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THE PRIME RESIDUE CLASS OF FINITELY ADDITIVE DISTRIBUTIONS UNIFORM ON THE INTEGERS

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Abstract. This note shows that the prime residue class of finitely additive uniform distributions on the integers is strictly more inclusive than previously studied classes: The class L of extension of limited relative frequency, the class S of shift invariant distributions, and the class R of extensions of residue classes mod m for all integers m.

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1. Distributions uniform on the integers

A distribution uniform on the integers $\mathbb{Z}$ gives each integer probability zero, but gives the set of all integers probability one. Therefore, only finitely additive probabilities that are not countably additive suffice. The specification that each integer has probability zero implies that each finite set has probability zero, and each cofinite set has probability one, but leaves undetermined the probability of every infinite set whose complement is also infinite.

Suppose C is a collection of subsets of a set Ω (here $\mathbb{Z}$) such that $\Omega \in C$. Let μ be a non-negative function on C such that $\mu(\Omega) = 1$. Then Theorem 1 of Kadane and O'Hagan (1995) gives a necessary and sufficient condition that μ can be extended to a finitely additive probability on the power set of Ω. Applying this result, they show that the class C of sets that have limiting relative frequencies can be so extended, where μ is taken to be the limiting relative frequency.

Schirokauer and Kadane (2007) study three classes C: The class L of extensions of limiting relative frequency, the class S of shift invariant functions μ, and the class R of finitely additive probabilities giving probability $1/m$ to each residue class mod m. They show that

$$L \subset S \subset R \tag{1}$$

where each of the inclusions above are strict.

2. The Prime Residue Class

Consider now the class PR of finitely additive probabilities on the integers that give probability $1/m$ to each residue class mod m where m is a prime number. Since this is a weaker condition than the condition defining the class R, we must have

$$L \subset S \subset R \subseteq PR \tag{2}$$

We now show by example that $R \subset PR$ (strictly).

Example:

To be effective, we must display a finitely additive probability that is an element of PR but is not an element of R. To this end, let $\mathbb{Z}_o$ be the odd numbers, and $\mathbb{Z}_4$ be those divisible by 4. Because L is known to be non-empty, so is R. Let $p \in R$.

For each set $A \subseteq \mathbb{Z}$, let

$$q(A) = p(A \cap \mathbb{Z}_o) + 2p(A \cap \mathbb{Z}_4). \tag{3}$$

We now show that q suffices.

To show that q is a finitely additive probability, observe:

$$q(A) = p(A \cap \mathbb{Z}_o) + 2p(A \cap \mathbb{Z}_4) \geq 0 \tag{4}$$

$$q(\mathbb{Z}) = 1/2 + 2(1/4) = 1. \tag{5}$$

Let $A \cap B = \emptyset$. Then

$$\begin{aligned} q(A \cap B) &= p(A \cap B \cap \mathbb{Z}_o) + 2p(A \cap B \cap \mathbb{Z}_4) \\ &= p(A \cap \mathbb{Z}_o) + p(B \cap \mathbb{Z}_o) + 2p(A \cap \mathbb{Z}_4) + 2p(B \cap \mathbb{Z}_4) \\ &= q(A) + q(B). \end{aligned} \tag{6}$$

Equations (4), (5) and (6) show that q is a finitely additive probability. Second, to show that q satisfies the residue class condition for all prime numbers, let $A_{k,m} = k + \mathbb{Z} \bmod m$, where $k \in \mathbb{Z}$ and $m > 0$. Consider first, when $m > 2$ is prime. Then

$$p(A_{k,m} \cap \mathbb{Z}_o) = 1/2m \tag{7}$$

$$p(A_{k,m} \cap \mathbb{Z}_4) = 1/4m. \tag{8}$$

Hence,

$$q(A_{k,m}) = 1/2m + 2(1/4m) = 1/m. \tag{9}$$

When $m = 2$,

$$p(A_{k,2} \cap \mathbb{Z}_o) = \begin{cases} 1/2 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases} \tag{10}$$

$$p(A_{k,2} \cap \mathbb{Z}_4) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 1/4 & \text{if } k \text{ is even} \end{cases}. \tag{11}$$

Hence

$$q(A_{k,2}) = \left\{ \begin{matrix} 1/2 + 2(0) & \text{if } k \text{ is odd} \\ 0 + 2(1/4) & \text{if } k \text{ is even} \end{matrix} \right\} = 1/2. \tag{12}$$

Therefore $q \in PR$.

Finally, to show $q \notin R$,

$$q(A_{2,4}) = 0 \neq 1/4. \tag{13}$$

Hence $q \notin R$. □

3. Conclusion

This result establishes PR as a fourth distinct sense of uniform finitely additive probabilities on the integers. We have shown

$$L \subset S \subset R \subset PR, \tag{14}$$

where each inclusion is strict.

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References

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