum_{j=1}^n \bar{\alpha}_j \bar{r}^j$ such that $0 \leq \bar{\alpha}_j$, $j = 1, \ldots, n$, and $\bar{\alpha}_h = 0$. Let $\bar{R}(\bar{x}) = \{\sum_{j=1}^n \alpha_j \bar{r}^j \,|\, 0 \leq \alpha_j \leq \bar{\alpha}_j,\, j = 1, \ldots, n\}$. Note that $\bar{R}(\bar{x}) = R(\bar{x}) \cap (\mathbb{R}^{n-1} \times \{0\})$. Let $\bar{\mathcal{I}}$ be the set of all points in $\bar{x} \in \mathbb{Z}^{n-1} \times \{0\}$ in the relative interior of some of the facets of P. We define $\bar{R} = \cup_{\bar{x} \in \bar{\mathcal{I}}} \bar{R}(\bar{x})$. Then $R \supseteq \bar{R} + \langle \hat{r} \rangle$. Hence, if for every $r \in \mathbb{R}^{n-1} \times \{0\}$ there exists $w \in \mathbb{Z}^{n-1} \times \{0\}$ such that $r + w \in \bar{R}$, it also holds that for every $r \in \mathbb{R}^n$ there exists $w^r \in \mathbb{Z}^{n-1} \times \{0\}$ such that $r + w^r \in R$.

Since $\mathbb{R}^{n-1} \times \{0\}$ is the lineality space of $\text{conv}(S)$, Theorem 6 implies that π_ℓ is the unique minimal lifting of ψ_B, and $\pi_\ell(r) = \psi(r + w^r)$.

The above property holds, for example, when $n = 2$ and P is an interval of length one (as seen in Section 3.1), when $n = 3$ and P is a maximal lattice-free triangle containing at least four points in $\mathbb{Z}^2 \times \{0\}$, or for general n when P is a unimodular transformation of $\text{conv}(0, (n-1)e_1, \ldots, (n-1)e_{n-1})$.

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