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# Full orders and their families of convex subgroups on locally nilpotent groups

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FULL ORDERS AND THEIR FAMILIES OF  
CONVEX SUBGROUPS ON LOCALLY  
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by

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## I. Introduction

If a group  $G$  has a full order relation,  $<$ , such that for each  $a, b, c \in G$  if  $a < b$  then both  $ac < bc$  and  $ca < cb$ , we say that  $(G, <)$  is an f.o. group, that it is fully ordered by  $<$ , and that  $<$  is a full order of  $G$ . If  $G$  is fully ordered by  $<$  and  $C \subseteq G$  is a subset such that when  $a, b \in C$ , then all  $x$  between  $a$  and  $b$  are also in  $C$ , we say that  $C$  is convex with respect to  $<$  or if there is no ambiguity simply that  $C$  is convex. We denote by  $C_<$  the family of all subgroups of  $G$  convex with respect to  $<$ . The family  $C_<$  is called the convex family of  $(G, <)$ .

Several authors have investigated properties of those families of subgroups of a group  $G$  which are the convex families determined by the various full orders of  $G$ . Among the investigators K. Iwasawa [3] has given conditions characterizing such families of subgroups. Another set of conditions on a family of subgroups of a group  $G$  which is necessary and sufficient that  $G$  admits a full order is stated

in Fuchs [1J pp. 51, 52, and attributed by him to L. S. Rieger and V. D. Podderiyugin.

In this paper we will investigate the convex families of full orders in torsion free locally nilpotent (TFLN) groups. Our main result (theorem 2) is that for an arbitrary full order  $<$  on the TFLN group  $G$ , the convex family,  $C_{<}$  is a central system. We remark that there are several examples known of full orders on non-TFLN groups which are not even normal systems. We use our result to obtain a characterization of those families of subgroups of TFLN groups which are convex families. This characterization is simpler than the general characterization given by Iwasawa. Secondly, we show that the Rieger-Podderiyugin conditions (theorem 1) do not characterize convex families, a question raised in Fuchs QJ.

## II. Results

A family of subgroups  $\mathfrak{S}$ , of a group  $G$  is a normal system if each  $F \in \mathfrak{S}$  is normal in  $G$ , if  $\mathfrak{S}$  is a chain with respect to set inclusion, and if  $\mathfrak{S}$  is closed with respect to arbitrary unions and intersections. A jump in  $\mathfrak{S}$  is a pair of elements  $C, D \in \mathfrak{S}$  such that  $C \subset D$  and no element of  $\mathfrak{S}$  is properly between  $C$  and  $D$ . Letting  $Z(H)$  denote the center of any group  $H$ , we say that  $\mathfrak{S}$  is a central system in group  $G$  if (1)  $G \in \mathfrak{S}$ , if  $\mathfrak{S}$  is a normal system of  $G$ , and if for each jump  $C, D$  in  $\mathfrak{S}$ ,  $D/C \in Z(G/C)$ .

Theorem 1: (Rieger, Podderiyugin as per Fuchs [3J]).

A group  $G$  is orderable if and only if it contains a family,  $\mathfrak{S}$ , of subgroups satisfying (i) - (v):

(i)  $\mathfrak{3}$  is a chain containing  $\{1\}$  and  $G$  and is closed under arbitrary unions and intersections.

(ii) If  $C \in \mathfrak{3}$  then for each  $g \in G$ ,  $g^{-1}Cg \in \mathfrak{3}$ .

(iii) If  $C, D$  is a jump in  $\mathfrak{3}$  then  $C$  is normal in  $D$  and  $D/C$  is isomorphic to a subgroup of the additive group of the real numbers.

(iv) If  $C, D$  is a jump in  $\mathfrak{3}$  then  $(N(C), N(C), D) \leq C$  where  $N(C)$  is the normalizer of  $C$  in  $G$ ,

(v) If  $C \in \mathfrak{3}$  and  $S(a)$  is the normal subsemigroup of  $G$  generated by  $a$  and if  $C \cap S(a)$  is non-empty then some conjugate of  $a$  is in  $C$ .

Furthermore there will be a full order,  $<_3$  of  $G$  such that  $\mathfrak{3} \neq C^\wedge$ .

Until now it was not known whether a full order  $<$  of  $G$  existed such that  $\mathfrak{3} = C_\leftarrow$ . However, by the example below this question is answered in the negative.

We now consider TFLN groups. It has been known for some time that all TFLN groups can be fully ordered (see Neumann (5); Graham [2] has another proof). Another result in Graham [2] is the following, Lemma: If  $G$  is TFLN and  $<$  is a full order of  $G$  then  $C_\leftarrow$  is a normal system of  $G$ .

Proof: By straightforward calculation if  $C \in C_\leftarrow$  and  $g \in G$  then  $g^{-1}Cg \in C$  also. Thus  $\langle g^{-1}Cg \rangle \subseteq C$  or  $g^{-1}Cg \not\subseteq C$ . To show that

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If  $x, y \in G$  then  $(x, y) = x^{-1}y^{-1}xy$ . If  $S$  and  $T$  are subgroups of  $G$  then  $(S > T) = \text{gp}(\{s, t \mid s \in S \text{ and } t \in T\})$ . Finally  $(S, T, U)$  means  $((ST)U)$ .

$g^{-1}Cg = C$  for all  $C \in \mathcal{C}$  and  $g \in G$  we assume otherwise. Without loss of generality we assume that there is a  $g \in G$  and a  $C \in \mathcal{C}$  such that  $g^{-1}Cg \neq C$ . Then there is an  $a \in C$  such that  $g^{-1}ag \neq a$  so for arbitrary positive  $n$  we have

$$(1) \quad g^{-n}ag^n \neq a / g^{-n}Cg^n.$$

Let  $C^* = g^{-m}Cg^m$ , let  $U(g;0) = a$ , and let  $(a,g;n+1) = ((a,g;n),g)$ .

Note that  $C_{i+1} \cap C_i \neq \emptyset$  and  $g^{-i}ag^i = a(a,g;i)$ . By induction on  $n$  the

identity (2) below can be established where  $\prod_{i=1}^n (a,g;i)$  is to be

**interpreted as some finite product of commutators  $(a,g;i)$  in which**

none of the  $i$ 's is greater than  $k$ . For example  $g^{-2}ag^2 = (a,g;1)^2 (a,g;2)$ .

$$(2) \quad g^{-n}ag^n = \left( \prod_{i=1}^n (a,g;i) \right) (a,g;n).$$

Since  $a(a,g;i) \in C_i$  it follows by induction on  $m$  and by (2) that  $(a,g;m) \in C_m$ . However  $G$  is locally nilpotent so the group generated by  $a$  and  $g$  is nilpotent say of class  $k$ . Then all commutators of length  $k+1$  and greater are the identity so from (2) we get

$$g^{-k-1}ag^{k+1} = \left( \prod_{i < k} (a,g;i) \right) (a,g;k+1) = \prod_{i < k} (a,g;i) \in CL_K.$$

But this contradicts (1) taking  $n = k$ , and the lemma is established.

Theorem 2: If  $G$  is TPLN and fully ordered by  $<$  then  $C_{\leq}$  is a central system.

Proof: Since  $\{1\}$  and  $G$  are convex they are in  $C$ . By the lemma  $C$  is a normal system. Now let  $C \triangleleft D$  be a jump in  $C_{\#}$ . All that remains is to show the  $D/C \subseteq Z(G/C)$ . First note that if for each  $g \in G$  the mapping  $\hat{g}: G/C \rightarrow G/C$  induced by conjugation by  $g$  is the identity

map on  $D/C$ , then for each  $d \in D$   $(dC)^{\wedge} = g^{-1}dgC = d(d,g)C$  hence it follows that  $(d,g) \equiv 1 \pmod{C}$  so that  $D/C \subseteq_z(G/C)$ .

Thus the problem reduces to showing that for each  $g \in G$ ,  $\hat{g}|_{D/C}$  is the identity map. Suppose on the contrary for some  $g \in G$ ,  $\hat{g}|_{D/C}$  is not the identity map. Then there is a  $d \in D$  such that  $(d^{\wedge}g) \not\equiv 1 \pmod{C}$ . Since  $G$  is TFLN one of  $(d^{\wedge}g; m) = 1$ , so for some first  $m$ , say  $n+2$ ,  $(d,g; n+2) \equiv 1 \pmod{C}$ . Then  $(d,g; n) \not\equiv (d,g; n+1) \pmod{C}$  and neither is congruent to 1 mod  $C$ . Let  $b = (d,g; n)$ .

Since  $(d,g; n+1)C^{\wedge} = (d,g; n+1)(d,g; n+2)C = (d^{\wedge}g; n+1)C$  and  $bC\hat{g} = b(d,g; n+1)C$  we have for all integers  $r \geq 0$  that  $bC\hat{g}^r = b(d,g; n+1)^r C$ . If  $r < 0$  then  $bC = bC\hat{g}^{-r} = (b(d,g; n+1)^{-r}C)\hat{g}^r = (bC)^{-r}(d,g; n+1)^{-r}C = (bC)^{-r}(d,g; n+1)^r C$  so for all  $r = 0, \pm 1, \pm 2, \dots$  we have

$$(3) \quad bC\hat{g}^r = b(d,g; n+1)^r C = b(b^{\wedge}g)^r C.$$

Now  $<$  on  $G$  induces a full order on  $G/C$  since  $C$  is convex\*. We will denote this induced order by  $<$  also\*. Furthermore, for each  $r \in \mathbb{Z}$   $\hat{g}^r$  is an order preserving automorphism of  $G/C$ .

Also  $(b,g)C^{\wedge} = (d,g; n+1)C^{\wedge} = (d^{\wedge}g; n+1)C = (b,g)C$  hence

for all  $r$ ,  $(b,g)C^{\wedge r} = (b,g)C$ . Now by the choice of  $b$ ,  $C^{\wedge} (b,g)C^{\wedge} bC$ .

Therefore either  $bC < (b,g)C$  or  $bC > (b,g)C$ . If  $bC < (b,g)C$  then

$bC^{\wedge r} < (b,g)C\hat{g}^r$  for all  $r$ . But  $bC\hat{g}^r = b(b,g)^r C = bC((b,g)C)^r < (b^{\wedge}g)C$

so  $bC < ((b,g)C)^{1/r}$  for all rational integers  $r$ . This is impossible

since neither is the identity and since  $D/C$  is an Archimedean ordered

group. A similar contradiction is reached if we assume that  $bC > (b,g)C$ .

Therefore the assumption that  $\wedge JD/C$  is not the identity leads to a contradiction and the theorem is established.

Using the conditions in theorem 1 with the added information from theorem 2 we can now characterize those systems of subgroups of TPLN group  $G$  which are the convex families with respect to the various full orders of  $G$ ,

Theorem 3: If  $G$  is TFLN the system  $\mathcal{O}$  of subgroups of  $G$  is the convex family of subgroups with respect to some full order of  $G$  if and only if:

(i)  $\mathcal{C}$  is a central system of  $G$ .

(ii) If  $C, D$  is a jump in  $\mathcal{C}$  the  $D/C$  is isomorphic to a subgroup of the additive group of real numbers.

Proof: If  $\mathcal{C}$  is the convex family of  $G$  with respect to some full order of  $G$  then (i) holds by theorem 2 and (ii) holds since in the induced order on  $D/C$ ,  $D/C$  is an Archimedean ordered group, hence isomorphic to a subgroup of the additive reals. (See Fuchs QJ P\* 45. The theorem is due O. Holder.)

On the other hand if (i) and (ii) hold for  $\mathcal{C}$  then one can define a full order on  $G$  as in Neumann [6J, theorem 2.3. (Strictly speaking the fuller generality of Graham [2J, theorem 23 is needed.) This full order is constructed so that the induced order on each jump factor  $D/C$  of  $\mathcal{C}$  is Archimedean, hence no new convex subgroups arise. Thus  $\mathcal{C}$  turns out to be the convex family of the newly defined full order of  $G$ .



III. Example

Let  $F$  be the free group on  $x$  and  $y$ ,  $P_4$  the fourth term in the lower central series of  $F$  and  $G = F/P_4$ . Group  $G$  is the free nilpotent group of class 3 on two generators\* Such groups are discussed in Kurosh [4]. In  $G$  let  $\mathcal{C}$  be the family  $\{G, G_{2j} [1]\}$ , where  $G_2$  is the second term in the lower central series, namely  $(G, G)$ . Since  $\mathcal{C}$  is a series of length 2 it cannot be a central series ( $G$  is nilpotent of class 3), so by theorem 2 it cannot be the convex family with respect to any full order of  $G$ , However we will now show that it does satisfy the Rieger-Podderyugin conditions. Thus these conditions do not characterize the convex families of a group which admits full orders.

On to the conditions of theorem 1:

- (i) Immediate,
- (ii) Since  $G_2$  is normal in  $G$ , this condition is satisfied.
- (iii)  $G/G_2$  and  $G_2$  are easily shown to be Abelian, They are known to be torsion free and finitely generated, hence they are isomorphic to subgroups of the additive reals,
- (iv) Since  $N(G_2) \neq G$  and since for the lower central series  $\{G_k\}$  of any group  $(G_{i+j} = G_i + G_j) \subseteq G_i + G_j$  we have  $(N(G_2), N(G_2), G) \in G_3 \in G_2$ , and  $(N(1), N(1), G_2) \in G^1 = \{1\}$ .

(v) Finally suppose that there is an  $a \in G$  such that  $S(a) = \{1, a, a^2, \dots, a^j\}$   $i = 1, 2, 4$ . Then we must show some conjugate of  $a$  is in  $G_i$ . We actually show that in each of the three cases  $a$  itself is in  $G_i$ .

If  $i = 1$ , then  $G_1 = G, \mathbb{1} = G$  and  $a \in G$ .

If  $i = 2$  and  $s(a) = 6$ , then for some finite non-empty set  $\{x_i \mid i = 1, 2, \dots, r\} \in G$   $\prod_{i=1}^r ax_i \in G^1$ , since all elements of  $S(a)$  are of this form. But then  $\prod_{i=1}^r ax_i = \prod_{i=1}^r (a(ax_i)) \in G^2$ . However  $(a, x_i) \in G_0$  for each  $i$  so  $\prod_{i=1}^r (a) = a^r \in G^1$ . But  $G/G_0$  is

torsion free whence  $a \in G_1$ .

If  $i = 4$  then  $G = [V$ . Then if  $S(a) \cap \{1^1 \wedge d\}$  there is a finite non empty set  $\{y_i \mid i = 1, \dots, s\} \in G$  such that  $\prod_{i=1}^s ay_i = 1$ . Thus  $\prod_{i=1}^s (a, y_i) = 1$ . Since each  $(a, y_i) \in G_0$  we have  $\prod_{i=1}^s (a, y_i) \equiv a^s \equiv 1 \pmod{G^1}$ . As before  $G/G^1$  is torsion free so  $a \in G^1$ . But then  $(a, y_i) \in G_3$  for each  $i$  so  $\prod_{i=1}^s (a, y_i) \equiv a^s \equiv 1 \pmod{G_3}$ . Since  $G^1/G^1$  is also torsion free it follows that  $a \in G_3$ . Then  $(a, y_i) \in G^1 = [V$  for each  $i$  so finally  $\prod_{i=1}^s (a, y_i) = a^s = 1$ . But  $G$  is torsion free so  $a = 1$  and we are done.

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