MANAGEMENT SCIENCE
Vol. 31, No. 12, December 1985
Printed in U.S.A.

NOTES

OPPOSITION OF INTEREST IN SUBJECTIVE BAYESIAN THEORY*

JOSEPH B. KADANE
Department of Statistics, Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213

A characterization is given of diametrically opposed interests between two players: either neither is a Bayesian, or both have a unique probability and utility function (up to the usual transformation) or both have many possible probabilities and utilities. In the second case, their utility functions must have representations that sum to zero, and they must have identical probability distributions on every uncertain event in the space. Implications of this result for negotiations and for game theory are discussed.
(BAYESIAN DECISION THEORY; GAME THEORY)

Recently Kadane and Larkey (1982) proposed that normatively optimal play in a game be viewed as a special case of subjective Bayesian decision theory. One consequence of viewing games in this way is that the distinction between constant-sum (or zero-sum) games and nonconstant-sum (or nonzero-sum) games does not matter. Here I return to the subject of the constant-sum (or zero-sum) relation between players, to explore what that means in the Bayesian context.

Without going beyond what may be observed, that is, preference on gambles, how may opposition of interest be defined? Start with a σ-field of events (Ω, a), and a set of rewards F. A gamble G over (Ω, a) and F specifies that if E_i occurs, the reward will be f_i, when $\{E_i\}$ are disjoint and exhaustive. Suppose that $\leqslant *^i$ is player i's preference relation over gambles. Then I define players 1 and 2 to have *opposed interests* on (Ω, a) and F when the following relation holds:

$$G \leqslant *^1 Q \quad \text{iff} \quad Q \leqslant *^2 G \tag{1}$$

for all gambles G and Q over (Ω, a) and F. Thus, players 1 and 2 have opposed interests just in those cases in which what is better for one is worse for the other. The question addressed here is what that may mean for their probabilities and utilities.

A player i's choices are *rationalized* by a probability P on (Ω, a) and a utility function U over F when

$$F \leqslant *^1 Q \quad \text{iff} \quad E_p U(G) \leqslant E_p U(Q) \tag{2}$$

where $E_p U(G) = \sum P(E_i) U(f_i)$.

When such a probability and utility exist, player i is said to be a Bayesian. There is a vast literature giving various assumptions about the preference relation $\leqslant *$ that entail the existence of a probability and utility satisfying (2), (for example, Savage 1954,

*All Notes are refereed.

Accepted by Robert L. Winkler; received April 30, 1984. This Note has been with the author 3 months for 1 revision.

0025-1909/85/3112/1586$01.25

DeGroot 1971). Let $\mathscr{S}^i$ be the set of probabilities P and utilities U rationalizing player i's choices.

THEOREM 1. *If players* 1 *and* 2 *have opposed interests, there is a* 1-1 *onto map between* $\mathscr{S}^1$ *and* $\mathscr{S}^2$ *given by* $U^2 = -U^1$ *and* $P^2 = P^1$.

PROOF. If one of $\mathscr{S}^1$ and $\mathscr{S}^2$ is empty, the other must be. If (P^1, U^1) rationalizes 1's choices, then $(P^1, -U^1)$ rationalizes 2's choices. If (P^1, U^1) and (P^{1*}, U^{1*}) both rationalize 1's choices and are mapped into (P^2, U^2), then $P^2 = P^1 = P^{1*}$ and $-U^2 = U^1 = U^{1*}$. Q.E.D.

COROLLARY 1. *If players* 1 *and* 2 *have opposed interests, then one and only one of the following conditions holds*

(a) *neither is a Bayesian.*

(b) *there is a unique probability distribution of* (Ω, a) *characterizing the beliefs of both players over* (Ω, a) *and utility functions* U^1 *and* U^2 *(unique up to the usual positive linear transformation) satisfying*

$$aU^1 + bU^2 = c \tag{3}$$

where $ab > 0$.

(c) *the probabilities and utilities of neither player is defined uniquely by preferences over* (Ω, a) *and* F, *but each pair* (P, U) *that rationalizes the choice of one player has a related pair* $(P, -U)$ *that rationalizes the choice of the other.*

The conditions of (b) have appeared together in the literature before, in DeGroot and Kadane (1980).

Consideration of opposed interests can also lead us to be curious about coincident interests. Players 1 and 2 have *coincident interests* on (Ω, a) and F when

$$G \leqslant^{*1} Q \quad \text{iff} \quad G \leqslant^{*2} Q \tag{4}$$

for all gambles G and Q over (Ω, a) and F. Analogously to Theorem 1, one can state

THEOREM 2. *If players* 1 *and* 2 *have coincident interests, there is a* 1-1 *onto map between* $\mathscr{S}^1$ *and* $\mathscr{S}^2$ *given by* $U^2 = U^1$ *and* $P^2 = P^1$.

The proof of Theorem 2 is obvious, and a corollary similar to Corollary 1 can be stated in which (3) would hold for $ab < 0$. Comparison of Theorems 1 and 2 shows something striking about opposition of interests compared to coincident interests: while the relation between utilities changes, the relation between probabilities does not. Hence players with opposed interests have identical beliefs about all the uncertain events in (Ω, a), but they evaluate the outcomes in F oppositely.

What is the relation of Theorem 1 to the Bayesian theory of constant-sum two person games? In Kadane and Larkey (1982), we argue that since I don't know what you will do, as a Bayesian I have some probability distribution over what you will do, and this implies optimal actions for me to take. When my probability coincides with your minimax strategy, any decision having positive probability under my minimax strategy is optimal for me, and so is any mixture of them, including my minimax strategy. It had been my hope that in studying opposition of interest I would be able to understand this relationship more deeply. Does Theorem 1 imply, as it appears to, that only if we agree about the probabilities of all events (including most particularly what you will do and what I will do in the game), does true opposition exist? I think not. For us to agree on the probability distribution of my action requires extraordinary gymnastics in the Bayesian foundations. One way of appreciating the difficulty is from the "Dutch Book" viewpoint (Kemeny 1955, Lehman 1955, Shimony 1955) according to which any person who decides which side of any bet to take either behaves as if they were a Bayesian, or can be made into a sure loser with a finite set of acceptable bets. But I can make anyone who bets with me about my own behavior a sure loser no matter which side they bet on. Thus I think the most comfortable stance to take is that since I decide what I shall do, I should not regard what I do as an uncertain event to me. Hence a player's σ-field a of uncertain events must, by my argument, exclude his own actions. Thus I doubt that opposition of interest has application to the Bayesian treatment of zero-sum two person games.

Aumann (1974) explored the possibility of using outside events on which players have differing probabilities to find mutually profitable agreements in situations in which $U^1 = -U^2$. The agreements he explores would require at least one player to tie his decision to the occurrence of the outside event on which the players have different probabilities. Thus the Aumann-style agreements set up situations in which the players legitimately have probabilities on their own behavior.

A second possibility opened by Theorem 1 is that it may afford some insight into a theory of bargaining. Raiffa (1982, Chapter 13) shows by example that differing probabilities may be exploited to make agreements possible. See also Sebenius (1984, p. 56, p. 144, and p. 120 ff.). As soon as events are discovered about which the parties in a negotiation have different probabilities, the possibility of mutually beneficial agreements is opened up, since their interests can no longer be opposed. Thus Raiffa (1982, esp. p. 183) suggests that linking an intractible issue (opposed interests) to other issues can lead to mutually acceptable agreements. For example, suppose in a labor negotiation that management wants low wages and labor wants high wages, and that conflict of interest certainly appears to hold. Labor may believe that the firm would make extraordinary profits under management's proposal, while management may believe that profits would not be high under that proposal. An agreement incorporating profit-sharing might be mutually acceptable, exploiting the differing beliefs of the players. In this larger space, conflict of interest no longer holds. Thus this theory may offer some insight into the search for mutually acceptable agreements.[1]

[1]This research is supported in part by the Office of Naval Research under Contract 014-82-K-0622. The author thanks Morris DeGroot and Teddy Seidenfeld for helpful discussions.

References

AUMANN, R. J., "Subjectivity and Correlation in Randomized Strategies," *J. Math. Economics*, 1 (1974), 67–96.

DEGROOT, M. H., *Optimal Statistical Decisions*, McGraw-Hill, New York, 1971.

——— AND J. B. KADANE, "Optimal Challenges for Selection," *Oper. Res.*, 28 (1980), 952–968.

KADANE, J. B. AND P. D. LARKEY, "Subjective Probability and the Theory of Games," *Management Sci.*, 28 (1982), 113–120.

KEMENY, J., "Fair Bets and Inductive Probabilities," *J. Symbolic Logic*, 20 (1955), 263–273.

LEHMAN, R. S., "On Confirmation and Rational Betting," *J. Symbolic Logic*, 20 (1955), 251–262.

RAIFFA, H., *The Art and Science of Negotiation*, Harvard University Press, Cambridge, Mass., 1982.

SAVAGE, L. J., *The Foundations of Statistics*, J. Wiley, New York, 1954.

SEBENIUS, J. K., *Negotiating the Law of the Sea*, Harvard University Press, Cambridge, Mass., 1984.

SHIMONY, A., "Coherence and the Axioms of Confirmation," *J. Symbolic Logic*, 20 (1955), 1–28.