A Geometric Paradigm Exposing High Gain Root Sensitivity of

Single-Input Single-Output Systems

T.R. Kurfess, M.L Nagurka

EDRC 24-60-91

A Geometric Paradigm Exposing High Gain Root Sensitivity of Single-Input Single-Output Systems

T.R. Kurfess and **MX. Nagurka**
Department of Mechanical Engineering
Carnegie Mellon University
Pittsburgh, PA 15213

Relationships useful for analysis and design exist between the magnitude of the classical root sensitivity function at high gain and the asymptotic behavior of eigenvalue magnitudes. These relationships are proven rigorously via mathematical analyses of closed loop single-input single-output systems whose eigenvalue magnitudes are predictable at high gain. More powerfully, the relationships are demonstrated via geometric arguments employing magnitude gain plots depicting eigenvalue magnitude as an explicit function of gain. Two theorems summarize the major results of high gain sensitivity magnitude behavior; a third theorem applicable for all gains relates root sensitivities to slopes of the magnitude gain plot.

Introduction

The **root** locus depicts the trajectories of the closed-loop eigenvalues as a parameter, such as gain, is varied. It is the set of curves representing the solutions of the closed-loop characteristic equation. As a parameter is changed, the eigenvalues proceed along the root loci at a specific parameter-dependent velocity known as the root sensitivity. Root sensitivity is the relative change of an eigenvalue with respect to a relative change in a system parameter for infinitesimally small changes in the eigenvalue and parameter. Usually, the parameter of interest is the forward loop gain.

The concept of root sensitivity is often introduced in basic classical controls textbooks as a means to identify the break points of a root locus plot *(e.g.,* Kuo, 1991). At the break points, corresponding to locations in the complex plane where real closed-loop eigenvalues become complex conjugates (break out points) or vice-versa (break in points), the root sensitivity is infinite. In fact, root sensitivity is a complex (not real) quantity, and at the break points it has an infinite magnitude. In addition to break point determination, root sensitivity has several important properties related to control system design. (The root sensitivity function is distinct from the frequency-domain sensitivity function, discussed in the robust controls literature.)

Root sensitivity with respect to gain (hereafter just called sensitivity) is defined mathematically as

$$S_k(s) = \frac{ds}{dk}\frac{k}{s} \qquad (1)$$

where s is the closed-loop eigenvalue location in the complex plane and k is the scalar gain in the forward loop cascaded with the single-input, single-output (SISO) plant transfer function g(s), as depicted in Figure L The classical definition of sensitivity normalizes ds/dk by dividing by s/k, where s is a function of k given by the closed-loop characteristic equation. The definition of sensitivity, being a function of the eigenvalue, may be a complex quantity having both magnitude and angle. From root locus analysis, it is well known that at high gains the poles - or system eigenvalues - either migrate toward infinite transmission zeros in a classical Butterworth configuration or migrate toward finite transmission zeros. Thus, at high gain the closed-loop eigenvalue angles are constant, resulting in an uninteresting asymptotic sensitivity angle. More intriguing are the relationships linking the sensitivity magnitude and the rate at which the eigenvalues tend toward finite or infinite transmission zeros. This report develops these relationships and explores the asymptotic behavior of root sensitivity for closed loop systems operating at high gain.

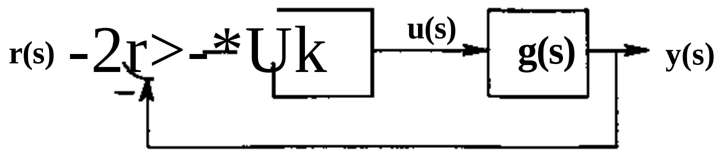

Figure 1. Closed-Loop SISO Negative Feedback Configuration.

In the following section, the eigenvalue magnitudes of three representative systems are related to incremental gain changes at high gains that are in turn connected to sensitivity magnitudes. The relationships are demonstrated conclusively in several examples. The examples are used as vehicles to introduce a geometric understanding using gain plots (Kurfess and Nagurka, 1991a) that offer an alternate visualization and distill the mathematical foundations employed in this report. Finally, three general theorems capture the essence of both the analytic and geometric arguments.

High Gain Behavior of Eigenvalues

This section examines the closed-loop system sensitivity magnitudes at high gains. The analysis is based on a unity feedback SISO system with a scalar gain amplifying the error signal as shown in Figure 1.

In general, a linear time-invariant SISO system may be characterized by the transfer function

$$g(s) = \frac{n(s)}{d(s)} = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{j=0}^{n} a_j s^j} \quad (2)$$

where n(s) is the transfer function numerator of order m in s with constant coefficients, bj (i=0, 1,..., m), and d(s) is the transfer function denominator of order n in s with constant coefficients, aj (j=0, 1,..., n) with $a_n=1$. Equation (2) represents the transfer function of an n^{th} order system. The characteristic polynomial, $\phi_{CL}(s)$, for the general system embedded in the closed-loop configuration of Figure 1 is

$$\phi_{CL}(s) = d(s) + kn(s) = \sum_{j=0}^{n} a_j s^j + k \sum_{i=0}^{m} b_i s^i \quad (3)$$

The roots of equation (3) **are the closed-loop** system **eigenvalues.**

Equation (3) can **be** written as

$$k = -\frac{d(s)}{n(s)} = -\frac{\sum_{j=0}^{n} a_j s^j}{\sum_{i=0}^{m} b_i s^i} \qquad (4)$$

From equation (4), as k-»oo at a specific **rate, the ratio of** d(s) **to** n(s) **must also** tend towards infinity **at the** same rate.

The magnitude of equation (4) is given by

$$k = \frac{|d(s)|}{|n(s)|} = \frac{\left|\sum_{j=1}^{n} a_j s^j\right|}{\left|\sum_{i=1}^{m} b_i s^i\right|} \qquad (5)$$

As k-*», there are two possibilities for the magnitude ratio of d(s) to n(s), *i.e.,*

$$d(s) \rightarrow \infty \qquad (6)$$

and/or

$$n(s) \rightarrow 0 \qquad (7)$$

In the literature, it is shown that both of these conditions are satisfied as k-><» (Ogata, 1990; Franklin, *etaU* 1991).

Eigenvalue Migration Towards Infinite Transmission Zeros

First the case given by expression (6) is addressed To satisfy condition (6) and the root locus angle criterion, n-m of the closed-loop system eigenvalues in the complex plane must tend towards a magnitude of infinity along asymptotes given by the Butterworth configuration (Kwakernaak and Sivan, 1972). These eigenvalues are termed the *infinite eigenvalues;* the remaining m eigenvalues are termed *finite eigenvalues.* For large gain, the highest order term in s for both d(s) and n(s) dominates since the infinite eigenvalues have magnitudes lsl-*». Thus, in the limit as k-*>o equation (5) may be approximated by

$$k = \left|\frac{s^n}{b_m s^m}\right| = \left|\frac{s^{n-m}}{b_m}\right| \qquad (8)$$

From equation (8), the asymptotic behavior of the eigenvalue magnitudes may be realized as

Islsfbmk)^1∧^11,,111 /m

Thus, high gain eigenvalues migrating toward infinite transmission zeros increase in magnitude at a rate inversely proportional to k^11''''1.

The power law relation of equation (9) yields the following expression for infinite eigenvalue magnitudes at high gains

$$\frac{d|s^*|}{dk^*} = \frac{i}{n-m} \qquad (10)$$

where, for convenience, the following notation is introduced

$$k^* = \log(k) \qquad (11)$$

$$s^* = \log|s| \qquad (12)$$

The generalization of equation (9) holds true for (minimum and non-minimum phase) SISO systems possessing n poles and m zeros (Kurfess and Nagurka, 1991b).

Eigenvalue Migration Towards **Unique Transmission Zeros**

This subsection considers the case given by expression (7). The derivation is slightly different from that developed for the infinite eigenvalues. Initially, unique zero locations are assumed; subsequently, the theory is extended to any SISO system. The derivation employs the magnitude criterion in conjunction with the limit given by equation (7). Clearly, for S=SJ (i=n-m+l, n-m+2, ..., n), ld(s)l must be finite and nonzero if no pole-zero cancellations occur. Thus, ln(s)l evaluated at any specific finite eigenvalue must approach zero as $k \to \infty$

$$\lim_{k\to\infty} \frac{|d(s)|}{|n(s)|} - \lim_{n(s)\to 0} \frac{|d(s)|}{|n(s)|} \qquad (13)$$

Furthermore, since the finite eigenvalues asymptotically approach a finite value, ld(s)l may be approximated as a nonzero constant, do. Thus as k-»oo, the magnitude criterion requires that ln(s)l evaluated at each finite eigenvalue must approach zero at a rate proportional to k

$$\lim_{k\to\infty} k = \lim \left(\frac{d}{T} V \right) \tag{14}$$

Equation (14) summarizes an important property of high gain finite eigenvalue behavior. At high gains, the gain is inversely proportional to the magnitude of n(s) In other words, the distance of a high gain eigenvalue from its matching finite transmission zero in the complex-plane changes at a rate inversely proportional to k.

An alternate perspective of the development above is to consider n(s) as the product of terms given by

$$n(s) = \prod_{i=1}^{m} (s + z_i) \tag{15}$$

where z\ is the i^1*1 zero location. The numerator n(s) evaluated at an admissible value of s tends to zero because one of its terms, S+ZJ, vanishes as s —» -zi. For that particular s, the other numerator terms are finite non-zero valued and do not cause n(s) —> 0. Since n(s) and k are inversely related in equation (5), n(s) approaches zero at a rate inversely proportional to the rate that k approaches infinity.

Eigenvalue **Migration Towards Transmission Zeros** in **General**

In the previous derivation, unique transmission zeros were assumed. This section extends the theory to multiple zeros identically located in the s-plane. Again, equation (S) must be satisfied as k -> «>. Therefore, n(s) -^ 0 at the same rate that k -» <». If a multiplicity of w transmission zeros exist at the same location, -zi, then as k —> <»

$$s -> -*i \tag{16}$$

and

$$n(s) -> 0 \tag{17}$$

at a rate proportional to r|d(s), where ri^s) is the distance sensitivity given by

$$n\ /_c\backslash_{Tld\{s\}\sim} - \frac{d(\log|s-z_i|)}{d(\log k)} \tag{18}$$

T|d(s) may be considered the sensitivity of the eigenvalue distance to the finite transmission zero with respect to gain. Equation (18) employs logarithms to clearly depict the power law relationship governing high gain behavior. Since there is a multiplicity of w finite transmission zeros, n(s) must approach zero at a rate proportional to $(T|d(s))^w$. In order for

equation (6) to hold, $|d(s)|$ must approach **zero (or** $s \to z_i$**) at a rate** proportional to $k^{-1/w}$, thus

$$|s - z_i| = c k^{-1/w} \tag{19}$$

where c is **a** constant

Sensitivity Analysis at High Gain

In the previous section, eigenvalue magnitude behavior at high gain was presented. This section employs those findings to analyze the root sensitivity magnitude at high gains.

Infinite Eigenvalue Sensitivity Relation

For infinite eigenvalues, equation (9) may be applied to write the ratio of the magnitudes of k and s in terms of k

$$\frac{|k|}{|s|} = \frac{k}{|s|} = \frac{k}{(b_m k)^{1/(n-m)}} = \frac{k^{(n-m-1)/(n-m)}}{(b_m)^{1/(n-m)}} \tag{20}$$

Equation (9) may also be used to determine the derivative of s with respect to k, *i.e.,*

$$\frac{d|s|}{dk} = \frac{b_m^{1/(n-m)}}{n-m}\, k^{\frac{1-(n-m)}{n-m}} \tag{21}$$

It is also possible to calculate the complete complex derivative of s with respect to k.

Equations (20) and (21) are the basis of the definition of sensitivity magnitude

$$|S_k| = \left|\frac{ds}{dk}\right|\left|\frac{k}{s}\right| = \left|\frac{1}{n-m}\, k^{\frac{1-(n-m)}{n-m}}\right| k^{\frac{n-m-1}{n-m}} \tag{22}$$

which simplifies to

$$|S_k| = \frac{1}{n-m} \tag{23}$$

as $k \to \infty$. To complete the proof for infinite eigenvalues equations (23) and (10) are equated, yielding

$$\lim_{k \to \infty} \left|\frac{d|s^*|}{dk^*}\right| = \lim_{k \to \infty} |S_k| \tag{24}$$

Each of the n-m poles that satisfies equation (9) represents an individual eigenvalue migrating towards an infinite transmission zero. If the first n-m eigenvalues are considered to be those with infinite magnitude at infinite gain, then each individual *infinite* eigenvalue, $s=s_i$ (i=1, 2,..., n-m), satisfies equation (9). However, there remain m *finite* eigenvalues that also may be used to satisfy equation (9). These m *finite* eigenvalues approach the finite transmission zeros at a rate which may be determined. The derivation (Kurfess and Nagurka, 1991c) is non-trivial, yielding results that are quite sensible but not intuitive, and is summarized in the following section.

Finite Eigenvalue Sensitivity Relation at Origin

This section discusses the closed-loop sensitivity magnitude at high gains for eigenvalues tending towards finite transmission zeros at the origin (differentiators).

It is assumed that there are w eigenvalues approaching w transmission zeros located at the origin (s=0). From equation (19) the derivative of $|s|$ with respect to k may be computed as

$$\frac{d|s|}{dk} = -\frac{c}{w} k^{-1-1/w} = -\frac{c}{w} k^{-1} k^{-1/w} z_i \tag{25}$$

Employing the definition of sensitivity given by equation (1), and using the relations of equations (25) and (19) with $z_i = 0$, the sensitivity of the finite transmission zeros approaching the origin may be written as

$$|S_k| = \left|\frac{k}{s}\frac{d|s|}{dk}\right| = \left\|\left[\frac{k}{c k^{-1/w}}\right]\left[-\frac{c}{w} k^{-1-1/w}\right]\right\| \tag{26}$$

which simplifies to

$$|S_k| = \frac{1}{w} \tag{27}$$

Thus, the sensitivity of the eigenvalues approaching the origin at high gain is the inverse of the number of non-unique zeros at s=0.

Finite Eigenvalue Sensitivity Relation for Transmission Zeros Not at Origin

For eigenvalues approaching transmission zeros not located at the origin, the derivation differs slightly. For this proof, a transmission zero located at $s = -z_i$ ($z_i \neq 0$) with multiplicity w is assumed.

As **the** gain approaches infinity, **the distance between the** eigenvalues and the transmission **zeros** is given by equation (19). Thus, the location of the eigenvalues is given by

$$|s| = z_i + ck^{-1/w} \tag{28}$$

Hence,

$$\frac{d|s|}{dk} = -\frac{c}{w} k^{-1-1/w} \tag{29}$$

The sensitivity of these eigenvalues may be determined from equations (28) and (29) as

$$|S_k| = \left|\frac{k}{|s|}\frac{d|s|}{dk}\right| = \left|\left[\frac{k}{z_i + ck^{-1/w}}\right]\left[-\frac{c}{w} k^{-1-1/w}\right]\right| = \frac{c}{w}\,\frac{1}{z_i + ck^{-1/w}}\,k^{-1/w} \tag{30}$$

As k tends to infinity, equation (30) and hence the sensitivity magnitude, tend toward zero. Clearly, equation (30) is singular when Zi=0. Thus, equation (30) must be used when analyzing sensitivities for eigenvalues approaching transmission zeros at the origin of the s-plane.

First Order Examples and Geometric Relationships

This section presents three different first-order examples of increasing complexity that demonstrate high gain behavior of the sensitivity function. The example problems include: a first order lag system, a lead-lag system, and a first order lag with a pure differentiator. The analysis of each system is followed by a geometric interpretation of the eigenvalue magnitude with respect to gain based on a new control analysis tool, namely, the magnitude gain plot.

First Order Lag Example

The first order system examined here is given by the transfer function

$$g(s) = \frac{1}{s+2} \tag{31}$$

embedded in the unity gain feedback configuration depicted in Figure 1. For this system, the closed-loop transfer function is

$$g_{CL}(s) = \frac{k}{s+2+k} \tag{32}$$

for which the magnitude of the closed-loop eigenvalue is given by

$$|s| = 2 + k \tag{33}$$

and the sensitivity magnitude is

$$|S_k| = \left|\frac{d|s|}{dk}\frac{k}{|s|}\right| = 1\left(\frac{k}{2+k}\right) \tag{34}$$

As k tends to infinity the sensitivity magnitude is

$$\lim_{k \to \infty} |S_k| = \frac{k}{2+k} = 1 \tag{35}$$

for the single eigenvalue proceeding along the negative real axis in the complex plane, as shown in the root locus plot of Figure 2a.

The behavior of the eigenvalue magnitude can be depicted graphically in a magnitude gain plot, as shown in Figure 2b. This plot graphs the magnitude of the eigenvalue as a function of gain using log-log scales, and offers a multitude of analysis and design properties (Kurfess and Nagurka, 199Id). At high gain, the slope (magnitude) of the magnitude gain plot is one, which is the high gain sensitivity magnitude given by equation (35).

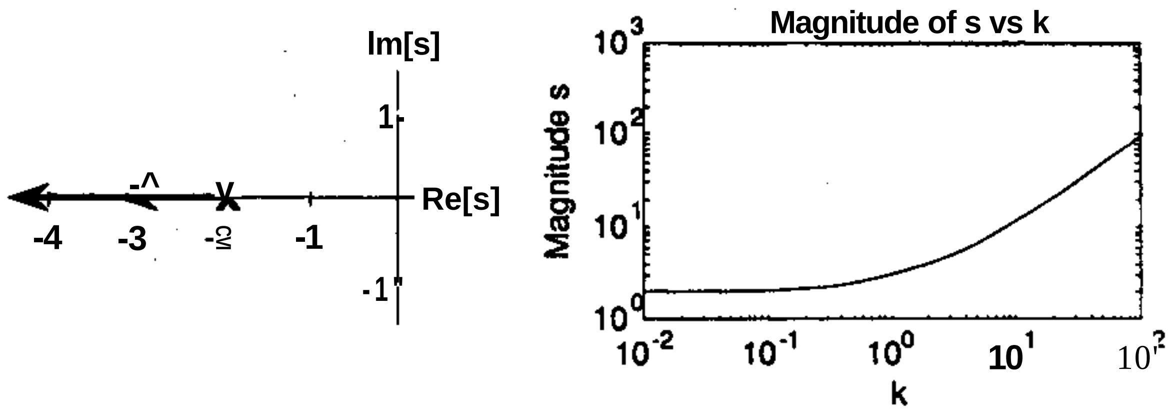

Figure 2a,b. Root Locus and Magnitude Gain Plot of Equation (31).

First-Order Lead-Lag Example

This example introduces a zero located at s = -1 to the transfer function of the previous example. The system is now given by the open-loop transfer function

g(s) = J ^ **(36)**

with a closed-loop transfer function of

$$G_{CL}(s) = \frac{k(s+1)}{s+2+k(s+1)} \tag{37}$$

and a characteristic equation of

$$\phi_{CL}(s) = (k+1)s + 2 + k = 0 \tag{38}$$

From the characteristic equation, the magnitude of s may be determined as a function of k

$$|s| = \frac{k+2}{k+1} \tag{39}$$

The derivative of equation (39) with respect to k is

$$\frac{d|s|}{dk} = \frac{-1}{(k+1)^2} \tag{40}$$

Combining equations (39) and (40) and taking the limit as k tends to infinity results in the following high gain sensitivity magnitude

$$\lim_{k\to\infty} |S_k| = \left|\frac{d|s|}{dk}\frac{k}{|s|}\right| = \lim_{k\to\infty}\left|\left(\frac{-1}{(k+1)^2}\right)\left(\frac{k(k+1)}{(k+2)}\right)\right| = 0 \tag{41}$$

Figure 3a,b show the root locus and the magnitude gain plot, respectively, for this lead-lag system. The slope magnitude of the gain plot is zero at high gain, corresponding to the high gain sensitivity magnitude.

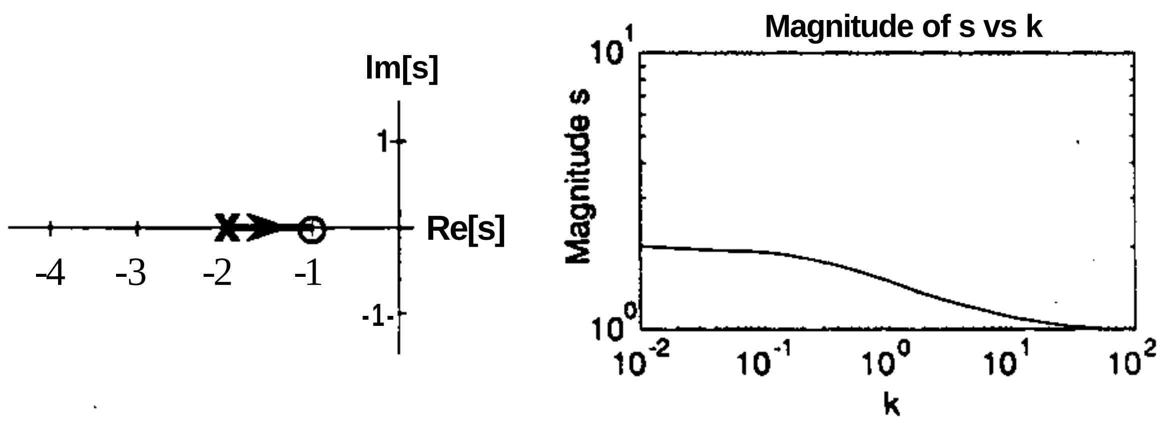

Figure 3a,b. Root Locus and Magnitude Gain Plot of Equation (36).

First-Order Lead-Lag Example with Zero at Origin

This example is similar to the previous example with the exception that the zero is now located at the origin of the s-plane. This requires the application of equation (27) since $|s|$ approaches zero as k approaches infinity, resulting in a value of zero in the denominator of equation (1) (the system sensitivity). For this example, the open-loop transfer function is given by

$$g(s) = \frac{s}{s+2} \tag{42}$$

corresponding to the closed-loop transfer function of

$$g_{CL}(s) = \frac{k\,s}{s+2+k\,s} \tag{43}$$

with a characteristic equation

$$\Phi_{CL}(s) = (k+1)s + 2 = 0 \tag{44}$$

From equation (44), the magnitude of s may be computed as

$$|s| = \frac{2}{k+1} \tag{45}$$

yielding a derivative of $|s|$ with respect to k

$$\frac{d|s|}{dk} = \frac{-2}{(k+1)^2} \tag{46}$$

Thus, the sensitivity magnitude as k tends to infinity is

$$\lim_{k\to\infty} |S_k| = \left|\frac{d|s|}{dk}\frac{k}{|s|}\right| = \left|\frac{2}{(k+1)^2}\cdot\frac{k(k+1)}{2}\right| = 1 \tag{47}$$

The root locus plot and magnitude gain plot are shown in Figure 4a,b, respectively, for this example. By inspection, the magnitude gain plot slope at high gains is -1, corresponding to a slope magnitude of unity which is the high gain sensitivity of this system. The negative slope of the magnitude gain plot suggests that the sensitivity magnitude does not capture all the information associated with the complex quantity S*.

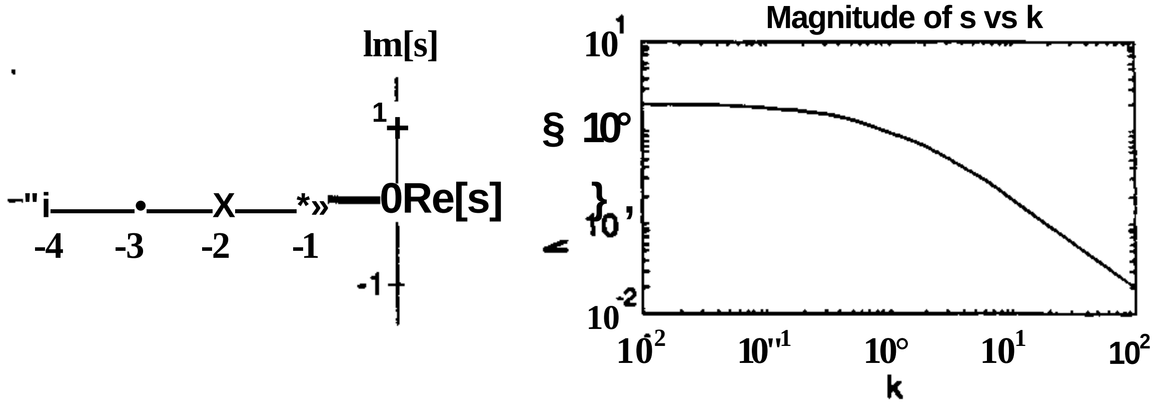

Figure 4a,b. Root Locus and Magnitude Gain Plot of Equation (42).

Second Order Example and Geometric Relationships

This section examines in detail a second-order system with a single transmission zero given by

$$g(s) = \frac{(s + 3)}{(s + 1)(s + 2)} \tag{48}$$

Figure 5 is the root locus plot of this transfer function. Each branch of the root locus starts at k=0 corresponding to a system open-loop pole (s=-1,-2), and asymptotically approaches either a finite (s=-3) or infinite ($s \to -\infty$) transmission zero.

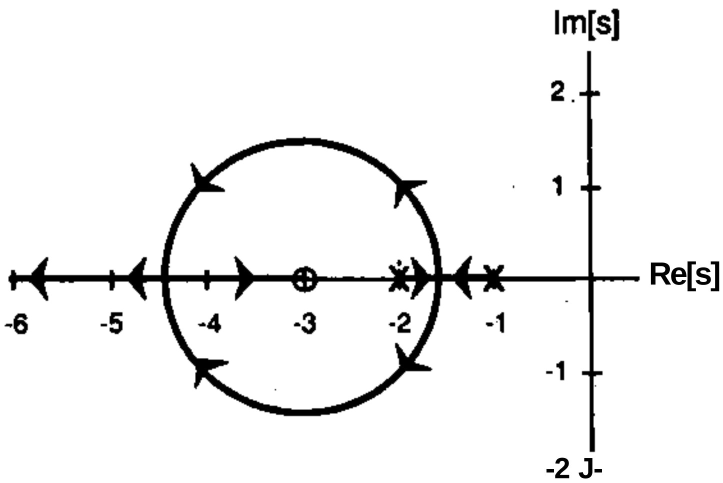

Figure 5. Root Locus Plot of Equation (48).

Figure 6 is the magnitude gain plot of the system of equation (48). Notice that the complex conjugate eigenvalues have magnitudes that are equal corresponding to a single segment in the plot. The magnitude gain plot shows the presence of two open-loop poles with magnitudes 1 and 2 at k = 0. As k —»<» it shows a single finite transmission zero with magnitude 3 and an asymptote tending toward an infinite transmission zero. It also •highlight the break points corresponding to points where branches leave or enter the real axis of the root locus. For this example, these break points occur at k « 0.17 and at k « 5.83. Between these break points the loci of the two branch points are not on the real axis and the corresponding single curve of the magnitude gain plot confirms that the trajectories are those of a complex conjugate pair.

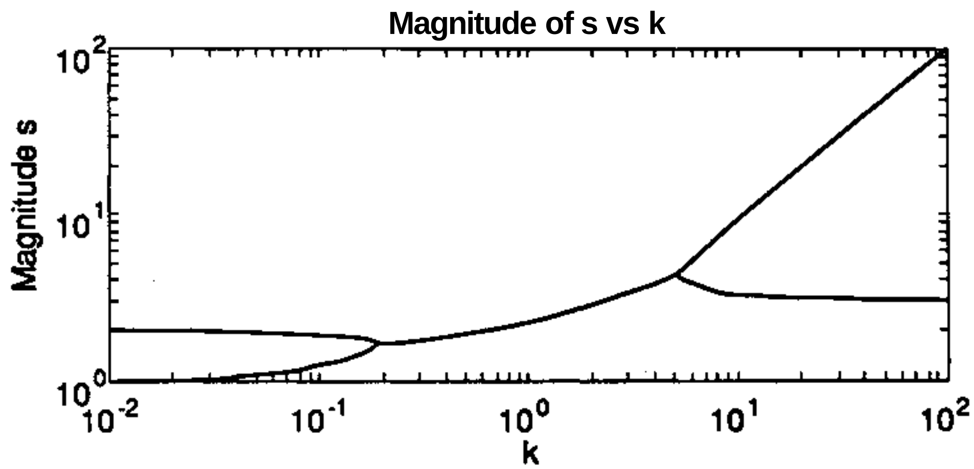

Figure 6. The Magnitude Gain Plot of Equation (48).

Although the mathematics for the analysis of this second order system analysis are tractable, they do become somewhat cumbersome. Sensitivity analyses of higher order systems become extremely challenging and are, therefore, not conducted often. The ease with which the high gain sensitivity magnitude may be calculated from the slope of the magnitude gain plot simplifies sensitivity determination significantly.

There are two separate sensitivities for this second order system, one for the eigenvalue approaching the transmission zero at s=-3, and one for the infinite eigenvalue proceeding along the negative real axis in Figure 5. From equation (30) the eigenvalue approaching the finite transmission zero has a high gain sensitivity magnitude of zero. Equation (10) may be used to compute the high gain sensitivity magnitude for the infinite eigenvalue as

$$\lim_{k \to \infty} \left| \frac{d|s^*|}{dk^*} \right| = \frac{1}{n-m} = \frac{1}{2-1} = 1 \tag{49}$$

Alternatively these high gain sensitivities may be determined directly by application of the definition, although this requires substantial calculations. First, the closed loop transfer function

$$g_{CL}(s) = \frac{(s+3)}{(s+1)(s+2)+k(s+3)} \tag{50}$$

must be computed resulting in the following characteristic equation

$$\Phi_{CL}(s) = s^2 + (3+k)s + 2 + 3k = 0 \tag{51}$$

The eigenvalues may be determined directly for this second order system via the quadratic formula as

$$s = \frac{-(3+k) \pm \sqrt{(3+k)^2 - 4(2+3k)}}{2} \tag{52}$$

As k tends towards infinity equation (52) may be approximated by

$$s \cong -\frac{1}{2}\left[(3+k) \pm (k-3)\right] \tag{53}$$

resulting in $s = \{-k, -3\}$ at high values of k.

For this second order system, the sensitivity must be determined for each individual eigenvalue. For the infinite eigenvalue, the high gain sensitivity magnitude is given by:

$$\lim_{k \to \infty} |S_k| = \left| \frac{d|s|}{dk} \frac{k}{|s|} \right| = \lim_{k \to \infty} \left| 1 \left(\frac{k}{k} \right) \right| = 1 \tag{54}$$

For the finite eigenvalue, the high gain sensitivity magnitude is

$$\lim_{k \to \infty} |S_k| = \left| \frac{d|s|}{dk} \frac{k}{|s|} \right| = \lim_{k \to \infty} \left| 0 \left(\frac{k}{3} \right) \right| = 0 \tag{55}$$

The two values given in equations (54) and (55) are the magnitudes of the slopes of the two high gain asymptotic lines in the plot of Figure 6. Equation (54) is the sensitivity magnitude for the infinite eigenvalue that is represented in the plot as a line with slope of one. This represents an eigenvalue having a magnitude tending towards infinity at the same rate that the gain k tends towards infinity. Equation (55) represents the high gain sensitivity magnitude for the eigenvalue approaching the finite transmission zero at s=-3.

This is depicted by the horizontal line on Figure 6. This sensitivity indicates that as k tends towards infinity, the magnitude of the eigenvalue approaches a constant finite value, and the eigenvalue is desensitized from the gain. The actual approach rate towards the transmission zero is kr^1 for this example, and is the topic of a separate paper (Kurfess and Nagurka, 1991b).

The sensitivity of the eigenvalue magnitude can be calculated numerically and graphed as a function of k, as depicted in Figure 7 for the system of equation (48). Figure 7 shows that the sensitivity magnitude tends to infinity for gains associated with eigenvalue break-in and break-out points. Furthermore, it suggests that at high gain values the eigenvalue migrating to the finite transmission zero has zero gain sensitivity, whereas the eigenvalue tending to the infinite zero has a constant sensitivity magnitude of unity. This corresponds to the analysis presented above. Figure 7 is not (to our knowledge) presented in classical controls textbooks, although it is a direct result of the numerical evaluation of the defining equation of sensitivity and is a very useful geometric representation of the eigenvalues of the closed-loop system.

The sensitivity, given by equation (1), can alternatively be defined as (Ur, 1960; Dorf, 1987)

$$S_k(s) \quad \frac{ds\;k}{dk\;s} \quad \frac{d(\ln(s))}{d(\ln(k))} \tag{56}$$

From equation (56), the sensitivity magnitude can be written as

$$|S_k(s)| = \left|\frac{d}{d}(Mkj)1\right|^{-} \; |d(\log(k))|^{-}{}_{dk}* \tag{57}$$

Thus, the sensitivity magnitude is equivalent to the magnitude of the slope of the magnitude gain plot. This is an important new result valid for all gains, although the focus of this report is directed to the asymptotic case of high gain.

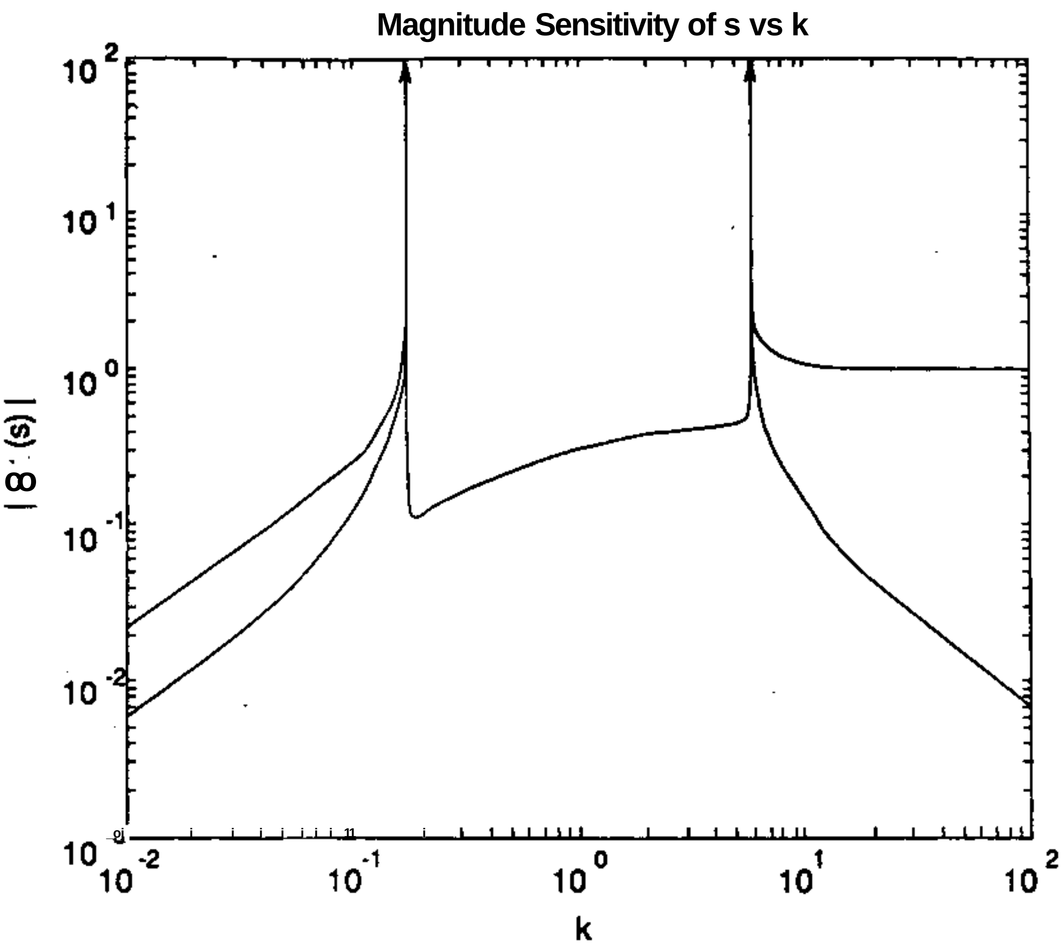

Figure 7. Sensitivity Magnitude Plot of Equation (48).

Conclusions

Although the root sensitivity function is introduced in the classical controls literature, it does not seem to be explained in complete detail. The magnitude gain plot provides an excellent graphical means by which the sensitivity may be evaluated by inspection. This report mathematically proves and graphically demonstrates a high gain relationship between the slope of the magnitude gain plot and the root sensitivity magnitude. The relationship provides a powerful procedure by which sensitivity magnitudes at high gain may be computed. Furthermore, three theorems may be surmised from this research:

Theorem 1

The sensitivity magnitude of any eigenvalue at high gain is in the range $0 \leq |S_k| \leq 1$.

Theorem 2

The sensitivity magnitude of any eigenvalue at high gain is the reciprocal of a positive integer (e.g., IS*I = H, i = 1,2, 3,...).

Theorem 3

For all gains the absolute value of the slope of the magnitude gain plot is the sensitivity function magnitude.

These theorems do not seem to appear in the controls literature.

This report has addressed primarily the sensitivity magnitude at high gains; several other aspects of the sensitivity remain to be reported. Since sensitivity is a complex function, the angle of its high gain behavior must also be analyzed. Such an analysis has proven to be quite interesting. One result of the analysis is that the sensitivity of eigenvalues approaching transmission zeros is negative due to the singularity of the sensitivity function at the origin. Other topics of current research include the asymptotic behavior of the sensitivity function at low gains, and general relationships between the sensitivity function and the gain plots for both single-variable systems and multi-variable systems.

References

DORF, R. C, 1987, *Modern Control Systems,* Addison Wesley, Reading, MA.

FRANKLIN, G.F., POWELL, J.D., and EMAMI-NAEINI, A., 1991, *Feedback Control of Dynamic Systems,* Second Edition, Addison Wesley, Reading, MA.

KUO, B.C., 1991, *Automatic Control Systems,* Fifth Edition, Prentice Hall, Englewood Cliffs, NJ.

KURFESS, T.R. and NAGURKA, M.L., 1991a, Understanding the Root Locus Using Gain Plots, *IEEE Control Systems Magazine,* in press (August).

KURFESS, T.R. and NAGURKA, M.L., 1991b, A General Theory for High Gain Asymptotic Behavior of Eigenvalues Approaching Finite and Infinite Transmission Zeros, *ASME Journal of Dynamic Systems, Measurement and Control, submitted.*

KURFESS, T. R. and NAGURKA, M. L., 1991C, High Gain Control System Design with Gain Plots, Technical Report #EDRC 24-48-91, Engineering Design Research Center, Pittsburgh, PA.

KURFESS, T.R. and NAGURKA, M.L.,1991d, Gain Plots: A New Perspective on the SISO Root Locus, ***ASME Journal of Dynamic Systems, Measurement and Control, submitted.***

OGATA, K., 1990, ***Modern Control Engineering,*** **Prentice Hall, Englewood Cliffs, NJ.**

UR, H., 1960, Root Locus Properties and Sensitivity Relations in Control Systems, ***IRE Transactions on Automatic Control,*** **Vol. AC-5, pp. 57-65.**