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by

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ON FINITE GROUPS WHOSE p-SYLOVI SUBGROUP IS A T. I. SET

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Throughout this note we let p be a fixed prime and let G be a finite group whose fixed p-Sylow subgroup P is a T. I. set (trivial intersection set)*. That is, the intersection of any two distinct conjugates of P is \{\}.

Denote |p| by p #. It is conjectured that if G has a faithful complex character X with \(\text{dim}(X) < p^{\frac{a}{2}} - 1\) then P \prec G. This has been confirmed in certain cases [4, page 287 and Lemma 4.2], [6, Theorem 4.33]. In fact under certain conditions it is sufficient to assume \#(1) < (p^a - 1)/2 [1, Theorem 3] > [6, Theorem 4.2], but in general the conclusion P \prec G does not hold under this weaker assumption because of the presence of Suzuki's simple groups.

Our purpose here is to use Brauer's theory of the correspondence between p-blocks of a subgroup of G and p-blocks of G [2], [3] together with a result of Gorenstein and Walter [5 § (46)] to obtain the theorem below which verifies the conjecture in the case that C(V) \subseteq N(P), where V is the group of p-regular elements of C(P). In particular for any counterexample of minimal order of the conjecture, we would have C(P) \triangleleft P \triangleleft Z(G).

The notation is standard. If H is a subgroup of G then N(H), C(H), and Z(H) denote the normalizer, centralizer, and center of H. Denote Z(G) by Z. All characters are over the complex field.

Assuming P is a T. I. set, let B be a p-block of G of defect \# = 0.

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and let \( D \) be a defect group of \( B \) with \( D \not\subseteq P \) and with \( |D| = p^d \).

Then \( N(D) \subseteq N(P) \), and the p-Sylow group of \( N(D) \) is normal in \( N(D) \).

Furthermore, by \([2, (80)]\) there is a block \( \widetilde{B} \) of \( N(D) \) which corresponds to \( B \) in the sense of \([2, (75)]\). The defect group of \( B \) is the p-Sylow group of \( N(D) \) \([2, (9F)]\) and is contained in \( D \) \([2, (8D)]\). We must have \( D = P \). Thus every p-block of \( G \) has defect 0 or full defect \( a \).

We know that

\[(1) \quad P \cap (P) = P \times V \]

where \( V \) is a group of order prime to \( p \). Then every p-block of \( P \cap (P) \) consists of the \( \overline{p} \) irreducible characters \( \chi^\alpha \) where \( \alpha \) is a fixed irreducible character of \( V \) and \( \alpha \) ranges over all the irreducible characters of \( P \). We shall denote this block by \( B(\chi^\alpha) \).

There is a one-to-one correspondence between the p-blocks of defect \( a \) of \( G \) and the classes \( \langle \chi^\alpha \rangle \) of irreducible characters of \( V \) associated in \( N(P) \) \([2, (12A)]\). Denote the block of \( G \) corresponding to \( \langle \chi \rangle \) by \( B(\chi) \), then, according to \([3 > (2D)]\),

\[(2) \quad B(\langle \chi \rangle) = B(\chi) \]

in the sense defined there. Every p-block of \( N(P) \) is of defect \( a \) and must be of the form \( B(\langle \chi \rangle) \) for some \( \chi \). We denote this block by \( \overline{B}(\langle \chi \rangle) \). Then \([3, (2C)]\) implies

\[(3) \quad \overline{B}(\langle \chi \rangle)^G = B(\chi) \]

**LEMMA 1** An irreducible character \( \chi \) of \( N(P) \) belongs to \( \overline{B}(\chi) \) if and only if \( \chi \) has \( \chi^\alpha \) as a constituent.
Proof, Let $\mathbb{X}$ be an algebraic number field of finite degree containing the $|N(P)|$-th roots of unity* Let $\mathfrak{o}$ be the ring of algebraic integers in $\mathbb{X}$, and let $\mathfrak{p}$ be a prime ideal of $\mathfrak{o}$ containing $p$.* If we apply (2) to $N(P)$, it follows from [2 > (12,2)] that for any $e \in (\mathfrak{o}^\times)$ and $v \in V$ we have

$$\frac{|N(P)|}{|C(v) \cap N(P)|} \psi(v) = \mathfrak{f}^M \pmod{p},$$

Here $w$ ranges over the elements of $V$ which are conjugate to $v$ in $N(P)$. Hence

$$\frac{M_{P\mathfrak{l}L \mathfrak{f} \mathfrak{i}L}}{|C(v) \cap N(P)|} \psi(l) \equiv \frac{M_{P\mathfrak{l}L \mathfrak{f} \mathfrak{i}L}}{|C(v) \cap N(P)|} \prod_{q \in \mathfrak{f}} \sum_{\mathfrak{g} \in \mathfrak{f}^{-1}} \mathfrak{f}_{j}(l) \pmod{p},$$

where $\mathfrak{f}_{j}$ ranges over the associates of $^V$ in $N(P)$ and $q$ is the number of these associates. But, since $V < N(P)$,

$$\psi \bigg| N(P) = \frac{\psi(l)}{q} \sum_{\mathfrak{g} \in \mathfrak{f}^{-1}} \mathfrak{f}_{j}(l),$$

for some class $C \mathfrak{g} \mathfrak{g} \mathfrak{g} \mathfrak{g} \mathfrak{g}$, where $q$ is the number of members of this class*.

These last two relations yield a congruence relating the values of $\psi f$ and its associates to those of $\psi$ and its associates. However, the irreducible characters of $V$ are linearly independent (mod $\mathfrak{f}$) [2, (3C)]. Therefore $\psi$ and $\mathfrak{f}$ are associates in $N(P)$, and the lemma follows from (k)*.

Let $D$ denote the set of $p$-singular elements of $G$ whose $p$-factor is in the fixed $p$-Sylow subgroup $\mathfrak{p}$ of $G$. Let $B$ be a $p$-block of $G$, and let $f \in B$. Then

$$\psi \bigg| N(P) = \sum_{j} a_{ij} \psi_j,$$
where the \( \tau_j \) are the irreducible characters of \( N(P) \) and the \( a_j \) are integers. Then according to [5 > (46)]

\[
(6) \quad \sum_{\psi_j} \psi_j \left( S \right) a_{ij} \psi_j D
\]

where we have summed only those terms for which \( \tau_j \in B \) and \( B = B \sim \) for some block \( B \) of \( N(P) \).

**Lemma 2.** If \( a \in B \) and \( t(l) < p \) then every constituent of \( P(l) \mid V \) is an associate in \( K(P) \) of \( \tau \). In particular, if \( t(l) = 1 \) then the kernel of \( O_f \) contains \( V \).

**Proofs.** For \( p \) such have an equation of the form (5). It follows from (3) and (6) that

\[
\sum_{\psi_j} \psi_j \left( S \right) a_{ij} \psi_j
\]

vanishes on \( P - \{1\} \). Hence \( \tau \mid P \) must be a multiple of the character of the regular representation of \( P \), so \( \tau \mid (l) \). Since \( \tau \) is not of defect \( 0 > a \), \( a \). Hence \( \tau \) is identically zero, and (5) and (6) have the same terms. The lemma now follows from Lemma 1.

In particular, \( B(l) \) is the principal block (containing the principal character \( 1Q \) of \( G \)).

**Proposition.** Suppose the \( p \)-Sylow subgroup \( P \) is a T.I. set. If \( G \) has an irreducible character \( BC \) such that \( X \mid V \) reducible and \( a \) \( 1/2 \)

\( \tau \mid (l) \) \( (p + 1) \) \( G \) has an \( 122121 \) \( 2HP \) \( V \) containing \( V \).

**Proof.** It follows from Lemma 2 that \( B(l) \) has a non-principal constituent in \( B(l) \), and that this constituent has \( V \) in its kernel.
REMARK* If $G$ has a non-principal character $\chi^G$ such that $X^V_N$ is irreducible then without use of the lemmas we see easily that $G$ has a normal subgroup $M^G$ containing either $P$ or $V$.

THEOREM. Suppose the $p$-Sylow subgroup $P$ of $G$ is a $T$. $l_o$ set and that $C(V) S K(P)$, if $G$ has a faithful character $\%$ all of whose constituents have degrees $< (p + 1)$, then $P^G$.

Proof* Suppose the statement is false and that $G$ is a counterexample of minimal order. If for every constituent $\%_o$ of $\%$, $X_o I^PV$ is irreducible then $Z(P) \subseteq Z(G)$ and $P < G$, which is not the case. Hence for some constituent $\%_o$ of $X > \%_o$, $I^P$ is reducible* Then $X , X_o$ has a constituent $p(, K, 1$ such that $l_P \leq P \in \frac{1}{P}. PV$. By Lemma 2 $V \leq K$, the kernel of $X_x$. Either $K N(P) = G$ or $P < K N(P)$. In the first case, $\% \perp J(P)$ is irreducible and then $P C^\perp K < QG$. By the minimality of $G > P.K < G$, which is not the case.

Thus $P < K N(P)$ Then $K H P = 1$ since $P^G$. Hence $K P = K X P$, so $V \leq K < V$, and $V < G$. Then $P A V < C(V) < J G$ so $P < G$. This is a contradiction and the proof is complete*

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