Carnegie Mellon University
Research Showcase @ CMU

Department of Civil and Environmental Engineering

Carnegie Institute of Technology

1993

Viscous flow-finite elasticity interaction : variational formulation and finite element approximation

Omar Ghattas
Carnegie Mellon University

Xiaogang Li

Carnegie Mellon University.Engineering Design Research Center.

Follow this and additional works at: http://repository.cmu.edu/cee

This Technical Report is brought to you for free and open access by the Carnegie Institute of Technology at Research Showcase @ CMU. It has been accepted for inclusion in Department of Civil and Environmental Engineering by an authorized administrator of Research Showcase @ CMU. For more information, please contact research-showcase@andrew.cmu.edu.

NOTICE WARNING CONCERNING COPYRIGHT RESTRICTIONS:
The copyright law of the United States (title 17, U.S. Code) governs the making of photocopies or other reproductions of copyrighted material. Any copying of this document without permission of its author may be prohibited by law.

Viscous Flow-Finite Elasticity Interaction: Variational Formulation and Finite Element Approximation

Omar Ghattas and Xiaogang Li

EDRC 12-61-93

Viscous Flow-Finite Elasticity Interaction: Variational Formulation and Finite Element Approximation[1]

^Omar **Ghattas and Xiaogang Li**

Engineering Design Research Center
and
Department of Civil Engineering
Carnegie Mellon University
Pittsburgh, PA 15213

Abstract. We consider the problem of the interaction of a stationary viscous fluid with an elastic solid that undergoes large deformation. The fluid is modeled by the stationary incompressible Navier-Stokes equations in an Eulerian frame of reference, while a Lagrangian reference frame and large deformation-small strain theory is used for the solid. A variational formulation of the problem is developed that insures satisfaction of continuity of interface tractions and velocities. The variational formulation is approximated by a Galerkin finite element method, yielding a system of nonlinear algebraic equations in unknown fluid velocities and pressures and solid displacements. A Newton-like method is introduced for solution of the discrete system. The method employs a modified Jacobian that enables decomposition into separate fluid and solid subdomains. This domain decomposition avoids possible ill-conditioning of the Jacobian, as well as the need to compute and store geometric coupling terms between fluid and interface shape. The capability of the methodology is illustrated by solution of a problem of the flow-induced large deformation of an elastic infinite cylinder.

Keywords. Nonlinear fluid-structure interaction, nonlinear aeroelasticity, Navier-Stokes equations, finite elasticity, variational methods, Galerkin method, finite elements.

1. Introduction

Despite its importance, the problem of modeling the nonlinear interaction of a viscous fluid with a solid undergoing large deformation has remained a challenging problems in mechanics. Its resolution is of significant practical importance to such disciplines as aerospace, marine, automotive, and wind engineering. Such problems may arise, for instance, in the large-amplitude vibration of flexible aerodynamic components such as high aspect ratio wings and turbine blades, in wind-induced deformation of towers, antennas, and lightweight bridges, in hydrodynamic flows around offshore structures, and in blood flow through the heart. The basic difficulty here lies when two-way coupling occurs between fluid and solid: viscous flow produces tractions which deform the solid, while deformation of the solid influences the flow physics and thus fluid tractions. Solid deformation influences the flow both by altering the fluid domain as well as by creating solid tractions that must be in equilibrium with the fluid tractions.

Because of its critical importance in aerospace applications, the problem of fluid-structure interaction has received considerable attention within the aerospace literature, where it is

[x]**This work has been supported by the Engineering Design Research Center, an NSF Engineering Research Center.**

known as aeroelasticity. Classical approaches based on linear theory are well established [1], [4]. Certain nonlinear aeroelasticity phenomena have been amenable to analytical and semi-analytical study, and significant understanding of the physics of these problems has been elucidated in recent years [5]. Recently, interest has increased in *computational aeroelasticity*, i.e. in developing methods for direct numerical approximation of the governing nonlinear partial differential equations of the fluid-solid system [9], [10], [11], [12], [6]. This interest has been motivated by advances in computational fluid dynamics and computational structural mechanics, and in the rapid growth in computational power.

Two approaches to computational aeroelasticity have emerged. Both approaches employ different numerical approximations in fluid and solid domains, typically finite difference or volume methods for the fluid and finite elements for the solid. Both approaches couple fluid and structural behavior *after* discretization.

The first approach iterates between 1) assuming a rigid solid about which to compute fluid flow, and 2) applying the resulting increment in fluid boundary tractions to deform the solid, thereby presenting a new flow domain to the fluid. Since each step of the above process is a *nonlinear* problem, a linearization is typically made. We refer to such techniques as *iterated physics* methods. In addition to their lack of rigor in enforcing traction continuity across the interface, iterated physics methods can be very time-consuming, due to their back-and-forth nature. On the other hand, these methods readily incorporate existing CFD and structural analysis codes, with the addition of a mechanism to transmit interface tractions between the two codes. Examples of this approach include [9], [10], [11], and [12].

The second approach to computational aeroelasticity couples the fluid and solid governing equations together into a single set of nonlinear algebraic equations. Again, since coupling is achieved after numerical approximation, continuity of interface tractions is not rigorously enforced. We refer to such methods as *algebraic coupling* methods. Such a method is reported in [6], with application to two-dimensional steady Navier-Stokes flow and a linearly-behaving structure. These methods have been criticized for resulting in possibly ill-conditioned Jacobian matrices of the coupled system, due to the disparity in solid and fluid behavior [12]. However, one ought to be able to apply various numerical linear algebraic devices to overcome this problem.

The lack of a unified framework for coupling viscous flow with large deformation elasticity has prompted us to develop a variational approach to the problem, which couples the two governing sets of equations before numerical approximation. We model the fluid by the stationary incompressible Navier-Stokes equations in an Eulerian frame of reference, while a Lagrangian reference frame and large deformation-small strain theory is used for the solid. Extensions to time-dependent and compressible problems and to non-Newtonian flows are discussed in the final section. Once coupled, we can systematically apply a numerical approximation—a Galerkin finite element method—to obtain a single set of nonlinear algebraic equations. This system of equations can be solved to simultaneously yield fluid velocities and pressures and solid displacements. If we are faced with ill-conditioning of the Jacobians of these equations, we can then perform *algebraic* decomposition into fluid and solid subproblems. Thus, we regard the issue of coupling as a *continuum mechanics* problem, to be established prior to discretization. Once coupled and discretized, we may

then seek appropriate numerical methods for solution, which may include linear algebraic devices such as domain decomposition to avoid ill-conditioning. Advantages of our method include: continuity of interface tractions and displacements is automatically enforced by the variational formulation; the weak form of the problem can be systematically translated into a unified finite element approximation; and, solvers that best treat the coupled problem can be devised.

The rest of this paper is organized as follows. In §2, we develop the variational form of the viscous flow-finite elasticity interaction problem. A finite element approximation is constructed in §3, while §4 introduces a modified Newton method for solution of the resulting discrete system. The method is illustrated in §5 through the solution of a problem of flow-induced deformation of an infinite, elastic cylinder. We conclude with some remarks in §6.

2. Variational formulation

In this section we develop a variational formulation of the fluid-solid interaction problem in the context of a stationary, viscous, incompressible, Newtonian fluid, modeled by the Navier-Stokes equations, interacting with an elastic solid in a Lagrangian frame of reference. We assume that the solid is capable of large deformations, but that strains are small—a reasonable assumption for problems arising in aerospace and civil engineering. Small deformation theory breaks down for problems in which deformations are on the order of the smallest solid dimension, e.g. shell thickness. Since many aeroelasticity problems fall into this category, we have incorporated large deformation theory into our model.

The finite nature of solid deformations implies a geometric dependence of the flow field on the solid deformation. Consider a solid of finite extent surrounded by an infinite fluid. Define *Q.F* as the fluid domain, *Us* as the *undeformed* solid domain, T^{l}_{F} as a boundary approximating the fluid far-field on which tractions are prescribed, F^{2}_{F} as the portion of the far-field fluid boundary over which velocity is prescribed, F^{l}_{s} as the undeformed solid boundary on which tractions are prescribed, F| as the undeformed solid boundary on which displacements are prescribed, and F/ as the interface between solid and fluid. The fluid unknowns are the pressure p, the velocity vector v, the stress tensor *ap,* and the rate of strain tensor d. In the solid, the unknowns are the displacement vector u, the Kirchhoff stress tensor S, and the Green strain tensor E. We shall have occasion to refer to the solid Eulerian stress tensor, which we denote <r$. Material constants are the fluid viscosity *fijr* and density /?, and the Lamé and shear moduli of the solid, A and *(is-* We take ṽ as the prescribed far-field fluid velocity, t̂p as the prescribed fluid traction, û as the prescribed solid displacement, and *is* as the prescribed solid traction. The fluid and solid body forces are denoted fp and fs. Define FF = $T^{l}_{F}urjp$ and *Fs* = F^UFl. We also define **n** as the unit outward normal to a deformed surface, and no as the unit outward normal to an undeformed surface.

The conservation of momentum, conservation of mass, constitutive law, and strain rate-velocity equations of the fluid are:

$$/9_F(v- V)\,v - V-<r_F = f_F \qquad \text{in } ft_F \tag{1}$$

$$V \cdot v = 0 \quad \text{in} \quad ft_F \tag{2}$$

$$<r_F = -pi + 2/z_F d \quad \text{in } Cl^f \tag{3}$$

$$d = \frac{1}{2}(Vv + Vv^T) \quad \text{in } n_F \tag{4}$$

The constitutive law, equilibrium equations, and strain-displacement relations of the solid are given by:

$$S = A \ \ tr(E)I + 2(i_s E \quad \text{in} \quad ft_5 \tag{5}$$

$$V\text{-}[(I + Vu)\text{-}S] = f_s \quad \dot{m}ils \tag{6}$$

$$E = \frac{i}{2}[Vu + Vu^T + Vu \cdot Vu^r] \quad \text{in } n_5 \tag{7}$$

At the interface, coupling between fluid and solid requires that tractions and velocity be continuous:

$$(Tj? \cdot n_F + as \cdot ns = 0 \quad \text{on } F/ \tag{8}$$

$$v = 0 \quad \text{on } T/ \tag{9}$$

Finally, the boundary conditions take the form

$$<r_F \cdot n = \hat{t}_F \quad \text{on } F_F^1 \tag{10}$$

$$[(I + Vu).S].no = \hat{t}_5 \quad \text{on } \wedge \tag{11}$$

$$v = \hat{v} \quad \text{on } F\wedge \tag{12}$$

$$u = \hat{u} \quad \text{on } T| \tag{13}$$

See for example [8] for derivations of the governing equations of fluid and solid. Notice that the consequence of the small strain assumption is to allow the use of Hooke's law for the solid constitutive relation (5).

We now proceed to establish the variational form of the problem. Let us assume, for simplicity of presentation, that the fluid and solid do not experience body forces, and that the fluid and solid prescribed tractions are zero, i.e. f_F, fs, $\hat{t}_F$, and $\hat{t}s$ are all zero. First, we substitute the strain rate-velocity relationship (4) into the fluid constitutive law (3), which is in turn substituted into the conservation of momentum equation (1). Then, multiplying the residual of the resulting equation by the test function w, integrating over the fluid domain, and applying the divergence theorem, we obtain the weak form of the conservation of momentum equation:

$$a(v, w) + 6(p, w) + c(v, v, w) = /_{\cdot} w \cdot cr_F \cdot n \, dF/(u) + /^{\cdot} w \cdot <r_F \cdot n \, dF_F(u) \tag{14}$$

where

$$a(v,w) = \int_{\Omega_F} \frac{\cdot}{2}(Vv + Vv^{\cdot}j : (Vw + Vw^{\cdot}1, \, dft_F(u) \tag{15}$$

$$6(p, w) = - /_{JOF} pV \cdot w <\!/ft_F(u) \tag{16}$$

$$c(v,v,w) = \int_{\Omega_F} /)W \cdot (v \cdot V) v \, dH_F(u) \tag{17}$$

Since we wish to consider problems in which the deformation may be large enough to influence the flow, we indicate the dependence of the fluid domain on the solid deformation in the definition of the domains of integration of the the bilinear functional a(-, •) and &(•, •) and the trilinear functional c(«, •, •). The second term on the right side of (14) can be rewritten as:

$$/ \quad w\text{-}<r_F\text{-}ndT_F(u) = / \quad w \bullet <r_F \bullet n \; dT^1_F(u) + / \quad w \bullet <r_F \bullet n \; dT^2_F(u) \qquad (18)$$
$$Jr_F \qquad\qquad Jr)_{,,} \qquad\qquad Jr]_{,,}$$

We shall require that the test function w satisfy the homogeneous essential boundary condition v = 0 on F^2_F, implying that that the second term on the right of (18) is zero. Furthermore, since i_F is zero, and in light of (10), the first term on the right side of (18) vanishes. Therefore,

$$/ \quad w.<r_F\text{-}ncn>(u) = 0 \qquad (19)$$
$$Jr_F$$

Next, we write the conservation of mass equation in weak form by multiplying (2) by the test function q and integrating over the domain of the fluid:

$$L_F \; q\nabla \cdot v \; cfft_F(u) = -6(_g, v) = 0 \qquad (20)$$

Again, note the dependence of the fluid domain, and thus weak form, on the solid deformation.

The weak form of the solid equilibrium equation is established by first substituting the expression for the Green's strain tensor (7) into the constitutive law (5), and then substituting the resulting expression for the Kirchhoff stress into the equilibrium equation. Multiplying the residual of the resulting equation by the test function r, integrating over the domain, and applying the divergence theorem, we obtain the weak form of the displacement form of the equilibrium equations:

$$\int_{\Omega_S} Vr:[(I+Vu).S(u)]dfi_5 =$$

$$f_{\_} \; r \bullet [(I + Vu) \bullet S(u)] \bullet n_0 \, dT_h + j \;\; r \bullet [(I + Vu) \bullet S(u)] \bullet n_0 \, dT_s \qquad (21)$$

where the relationship between Kirchhoff stress and displacement is given by:

$$S(u) = \hat{\ }A \; tr \; [Vu + Vu^T + Vu \bullet Vu^T] \; I + /x_5 \; [Vu + Vu^T + Vu \bullet Vu^T] \qquad (22)$$
$$2$$

Notice that, since we are in a Lagrangian frame of reference, the unit normal is with respect to the *undeformed* geometry, and the interface is between fluid and undeformed solid, denoted $F/_o$. The solid boundary *Fs* consists of the portion on which displacements are specified, F|, and the portion on which tractions are specified, F^. Thus, the second term on the right of (21) can be rewritten as:

$$\int_{\Gamma_S} r \cdot [(I + \nabla u) \cdot S(u)] \cdot n_0 \, d\Gamma_S =$$

$$\int_{\Gamma^1_S} r \bullet [(I + Vu) \bullet S(u)] \bullet n_0 \, dT\backslash + jf_f \; r. \; [(I + Vu) \bullet S(u)] \bullet n_0 \, X^*_S \qquad (23)$$
$$\Gamma_S$$

Since *is* is zero, and in light of (11), the first term on the right of this equation is zero. Furthermore, we shall require that the test function r satisfy the homogeneous essential boundary condition u = 0 on F|. Therefore, the second term vanishes over F|. Thus,

$$/ \ r \bullet [(I + Vu) \bullet S(u)] \bullet n_0 \, dT_s = 0 \tag{24}$$

The first term on the right of (21) can be transformed to the deformed geometry by noting that (I + Vu) • S = T, the Lagrangian stress tensor. The resultant surface traction acting on an element of undeformed geometry, 5o, in terms of the Lagrangian stress, is related to the resultant surface traction acting on an element of the deformed geometry, 5, in terms of the Eulerian stress, by the identity:

$$T \bullet n_0 \, dT_{Io} = a_s \bullet n \, dTi \tag{25}$$

Thus, provided

$$\mathbf{r(r_{/o})=w(r_7)} \tag{26}$$

i.e. the restrictions of the test functions r to the undeformed interface is equal to the restriction of w to the deformed boundary, the first term on the right side of (21) can be rewritten as

$$/ \ r \bullet [(I + Vu) \bullet S(u)] \bullet n_0 \, dT_{Io} = / \ w \bullet <r_s \bullet n \, dT_r \tag{27}$$

To simplify the right side of (21), we separate S into S^L, a tensor that depends linearly on displacement, and S^, one whose dependence is nonlinear, in fact quadratic:

$$S^L = A \ tr \ (Vu) \ I + IAS \ (Vu + Vu^T) \tag{28}$$

$$S^N = \wedge A \ tr \, (Vu \bullet Vu^T) \, I + /x_5 \, (Vu \bullet Vu^T) \tag{29}$$

so that (22) can be rewritten as $S = S^L + S^N$. Thus, the domain integral on the left side of (21) can be rewritten as the sum of terms that depend linearly, quadratically, and cubically on the derivatives of u:

$$/ \ Vr : [(I + Vu) \bullet S(u)] \, dSl_s = <i(u, r) + e(u, u, r) + /(u, u, u, r) \tag{30}$$

Jn,

where we define the bilinear form

$$<f(u,r)= \int_{\Omega_s} Vr : S^L(u) \, dti_s \tag{31}$$

the trilinear form

$$\mathbf{e(u, u, r) = \int_{Jtts} Vr : [s^N(u) + Vu \bullet S^L(u)] \, dQ,} \tag{32}$$

and the quadrilinear form

$$\mathbf{/(u, u, u, r) = \int_{Js} Vr : [Vu \bullet S^*(u)] \, dil_s} \tag{33}$$

We are in a position now to write the interface condition. Adding equations (14) and (21), and making use of (19), (24), and (27), gives:

$$a(u, v, w) + fc(u,p, w) + c(u, v, v, w) + d(u, r) + e(u, u, r) + /(u, u, u, r) =$$

$$/ \quad w-(<r_F-n + <r_{\leq}j-n) \quad </T/(u) \qquad (34)$$

The right side of this equation is just zero, in view of the continuity of traction condition (8).

We now state the unified variational form of the viscous flow-finite elasticity interaction problem: Find v G Hj (ft), p G L^2 (ft), and u G HJ (ft) such that:

$$a(v, w) + 6(p, w) + c(v, v, w) + d(u, r) + e(u, u, r) + /(u, u, u, r) = 0$$

$$\text{for all } w \text{ G } Hj\,(ft_F) \text{ and } r \text{ G } H^*_0\,(ft_5) \qquad (35)$$

$$6\,(q, v) = 0 \qquad \text{for all } q \text{ e } L^2\,(ft_F) \qquad (36)$$

where the functional a(-, •, •), &(•, •, •), c(«, •, •, •), d(-, •), e(-, •, •), and /(•, •, •, •) are defined by expressions (15), (16), (17), (31), (32), and (33), respectively. Here, $Hj(ft_F)$ is the Sobolev subspace of all functions having one square integrable derivative over *ftp* and that vanish on F^2_F, L^2 (ft_F) is the space of functions that are square integrable over ft/r, and HJ (fts) is the Sobolev subspace of all functions having one square integrable derivative over fts and that vanish on *T%*. The essential boundary conditions v = $\hat{v}$ and u = $\hat{u}$ must be enforced on Y^2_F and F|, respectively.

3. Finite element approximation

Let us define the finite element approximations v^, p^, and u^:

$$V^* = \sum_{i=1}^{n_v} >,(x)v,- \qquad (37)$$

$$PA = \sum_{j=1}^{n_p} xi(x)_{Pi} \qquad (38)$$

$$^{u}h. = \sum_{k=1} X\wedge \wedge^*(^{X}W \qquad (39)$$

where v_i, pj, and *Uk* are approximations of velocity, pressure, and displacement at nodes *i. j*, and A; respectively. The basis function families $<f>_{\{}$, *Xjj* ^{a}^d *rj)*$_k$ define finite element spaces V/t, *Vh** and *Uh* for velocity, pressure, and displacement, respectively:

$$V^* = \text{span}j\wedge,\ldots,\wedge\} \qquad (40)$$

$$P_{fc} = \text{span}\{xi,\ldots,Xn_p\} \qquad (41)$$

$$U_k = \text{span}\hat{}t,\ldots,\hat{}\} \tag{42}$$

Let V_h C H£(ft$_F$), *Vh* C *LftSl$_F$),* and U_h C Hj(n$_5$), i.e. the finite element spaces V*, *Vhi* and Z4 are subspaces of the infinite dimensional spaces in (35) and (36). In order to satisfy the condition (26), we require that fluid velocity and solid displacement shape functions be identical, when restricted to the interface between solid and fluid. An example of this is given by combining quadratic triangles in the solid with the Taylor-Hood element in the fluid. The Taylor-Hood element employs a quadratic approximation of velocity in conjunction with a linear approximation of pressure; thus, displacement and velocity shape functions are identical on the interface.

Applying the Galerkin method to the problem (35)-(36) yields the discrete problem: Find V* *e Vh, Ph* 6 *Vh,* and 11^ G *Uh* such that

$$a(\mathbf{v}_h,\mathbf{w}_h)+b(p_h,\mathbf{w}_h)+c(\mathbf{v}_h,\mathbf{v}_h,\mathbf{w}_h)+d(\mathbf{u}_h,\ r_h)+e(u_h,\ u_h,\ \text{r}\hat{})+/(\text{u}_{/1},\ \text{u}\hat{},\ \text{u}\hat{},\ r_k)\ =\ 0$$

$$\text{forallWfc}\,eV_h\text{and}\,r_h\text{€}S_k \tag{43}$$

and

$$6(_{g,V/l}) = 0 \qquad \text{for all } q\ eV_h \tag{44}$$

The discrete problem (43)-(44) is a system of nonlinear algebraic equations. To show the explicit form of these equations, let us first distinguish between nodes lying in the interior and those on the interface. Let

$$\begin{aligned} n_v &= n^v_F + n^v j \\ n_v &= n^p_F + n^p j \\ \text{ft}^* &= \hat{}^s\text{s}+^n/^s \end{aligned} \tag{45}$$

where the subscript *F* indicates the number of nodes belonging strictly to the fluid domain, *S* the number of nodes belonging strictly to the solid domain, and / the number of nodes belonging to the interface. So, for example, the n_v velocity nodes are composed of n^v_F fluid domain nodes as well *as n)* interface nodes. Notice that the satisfaction of condition (26) implies that the number of velocity and displacement interface nodes are equal. Let us call this number n/:

$$nj \equiv n^v p = \text{n£} \tag{46}$$

We can now elucidate the structure of the discrete problem (43)-(44): In the fluid, we have the n^v_F discrete conservation of momentum equations

$$\sum_{i=1}^{n_v} a(\phi_i,\phi_\ell)v_i + \sum_{j=1}^{n_p} b(\chi_j,\phi_\ell)p_j + \sum_{t>=1}^{n_v} \text{c}(\text{<k,</>}_r,4\text{>,})_{\text{V},\text{-t},r} = 0 \qquad * = 1,\ldots,\text{n £} \tag{47}$$

and the n^p_F discrete conservation of mass equations

$$\sum_{i=1}^{n_v} b(\chi_m,\phi_i)v_i = 0 \qquad \text{m} = 1,\ldots,\text{n j}. \tag{48}$$

In the solid, the n£ discrete equilibrium equations are given by

$$\sum_{k=1}^{n_u} d(\psi_k, \psi_n)u_k + \sum_{k,s=1}^{n_u} e(\psi_k, \psi_s, \psi_n)u_k u_s + \sum_{k,s,t=1}^{n_u} f(\psi_k, \psi_s, \psi_t, \psi_n)u_k u_s u_t = 0$$

$$n = 1, \ldots, n£ \tag{49}$$

Finally, on the interface, we have the n/ discrete traction continuity equations:

$$\sum_{i=1}^{n_v} a(\phi_i, \phi_y)v_i + \sum_{j=1}^{n_p} b(\chi_j, \phi_y)p_j + \sum_{i,r=1}^{n_v} c(\phi_i, \phi_r, \phi_y)v_i v_r + \sum_{k=1}^{n_u} d(\psi_k, \psi_y)u_k +$$

$$\sum_{k,s=1} e(\psi_k, \psi_s, \psi_y)u_k u_s + \sum_{k,s,t=1}^{n_u} f(\psi_k, \psi_s, \psi_t, \psi_y)u_k u_s u_t = \quad y = 1, \ldots, n_7 \tag{50}$$

and the n^ discrete conservation of mass equations

$$\sum_{i=1}^{n_v} b(\chi_z, \phi_{,>,\bullet} = 0 \qquad * = 1, \ldots, nj \tag{51}$$

Let us define vectors of unknown nodal quantities: let V/r G ^^ represent the fluid nodal velocities, pjr 6 3£^ the fluid pressures, V/ € 9f^/ the interface velocities, p/ G 3ft? the interface pressures, us G 3ft^ the solid displacements, and u/ G 3i^/ the interface displacements. We can rewrite the discrete equations (47)-(51) symbolically *as*

$$\begin{aligned} hMv_F, p_F, v/, p/, u/) &= 0 \\ hjp(v_F, v/, u/) &= 0 \\ h5(u_5, u/) &= 0 \\ h/(v_F, p_F, v_7, p/, u_5, u_7) &= 0 \\ h/(v_F, v/, u/) &= 0 \end{aligned} \tag{52}$$

where h_F^v G 3ft^ represents conservation of momentum in the fluid, h_F^p G 3£^ conservation of mass in the fluid, h£ G 3?^ equilibrium in the solid, h/ G 5R^/ continuity of interface tractions, and h/ G 3?^/ conservation of mass on the interface. It appears that we have $n_v + n_p + n_u$ — nj equations in $n_v + n_p + n_u$ unknowns. However, the continuity of interface velocity condition (9) implies that v/ = 0, and we are thus left with an equal number of equations and unknowns upon enforcing this condition in (52).

Note that, in addition to h^, the fluid and interface equations h£, h_F^p, h^, and h/ depend on the interface displacements u/. This is implied in the domain of integration of the functionals a(«, •) (15), &(•, •) (16), and c(-, •, •) (17), i.e. in the dependence of the flow on the interface geometry.

4. Solution of the discrete system

We discuss in this section a Newton-like method for solving the system of nonlinear algebraic equations (52). Our discussion will be kept brief; a more extensive discussion of this and other solution methods for finite element approximations of viscous flow-finite elasticity interaction will be presented in the future.

Let us first begin by rewriting (52) as

$$\begin{aligned} h_F(x_F, X/) &= 0 \\ h_5(x_5, x/) &= 0 \\ h/(xF, x\$, x/) &= 0 \end{aligned} \tag{53}$$

where

$$h_F = \begin{bmatrix} h^h_F \\ h^p_F \end{bmatrix} \qquad x/r = \begin{bmatrix} x_F \\ p_F \end{bmatrix} \qquad xs = us \qquad x/ = us \tag{54}$$

Note that the fluid equations *hp* include the equations for conservation of mass on the interface, h/, and the fluid variables *xp* include the interface pressures p/. Accordingly, the interface variables consist only of the interface displacements. The reason for this choice of partitioning will become apparent.

Newton's method for the nonlinear system h(x) = 0 consists of iterating on solution of the linear system:

$$J(x^*)(x^{*+1} - x^*) = -h(x^*) \tag{55}$$

until convergence, given an initial iterate x°. Here, J is the Jacobian of **h** with respect to x. A Newton step for the discrete system (53) takes the form:

$$\begin{bmatrix} J^k_{FF} & 0 & J^k_{FJ} & 1 & f & \&^*F \\ 0 & J^*_{s_S} & J^*_{s,} & \Big| & \backslash & Axs \\ X^k_{IF} & J^{Tfc}_{/S} & J^{Tfc}_{//_F} & "T" & J^{Tfc}_{/7_S} & J \end{bmatrix} \begin{bmatrix} \backslash & A & \backslash \\ & \wedge & X \end{bmatrix} = - \begin{bmatrix} h^k_F \\ h^*_s \\ I \end{bmatrix} \tag{56}$$

where

$$Z\backslash X — X \quad - X \tag{Ol}$$

Here, the superscript *k* indicates evaluation of the residual **h** and the Jacobian J at the point x*, and the interface-interface coupling matrix J// includes contributions from both solid and fluid:

$$J// = J//_F + J//_5 \tag{58}$$

The Newton iteration (56) entails two difficulties. First, the Jacobian matrix, because of the disparity between fluid and solid behavior, can be very ill-conditioned. Second, the coupling terms between fluid and interface variables in general render the matrices ***JFI*** and $J//_F$ dense. The density of these matrices is a consequence of the dependence of the domains of integration of a(-,-), &(-,•)> and c(-,-,«) on the interface displacements. In the case of Jjr/, all fluid nodal velocities and pressures may be coupled to all interface nodal

displacements, since a change in any interface displacement potentially moves the fluid mesh everywhere. The matrix $J//_F$ derives its density from the fact that the interface traction continuity equation (50) includes contributions from the first layer of fluid elements, which change with a movement in the interface. Thus, the potential exists for coupling between all interface variables. $J//_5$ contains nonzeroes contributed by the solid terms in (50), i.e. the terms involving c?(«, •), e(-, •, •), and /(•,-,-,•)• These are just the standard solid stiffness coupling terms, so the coupling is local in nature.

The exact sparsity pattern depends on the moving mesh scheme employed, but in general, the storage requirements and arithmetic complexity associated with J/rj and $J//_F$ can be quite severe. Therefore, we consider a *modified* Newton's method obtained by ignoring the fluid-interface coupling matrix J/r/ and the contribution of the fluid to the interface-interface coupling matrix, $J//_F$. The resulting Jacobian in (56) becomes block-lower triangular. Thus, the fluid variables can be found by solving the linear system

$$3^k_{FF} A x_F = -h £ \tag{59}$$

for Axjr. The change in the displacements (both interior and interface) can then be found by solving:

$$\left[J^k_{IS} \quad J^k_{II_S} \quad J \right] \Delta x_I \quad J^- \quad - \quad h \} + J^k_{IF} x_F \; / \tag{60}$$

This method avoids the ill-conditioning associated with the coupled problem by employing a "domain decomposition" into separate fluid and solid subdomains. Large storage requirements associated with geometric coupling matrices are avoided by ignoring these terms while constructing the Jacobian. However, since the residual in (56) is calculated correctly, we are guaranteed that, if the method converges, it must converge to the correct solution. This can be seen from (55): the only way that Ax can be zero is for h to be zero, provided only that J is nonsingular, regardless of whether or not it represents the true Jacobian. The price we pay for this modified Jacobian is that we must give up the Newton guarantee of local quadratic convergence.

We now establish that the modified Jacobian is indeed nonsingular. First, the fluid step (59) can be seen to be just a Newton step for the Navier-Stokes equations, with a rigid boundary given by the current deformed interface, and a no-slip boundary condition imposed on the interface. Thus, the linear system (59) has a unique solution (provided of course that we are away from singular bifurcation or turning points). Second, the solid step (60) can also be regarded as a Newton step for the solid equilibrium equations, with imposed traction boundary conditions on the interface given by the current estimate of the fluid tractions. It too must have a unique solution (provided again we are away from buckling points). Thus, the solution of (56) is unique, and the approximate Jacobian is nonsingular.

5. Example: flow-induced deformation of an infinite elastic cylinder

We have built a code that implements the finite element approximation of §3 in two-dimensions, and solves the resulting nonlinear algebraic system using the modified Newton

method described in §4. We employ a continuation strategy to help globalize the solution. Our code discretizes both solid and fluid with triangular elements, and uses quadratic shape functions for velocity and displacement, and linear shape functions to interpolate pressure. The Taylor-Hood element pair is known to satisfy the Ladyzhenskaya-Babuska-Brezzi stability condition, e.g. [2], and the choice of quadratic triangles for the solid insures the satisfaction of the interface compatibility condition (26). The Taylor-Hood element produces errors of order h^3 for velocity and h^2 for pressure [7], while quadratic triangles for elasticity problems are third-order accurate for displacements [3] (provided the solution is sufficiently smooth). Currently, we keep the mesh topology fixed throughout the iterations, moving the fluid mesh in response to solid deformations according to a known mapping function. We plan to incorporate adaptive refinement strategies in the future; these may be necessary for very large deformation problems.

In order to illustrate our methodology, we now present results of a physical problem solved by our code. The problem is viscous flow about an infinite elastic cylinder. The problem thus is two-dimensional. Symmetry is exploited, so the computational domain consists of only the upper half of the cylinder and surrounding fluid. Flow is from left to right. The upstream support is free to translate horizontally, while the downstream support is fixed both horizontally and vertically. The Reynolds number of the fluid flow is 200, while the solid has $\nu = 0.3$. Two elements are used in the thickness direction of the cylinder. The (undeformed) mesh is shown in Figure 1. Figure 2 shows a portion of the converged flow field near the cylinder, assuming a nearly rigid cylinder ($E = 100000$, thickness of 0.1). The resulting deformation is negligible and does not affect the flow field.

Figure 3 shows the resulting deformation and a portion of the flow field when the cylinder is more flexible ($E = 7500$, thickness of 0.022) and thus undergoes large deformation. The initial shape of the cylinder is shown in addition to the deformed shape. The flow field depicted corresponds to the converged solution, i.e. to the deformed shape. Clearly, the flow fields are quite different. Convergence was obtained in a total of 12 continuation steps, each requiring on average 6 iterations. The continuation strategy employed is quite conservative, first increasing Reynolds number to the desired value, then decreasing elastic modulus and thickness. We will be incorporating some fixed-point type methods in the future to make the globalization more automatic.

6. Concluding remarks

We have developed a methodology for numerical approximation of the interaction of a stationary viscous fluid with a elastic solid that undergoes large deformation. The fluid is modeled with respect to an Eulerian frame of reference by the stationary incompressible Navier-Stokes equations, while a Lagrangian reference frame and large deformation-small strain theory is used for the solid. A variational formulation of the problem is developed that insures satisfaction of continuity of interface tractions and velocities. The variational formulation is approximated by a Galerkin finite element method, yielding a system of nonlinear algebraic equations in unknown fluid velocities and pressures and solid displacements. A Newton-like method is introduced for solution of the discrete system. The method employs

a modified Jacobian that enables decomposition into separate fluid and solid subdomains. This domain decomposition avoids possible ill-conditioning of the Jacobian, as well as the need to compute and store geometric coupling terms between fluid and interface shape. The method is illustrated by solution of a problem of flow-induced deformation of an elastic cylinder.

In addition to providing a unified framework for solving nonlinear fluid-structure interaction problems, the methodology we have presented allows sensitivity analysis to be performed in a straightforward manner, by expressing system behavior as a single set of algebraic equations. Sensitivity analysis is essential for efficient design optimization, which is the ultimate objective of our work. For this reason, we have focused on a steady-state model of the system. However, the extension of our variational formulation and finite element method to time-dependent problems can be straightforwardly achieved by adding the appropriate inertial force terms. Spatial discretization by finite elements then yields a system of ordinary differential equations, which can be integrated in time given appropriate initial conditions.

We have also assumed the fluid to be incompressible. Given the importance of aeroelastic effects in aircraft in the transonic flow regime, a useful extension of our work would be to relax this incompressibility assumption. This entails the addition of conservation of energy and state equations, as well as temperature and density variables, which complicate the numerical approximation. However, we remark that the coupling mechanism is independent of compressibility—continuity of interface tractions is affected through a weak formulation of the stress divergence terms in the conservation of momentum equations. Thus, our fluid-solid coupling for the compressible problem goes through in the same fashion.

Another point worth mentioning is that fluid-solid coupling is independent of the fluid constitutive law; thus non-Newtonian fluid models (important for cardiac blood flows, for example) can be readily accommodated in the variational formulation. An additional consequence is that algebraic turbulence models that modify the viscosity coefficient to include a term that depends on strain rate, thus rendering the constitutive relations nonlinear, permit incorporation into our model quite readily.

References

[1] R.L. Bisplinghoff and H. Ashley. *Principles of Aeroelasticity.* Wiley, 1962.

[2] J. Boland and R. Nicolaides. Stability of finite elements under divergence constraints. *SIAM Journal on Numerical Analysis,* 20:722-731, 1983.

[3] P. Ciarlet. *The Finite Element Method for Elliptic Problems.* North-Holland, Amsterdam, 1978.

[4] E.H. Dowell, H.C. Curtiss, R.H. Scanlan, and F. Sisto. *A Modern Course in Aeroelasticity.* Sijthoff and Noordhoff, the Netherlands, 1980.

[5] E.H. Dowell and M. Ilgamov. *Studies in Nonlinear Aeroelasticity.* Springer Verlag, 1988.

[6] F.F. Felker. Direct solution of the two-dimensional Navier-Stokes equations for static aeroelasticity problems. *AIAA Journal,* 31(1):148-153, 1993.

[7] M. Gunzburger. *Finite Element Methods for Viscous Incompressible Flows.* Academic Press, 1989.

[8] M.E. Gurtin. *Introduction to Continuum Mechanics.* Academic Press, 1981.

[9] G.P. Guruswamy. Unsteady aerodynamic and aeroelastic calculations for wings using the Euler equations. *AIAA Journal,* 28(3):461-469, 1990.

[10] G.P. Guruswamy. Vortical flow computations on swept flexible wings using Navier-Stokes equations. *AIAA Journal,* 28(12):2077-2084, 1990.

[11] G.P. Guruswamy. Vortical flow computations on a flexible blended wing-body configuration. In *32nd Structures, Structural Dynamics, and Materials Conference,* pages 719-734. AIAA, 1991.

[12] G.P. Guruswamy. Coupled finite-difference/finite element approach for wing-body aeroelasticity. In *Fourth AIAA/USAF/NASA/OAI Symposium on Multidisciplinary Analysis and Optimization,* pages 1-12, 1992.

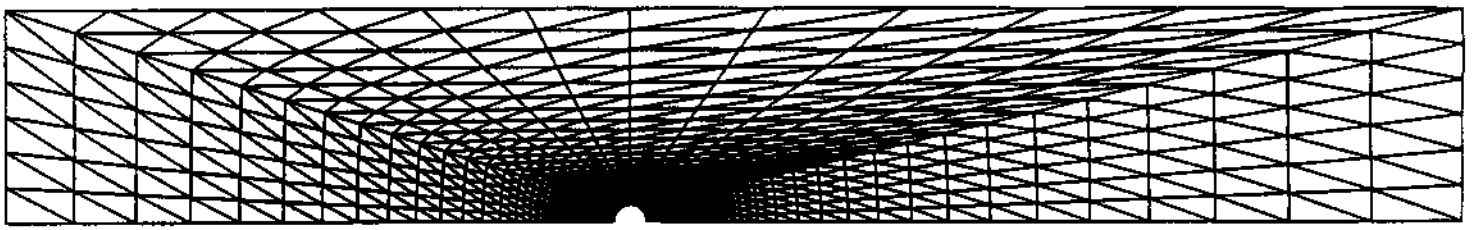

Figure 1: Geometry and mesh.

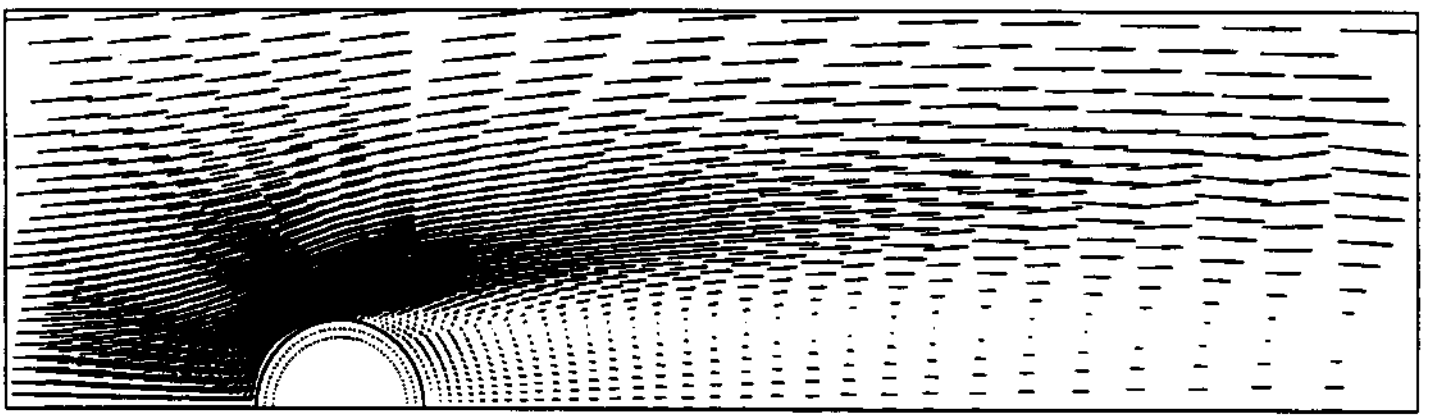

Figure 2: Viscous flow about rigid cylinder.

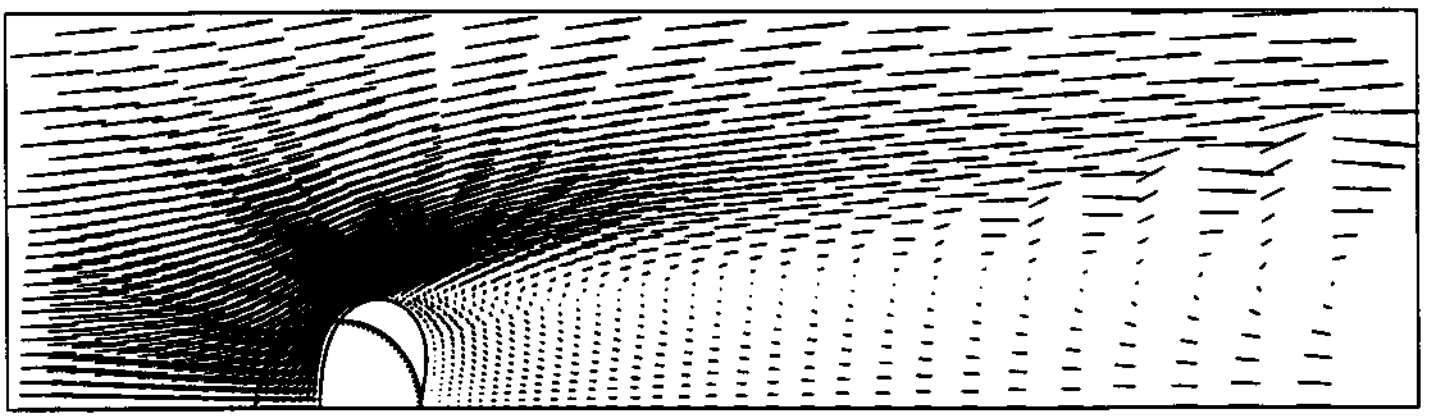

Figure 3: Viscous flow about elastic cylinder.