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**A General Theory Quantifying
the Root Sensitivity Function**

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by

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Abstract - In this report, we present a geometric method for representing the classical root sensitivity function of linear time-invariant systems. The method employs gain plots that expand the information presented in the root locus plot in a manner that permits determination of both the real and imaginary components of the root sensitivity function by inspection.

I. The Root Sensitivity Function

In classical control theory the root sensitivity, S_{p_i} is defined as the relative change in the system roots or eigenvalues, λ_i ($i = 1, \dots, n$), with respect to a system parameter, p . Most often, the parameter analyzed is the forward proportional controller gain, k . The root sensitivity with respect to gain is given by

$$S = \frac{d\lambda(k)/\lambda(k)}{dk/k} = \frac{d\lambda(k)}{dk} \frac{k}{\lambda(k)} \quad (1)$$

Since the eigenvalues may occur as complex conjugate pairs, S^* may be complex.

Equation (1) is often introduced in determining the break points of the Evans root locus plot for single-input single-output systems. At the break points, S^* becomes infinite as at least two of the n system eigenvalues undergo a transition from the real domain to the complex domain or vice versa. This transition causes an abrupt change in the relation between the eigenvalue angle ZX and gain k yielding an infinite eigenvalue derivative with respect to gain [1].

The root sensitivity function S_k is a measure of the effect of parameter variations on the eigenvalues. It is important since one of the key objectives of feedback control theory is to reduce system sensitivity to variations in system parameters. For example, the control system of a robot should be relatively insensitive to the payload carried by the arm for the recommended payload range. If the robot's performance is sensitive to payload variations, then the control system is not robust and performance is difficult to guarantee. In this case, S_m , where m is the payload mass, should be relatively small over the operational range of m . Such considerations are critical if control designers are to develop high performance, robust, closed-loop systems.

In this report, we present a geometric technique for computing and understanding S_k . The technique relies on a set of plots called gain plots (GPs) [2] that are an alternate visualization of the Evans root locus plot. In particular, we prove that the slopes of the GPs are directly related to the real and imaginary components of S^* . In an example problem, a PD controller is implemented on a standard second order plant demonstrating the importance of S^* in control system design.

II. Root Sensitivity Analysis

In this section, we derive the complex root sensitivity function by employing a polar representation of the eigenvalues in the complex plane. We proceed by positing three assumptions:

Assumption 1: The systems analyzed are lumped parameter, linear time-invariant (LTI) systems.

Assumption 2: There are no eigenvalues at the origin of the s-plane, i.e.,

$$X_i \neq 0, \quad \forall i = 1, \dots, n \quad (2)$$

However, the eigenvalues may be arbitrarily close to the origin singularity.

Assumption 3: The forward scalar gain, k , is real and non-zero, i.e., $k \in \mathbb{R}$, $k \neq 0$.

Based on these assumptions, we draw the following theorems.

Theorem 1: The real component of the sensitivity function is given by

$$\operatorname{Re}\{S_k\} = \frac{d \ln |\lambda(k)|}{d \ln(k)} \quad (3)$$

Theorem 2: The imaginary component of the sensitivity function is given by

$$\operatorname{Im}\{S_k\} = \frac{d \angle \lambda(k)}{d \ln(k)} \quad (4)$$

where $\angle X$ is the eigenvalue angle.

Proof: Equation (1) may be rewritten [3,4] in terms of the derivatives of natural logarithms as

$$S_k \sim \frac{d \ln |X(k)| + j \frac{d \angle X(k)}{d \ln(k)}}{d \ln(k)} \quad (5)$$

The natural logarithm of the complex value, X , may be written as the sum of the logarithm of the magnitude of X and the angle of X multiplied by $j = \sqrt{-1}$. Thus, (5) becomes

$$S_k = \frac{d[\ln |X(k)| + j \angle X(k)]}{d \ln(k)} \quad (6)$$

Since j is a constant, (6) may be rewritten as

$$S^* \approx \frac{d \ln |M(k)|}{d \ln(k)} + j \frac{d \angle \lambda(k)}{d \ln(k)} \quad (7)$$

The complex root sensitivity function is now expressed with distinct real and imaginary components employing the polar form of the eigenvalues. It follows from *Assumptions 2* and *3* that the denominator of (5) is purely real. (In general, most parameters studied are real and this proof is sufficient. If, however, the parameter analyzed is complex, it is a straightforward task to extend the above analysis.)

The proof is completed by taking the real and imaginary components of (7), yielding (3) and (4). It is interesting to note that the Cartesian representation of S^* is related to the polar representation of λ .

III. Geometric Relations to Gain Plots

The gain plots are an alternate graphical representation of the Evans root locus plot. They explicitly graph the eigenvalue magnitude vs. gain in a magnitude gain plot (MGP) and the eigenvalue angle vs. gain in an angle gain plot (AGP). The MGP employs a log-log scale whereas the AGP uses a semi-log scale (with the logarithms being base 10.) Although we use gain as the variable of interest, it should be noted that any parameter may be used in the geometric analysis.

Theorem 3: The slope of the MGP is the real component of S^* .

Proof: The MGP slope, M_m , is

$$M_m \equiv \frac{d \log(|X(k)|)}{d \log(k)} \quad (8)$$

which may be rewritten as

$$M_m \equiv \frac{d[\log(e) \ln(|Mk|)]}{d[\log(e) \ln(k)]} = \frac{d \ln(|X(k)|)}{d \ln(k)} \quad (9)$$

corresponding to (3).

Theorem 4: The slope of the AGP is linearly related to the imaginary component of S^* by the constant, $(\log(e))^{-1}$.

Proof: The AGP slope, M_a , is

$$M_a = \frac{d \angle \lambda(k)}{d \log(k)} \quad (10)$$

which may be rewritten as

$$\frac{d \angle \lambda(k)}{d [\log(e) \ln(k)]} = \frac{1}{\log(e)} \frac{d \angle \lambda(k)}{d \ln(k)} \quad (11)$$

proving that M_a is proportionally related to (4) by $(\log(e))^{-1}$.

IV. Example

In this example, we consider the plant

$$g_p(s) = \frac{1}{s^2 + 6s + 18} \quad (12)$$

with a PD compensator

$$g_c(s) = k(s+1) \quad (13)$$

giving the loop-transmission transfer function

$$g(s) = g_p(s) g_c(s) = \frac{k(s+1)}{s^2 + 6s + 18} \quad (14)$$

The root locus plot shown in Figure 1 portrays the effect of varying gain k . An alternate visualization is shown in the MGP and AGP of Figures 2a,b, respectively. These figures show that the eigenvalues are either completely real or are complex conjugate values. The real and imaginary components of s are plotted as functions of gain in Figures 3a,k. These are the slopes of the MGP and AGP, respectively.

Figures 2a,b and 3a,b show that the break point occurs at $k_{bp} \approx 3.21$. In particular, Figures 3a,b highlight the infinite values of the real and imaginary eigenvalue components at k_{bp} . Below k_{bp} the closed-loop eigenvalues follow a circular trajectory about the point $s = -1$ as demonstrated in Figure 1.

Figures 2a and 3a show that the eigenvalues have coincident trajectories below k_{bp} . Above k_{bp} one eigenvalue migrates to the finite transmission zero at $s = -1$ while the other eigenvalue migrates to the infinite zero at $-\infty$. At high gains the real components of s are zero and one, respectively [51]. Thus, at high gains the finite transmission zero is desensitized from k , and there is a unity magnitude relation between k and the finite eigenvalue. In contrast to Figures 2a and 3a, Figures 2b and 3b show multi-valued eigenvalue trajectories above k_{bp} and coincident trajectories below k_{bp} . This is expected

since the eigenvalues are either purely real or complex conjugates. Finally, since the eigenvalues are purely real above k_{bp} , the imaginary component of S^* is zero for $k > k_{bp}$.

V. Closing

The concept of root sensitivity in classical controls is often introduced to emphasize the high "sensitivity" of eigenvalues with respect to a system parameter such as gain near the break-points. Normally, the root sensitivity function is not discussed as a complex quantity in control system analysis and design. Here, we have derived and demonstrated a powerful means to visualize the root sensitivity function via the gain plots. The slopes of the gain plots provide a direct measure of the real and imaginary components of the root sensitivity. These slopes are available by inspection, and offer the control system designer important information for selection of appropriate system parameters such as gain.

References

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- [4] B.C. Kuo, *Automatic Control Systems*, Fifth Edition, Prentice Hall, Englewood Cliffs, NJ, 1991.
- [5] T. R. Kurfess, M. L. Nagurka, "A Geometric Paradigm Exposing High Gain Root Sensitivity of Single-Input Single-Output Systems," Technical Report #EDRC 24-60-91, Engineering Design Research Center, Pittsburgh, PA, 1991.

Figures

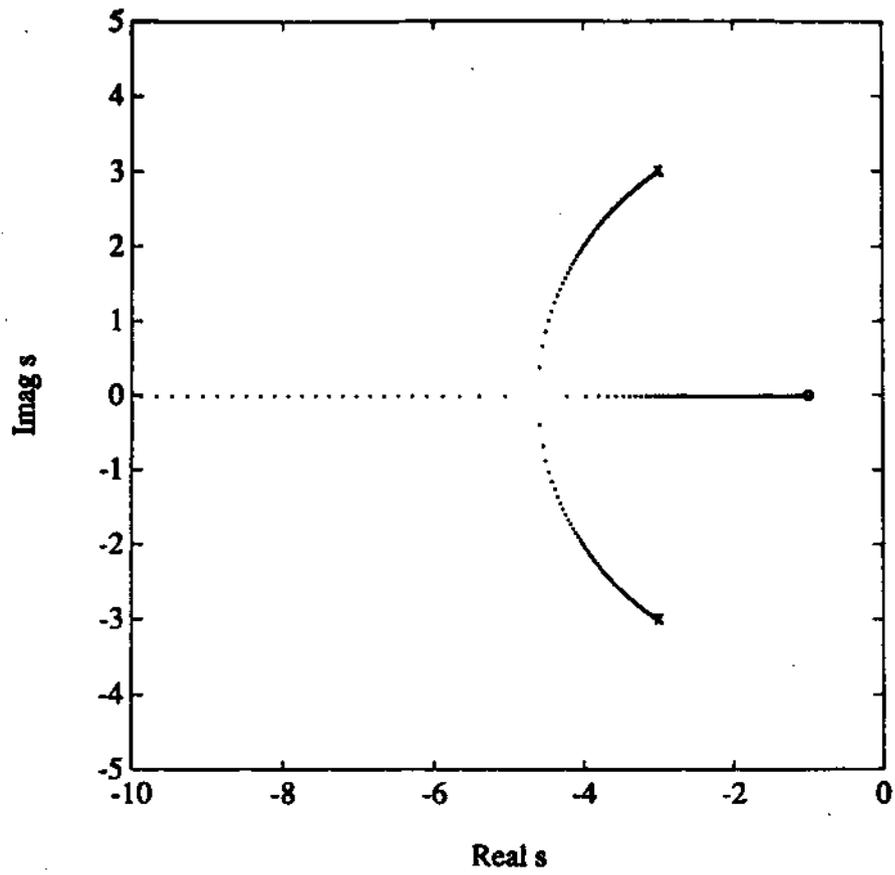


Figure 1. Root Locus Plot for Equation (14).

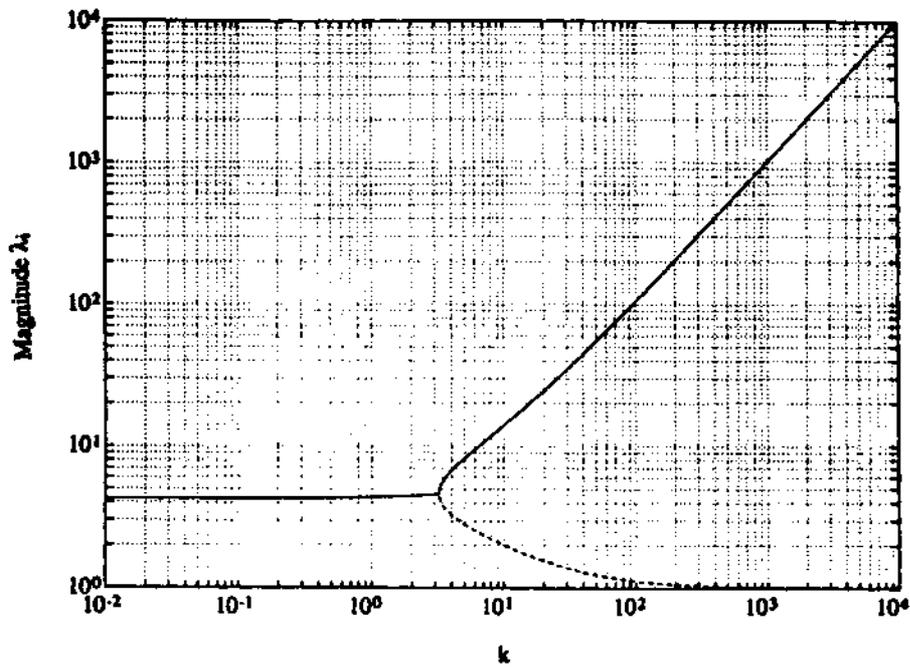


Figure 2a. Magnitude Gain Plot for Equation (14).

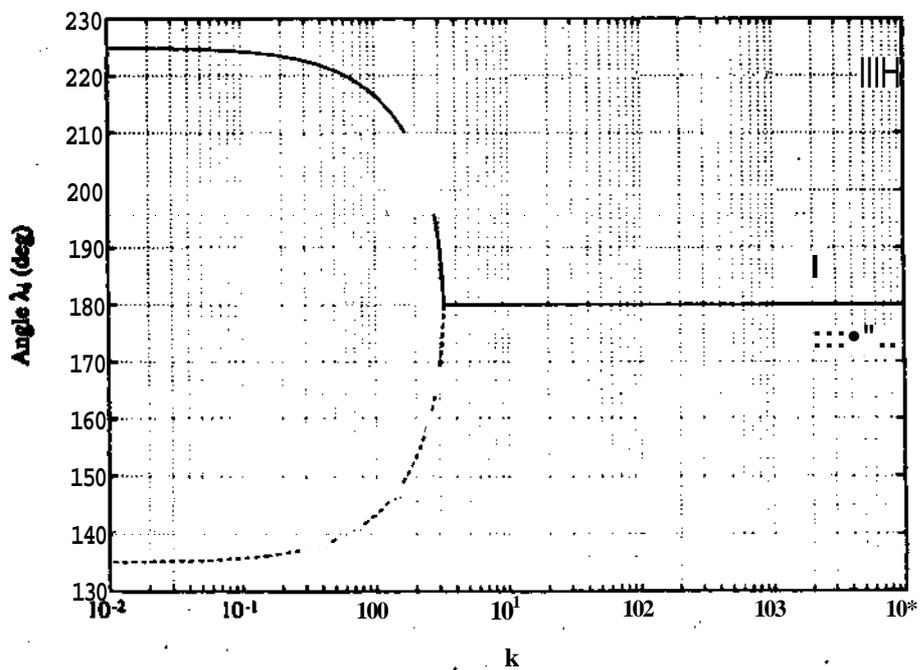


Figure 2b. Angle Gain Plot for Equation (14).

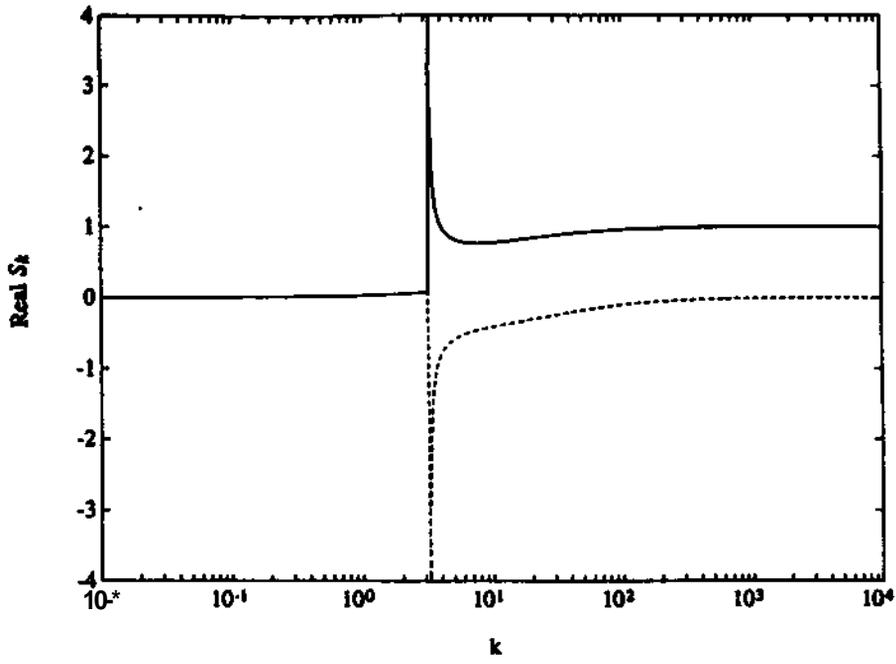


Figure 3a. Real Component of S^* for Equation (14).

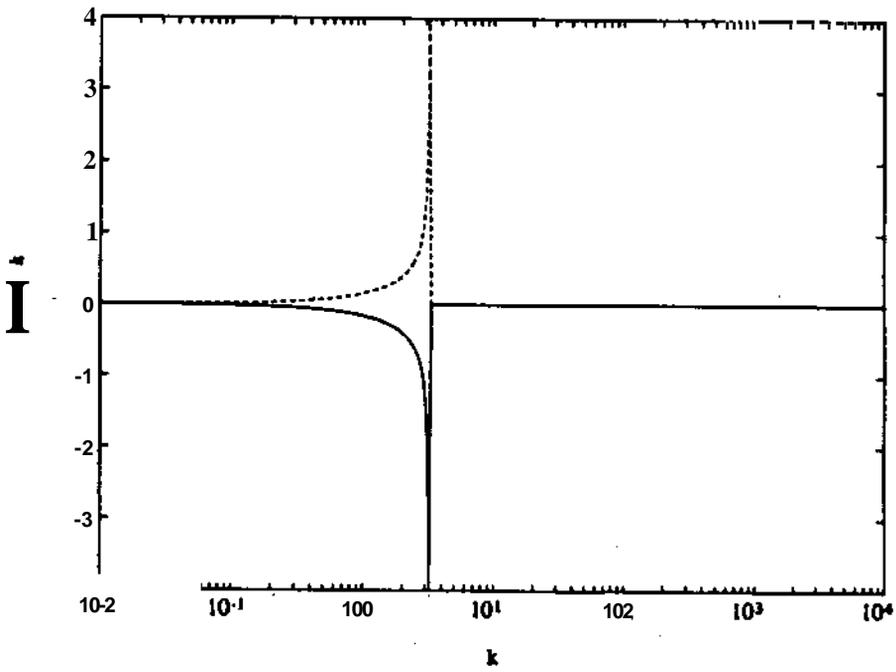


Figure 3b. Imaginary Component of S^* for Equation (14).