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An ordinal analysis of admissible set theory using recursion on ordinal notations*

Jeremy Avigad

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Abstract

The notion of a function from $\mathbb{N}$ to $\mathbb{N}$ defined by recursion on ordinal notations is fundamental in proof theory. Here this notion is generalized to functions on the universe of sets, using notations for well-orderings longer than the class of ordinals. The generalization is used to bound the rate of growth of any function on the universe of sets that is Σ_1-definable in Kripke-Platek admissible set theory with an axiom of infinity. Formalizing the argument provides an ordinal analysis.

1 Introduction

In informal proof-theoretic parlance, the definition of a set of objects is said to be *impredicative* if it makes reference to a collection of sets that includes the set being defined. A classic example arises if one takes the real numbers to be lower Dedekind cuts of rationals, and then defines the least upper bound of a bounded set of reals to be the intersection of *all* the upper bounds. A theory is said to be (prima facie) impredicative if its intended interpretation depends on such a definition.

The circularity implicit in an impredicative theory poses problems for its ordinal analysis, since the goal of ordinal analysis is to measure the theory's strength in terms of well-founded ordinal notations — that is, "from the bottom up." For that reason, the first ordinal analyses of impredicative theories, due to Takeuti, Buchholz, and Pohlers were a landmark (see the discussion in the introduction to [7]). Another important step was the move to studying fragments of set theory instead of second-order arithmetic, carried out by Jäger [12, 13, 14], providing a more natural framework for the analysis of impredicativity.

In this paper I will discuss the ordinal analysis of Kripke Platek admissible set theory with an axiom of infinity, henceforth denoted $KP\omega$. This theory has the same strength as the theory ID_1 of one arithmetic inductive definition,

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or Π^1_1-CA^-, based on Π^1_1 comprehension without parameters. By now ordinal analyses of $KP\omega$ using Gentzen-Schütte cut-elimination methods are very polished; see, for example, [12, 14, 16, 17, 19, 21]. My goal here is to develop an alternative, complementary approach, in which the emphasis is on *computations* instead of *derivations*. This paper can be read as a sequel to [2], but can also be read independently.

Much of the effort here is devoted to drawing together notions and methods from a number of sources. In Section 2, I review the axioms of Kripke-Platek set theory. In Section 3, I discuss the primitive recursive set functions of Jensen and Karp [15], and an axiomatization thereof due to Rathjen [20]. Section 4 presents a characterization, due to a number of authors independently, of those ordinals α for which L_α (or V_α) is closed under the primitive recursive set functions.

In Section 5, I present a definition of the Howard-Bachmann ordinal, using a version of the Feferman-Aczel functions. With this definition in hand, we can state the main results of the ordinal analysis of $KP\omega$, including a bound on the rate of growth of the theory's Σ_1-definable functions.

In [2], I defined a collection of functions from $\mathbb{N}$ to $\mathbb{N}$ using iterative computations that "count down" from a given ordinal notation; this is one of many equivalent characterizations of ordinal recursion. In Section 6, I lift this notion to functions on the universe of sets, using set notations for a well-ordering that is longer than the class of ordinals.

With all these pieces in place, the ordinal analysis itself takes place in Sections 7 and 8. Section 7 shows that one can eliminate foundation axioms in favor of iterated computations below $\varepsilon_{\Omega+1}$, in a manner similar to the way in which one can eliminate induction over the natural numbers in favor of iterated computations below ε_0. The main novelty in this paper is a combinatorial argument in Section 8, which shows that one can interpret such iterated computations, involving a Skolem function for the Δ_0 collection schema, in the constructible hierarchy below the Howard-Bachmann ordinal. This lemma provides a semantic analogue of a proof-theoretic "collapsing" argument, but maintains the thematic emphasis on iterated computations instead of infinitary derivations.

For a survey of more recent developments in ordinal analysis, see, for example, [17, 22].

2 Kripke Platek set theory

We will take the language of set theory to consist of a single binary relation symbol $\in$, with $x = y$ defined by $\forall z\ (z \in x \leftrightarrow z \in y)$. A formula is said to be Δ_0 if every quantifier is bounded, i.e. of the form $\exists x \in y$ or $\forall x \in y$, where these are interpreted in the usual way. A formula is said to be Σ_1 (resp. Π_1) if it is of the form $\exists \vec{y}\ \varphi$ (resp. $\forall \vec{y}\ \varphi$), where φ is Δ_0. The classes Σ_n and Π_n are defined analogously.

The axioms of KP are as follows:

1. Extensionality: $x = y \rightarrow (x \in w \rightarrow y \in w)$

2. Pair: $\exists x\ (x = \{y, z\})$
3. Union: $\exists x\ (x = \bigcup y)$
4. Δ_0 separation: $\exists x\ \forall z\ (z \in x \leftrightarrow z \in y \wedge \varphi(z))$ where φ is Δ_0 and x does not occur in φ
5. Δ_0 collection: $\forall x \in z\ \exists y\ \varphi(x, y) \rightarrow \exists w\ \forall x \in z\ \exists y \in w\ \varphi(x, y)$, where φ is Δ_0
6. Foundation: $\forall x\ (\forall y \in x\ \varphi(y) \rightarrow \varphi(x)) \rightarrow \forall x\ \varphi(x)$, for arbitrary φ

In 2 and 3, "$x = \{y, z\}$" and "$x = \bigcup y$" abbreviate the usual representations in the language of set theory. In 4–6, the formula φ may have free variables other than the ones shown. The foundation axiom as presented here is classically equivalent to the assertion that every nonempty definable class of sets has an $\in$-least element. The theory KP^- arises if one replaces the foundation schema with the single instance expressing foundation for sets, where $\varphi(x)$ is just the formula $x \in z$. Below we will consider the restriction of the foundation schema to Π_n formulae, and we will use Π_n-*Foundation* to denote this restriction.

Let L denote the constructible hierarchy of sets. Ordinals α such that L_α models KP are called *admissible*, with ω being the least such. We will use $KP\omega$ ($KP\omega^-$, etc.) to denote the result of adding an axiom of infinity

$$\exists x\ (\emptyset \in x \wedge \forall y \in x\ (y \cup \{y\} \in x))$$

to the corresponding theories above. The least admissible ordinal above ω is the least non-recursive ordinal, also called the Church-Kleene ordinal, ω_1^{ck}. For more about KP see, for example, [6, 16, 17, 19, 20].

3 The primitive recursive set functions

In this section we will define the primitive recursive set functions, and consider axiomatizations thereof. For the moment, we will think of these functions as class functions defined over a fixed universe $\langle V, \in \rangle$ of ZF set theory, generalizing the primitive recursive functions on the natural numbers. However, it will be a recurring theme of this paper that one can find more meager interpretations; for example, for each regular κ, the κth level of the cumulative hierarchy, V_κ, is closed under the primitive recursive set functions, as well as L_α, for any admissible α. In the next section we will, in fact, characterize the ordinals α for which L_α is so closed.

The collection of *primitive recursive set functions*, denoted *Prim*, is the collection of functions from V to V (of various arities) defined inductively by the following clauses:

- For each natural number n and $i \leq n$, the projection function defined by $p_{n,i}(x_1, \ldots, x_n) = x_i$ is in *Prim*.

- The constant 0 is in *Prim* (as a 0-ary function).
- The function m defined by $m(x, y) = x \cup \{y\}$ is in *Prim*.
- The function c defined by

$$c(x, y, u, v) = \begin{cases} x & \text{if } u \in v \\ y & \text{otherwise} \end{cases}$$

  is in *Prim*.
- *Composition:* If $h, g_1, \ldots, g_k$ are of appropriate arities and in *Prim*, then so is the function f, defined by

$$f(\vec{x}) = h(g_1(\vec{x}), \ldots, g_k(\vec{x})).$$

- *Primitive recursion:* If $h(w, z, \vec{x})$ is in *Prim*, then so is the function f, defined by

$$f(z, \vec{x}) = h(\bigcup\{f(u, x) \mid u \in z\}, z, \vec{x}).$$

The notion of a primitive recursive set function is due to Jensen and Karp, generalizing Kino and Takeuti's notion of a primitive recursive function on the ordinals. In [15], the formulation above is attributed to Gandy. (See also [20].)

A relation R on sets is said to be primitive recursive if its characteristic function, χ_R, is. The collection of primitive recursive functions and relations is remarkably robust. The collection of primitive functions contains pairing and projection functions, union, intersection, cartesian product, transitive closure, and rank; various operations on ordinary functions (represented as sets of ordered pairs) like domain, range, application, and restriction; and some basic operations on ordinals like addition, multiplication, and exponentiation. The collection of primitive recursive functions is further closed under definition by cases. Similarly, the collection of primitive recursive relations contains, for example, the element-of and subset-of relations, and is closed under boolean operations and bounded quantification.

The definition of the collection of primitive recursive functions can be relativized by adding functions $f_1, \ldots, f_k$ to the stock of initial functions (and assuming the universe of sets under consideration is suitably closed). The resulting collection is denoted $Prim[f_1, \ldots, f_k]$.

Rathjen [20, Section 6] introduces an axiomatic theory, *PRS*, that characterizes the primitive recursive set functions. In addition to the $\in$ symbol, the language of *PRS* has function symbols corresponding to the inductive definition of *Prim*. The axioms of *PRS* are extensionality, pair, union, the foundation axiom for sets, and the schema of Δ_0 separation, together with the natural rendering of the defining equations above in the language of set theory. Note that Δ_0 collection is *not* one of the axioms of *PRS*.

I have already noted that many common functions on the universe of sets are primitive recursive; it is further the case that their general properties can be

verified axiomatically in PRS. I will assume that finite sequences are represented in such a way that operations like concatenation, projections, and so on are primitive set recursive. If s is a sequence, $\mathit{length}(s)$ denotes the length of the sequence, $\mathit{last}(s) = \mathit{length}(s) - 1$ denotes the index of the last element in the sequence, and $(s)_0, \ldots, (s)_{\mathit{last}(s)}$ denote the elements.

Of course, the axiomatic theory can be relativized to arbitrarily many free function variables $f_1, \ldots, f_k$, of various arities; I will denote the resulting theory by $PRS[f_1, \ldots, f_k]$.[1]

We will mostly be interested in the primitive recursive set functions relativized to the constant ω. That is, we will consider theories with an additional constant symbol, ω, described by a defining axiom, (ω), which asserts that: ω is transitive, and linearly ordered by $\in$; ω contains $\emptyset$, and is closed under the successor function $x \mapsto x \cup \{x\}$; and no element of ω contains $\emptyset$ and is so closed.

Below, we will need a universally axiomatized theory that includes $PRS[\omega]$. To that end, let us add a function symbol μ, with defining axiom

$$y \in x \rightarrow \mu(x) \in x \wedge y \notin \mu(x). \tag{μ}$$

The axiom states that if x is a nonempty set, $\mu(x)$ returns an $\in$-least element of x. For example, if one restricts one's attention to the constructible hierarchy, one can interpret $\mu(x)$ as returning the least element of x in the standard ordering of L. The following definition will be notationally convenient:

Definition 3.1 *Let $PRS\omega$ denote the theory $PRS[\omega, \mu] + (\omega) + (\mu)$.*

The fact we need is the following:

Proposition 3.2 *$PRS\omega$ has a set of universal axioms.*

The proposition follows from the following two lemmata:

Lemma 3.3 *$PRS\omega$ has a set of Π_1 axioms.*

Proof. Using the explicit function symbols for pairing and union, one can eliminate the existential quantifiers in the pairing and union axioms, and, with a little effort, rewrite the matrix so that it is Δ_0. Similarly, if $\varphi(x, \vec{z})$ is Δ_0, it is also primitive recursive, and hence so is

$$\{x \in y \mid \varphi(x, \vec{z})\} = \bigcup_{x \in y} f(x, \vec{z}),$$

where $f(x, \vec{z})$ is equal to $\{x\}$ if $\varphi(x, \vec{z})$ and $\emptyset$ otherwise. This takes care of the existential quantifier in the separation axiom, and μ handles foundation for sets. Finally, the defining axioms for the primitive recursive set functions are already Π_1, provided we use an explicit symbol for $\bigcup\{f(u, x) \mid u \in z\}$ in the clause for primitive recursion. $\square$

[1] Note that our $PRS[G]$ is different from Rathjen's PRS^G. In Rathjen's system, one assumes G is Δ_0-definable, and the axioms of PRS^G include an axiom asserting that G satisfies its Δ_0 definition. In contrast, we are just adding G "freely."

Lemma 3.4 *There is a set of universal consequences of $PRS\omega$, over which every Δ_0 formula is equivalent to a quantifier-free one.*

Proof. By recursion on Δ_0 formulae $\varphi(\vec{z})$ define primitive recursive relation symbols $R_\varphi(\vec{z})$, and universal consequences of $PRS\omega$ which entail $\varphi(\vec{z}) \leftrightarrow R_\varphi(\vec{z})$. The cases for atomic formulae $s \in t$ and boolean operations are straightforward. To handle an existential quantifier $\exists x \in y\ \theta(x, \vec{z})$, let

$$R_{\exists x \in y\ \theta(x,\vec{z})}(\vec{z}) \equiv R_{\theta(x,\vec{z})}(\mu(\{x \in y \mid R_{\theta(x,\vec{z})}(x, \vec{z})\}), \vec{z}),$$

and add the sentence

$$\forall x, \vec{z}\ (x \in y \wedge R_{\theta(x,\vec{z})}(x, \vec{z}) \rightarrow R_{\theta(x,\vec{z})}(\mu(\{x \in y \mid \theta(x, \vec{z})\}, \vec{z}), \vec{z})).$$

□

Proof of Proposition 3.2. To obtain a universal axiomatization of the theory $PRS\omega$, use the universal set of sentences given by Lemma 3.4, together with the universal "translations" of the axioms from Lemma 3.3. □

4 Ordinal bounds

Our goal in this section is to get a sense of the rate of growth of the primitive recursive set functions, and characterize the ordinals α such that L_α is closed under these functions. The characterization presented below has been obtained by a number of authors independently: Stanley Wainer informs me that it can be found in a handwritten manuscript by Gandy; it appears in Schütte [23]; and was later rediscovered by Cantini (see [11], where Schütte is credited with the result). Proofs are included here for completeness.

First, we need to introduce the hierarchy of Veblen functions on the ordinals. If f is a continuous, nondecreasing function on the ordinals, then so is the function f' which enumerates the fixed points of f, that is, the set $\{\alpha \mid f(\alpha) = \alpha\}$. The hierarchy of Veblen functions is defined by letting $\varphi_0(\beta) = \omega^\beta$, for each α letting $\varphi_{\alpha+1} = \varphi'_\alpha$, and, at limit stages λ, letting φ_λ enumerate the simultaneous fixed points of φ_γ, for $\gamma < \lambda$. Then we have that $\varphi_\alpha(\beta) < \varphi_\gamma(\delta)$ if and only if one of the following holds:

- $\alpha < \gamma$ and $\beta < \varphi_\gamma(\delta)$,
- $\alpha = \gamma$ and $\beta < \delta$, or
- $\alpha > \gamma$ and $\varphi_\alpha(\beta) < \delta$.

Instead of taking the Veblen functions and their indices to range over the entire universe of ordinals, one can just as well restrict one's attention to any uncountable regular cardinal, Ω. For more information see [16, 17].

The primitive recursive *ordinal* functions, denoted $Prim_O$, are obtained by restricting the domains of the functions in the defining schemata of Section 3 to ordinals, and replacing $m(x, y)$ by $s(x) = x \cup \{x\}$. Clearly, every primitive recursive ordinal function can be viewed as the restriction of a primitive recursive set function to the ordinals.

The following proposition provides a lower bound on the rate of growth of the functions in $Prim_O[\omega]$.

Lemma 4.1 *For each natural number i, the Veblen function φ_i is in $Prim_O[\omega]$.*

Proof. Use induction on i. For $i = 0$, $\varphi_0(\beta) = \omega^\beta$ is primitive recursive in ω. If f is any increasing, continuous function, then the αth iterate of f on β, defined by

$$Iterate_f(\alpha, \beta) = f(\bigcup_{\gamma<\alpha} \max(\beta, Iterate_f(\gamma, \beta)))$$

is primitive set recursive; and then so is f', defined by

$$f'(\alpha) = \bigcup_{\gamma<\alpha} (Iterate_f(\omega, f'(\gamma))).$$

This completes the proof. □

The following lemma provides the corresponding upper bound. In its proof, "#" and $\sum$ refer to the symmetric sum, which is monotone in its arguments (see [16, 17]).

Lemma 4.2 *Let $f(x_1, \ldots, x_k)$ be an element of $Prim_O[\omega]$. Then there is a natural number i such that for every $\alpha_1, \ldots, \alpha_k$, $f(\alpha_1, \ldots, \alpha_k) \leq \varphi_i(\max(\alpha_1, \ldots, \alpha_k))$.*

Proof. Use induction on the defining schemata for the primitive recursive ordinal functions to show that for each such function f there is a j such that for each $\alpha_1, \ldots, \alpha_k$, $f(\alpha_1, \ldots, \alpha_k) \leq \varphi_j(\alpha_1 \# \ldots \# \alpha_k + 1)$. Letting $i = j + 1$ yields the conclusion of the lemma, since $\varphi_{j+1}(\max(\alpha_1, \ldots, \alpha_k))$ contains each of $\alpha_1, \ldots, \alpha_k$, and is closed under # and φ_j.

For 0, projections, c, and m, one can take $j = 0$. For composition, if j is such that the conclusion of the lemma holds for $h, g_1, \ldots, g_l$, then

$$\begin{aligned} h(g_1(\vec{x}), \ldots, g_l(\vec{x})) &\leq \varphi_j(\varphi_j(\sum \vec{\alpha} + 1) \# \ldots \# \varphi_j(\sum \vec{\alpha} + 1) + 1) \\ &\leq \varphi_{j+1}(\sum \vec{\alpha} + 1) \end{aligned}$$

since $\varphi_{j+1}(\sum \vec{\alpha} + 1)$ contains $\sum \vec{\alpha} + 1$ and is closed under φ_j. Finally if, j is such that the conclusion of the lemma holds for h and f is defined from h using primitive recursion, then, fixing $\vec{\alpha}$, we can use induction on β to show $f(\beta, \vec{\alpha}) \leq \varphi_{j+1}(\beta \# \sum \vec{\alpha} + 1)$. In the induction step we have

$$\begin{aligned} f(\beta, \vec{\alpha}) &\leq \varphi_j((\sup_{\beta'<\beta} \varphi_{j+1}(\beta' \# \sum \vec{\alpha} + 1)) \# \beta \# \vec{\alpha} + 1) \\ &\leq \varphi_j(\varphi_{j+1}(\beta \# \sum \vec{\alpha}) \# \beta \# \vec{\alpha} + 1) \\ &\leq \varphi_{j+1}(\beta \# \sum \vec{\alpha} + 1), \end{aligned}$$

since the latter contains each of $\beta, \vec{\alpha}, \varphi_{j+1}(\beta \# \sum \vec{\alpha})$ and is closed under φ_j. $\square$

The Stability Theorem of [15] allows us to transfer these bounds from the primitive recursive *ordinal* functions to the primitive recursive *set* functions. For our purposes, it suffices to restrict our attention to L, but analogous results hold for the other hierarchies discussed in [15].

Suppose $f(x_1, \ldots, x_n)$ is an n-ary function from L to L whose graph is defined by a first-order formula $\mathcal{D}_f(x_1, \ldots, x_n, y)$. Let h be a function from the class of ordinals of L to itself. Then the function h is said to L-stabilize $\mathcal{D}_f$ if, for every α and $a_1, \ldots, a_n$ in L_α, the following two conditions hold:

- $f(a_1, \ldots, a_n) \in L(h(\alpha))$; and
- whenever $\beta > h(\alpha)$ and $b \in L_\beta$, $f(a_1, \ldots, a_n) = b \leftrightarrow L_\beta \models \mathcal{D}_f(a_1, \ldots, a_n, b)$.

Now suppose $g_1, \ldots, g_k$ are functions from L to L defined by formulae $\mathcal{D}_{g_1}, \ldots, \mathcal{D}_{g_k}$. Then to each set function f that is primitive recursive in $g_1, \ldots, g_k$, one can assign a definition $\mathcal{D}_f$ that is Σ_1 in $\mathcal{D}_{g_1}, \ldots, \mathcal{D}_{g_k}$. Specializing the Stability Theorem to the constructible hierarchy yields the following:

Lemma 4.3 *Suppose $g_1, \ldots, g_k$ and $\mathcal{D}_{g_1}, \ldots, \mathcal{D}_{g_k}$ are as above, and each $\mathcal{D}_{g_i}$ is L-stabilized by an ordinal function h_i. Then for each $f \in Prim[g_1, \ldots, g_k]$, $\mathcal{D}_f$ is stabilized by a function in $Prim_O[h_1, \ldots, h_k]$.*

This gives us the bound we want.

Theorem 4.4 *Let $f(x_1, \ldots, x_k)$ be an element of $Prim[\omega, \mu]$. Then there is a natural number i such that the following holds: for every α and every sequence of elements $a_1, \ldots, a_k$ in L_α, $f(a_1, \ldots, a_k) \in L_{\varphi_i(\alpha)}$.*

Proof. Both ω and μ are defined by Δ_0 formulae, say, $\mathcal{D}_\omega$ and $\mathcal{D}_\mu$, respectively, where $\mathcal{D}_\omega$ is stabilized by $\omega + 1$ and $\mathcal{D}_\mu$ is stabilized by the identity function on the ordinals. By Lemma 4.3, for each $f \in Prim[\omega, \mu]$, $\mathcal{D}_f$ is stabilized by a function in $Prim_O[\omega]$. By Lemma 4.2, this stabilizing function is bounded by one of the φ_i. $\square$

In the absence of ω, of course, L_ω provides a model of PRS. This provides a precise characterization of the ordinals α for which L_α is a model of PRS.

Corollary 4.5 *The following statements are equivalent:*

1. *L_α is closed under the primitive recursive set functions (with or without μ).*
2. *α is closed under the primitive recursive ordinal functions.*
3. *α is either ω or of the form $\varphi_\omega(\beta)$, for some β.*

Adding ω and μ, we have a model of $PRS\omega$.

Theorem 4.6 *Suppose $PRS\omega$ proves $\forall x\ \exists y\ \varphi(x,y)$, where φ is Σ_1. Then for some natural number i, we have $\forall x \in L_\alpha\ \exists y \in L_{\varphi_i(\alpha)}\ \varphi(x,y)$.*

Proof. Suppose $PRS\omega$ proves $\forall x\ \exists y\ \varphi(x,y)$, with φ a Σ_1 formula. Since $PRS\omega$ has a universal axiomatization and $\varphi(x,y)$ is provably equivalent to an existential formula, we can apply Herbrand's theorem and conclude that there is a function symbol f such that $\forall x\ \varphi(x, f(x))$ is provable as well. □

A function from L to L is said to be Σ_1-definable in a theory T if the graph of f is defined by a Σ_1 formula $\varphi(x,y)$ such that T proves $\forall x\ \exists! y\ \varphi(x,y)$.

Corollary 4.7 *If f is a Σ_1-definable function of $PRS\omega$, then there is a natural number i such that for every α and $x \in L_\alpha$, $f(x) \in L_{\varphi_i(\alpha)}$. In particular, every element of L that is Σ_1-definable in $PRS\omega$ is an element of $L_{\varphi_\omega(0)}$.*

Now let α be any primitive recursive notation system. For the purposes of this paper, I will take the theory of "ramified analysis up to α," $RA(\prec\alpha)$, to be the subsystem of second-order arithmetic consisting of quantifier-free axioms defining the symbols in the language of arithmetic, the schema of comprehension for arithmetic sets, induction for sets, and for each β less than α, and axiom asserting that for every set X there is a transfinite jump hierarchy of length β, starting with X. Using such jump hierarchies, one can model the construction of segments of L, relative to any set X. (For limit ordinals α, $RA(\prec\alpha)$ is inter-interpretable with the theory $(\Pi^1_0\text{-}CA)_{\prec\alpha}$, which allows iterated arithmetic comprehension of length β for each β less than α.)

Any Π^1_1 sentence in the language of arithmetic has a natural translation into the language $PRS\omega$. Given any particular proof of such a sentence in $PRS\omega$, one can find a $\beta \prec \varphi_\omega(0)$ large enough so that $RA(\prec\varphi_\omega)$ proves that for each set X, one can construct the β-many levels of the L hierarchy, starting from X; and in this segment of L, one can model all the function symbols occuring in the proof. As a result, we have:

Theorem 4.8 *$PRS\omega$ is conservative over $RA(\prec\varphi_\omega(0))$ for Π^1_1 sentences, in the sense above.*

Using the standard methods of predicative proof theory, one can conclude the following:

Corollary 4.9 *The proof-theoretic ordinal of $PRS\omega$ is $\varphi_{\varphi_\omega(0)}(0)$.*

See, for example, [2, 16, 17, 21] for further discussion of the notion of a "proof-theoretic ordinal."

Cantini [10] and later [11] shows that Theorem 4.6 holds even if one adds the schema of Σ_1 foundation to $PRS\omega$. This strengthening can also be obtained from [20], where a standard cut-elimination shows that adding the schema of Σ_1 foundation to $PRS\omega$ yields an extension that is conservative for Π_2 sentences in the language of set theory. As a result, Theorems 4.6 and 4.8 and their corollaries

hold with $KP\omega^- + \Sigma_1$-*Foundation* in place of $PRS\omega$. (See [11] and [20] for even stronger results involving forms of dependent choice and Π_1 foundation.) We will not need these facts below.

We would like to extend the last two theorems and their corollaries to the full theory $KP\omega$. To do so, we first need to describe some faster-growing functions on constructible hierarchy.

5 The Howard-Bachmann ordinal

For the moment, let Ω denote the first uncountable regular cardinal, and suppose we have defined the Veblen hierarchy on Ω. We can extend this hierarchy, and define even faster-growing functions from Ω to Ω. For example, we can diagonalize and define φ_Ω to be the function which maps α to $\varphi_\alpha(0)$, and then continue defining $\varphi_{\Omega+1}, \varphi_{\Omega+2}, \ldots$ as before. Let $\varepsilon_{\Omega+1}$ denote the $\Omega + 1$st ε-number, i.e. the limit of the sequence

$$\Omega, \Omega^\Omega, \Omega^{(\Omega^\Omega)}, \ldots$$

If one extends the Veblen hierarchy in a reasonable way, one obtains the Howard-Bachmann ordinal as $\varphi_{\varepsilon_{\Omega+1}}(0)$.

A framework due to Feferman and Aczel provides a neat (and more extendable) way of describing this ordinal. First, note that any ordinal $\alpha < \varepsilon_{\Omega+1}$ can be written in Cantor normal form to the base Ω, i.e. in the form

$$\alpha = \Omega^{\alpha_1}\beta_1 + \ldots \Omega^{\alpha_k}\beta_k$$

where $\alpha > \alpha_1 > \ldots > \alpha_k$ and each β_k is an element of Ω. I will call the coefficients $\beta_1, \ldots, \beta_k$, together with the coefficients occuring in the Cantor normal form expansions of $\alpha_1, \ldots, \alpha_k$, the *components of* α.

By transfinite induction up to $\varepsilon_{\Omega+1}$, define functions $C_\alpha : \Omega \to \mathcal{P}(\Omega)$ and $\theta_\alpha : \Omega \to \Omega$, as follows:

$$\begin{aligned} C_\alpha(\beta) &= \text{the closure of } \{0,1\} \cup \beta \text{ under } + \text{ and} \\ &\quad\ \text{the functions } \theta_\gamma, \text{ where } \gamma < \alpha \text{ and the} \\ &\quad\ \text{components of } \gamma \text{ are in } C_\alpha(\beta) \\ \theta_\alpha &= \text{the enumerating function of } \{\delta \mid \delta \notin C_\alpha(\delta) \wedge \alpha \in C_\alpha(\delta)\}. \end{aligned}$$

Roughly, $C_\alpha(\beta)$ is the collection of ordinals that can be expressed using 0, 1, elements of β, $+$, and previously defined θ functions whose indices have components in $C_\alpha(\beta)$; and θ_α enumerates the ordinals δ that are "inaccessible" from below. A more general notation system is described in detail in [16]; the fragment used here is also treated briefly in [19] and [21].

One can describe the set of ordinals less than $\theta_{\varepsilon_{\Omega+1}}(0)$ more explicitly. Every ordinal other than 0 can be written uniquely as a sum of ordinals $\theta_{\alpha_i}(\beta_i) + \ldots + \theta_{\alpha_k}(\beta_k)$, where all the β_i and all the components of the α_i are less than α. In general, one has $\theta_\alpha(\beta) < \theta_\gamma(\delta)$ if and only if one of the following holds:

- $\alpha < \gamma$, $\beta < \theta_\gamma(\delta)$, and all the components of α are less than $\theta_\gamma(\delta)$
- $\alpha = \gamma$ and $\beta < \delta$
- $\gamma \leq \alpha$ but either δ or some component of γ is greater than or equal to $\theta_\alpha(\beta)$.

Since $\theta_{\varepsilon_{\Omega+1}}(0)$ is exactly the set of ordinals that can be represented by explicit notations, our definition is stable under any reinterpretation of Ω that is suitably closed; for example, we can take Ω to be any admissible ordinal greater than ω, or even the Howard-Bachmann ordinal itself.

We can now state the analogous versions of Theorems 4.6 and 4.8. The first follows immediately from Jäger [12].

Theorem 5.1 *Suppose $KP\omega$ proves $\forall x\, \exists y\, \varphi(x, y)$, where φ is Σ_1. Then there is an ordinal $\alpha < \varepsilon_{\Omega+1}$ such that for every β, we have $\forall x \in L_\beta\, \exists y \in L_{\theta_\alpha(\beta)}\, \varphi(x, y)$.*

Theorem 5.2 *$KP\omega$ is conservative over $RA(\prec\theta_{\varepsilon_{\Omega+1}}(0))$ for Π^1_1 sentences of arithmetic.*

The analogues of Corollaries 4.7 and 4.9 follow from these. The next three sections are devoted to proving Theorem 5.1; Theorem 5.2 can be obtained by formalizing the arguments.

6 Set recursion on ordinal notations

In the language of arithmetic, variables are assumed to range over elements of ω, but we can nonetheless define *notations* for a larger ordinal, ε_0, and describe the induced ordering in a primitive recursive way. This plays a role in the ordinal analysis of arithmetic, and allows us, for example, to define functions from $\mathbb{N}$ to $\mathbb{N}$ by recursion along these notations.

In an entirely analogous way, we would like to define a class of notations for $\varepsilon_{\Omega+1}$ in the language of set theory, where Ω is the order-type of the class of ordinals. The precise details of the coding are unimportant, but, for concreteness, we will use the representation in [19, Section 4]. The class of ordinal notations OR and the ordering $\prec$ on these notations are defined by simultaneous recursion, as follows:

- $\emptyset \in OR$.
- If $\alpha_1, \ldots, \alpha_k$ are ordinals, $s_1, \ldots, s_k \in OR$, and $s_1 \succ \ldots \succ s_k$, then
$$\langle k, \langle s_1, \alpha_1\rangle, \ldots, \langle s_k, \alpha_k\rangle\rangle \in OR.$$
Informally, we will write this as $\hat{\Omega}^{s_1}\alpha_1 + \ldots + \hat{\Omega}^{s_k}\alpha_k$.
- If $s \in OR$ and $s \neq \emptyset$, then $0 \prec s$.

- If $s = \hat{\Omega}^{s_1}\alpha_1 + \ldots + \hat{\Omega}^{s_k}\alpha_k$ and $t = \hat{\Omega}^{t_1}\beta_1 + \ldots + \hat{\Omega}^{t_l}\beta_l$, then $s \prec t$ if and only if one of the following holds:
  1. $k < l$ and for for all i, $1 \leq i \leq k$, $\alpha_i = \beta_i$
  2. There is an $m \leq k, l$ such that for all $i < m$, $s_i = t_i$ and $\alpha_i = \beta_i$, and either $s_m \prec t_m$ or $s_m = t_m$ and $\alpha_m < \beta_m$.

Proposition 6.1 *The predicate OR and the relation $\prec$ are primitive set recursive, and the relevant properties can be derived in PRS.*

Proof. Rathjen [19] proves this with "Σ_1-definable in $KP^- + \Sigma_1$-F*oundation*" in place of "primitive set recursive," but it is not difficult to verify that the definitions he gives are, in fact, primitive set recursive. □

Henceforth, to avoid confusion, we will use $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \ldots$ to range over ordinal *notations*, and α, β, γ to range over *ordinals*. We will also now take the θ functions of the last section to be indexed by ordinal *notations* for $\varepsilon_{\Omega+1}$, so that $\theta_{\hat{\alpha}}(\beta)$ denotes an ordinal. I will use $\hat{\Omega}$ to denote the ordinal notation for Ω, but I will blur the distinction between ordinary ordinals and their notations, and write "+" for the defined operation on notations. So, for example, $\hat{\Omega}^\omega$, $\hat{\Omega}\cdot\omega$, and $\hat{\Omega} + 1$ denote the obvious notations. In general, ordinal notations are abstract objects, elements of the universe of sets; but note that some notations, like the ones just listed, are also denoted by constants in the language of $PRS\omega$.

We would like to introduce a notion of recursion along ordinal notations. When it comes to notations in the natural numbers, there are many ways to characterize the ordinal recursive functions, one of which is presented in [2]. It is this particular characterization that we will now lift to the universe of sets.

If $\hat{\alpha}$ is a notation, an $\hat{\alpha}$-*recursive* functional $F(x_1, \ldots, x_l, f_1, \ldots, f_k)$, where $x_1, \ldots, x_l$ are sets and $f_1, \ldots, f_k$ are functions on the universe of sets, is given by functions $\mathit{start}(x_1, \ldots, x_l)$, $\mathit{next}(q, u_1, \ldots, u_k)$, $\mathit{query}_1(q)$, $\ldots$, $\mathit{query}_k(q)$, $\mathit{norm}(q)$, and $\mathit{result}(q)$, all in $\mathit{Prim}[\omega, \mu]$. Informally, these data describe the functional whose values are computed in the following way: on input $x_1, \ldots, x_l$, the algorithm begins in state $\mathit{start}(x_1, \ldots, x_l)$. As long as the norm of the current state q is less than $\hat{\alpha}$ and the norm of the previous state, the algorithm queries the functions $f_1, \ldots, f_k$ at $\mathit{query}_1(q), \ldots, \mathit{query}_k(q)$, respectively. Based on the current state, q, and the responses $u_1, \ldots, u_k$ to these queries, the algorithm then proceeds to the next state, $q' = \mathit{next}(q, u_1, \ldots, u_k)$. If $\mathit{norm}(q')$ is not less than $\mathit{norm}(q)$, the computation halts and returns $\mathit{result}(q')$; otherwise, the algorithm proceeds to the next iteration, with q' in place of q.

More formally, s is a *computation sequence* for F at the values $\vec{x}, \vec{f}$ if s is a sequence $\langle s_0, s_1, s_2, \ldots, s_m \rangle$ satisfying the following: $s_0 = \mathit{start}(\vec{x})$; for every $i < m$, $s_{i+1} = \mathit{next}(s_i, f_1(\mathit{query}_1(s_i)), \ldots, f_k(\mathit{query}_k(s_i)))$; and either $m = 0$ and $\mathit{norm}(s_0) \not\prec \hat{\alpha}$, or $m > 0$, $\mathit{norm}(s_0) \prec \hat{\alpha}$, $\mathit{norm}(s_{i+1}) \prec \mathit{norm}(s_i)$ for every $i < m - 1$, and $\mathit{norm}(s_m) \not\prec \mathit{norm}(s_{m-1})$. F is *defined* at $\vec{x}, \vec{f}$ if there is a computation sequence s for F at $\vec{x}, \vec{f}$, and in that case, the value of $F(\vec{x}, \vec{f})$ is said to be $\mathit{result}(s_{\mathit{last}(s)})$.

A functional is said to be $\prec\hat{\beta}$*-recursive* if it is $\hat{\alpha}$-recursive for some $\hat{\alpha}\prec\hat{\beta}$. Reference to a $\prec\hat{\beta}$-recursive functional $F(x_1,\ldots,x_l,f_1,\ldots,f_k)$ in the context of an axiomatic theory extending $PRS\omega$ should always be interpreted in terms of the symbols denoting $\hat{\alpha}_F$, $start_F$, $next_F$, etc., and function symbols $f_1,\ldots,f_k$. One has to be careful, since such a theory may be too weak to prove that F is everywhere defined. The notation $F(\vec{x},\vec{f})\downarrow= y$ abbreviates the assertion that there is a computation sequence s for F at $\vec{x},\vec{f}$ with $result_F(s_{last(s)}) = y$. Note that this assertion is Σ_1.

Using the methods of [2] it is not difficult to show that if $\hat{\beta}$ is closed under addition, then the $\prec\hat{\beta}$-recursive functionals are closed under composition. Also, terms and quantifier-free formulae in the language of $PRS\omega$ with additional function symbols $\vec{f}$ can be evaluated with finitely many queries to $\vec{f}$, and so are $\prec\omega$-recursive in $\vec{f}$.

If $F(\vec{x},\vec{f})$ is an $\hat{\alpha}$-recursive functional and $F(\vec{x},\vec{f})\downarrow= y$, then y is ultimately obtained from finitely many applications of functions in $Prim[\omega,\mu]\cup\{\vec{f}\}$. So if, for some δ, L_δ is closed under these functions, it is also closed under F. In the particular case when there are no function arguments, we see that that our new version of recursion on ordinal notations provides a hierarchy of functions on L_δ for any primitive recursively closed ordinal $\delta > \omega$, much the way that the original, finitary notion of recursion on ordinal notations provides hierarchies of functions from $\mathbb{N}$ to $\mathbb{N}$.

Two modifications of the notions introduced above will be needed in the sequel. First, we need to define a *partial computation sequence* of F at $\vec{x},\vec{f}$ to be a proper initial segment of a computation sequence, i.e. a sequence satisfying the definition above except that the computation has not yet halted, so the norm of the last state is less than the norm of the previous one. Second, we need to adapt our terminology to be able to discuss computations of $F(x,f)$ where f is an ordinary function (i.e. set of ordered pairs) in the universe under consideration, rather than a class function on the entire universe. If u is such a function, simply say that F is defined at x,u if there is a computation sequence s for F at x,u, such that every value queried happens to be in the domain of u, i.e. for each $i < last(s)$, $query_F((s)_i) \in dom(u)$.

7 Eliminating foundation

Define a sequence of ordinal notations by $\hat{\delta}_1 = \hat{\Omega}^\omega$ and $\hat{\delta}_{i+1} = \hat{\Omega}^{\hat{\delta}_i}$. Let $L^{\vec{f}}$ denote the language of $PRS\omega$ augmented by new function symbols $\vec{f}$. Note that this language differs from the language of $PRS[\omega,\mu,\vec{f}]$, in that here we are adding these function symbols alone; so there are no symbols for functions defined from these using composition or primitive set recursion. Say that a formula is $\forall_n^{\vec{f}}$ if it consists of at most n quantifiers starting with a universal one, followed by a quantifier-free formula in $L^{\vec{f}}$. In this section we will show that one can eliminate foundation for $\forall_n^{\vec{f}}$ formulae in favor of $\prec\hat{\delta}_n$ recursion.

Theorem 7.1 *Let $\vec{f}$ be a sequence of function variables, and suppose*

$$PRS\omega + \forall_n^{\vec{f}}\text{-}Foundation \vdash \forall x\ \exists y\ \varphi(x, y, \vec{f}),$$

where $\varphi(x, y, \vec{f})$ is an existential formula of $L^{\vec{f}}$. Then there is a $\prec\hat{\delta}_n$-recursive function $F(x, \vec{f})$ such that

$$PRS\omega \vdash \forall x, y\ (F(x, \vec{f}) \downarrow = y \rightarrow \varphi(x, y, \vec{f})).$$

As a corollary, whenever $PRS\omega + Foundation$ proves $\forall x\ \exists y\ \varphi(x, y, \vec{f})$, with φ existential, there is a $\prec\varepsilon_{\Omega+1}$-recursive functional witnessing the conclusion, provably in $PRS\omega$. The situation here is entirely analogous to that of Peano arithmetic, where $PRS\omega + Foundation$, $PRS\omega$, and $\varepsilon_{\Omega+1}$ are replaced by PA, PRA, and ε_0, respectively. The corresponding proof can be obtained by adapting the methods that have been developed for arithmetic, including the usual Gentzen-Schütte techniques [16, 17], or, for example, methods developed in [2, 3, 8]. A purely model-theoretic proof is possible, combining the methods of [8] and [4]. An informal statement of Theorem 7.1 is alluded to in [18].

By pairing existential quantifiers, to prove Theorem 7.1 it suffices to assume that $\varphi(x, y, \vec{f})$ is quantifier-free. For the sake of completeness, I will sketch a proof, along the lines of [2].

The first step is to add Skolem functions to embed $PRS\omega + \forall_n^{\vec{f}}\text{-}Foundation$ in a universally axiomatized theory. To that end, for each existential formula $\exists y\ \eta(y, \vec{z}, \vec{f})$, add a function symbol $wit_\eta^{\vec{f}}(\vec{z})$ with defining axiom

$$\forall y, \vec{z}\ (\eta(y, \vec{z}, \vec{f}) \rightarrow \eta(wit_\eta^{\vec{f}}(\vec{z}), \vec{z}, \vec{f})).$$

This has the net effect of making every existential formula in $\vec{f}$ equivalent to one that is quantifier free. Iterate the process by adding function symbols for existential quantifiers in the new language. Let $(wit_{n-1}^{\vec{f}})$ denote the set of axioms that arise from iterating the process $n-1$ times, and let $\forall_1^{\vec{f}, wit_{n-1}^{\vec{f}}}$ denote the class of formulae that are universal in the new language. Then we have:

Lemma 7.2 *For each n, the theory $PRS\omega + \forall_n^{\vec{f}}\text{-}Foundation$ is included in the theory $PRS\omega + (wit_{n-1}^{\vec{f}}) + \forall_1^{\vec{f}, wit_{n-1}^{\vec{f}}}\text{-}Foundation$.*

We can embed the latter in a universally axiomatized theory by adding Skolem functions for $\forall_1^{\vec{f}, wit_{n-1}^{\vec{f}}}\text{-}Foundation$. This schema is equivalent to the schema which asserts, for each sequence of function symbols $\vec{wit}_{n-1}^{\vec{f}}$ and each existential formula $\exists y\ \theta(x, y, \vec{z}, \vec{f}, \vec{wit}_{n-1}^{\vec{f}})$, that if any x satisfies this formula, there is an $\in$-least such x. Note that since $\theta(x, y, \vec{z}, \vec{f}, \vec{wit}_{n-1}^{\vec{f}})$ is quantifier-free, there is a term $t_\theta(x, y, \vec{z}, \vec{f}, \vec{wit}_{n-1}^{\vec{f}})$ such that

$$\theta(x, y, \vec{z}, \vec{f}, \vec{wit}_{n-1}^{\vec{f}}) \leftrightarrow t_\theta(x, y, \vec{z}, \vec{f}, \vec{wit}_{n-1}^{\vec{f}}) = 1$$

is provable in $PRS\omega$. Let $norm(u,v)$ be the function

$$norm(u,v) = \begin{cases} rank((u)_0) & \text{if } v = 1 \\ \hat{\Omega} & \text{otherwise,} \end{cases}$$

and let $norm_\theta(u, \vec{z}, \vec{f}, \vec{wit}^{\vec{f}}_{n-1})$ denote the term $norm(u, t_\theta((u)_0, (u)_1, \vec{z}, \vec{f}, \vec{wit}^{\vec{f}}_{n-1}))$. Then the $\in$-least element principle for $\exists y\ \theta(x, y, \vec{z}, \vec{f}, \vec{wit}^{\vec{f}}_{n-1})$ is equivalent to saying that there is a value $min_\theta(\vec{z})$ minimizing $norm_\theta(\cdot, \vec{z}, \vec{f}, \vec{wit}^{\vec{f}}_{n-1})$, i.e.

$$\forall \vec{z}, w\ (norm(min_\theta(\vec{z}), \vec{z}, \vec{f}, \vec{wit}^{\vec{f}}_{n-1}) \preceq norm(w, \vec{z}, \vec{f}, \vec{wit}^{\vec{f}}_{n-1})).$$

Let $PRS\omega + wit^{\vec{f}}_{n-1} + min^{\vec{f}}_n$ denote the theory obtained by adding to $PRS\omega + wit^{\vec{f}}_{n-1}$ new function symbols min together with the axioms above. Then we have

Lemma 7.3 *For each n, the theory $PRS\omega + (wit^{\vec{f}}_{n-1}) + \forall_1^{\vec{f}, wit^{\vec{f}}_{n-1}}$-Foundation is included in $PRS\omega + (wit^{\vec{f}}_{n-1}) + (min^{\vec{f}}_n)$.*

Having embedded everything into a universal theory, we can begin the ordinal analysis, following the pattern of [2].

Lemma 7.4 *Suppose $PRS\omega + (wit^{\vec{f}}_{n-1}) + (min^{\vec{f}}_n)$ proves $\forall x\ \exists y\ \varphi(x, y, \vec{f})$, where φ is quantifier-free. Then there are a $\prec\hat{\Omega}^\omega$-recursive function F and a sequence of function symbols $\vec{wit}^{\vec{f}}_{n-1}$ of $PRS\omega + (wit^{\vec{f}}_{n-1})$ such that $PRS\omega + (wit^{\vec{f}}_{n-1})$ proves*

$$\forall x, y\ (F(x, \vec{f}, \vec{wit}^{\vec{f}}_{n-1}) \downarrow = y \rightarrow \varphi(x, y, \vec{f})).$$

Proof (sketch). As in the proof of Lemma 8.10 of [2]. By Herbrand's theorem, there is a sequence of terms $t_1(\vec{x}, \vec{f}, \vec{wit}^{\vec{f}}_{n-1}, \vec{min}^{\vec{f}}_n), \ldots, t_k(\vec{x}, \vec{f}, \vec{wit}^{\vec{f}}_{n-1}, \vec{min}^{\vec{f}}_n)$ such that there is a quantifier-free proof of $\bigvee_i \varphi(t_i)$ from instances of the universal axioms. The sequence of terms appearing in this proof can (provably) be evaluated with a single instance of minimization below $\hat{\Omega}^\omega$; another application of Herbrand's theorem allows one to extract a specific function F. $\square$

Lemma 7.5 *Let $m \geq 1$, and let $\hat{\alpha}$ be infinite and closed under multiplication. Suppose F is a $\prec\hat{\alpha}$-recursive and $\vec{wit}^{\vec{f}}_m$ is a sequence of function symbols such that $PRS\omega + (wit^{\vec{f}}_m)$ proves*

$$\forall x, y\ (F(x, \vec{f}, \vec{wit}^{\vec{f}}_m) \downarrow = y \rightarrow \varphi(x, y, \vec{f})).$$

Then there are a $\prec\hat{\Omega}^{\hat{\alpha}}$-recursive function G and a sequence of function symbols $\vec{wit}^{\vec{f}}_{m-1}$ such that $PRS\omega + (wit^{\vec{f}}_{m-1})$ proves

$$\forall x, y\ (G(x, \vec{f}, \vec{wit}^{\vec{f}}_{m-1}) \downarrow = y \rightarrow \varphi(x, y, \vec{f})).$$

Proof (sketch). As in the proof of Lemma 9.2 in [2], using Ω in place of ω and the rank of a set as its norm. Applying Herbrand's theorem to the hypothesis of the lemma, one obtains specific instances of the axioms at level m which yield the conclusion. One can then design a $\prec\hat{\Omega}^{\hat{\alpha}}$ recursive functional which returns both a suitable approximation to the relevant function symbols of level m and a computation sequence for F at that approximation. The construction has the flavor of a finite injury priority argument: one starts with empty approximations to the witness functions at level m, and carries out the computation of F. If the computation yields values which falsify the witness axioms, one updates the witness functions and recomputes F. An appropiate assignment of ordinals to these computations shows that this process always terminates. (See also [3, 8, 18] for similar arguments.) $\square$

For the proof of Theorem 7.1, embed $PRS\omega + \forall^{\vec{f}}_n$-*Foundation* in a universal theory using Lemmata 7.2 and 7.3, apply Lemma 7.4, and then apply Lemma 7.5 $n-1$ times.

8 Eliminating collection

The essential difference between $KP\omega$ and $PRS\omega + Foundation$ is the schema of Δ_0 collection. To complete our ordinal analysis, then, we need only add Skolem functions for the collection schema to the latter theory, and then figure out how to eliminate them.

Remember that an instance of Δ_0 collection is of the form

$$\forall v, \vec{z}\, (\forall x \in v\ \exists y\ \theta(x, y, \vec{z}) \to \exists w\ \forall x \in v\ \exists y \in w\ \theta(x, y, \vec{z}))$$

where θ is Δ_0 with the free variables shown. By combining quantifiers, collection for Σ_1 formulae follows. On the other hand, by choosing a suitably universal formula θ, we can reduce this to a single instance of collection, where $\theta(x, y, z)$ has only a single parameter z.

We can rewrite this instance of collection as

$$\forall v, z\ (\exists x\ (x \in v \wedge \forall y\ \neg\theta(x, y, z)) \vee \exists w\ \forall x \in w\ \exists y \in v\ \theta(x, y, z)).$$

We can then bring quantifiers to the front, combine $\exists x$ and $\exists w$ into a single existential quantifier, pair v and z, and Skolemize. Letting $coll(u)$ be a new function symbol, the collection axiom then follows from the Π_1 assertion

$$\forall u, y\ ((coll(u) \in (u)_0 \wedge \neg\theta(coll(u), y, (u)_1)) \vee \forall x \in u\ \exists y \in coll(u)\ \theta(x, y, (u)_1)).$$

In other words, for any v and z, $coll(\langle v, z\rangle)$ is supposed to return either a value x satisfying $x \in v \wedge \forall y\ \neg\theta(x, y, z)$, or a value w satisfying $\forall x \in u\ \exists x \in w\ \theta(x, y, z)$. Let $Coll'(u, y, c)$ denote the primitive recursive relation

$$(c \in (u)_0 \wedge \neg\theta((u)_0, y, (u)_1)) \vee \forall x \in u\ \exists y \in c\ \theta(x, y, (u)_1),$$

which says that "c is a sound interpretation of $coll(u)$ at y." Let $(Coll)$ be the universal axiom $\forall u, y\ Coll'(u, y, coll(u))$.

Lemma 8.1 *For every n, the theory $KP\omega^- + \Pi_{n+1}$-Foundation is included in the theory $PRS\omega + (Coll) + \forall_n^{coll}$-Foundation.*

Proof. Since $KP\omega^-$ is included in $PRS\omega + (Coll)$, we only need to show that in the presence of $(Coll)$ every Σ_1 formula is equivalent to one that is quantifier-free. But note that when the formula $\theta(x, y, \vec{z})$ in the collection schema does not depend on x, the result is of the form

$$\forall \vec{z}\, (\exists y\ \psi(y, \vec{z}) \to \exists w\ \exists y \in w\ \psi(y, \vec{z})),$$

where ψ is Δ_0. Since we chose θ in $(Coll)$ to be universal, there will be a primitive recursive set function f such that the assertion

$$\forall y, \vec{z}\, (\psi(y, \vec{z}) \to \exists y \in coll(f(\vec{z}))\ \psi(y, \vec{z})).$$

So $\exists y\ \psi(y, \vec{z})$ is provably equivalent to $\exists y \in coll(f(\vec{z}))\ \psi(y, \vec{z})$, which can be expressed as a quantifier-free formula. □

Since the theory mentioned in the conclusion of Lemma 8.1 is universal, we can apply Theorem 7.1.

Lemma 8.2 *Suppose $PRS\omega + (Coll) + \forall_n^{coll}$-Foundation proves $\forall x\ \exists y\ \varphi(x, y)$, where φ is Δ_0. Then there is a $\prec\hat{\delta}_n$-recursive functional F such that $PRS\omega$ proves*

$$\forall x, y\ (F(x, coll) \downarrow = y \wedge Coll'((y)_0, (y)_1, coll((y)_0)) \to \varphi(x, y)).$$

In other words, the conclusion states that for every x, the functional F returns either a value y satisfying $\varphi(x, y)$, or a witness to the failure of the *coll* function; and this fact is provable in $PRS\omega$.

Proof. If $PRS\omega + (Coll) + \forall_n^{coll}$-*Foundation* proves $\forall x\ \exists y\ \varphi(x, y)$, then, by the deduction theorem, $PRS\omega + \forall_n^{coll}$-*Foundation* proves $\neg(Coll) \vee \forall x\ \exists y\ \varphi(x, y)$. We can bring quantifiers to the front, pair the existential quantifiers, and apply Theorem 7.1. The result is a $\prec\hat{\delta}_n$-recursive functional $G(x, coll)$ such that $PRS\omega$ proves

$$G(x, coll) \downarrow = y \to \neg Coll'((y)_0, (y)_1, coll((y)_0)) \vee \varphi(x, (y)_2).$$

Let $F(x, coll)$ be the $\prec\hat{\delta}_n$-recursive functional defined by

$$F(x, coll) \simeq \begin{cases} \langle (G(x, coll))_0, (G(x, coll))_1 \rangle & \text{if } \neg Coll'((y)_0, (y)_1, coll((y)_0))) \\ (G(x, coll))_2 & \text{otherwise} \end{cases}$$

This completes the proof. □

We are now ready to prove the following refined version of Theorem 5.1, due to Rathjen [20].

Theorem 8.3 *Suppose $KP\omega^- + \Pi_{n+1}$-Foundation proves $\forall x\ \exists y\ \varphi(x,y)$, where φ is Σ_1. Then there is an ordinal notation $\hat{\alpha} \prec \hat{\delta}_n$ such that for every ordinal β, we have $\forall x \in L_\beta\ \exists y \in L_{\theta_{\hat{\alpha}}(\beta)}\ \varphi(x,y)$.*

Once again, by pairing quantifiers, we can assume without loss of generality that φ is Δ_0. To prove the theorem, we only need to show that we can carry out the computation alluded to in the conclusion of Lemma 8.2, finding a suitable interpretation for *coll*. The following lemma makes this precise.

Lemma 8.4 *Suppose $F(x,f)$ is $\hat{\alpha}$-recursive, and $x \in L_\gamma$. Then there is a pair $\langle s,m\rangle \in L_{\theta_{\omega+\hat{\alpha}}(\gamma)}$ such that*

- *m is a function,*
- *s is a computation sequence for F at x, m, and*
- *if $result_F((s)_{last(s)}) = y$ and $(y)_0 \in dom(m)$, then $Coll'((y)_0,(y)_1,m((y)_0))$.*

Proof of Theorem 8.3 from Lemma 8.4. Suppose $KP\omega^- + \Pi_{n+1}$-*Foundation* proves $\forall x\ \exists y\ \varphi(x,y)$, where φ is Δ_0. By Lemma 8.1 and Lemma 8.2 there are a notation $\hat{\alpha} \prec \hat{\delta}_n$ and an $\hat{\alpha}$-recursive functional F such that $PRS\omega$ proves

$$\forall x,y\ (F(x,coll)\downarrow= y \wedge Coll'((y)_0,(y)_1,coll((y)_0)) \rightarrow \varphi(x,y)).$$

We can assume that before the computation of F returns a result, y, it queries *coll* at $(y)_0$, since otherwise we can replace F with a $1+\hat{\alpha}$-recursive functional that does so at the last step. Now we know that for every x in L_β there a pair $\langle s,m\rangle$ satisfying the conclusion of Lemma 8.4. Let $y = result_F(s)$; then, in particular, for there is a natural number i such that y is an element of $L_{\varphi_i(\theta_{\omega+\hat{\alpha}}(\gamma))}$. Set $\hat{\alpha}' = \omega + \hat{\alpha} + 1 \prec \hat{\delta}_n$. Then $F(x,m)\downarrow= y$ and $y \in L_{\theta_{\hat{\alpha}'}(\gamma)}$. By the conclusion of Lemma 8.4, we have $Coll'((y)_0,(y)_1,m((y)_0))$, and hence $\varphi(x,y)$. $\square$

We would like to prove Lemma 8.4 by induction on $\theta_{\omega+\hat{\alpha}}(\gamma)$. In fact, we need a slightly stronger induction hypothesis, to the effect that given a partial computation sequence and a partial determination of m in L_γ, these can be extended to an appropriate pair $\langle s,m\rangle \in L_{\theta_{\omega+\hat{\alpha}}(\gamma)}$. The precise statement of this is in Lemma 8.6 below. Before presenting this lemma, it will be notationally convenient to introduce slight variants of the θ functions, defined by $\bar{\theta}_{\hat{\beta}}(\delta) = \theta_{\omega+\hat{\beta}}(\delta)$. The relevant properties of the $\bar{\theta}$ hierarchy are summarized in the next lemma.

Lemma 8.5 *For all ordinal notations $\hat{\beta}$ and $\hat{\beta}'$, and all ordinals δ:*

1. *$\delta \leq \bar{\theta}_{\hat{\beta}}(\delta)$.*
2. *For each natural number i, $\bar{\theta}_{\hat{\beta}}(\delta)$ is closed under the Veblen function φ_i.*

3. *If $\hat{\beta}' \prec \hat{\beta}$ and the components of $\hat{\beta}'$ are less than $\bar{\theta}_{\hat{\beta}}(\delta)$, then $\bar{\theta}_{\hat{\beta}}(\delta)$ is closed under $\bar{\theta}_{\hat{\beta}'}$.*

4. *If $\hat{\beta}' \prec \hat{\beta}$ and $\hat{\beta}' \in L_{\bar{\theta}_{\hat{\beta}}(\delta)}$, then $\bar{\theta}_{\hat{\beta}}(\delta)$ is closed under $\bar{\theta}_{\hat{\beta}'}$.*

Proof. The first three clauses follow immediately from the definition of the $\bar{\theta}$ functions and the ordering properties of the θ functions, once we note that for natural numbers i, $\theta_i = \varphi_i$. The fourth clause follows immediately from the third: if $\hat{\beta}' \in L_{\bar{\theta}_{\hat{\beta}}(\delta)}$, the components of $\hat{\beta}'$ are also elements of $L_{\bar{\theta}_{\hat{\beta}}(\delta)}$, and hence less than $\bar{\theta}_{\hat{\beta}}(\delta)$. □

Lemma 8.6 *Suppose F is a $\prec\varepsilon_{\hat{\Omega}+1}$-recursive function, and $x \in L$. Suppose also that $\delta, \hat{\beta}, n$, and t are elements of L, such that*

- *δ is an ordinal,*
- *$\hat{\beta}$ is an ordinal notation,*
- *n is a function,*
- *t is a partial computation sequence for F at x, n,*
- *$\langle t, n\rangle \in L_\delta$, and*
- *$norm_F((t)_{last(t)}) = \hat{\beta}$.*

Then are s, m in L such that

- *m is a function extending n,*
- *s is a computation sequence for F at x, m,*
- *$\langle s, m\rangle \in L_{\bar{\theta}_{\hat{\beta}+1}(\delta)}$, and*
- *if $result_F((s)_{last(s)}) = y$ and $(y)_0$ is an element of $dom(m) - dom(n)$, then $Coll'((y)_0, (y)_1, m((y)_0))$.*

Remember that m is our approximation to the collection function. The last clause says that if m fails to pass the test at the end, it is the fault of n; or conversely if, at $(y)_0$, n is a sound approximation to the collection function for the values queried in t, then m is sound for the values queried in s.

Proof of Lemma 8.4, assuming Lemma 8.6. Suppose as in the statement of Lemma 8.4 that F is $\hat{\alpha}$-recursive and $x \in L_\gamma$. Let

- $t = \langle start_F(x)\rangle$,
- $n = \emptyset$, and
- $\hat{\beta} = norm_F(start_F(x))$.

If $\hat{\beta} \not\prec \hat{\alpha}$, then t is a computation sequence, with

$$\mathit{result}_F(\mathit{start}_F(x)) \in L_{\bar{\theta}_0(\gamma)} = L_{\theta_\omega(\gamma)} \subseteq L_{\theta_{\omega+\hat{\alpha}}(\gamma)}.$$

Otherwise, apply Lemma 8.6, to get a pair $\langle s, m\rangle$ such that $\langle s, m\rangle \in L_{\bar{\theta}_{\hat{\beta}}(\bar{\theta}_0(\gamma))}$. Then we have

$$\mathit{result}_F((s)_{\mathit{last}(s)}) \in L_{\bar{\theta}_0(\bar{\theta}_{\hat{\beta}}(\bar{\theta}_0(\gamma)))} \subset L_{\bar{\theta}_{\hat{\beta}+1}(\gamma)} = L_{\theta_{\omega+\hat{\beta}+1}(\gamma)} \subseteq L_{\theta_{\omega+\hat{\alpha}}(\gamma)},$$

as required. □

Proof of Lemma 8.6. The proof is by induction on $\bar{\theta}_{\hat{\beta}}(\delta)$. Let t and n be as in the statement of the lemma; so t is a partial computation sequence for F at x, n and n is a partial approximation to *coll*. Let $q = (t)_{\mathit{last}(t)}$ be the last state in the partial computation sequence, and let $u = \mathit{query}_F(q)$ be the next query to *coll*. We need to assign an appropriate value to $\mathit{coll}(u)$ and extend the computation one step.

Case 1. u is already in the domain of n. In this case, there is little to do, since n already commits us to a value for $\mathit{coll}(u)$. Set

- $q' = \mathit{next}_F(q, n(u))$,
- $\beta' = \mathit{norm}_F(q')$, and
- $t' = t^\frown\langle q'\rangle$.

If $\hat{\beta}' \not\prec \hat{\beta}$, then we are done: let $s = t'$ and let $n = m$. Then s is a computation sequence for F at x, n, and since $\bar{\theta}_{\hat{\beta}+1}(\delta)$ is closed under the first ω-many Veblen functions, we have $\langle s, t\rangle \in L_{\hat{\theta}_{\beta+1}(\delta)}$.

If $\hat{\beta}' \prec \hat{\beta}$, apply the induction hypothesis to t', n, with $\delta' = \theta_0(\delta)$. This yields elements s, n satisfying the conclusion of the lemma, with

$$\langle s, n\rangle \in L_{\bar{\theta}_{\hat{\beta}'+1}(\bar{\theta}_0(\delta))} \subseteq L_{\bar{\theta}_{\hat{\beta}+1}(\delta)}.$$

Case 2. Otherwise, u is not in the domain of n, and we need to find an appropriate value for $\mathit{coll}(u)$. Our strategy is as follows: divide up the portion of L between δ and $\bar{\theta}_{\hat{\beta}+1}(\delta)$ into two parts: the part between δ and $\bar{\theta}_{\hat{\beta}}(\delta)$, and the part between $\bar{\theta}_{\hat{\beta}}(\delta)$ and $\bar{\theta}_{\hat{\beta}+1}(\delta)$. If possible, we will extend n to n' so that the first disjunct of $\mathit{Coll}'(u, (y)_1, n'(u))$ holds for any value y that the computation may return, and then we will finish up the computation in the first part. Otherwise, we will satisfy the second disjunct of $\mathit{Coll}'(u, (y)_1, n'(u))$, and finish up the computation in the second part.

Case 2a. Suppose we have

$$\exists x \in (u)_0\ \forall y \in L_{\bar{\theta}_{\hat{\beta}}(\delta)}\ \neg\theta(x, y, (u)_1).$$

Then we can guarantee that the first disjunct of Coll' will be satisfied at u by assigning $\mathit{coll}(u)$ an element of $(u)_0$, as follows.

Let x be any element of $(u)_0$ witnessing the formula above, for example, the least such element in the standard ordering of L. Let

- $n' = n \cup \{\langle u, x\rangle\}$,
- $q' = next_F(q, x)$,
- $\beta' = norm_F(q')$, and
- $t' = t\hat{\ }\langle q'\rangle$.

If $\hat{\beta}' \not\prec \hat{\beta}$ let $s = t'$ and $m = n'$. Then, as above, s is a computation sequence for F at x, m, with $\langle s, m\rangle$ in $L_{\bar{\theta}_{\hat{\beta}}(\delta)}$. If $result_F((s)_{last(s)}) = y$, then $(y)_0$, and $(y)_1$ are also in $L_{\bar{\theta}_{\hat{\beta}}(\delta)}$. If $(y)_0 \in dom(n) - dom(m)$, then $(y)_0$ can only be u, and our choice of x above guarantees that $Coll(u, (y)_1, m(u))$ holds.

If $\hat{\beta}' \prec \hat{\beta}$, then we can apply the inductive hypothesis to t', n', with $\delta' = \bar{\theta}_0(\delta)$. Note that since $\beta' \in L_{\bar{\theta}_0(\delta)}$, the components of β' are less than $\bar{\theta}_{\hat{\beta}}(\delta)$. The result is a pair $\langle s, m\rangle$ with m extending n' and

$$\langle s, m\rangle \in L_{\bar{\theta}_{\hat{\beta}'}(\delta')} = L_{\bar{\theta}_{\hat{\beta}'}(\bar{\theta}_0(\delta))} \subset L_{\bar{\theta}_{\hat{\beta}'+1}(\delta)} \subseteq L_{\bar{\theta}_{\hat{\beta}}(\delta)} \subset L_{\bar{\theta}_{\hat{\beta}+1}(\delta)}.$$

Once again, our choice of x guarantees that if $result_F((s)_{last(s)}) = y$ and $(y)_0 = u$, then $Coll(u, (y)_1, m(u))$ holds, and the other values of $dom(m) - dom(n)$ are sound by the inductive hypothesis.

Case 2b. Otherwise, we have

$$\forall x \in (u)_0\ \exists y \in L_{\bar{\theta}_{\hat{\beta}}(\delta)}\ \theta(x, y, (u)_1).$$

In this case, we can guarantee that the second disjunct of $Coll'$ will be satisfied at u, by assigning $coll(u) = L_{\bar{\theta}_{\hat{\beta}}(\delta)} \in L_{\bar{\theta}_{\hat{\beta}}(\delta)+1}$, as follows. Let

- $n' = n \cup \{\langle u, L_{\bar{\theta}_{\hat{\beta}}(\delta)}\rangle\}$,
- $q' = next_F(q, L_{\bar{\theta}_{\hat{\beta}}(\delta)})$,
- $\beta' = norm_F(q')$, and
- $t' = t\hat{\ }\langle q'\rangle$.

Yet again, if $\hat{\beta}' \not\prec \hat{\beta}$, let $s = t'$ and $m = n'$. Then s is a computation sequence for F at x, m, with

$$\langle s, n\rangle \in L_{\bar{\theta}_0(\bar{\theta}_{\hat{\beta}}(\delta)+1)} \subset L_{\bar{\theta}_{\hat{\beta}+1}(\delta)},$$

and our choice of $m(u)$ guarantees that for any y, $Coll'(u, (y)_1, m(u))$ will hold.

If $\hat{\beta}' \prec \hat{\beta}$, then we can apply the inductive hypothesis to t', n', with $\delta' = \bar{\theta}_0(\bar{\theta}_{\hat{\beta}}(\delta) + 1)$. Since we have $\hat{\beta}' \in L_{\bar{\theta}_0(\bar{\theta}_{\hat{\beta}}(\delta)+1)} \subseteq L_{\bar{\theta}_{\hat{\beta}+1}(\delta)}$, the components of β' are less than $\bar{\theta}_{\hat{\beta}+1}(\delta)$. The result is a pair $\langle s, m\rangle$ with m extending n' and

$$\langle s, m\rangle \in L_{\bar{\theta}_{\hat{\beta}'}(\delta')} = L_{\bar{\theta}_{\hat{\beta}'}(\bar{\theta}_0(\bar{\theta}_{\hat{\beta}}(\delta)+1))} \subset L_{\bar{\theta}_{\hat{\beta}+1}(\delta)}.$$

Once again, our choice of $n'(u)$ guarantees that for any y, $Coll'(u, (y)_1, m(u))$ will hold, and the inductive hypothesis takes care of any other values of $dom(m) - dom(n)$. $\square$

The proof we have just seen is a semantic analogue of a proof-theoretic collapsing argument. The "true" computation of $F(x, coll)$ takes place most naturally in L_α, where α the least admissible ordinal above x; but we have shown that if β is less than α then, as x ranges over L_β, we can find reasonable approximations to the computation of $F(x, coll)$ in a segment of the constructible hierarchy bounded strictly below α.

Similar arguments are used for the ordinal analysis of Σ^1_1-AC in [5] and [18]. The construction is easier there, since one only has to deal with induction on the set of natural numbers (and hence notations in the set of natural numbers), which remain fixed throughout the construction. In the case of $KP\omega$, the circularity becomes evident: one needs to deal with foundation on the universe of sets (and hence notations in the universe of sets), while the universe of sets in the final model depends on the outcome of the construction.

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