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ON MIXED FINITE ELEMENT METHODS
FOR FIRST ORDER ELLIPTIC SYSTEMS

by

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ON MIXED FINITE ELEMENT METHODS FOR FIRST ORDER

ELLIPTIC SYSTEMS*

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ABSTRACT

A physically based duality theory for first order elliptic systems is shorn to be of central importance in connection with the Galerkin finite element solution of these systems. Using this theory in conjunction with a certain hypothesis concerning approximation spaces, optimal error estimates for Galerkin type approximations are demonstrated. An example of a grid which satisfies the hypothesis is given and numerical examples which illustrate the theory are provided.

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I. Introduction

Mixed finite element methods are finite element approximations based on stationary variational principles as contrasted with those based on extremal principles which yield strict maxima or minima. Remarkable progress has been made in the finite element theory for elliptic boundary value problems, especially for those methods which are based on extremal principles. However, there still exists a gap in the theory as it concerns mixed methods. In particular, error estimates presently available in the literature often predict rates of convergence well below those observed in computations. The purpose of this paper is to develop a sharp theory for mixed finite element methods in the context of approximate solutions to the Poisson equation.

The fact that a particular variational principle is stationary in nature has serious implications for finite element approximations. For example, it is well known that finite element approximations based on the Dirichlet Principle will be, in a suitable sense, unconditionally stable and their convergence depends only on the ability to approximate in the finite element spaces [7]. These are not true for methods based on stationary principles. For instance, for the Galerkin method based on the Kelvin Principle considered in this work, we shall find that to obtain stability and convergence certain conditions must be satisfied which restrict the type of grids that can be used. The theory of the present work contains both necessary and sufficient conditions for the stability and convergence of mixed finite element methods derived from the Kelvin Principle.

We begin by stating the boundary value problem to be considered and some equivalent variational formulations. Let \( \Omega \) be a bounded region in \( \mathbb{R}^n \) whose boundary \( \Gamma \) consists of two parts, \( \Gamma_D \) and \( \Gamma_N \). Given (*)

(*) The space \( H^r(\Omega) \) denotes the Sobolev space of order \( r \), \( \| \cdot \|_r \) denotes the norm on this space [1], [4].
we seek a real valued function \( \phi_0 \) satisfying

1. \( \Delta \phi_0 = f_0 \) in \( \Omega \)
2. \( \phi_0 = 0 \) on \( \Gamma_D \)
3. \( \nabla \phi_0 \cdot \nu = 0 \) on \( \Gamma_N \),

where \( \nu \) is the outer normal to \( \Gamma \). Alternatively, find \( \phi_0 \) and the \( \mathbb{R}^n \) valued function \( u_0 \) satisfying

4. \( \text{div}(u_0) = f_0 \) in \( \Omega \)
5. \( \nabla \phi_0 \cdot u_0 = 0 \) in \( \Omega \)
6. \( \phi_0 = 0 \) on \( \Gamma_D \)
7. \( u_0 \cdot \nu = 0 \) on \( \Gamma_N \).

The classical Dirichlet Principle uses the spaces

8. \( S = H^1(\Omega) \), \( S_0 = \{ \psi \in S : \psi = 0 \text{ on } \Gamma_D \} \),

and asserts that the solution \( \phi_0 \) of (1)-(3) minimizes

9. \( \int_\Omega \left\{ \frac{1}{2} \nabla \psi \cdot \nabla \psi + f_0 \psi \right\} \)

over \( \psi \in S_0 \). Observe that if

10. \( v_0 = \nabla S_0 \)

this is equivalent to minimizing
The Kelvin Principle is in some sense dual to the Dirichlet Principle with \( \text{div} \) being the dual of \( V \). In this setting we let

\[
(11) \quad \mathcal{E}^* - \mathcal{E}^B, \quad ^{v} - \{ v \in V \mid v \cdot n = 0 \text{ on } \partial \Omega \},
\]

and the Kelvin Principle asserts that \( u_0 \) minimizes

\[
\int_a \{ \frac{1}{2} v \cdot v + \epsilon_n \phi \}
\]

over \( C_{p,v} \times S_0 \times \mathcal{I}_0 \) subject to the constraint

\[
\mathcal{E}^* - \mathcal{E}^B, \quad ^{v} - \{ v \in V \mid v \cdot n = 0 \text{ on } \partial \Omega \}.
\]

The scalar \( \phi_0 \) enters into the Kelvin Principle as a Lagrange multiplier, i.e., an equivalent statement of the Kelvin Principle is the following.

Let

\[
(12) \quad S_0 - \text{div}(V_0),
\]

then find

\[
(\phi_0, u_0) \in S_0 \times V_0
\]

satisfying

\[
(13) \quad \int_\Omega \left[ -\frac{1}{2} \psi \cdot v + \psi \text{div}(v) + \phi \text{div}(u) \right] = \int_\Omega \phi_0 \psi
\]

for all \( \Omega \).
In the fluid dynamic context [6], the Dirichlet Principle asserts that among all irrotational fields the one that minimizes the kinetic energy is the incompressible field. Dually, the Kelvin Principle asserts that among all incompressible fields, the field that minimizes the kinetic energy is irrotational.

One uses the Dirichlet Principle in computations as follows. Let for example

\[ S^h \subset \mathcal{S} \]

denote the space of continuous piecewise linear functions on some triangulation of \( \Omega \) and let

\[ S^h_0 = \{ \psi^h \in S^h : \psi^h = 0 \text{ on } \Gamma_D \} \]

Compute the minimum of (9) as \( i > \) ranges over \( S^h_j \) instead of all of \( S_0 \). The point where the minimum is achieved and if \( u_h = v^h \) then it is well known that

\[ \| \phi_0 - \phi_h \|_0 \leq c h^2 \| \phi_0 \|_2 \]

(14)

\[ \| u_0 - u_h \|_0 \leq c h \| u_0 \|_1 \]

(15)

(see [1], [7]).

The Kelvin principle is in some sense a dual to the Dirichlet principle with the greatest stress being placed on the vector \( \mathbf{v} \); i.e., in this method the \( \mathbf{v} \) is represented in terms of piecewise linear functions and presumably errors of the form

\[ \| u_0 - u_h \|_0 \leq c h^2 \| u_0 \|_2 \]

(see [1], [7]).
are obtained.

More precisely, we compute \( \{h \cdot 2L_h\} \) letting

\[
J^c_l \quad \text{denote the finite dimensional space of } \mathbb{R}^n \text{ valued continuous piecewise linear functions, and letting}
\]

\[
J^c_l \quad \text{denote the finite dimensional space of } \mathbb{R}^n \text{ valued continuous piecewise linear functions, and letting}
\]

\[
j_j = \{v^h_0 \in V^h_0 : v^h_0 \neq u^h\} \text{ on } T_j
\]

and

\[
S_j = \text{div}(J/|j|)
\]

The pair

\[
\{\phi_h, v_h\} \in S^h_0 \times v^h_0
\]

is determined by requiring that (13) hold for all \( ty, v \in S^h_0 \times v^h_0 \) with (21) replacing \( \{\}^1 \).

Unfortunately, (16)–(17) are in general not true without further conditions on the subspace \( v^h_0 \). In subsequent sections we shall give necessary and sufficient conditions for results of the type (16)–(17) to be valid.

Previous work on this problem [2], [3], [5], [8] is based on the Babuska–Brezzi condition, i.e.,

\[
\sup_{v^h_0} \frac{1}{2} \left( L \text{div } v^h + h v^h_0 \cdot H_0 \right) - \frac{1}{2} \left( L \text{div } v^h + h v^h_0 \cdot H_0 \right)
\]
This type of condition leads to an error estimate of the form

\[(23) \quad \| \mathbf{S}_h \|_0 \leq C h \| u_0 \|_2 \]

\[(24) \quad \| \phi_0 - \phi_h \|_0 \leq C h \| u_0 \|_2 \]

for piecewise linear elements. This is clearly unsatisfactory since it implies there is no advantage in using the Kelvin Principle except, perhaps, for the fact that the Dirichlet boundary conditions are natural in this context.*

Our theory indicates that for a certain class of grids the optimal accuracy (16) is achieved. These grids satisfy the Grid Decomposition Property defined in the next section. The latter is necessary and sufficient for stability and optimal accuracy. Incidentally, there is a dual of this property for the Dirichlet principle, but it reduces to a requirement that the space \( S^h \) contains the constant function \( f \). This property is possessed by all known finite element spaces.

These results have been generalized [9] to include other physical situation described by equations related to the Navier-Stokes or Maxwell equations.

II. The Discrete Kelvin Principle

To formulate the discrete approximation we let

\[(1) \quad V = H^0, \quad W = \{ v \in V : v \cdot n = 0 \text{ on } T_i \}. \]

The next step is to let

\[(2) \quad j^C \subseteq \mathbb{F} \]

be a finite dimensional space and

\[(3) \quad \mathbb{F} = \{ v^h \in \mathbb{F} : v^h \cdot n = 0 \text{ on } T_j \}. \]

Then letting
the discrete Kelvin Principle requires us to compute

\[ s^h_0 = \text{div} \nu^h_0 \]

satisfying

\[ \{ \psi^h, \nu^h \} \in S^h_0 \times \nu^h_0 \]

for all

\[ \{ \psi^h, \nu^h \} \in S^h_0 \times \nu^h_0. \]

Once a basis for \( S^h_0 \times \nu^h_0 \) has been chosen, (6) reduces to a system of \( N \) algebraic equations, where \( H \) is the dimension of this space.

We shall assume that \( \psi^h, \nu^h \) satisfy the following property:

**Approximation property.** There exists \( \ell \geq 1 \) and a constant \( 0 < C < \infty \) (independent of \( h \)) such that for each \( z \in L^2(\Omega) \) the approximate function \( \tilde{\nu}^h \in \mathcal{R}(\psi) \) satisfies

\[ \| \tilde{\nu}^h - \nu \|_{1,2} \leq C h^\ell \| \nu \|_{1,2}; \]

and for each \( \psi \in S^h_0 \) the approximate \( \hat{\psi}^h \in S^h_0 \) satisfies

\[ \| \hat{\psi}^h - \psi \|_0 \leq C h^{k+1} \| \psi \|_{k+1}. \]

In addition the estimate (7)–(8) hold for any \( 0 < k \leq k^2 \).

This property is valid for spaces of piecewise polynomial functions of degree \( k-1 \). For example, \( k = 2 \) with linear elements. The error estimate (7) is standard (see [1], [7]). The space \( S^h_0 \) in this case is contained in
the space of all piecewise constant functions. It may be strictly contained in the latter but there are always enough functions in $S_0^h$ to achieve (8) for any $\psi \in S_0$. This is discussed in Section 5.

We are now prepared to introduce the Grid Decomposition Property. To motivate it let us recall that any $v \in V_0$ can be decomposed as

$$v = \nabla\xi + z \quad (9)$$

where

$$\text{div}(z) = 0 \quad (10)$$

and

$$\int_{\Omega} z \cdot \nabla\xi = 0 \quad (11)$$

Indeed, we construct $\xi$ by solving

$$\Delta\xi = \text{div}(v) \text{ in } \Omega \quad (12)$$

$$\xi = 0 \text{ on } \Gamma_D \quad (13)$$

$$\nabla\xi \cdot v = 0 \text{ on } \Gamma_N \quad (14)$$

and then determine $z$ by

$$z = v - \nabla\xi \quad (15)$$

Observe that if

$$v = \nabla\xi \quad (16)$$

then from the theory of partial differential equations [4]

$$\|v\|_0 \leq C \|\text{div}(v)\|_1 \quad (17)$$
The Grid Decomposition Property requires that this hold on \( \psi_0^h \). More precisely, we have the following.

**Definition 1.** \( \psi_0^h \) satisfies the GDP with constant \( 0 < C_G < \infty \)

if and only if for each

\[ \psi_h \in \psi_0 \]

there exist

\[ \psi_h, z_h \in \psi_0 \]

satisfying

\[ \psi_h = \psi_h + z_h \]

with

\[ \text{div}(z_h) = 0, \quad \int \psi_h \cdot z_h = 0, \quad \|\psi_h\|_0 \leq C_G \|\text{div} \psi_h\|_1 \cdot \]

Observe that GDP is related to the way \( \text{div} \) is represented in the discrete problem. Indeed, it states that if \( \text{div}(\psi_h) \) is small for any \( \psi_h \in \psi_0^h \), then the projection of \( \psi^h \) onto the orthogonal complement of

\[ N_h(\text{div}) = \{ z_h \in \psi_0^h : \text{div}(z_h) = 0 \text{ in } \Omega \} \]

is also small, i.e. \( z_h \) in (21) is truly the divergence free part of \( \psi_h \). In the next section we shall show that GDP is sufficient for optimal accuracy. Here we shall show that it is necessary and sufficient for stability.
Definition 2. The duCate, Kelvin problem is Stable with constant \(0 < c < \cdot\)

II and only if the \(\gamma\) elliptic holds, let

\[
\begin{align*}
\text{be given and let } v_h & \text{ minimize } \| v \|_h^2, \\
\text{satisfying} & \quad v_h \in V_0^b.
\end{align*}
\]

(24)

Then

\[
\begin{align*}
\text{Minimize } f_h & \in S_0^b, \\
\text{subject to } v_h - v, \\
\text{subject to } & \|v\|_h^2 = \|v\|_h^2.
\end{align*}
\]

(25)

(26)

Theorem 1. The duCate, Kelvin problem is Stable with constant \(c\).

Proof. Let \(\gamma \cap (\text{div})\) be the orthogonal complement of \(W_h(\text{div})\). Thus

First suppose GPP holds, i.e., any \(\in \mathcal{N} / j\) can be written

(28)

where

(29)

Moreover, let \(\gamma \cap (\text{div})\) satisfy

(30) \(\|H^\gamma H\| \leq \text{mnl. subject to } v, \in \mathcal{N} / j\) and \(\text{div}(v) = t\).
where \( f_c \in \mathcal{S}^\ell - \text{div}(1/j_j) \) is given. We want to show that

\[
\begin{align*}
\mathcal{K} & \quad \mathcal{II} \quad \mathcal{I}_0 \quad \mathcal{S} \quad \text{IM} \quad \mathcal{J} \\
& < 3 <
\end{align*}
\]

To do this we write \( v^+ \) as in (28)-(29). The data is that \( z^+ = \ell \) and so

\[
\| v_h \|_0 - \| v_h \|_0 \leq c_c \| \text{div} v_h \|_1 - c_c \| \ell_h \|_1.
\]

To see this observe that for any real number \( \delta \), \( v^+ + \delta z^+ \) is in \( V^+ \) and

\[
\text{div}(v^+ + 5z^+) = \text{div}(v^+) - i^h.
\]

Thus as \( v^+ \to \ell \) over \( \Omega \) we have

\[
\int_\Omega (v_h + \delta z_h) \cdot (v_h + \delta z_h) \geq \int_\Omega v_h \cdot v_h,
\]

i.e.,

\[
2\delta \int_\Omega z_h \cdot v_h \geq -\delta^2 \int_\Omega z_h \cdot z_h.
\]

Since \( \delta \) is arbitrary we necessarily have

\[
\int_\Omega z_h \cdot v_h = 0.
\]

But \( v^+ - w^+ + 2z^+ \) and \( w^+ \) is orthogonal to \( z^+ \). This means

\[
\int_\Omega \frac{^\wedge \cdot ^\wedge}{\Omega} - \frac{(^\wedge - w^\wedge) \cdot ^\wedge}{\Omega} = 0.
\]

Conversely, assume that the Kelvin problem is stable (with constant \( C_{\ell^j} \)) and let \( v^+_h \in V^+_0 \) be given. By (27), we can always write
(37) \[ \varphi_h = \varphi_n + \varphi_h. \]

where

(38) \[ \varphi_h \in \mathcal{H}_h(\text{div}), \quad \varphi_n \in \mathcal{H}_n(\text{div}). \]

He wants to select \( \varphi \) such that

\[ \| \varphi \|_0 \leq C \| \text{div} \varphi \|_1. \]

To do this we solve a Kelvin problem. More precisely, let

\[ f_h = \text{div}(\varphi_n), \]

and let \( \tilde{\varphi}_h \) minimize \( \| \tilde{\varphi} \|_0 \) subject to

\[ \tilde{\varphi}_h \in \mathcal{V}_h, \quad \text{div}(\tilde{\varphi}_h) = f_h. \]

By (31) \( \tilde{\varphi}_h \) is playing the role of \( \varphi_n \) in this inequality)

\[ \| \varphi_n \|_0 \leq C \| f_h \|_1. \]

and also

\[ \quad ** - \varphi_n - 2 \in \mathcal{W}_h(\text{div}). \]

Therefore the result is proved.

In one spatial dimension \( (n=1) \) all finite element spaces satisfy GDP, the proof being exactly the same as for the space \( J/Q \) i.e.,

(9)-(15). In two dimensions, however, this is no longer true. For example if linear elements in triangles are used, the GDP is valid for the criss-cross grid in Figure 1a but fails for the directional grids in Figure 1b and 1c. The GDP also fails for bilinear elements in the rectangles of Figure 1d. That the GDP is valid for the criss-cross grid is established in section 5.
III. Error Estimates

The major theorem of this paper is the following:

Theorem 2. Let GDP hold $u \tau h$ constant $C$, and the approximation property [(7)–(S), section 2] (i.e., that $\tau M U$ a constant $c$ depending only on $C$ and $C_u$). Hence that:

$$\| \psi_0 - \psi_h \|_0 \leq \| \phi_k \|_k + \| u \|_k$$

and

$$\| \psi_h - \psi_{h-1} \|_0 \leq \| \phi_k \|_k + \| u \|_k$$

The key identity that will be used repeatedly is

(1) \[ \int_\Omega \{ u_0 \psi^{\tau h} + \text{div}(v) - H \}, \psi \text{div}(u_0) + i' C u - v \text{div}(v^{\tau h}) + \psi \text{div}(u_0) \].

This is valid for all $\psi^{\tau h}, \psi \in \tilde{\Omega} \times J$ (since both sides are equal to $\int_\Omega \psi^{\tau h}$ by $f^{13}$ section 1] and [(6), section 2]).

Lemma 1. Fan alt $w^h \in \psi jj$

(2) \[ \| \text{div}(u_0 - u_h) \|_0 \leq \| \text{div}(u_0 - u_\psi) \|_0 \]

In particular,

(3) \[ \| \text{div}(u_0 - u_h) \|_0 \leq \| \text{div}(u_0 - u_{h_1}) \|_0 \]

where $u_{h_1}$ is the function In [(7), section 2].

Proof. Let $\psi_h = 0$ in (1). Then

(4) \[ \int_\Omega \text{div}(u_0 - u_h) \psi_h = 0 \]
for all \( \psi^h \in \mathcal{S}^* - \text{div}^* \). Let \( \psi^h = \text{div}\mathcal{C}\psi^h \). Then (4) gives (2).

**Lemma 2.**

(5) \[ ||\text{div}(u^\perp)\|_{-1} \leq C.h ||\text{div}(u_0-w_h)\|_0 \]

**Proof.** Solve

\[-A\xi + \xi - \text{div}\mathcal{C}\psi^h \text{ in } \Omega\]

\[ C = 0 \text{ on } T.\]

Then

(6) \[ ||\mathcal{C}||_{\mathcal{E}_x} \leq ||\text{div}^\perp(u^\perp)\|_{-1} \]

But

(7) \[ \mathcal{C}||J - \{\psi^h, \psi^h \not\in \mathcal{S}^2\} - \int_\Omega \xi \text{div}(u_0-w_h). \]

We note that if \( h \ll 0 \) in (1)

(8) \[ \int_\Omega \text{div}^\perp(u^\perp) - 0 \text{ for all } \psi^h \in \mathcal{S}^h_0. \]

Thus letting \( \psi^h = \nabla_h \)

(9) \[ ||\xi||^2_{L^1} = \int_\Omega (\xi - \nabla_h)^2 \text{div}(u_0-w_h) \]

\[ \leq ||\xi - \nabla_h||_0 \int_\Omega \text{div}(u_0-w_h) ||_0. \]

Using the approximation property \( \mathcal{F}(8) \), section 2] with \( k - 1 \) we can choose \( h \)
\[ \epsilon \text{ such that} \]

-14-
Thus (5) follows from (6), (9), and (10).

Observe that $\text{div}(u_0 - u_h)$ is optimal in $\| \cdot \|_1$, i.e.

$\| \text{div}(u_0 - u_h) \|_{-1} \leq C_A h \| \text{div}(u_0 - u_h) \|_0 \leq C_A h \| \text{div}(u_0 - \hat{u}_h) \|_0 \leq C_A h \| u_0 - \hat{u}_h \|_1 \leq C_A^2 h^k \| u_0 \|_k$. (Lemma 2)

So far GDP has not been used, however from this point on it will play a crucial role. In particular, write

$u_h - \hat{u}_h = \tilde{u}_h + z_h$

where

$\| \text{div}(z_h) \| = 0$, $\int \tilde{u}_h \cdot z_h = 0$, $\| \tilde{u}_h \|_0 \leq C_G \| \text{div}(u_h - \hat{u}_h) \|_{-1}$.

Note that for all $v \in V$

$\| \text{div} v \|_{-1} \leq \| v \|_0$.

Indeed

$\| \text{div} v \|_{-1} = \sup_{\psi \in H^1_0(\Omega)} \frac{\int (\text{div} v) \psi}{\| \psi \|_1} = \sup_{\psi \in H^1_0(\Omega)} \frac{-\int v \cdot \nabla \psi}{\| \psi \|_1}$

$\leq \sup_{\psi \in H^1_0(\Omega)} \frac{\| v \|_0 \| \nabla \psi \|_0}{\| \psi \|_1} \leq \| v \|_0$.

Thus

$\| \tilde{u}_h \|_0 \leq C_G \{ \| \text{div}(u_0 - \hat{u}_h) \|_{-1} + \| \text{div}(u_0 - u_h) \|_{-1} \}$
and so

\[ \| u_h \|_0 \leq C \left( \| u_0 - \hat{u}_h \|_0 + \| \text{div}(u_0 - u_h) \|_1 \right). \]

Thus it is sufficient to obtain a similar bound for \( z_h \). In particular, letting \( \psi^h = 0 \) and \( \nu^h = z_h \) in (1) we obtain

\[ \int_{\Omega} u_h \cdot z_h = \int_{\Omega} u_0 \cdot z_h \]

and so

\[ \int_{\Omega} z_h \cdot z_h = \int_{\Omega} (u_h - \hat{u}_h) \cdot z_h = \int_{\Omega} (u_0 - \hat{u}_h) \cdot z_h. \]

This gives

\[ \| z_h \|_0 \leq \| u_0 - \hat{u}_h \|_0, \]

then (11), (12), (14) and (17) give

\[ \| u_h - \hat{u}_h \|_0 \leq C h^k \| u_0 \| \]

and from the triangle inequality we obtain the first part of Theorem 2,

\[ \| u_h - u_0 \|_0 \leq C h^k \| u_0 \| \]

To estimate \( \phi_0 - \phi_h \) we let \( \psi^h = 0 \) in (1) to get

\[ \int_{\Omega} \{ \phi_h \text{div} \nu^h \} = \int_{\Omega} \{ \phi_0 \text{div} \nu^h + \nu^h : (u_0 - u_h) \}. \]

Let \( \hat{\phi}_h \in S^h_0 \). Then

\[ \int_{\Omega} \{ (\phi_h - \hat{\phi}_h) \text{div} \nu^h \} = \int_{\Omega} \{ (\phi_0 - \hat{\phi}_h) \text{div} \nu^h + \nu^h : (u_0 - u_h) \}. \]

Now let \( \hat{\phi}_h \) be the function in [(8), section 2] with \( \phi = \phi_0 \).

Since \( S^h_0 = \text{div}(\nu^h_0) \) there is a \( \nu_h \in \nu^h \) such that

\[ \phi_h - \hat{\phi}_h = \text{div}(\nu_h). \]
We now use GDP to write

\[ \mathbf{v}_h = \mathbf{v}_h + \mathbf{e}_h, \]

with

\[ \text{div} \mathbf{v} = 0, \quad \text{div} \mathbf{e} = 0, \quad || \mathbf{v}_h ||_0 \leq c || \phi - \hat{\phi}_h ||_1. \]

Letting \( \mathbf{v}_h = \mathbf{v}_h \) in (20) we obtain

\[ \int_{\Omega} |\phi - \hat{\phi}_h|^2 \leq \| \phi - \hat{\phi}_h \|_0 \| \phi - \hat{\phi}_h \|_0 + || \mathbf{v}_h ||_0 \| u_0 - u_h \|_0 \]

\[ \leq \| \phi - \hat{\phi}_h \|_0 \| \phi - \hat{\phi}_h \|_0 + c || \phi - \hat{\phi}_h ||_0 \| u_0 - u_h \|_0. \]

Thus

\[ \| \phi - \hat{\phi}_h \|_0 \leq \| \phi - \hat{\phi}_h \|_0 + c || u_0 - u_h \|_0. \]

The second part of Theorem 2 now follows from an application of the triangle inequality. Thus Theorem 2 is proved.

With linear elements on the criss-cross grid, Theorem 2 asserts that the \( L^2 \) error in \( (H^\infty)^n \) is of \( O(h^2) \). This sharpens the \( O(h) \) estimate found in [2] and [5]. The \( L^2 \) error in \( \langle Q^{-} \rangle \) is \( O(h) \), the same as predicted in [2] and [5]. However, if in (20) we choose \( \phi \) to be the best \( L^2 \) approximations in \( V \), then, since \( \text{div} \mathbf{e} \mathbf{v} = \mathbf{e} \mathbf{v} \), the first term on the right hand side of (20) vanishes. We are then led to the conclusion that

\[ N_h^* \hat{\mathbf{v}}_h \|_0 = O(h^2), \]

i.e. the mean value of \( \phi_h \) over a given triangle is actually approximated to \( O(h^2) \). This phenomena is illustrated in the numerical examples of section 4.
IV. **Numerical Results**

In this section we briefly report the results of computations based on the Kelvin principle. These results give evidence of the essential role played by the GDP. The examples of this section deal with the Poisson equation [(1), section 1]. An equivalent first order system is given by [(4)-(5), section 1].

We first consider results for the mixed data problem depicted in Figure 2a using the directional grid illustrated in Figure 1b. The particular problem considered has an exact solution given by

$$\text{sin}(irx/2)\text{cos}(iry)$$

Figure 3 displays the $L^2$ error of the approximate solution for $\psi$ and the components $u$ and $v$ of $\text{grad} \psi$. The figure indicates that the $L^2$ errors in $u$ and $v$ remains roughly constant and the $L^2$ error in $\psi$ grows linearly as the size of the grid is reduced.

We recall from section 2 that the GDP is necessary and sufficient for the stability of the Kelvin approximation. The results shown in Figure 3 indicate that for the directional grid the "constant" $C_0$ appearing in the definition of the GDP in fact grows like $h$, where $h$ is a measure of the grid size. As a result, all accuracy in the approximation to $ju$ is lost, and the approximation in $\psi$ actually becomes unbounded. These results, and those below concerning the criss-cross grid give evidence of the importance of the GDP.

The directional grid used to generate the results of Figure 3 does not satisfy the GDP. However, Lemma 1 of section 3 is independent of this property of the grid. In the context of the directional grid, that
lemma shows that the divergence of the error in the approximation to \( u \) should be \( O(h) \). This result is confirmed in Figure 3 where that divergence is graphed as a function of \( h \). As is evident from the figure, the divergence of the error in \( u \) is indeed \( O(h) \) even though the error in \( u_x \) itself is \( O(1) \).

We now consider results using the "criss-cross" grid illustrated in Figure 1a. Figure 4 displays the \( L^2 \) errors of the approximate solutions for \( u \) and \( v \). Results are given for the mixed data problem with exact solution given by (1) and for a Dirichlet problem (see Figure 2b) with exact solution

\[
4 - \sin (\pi x) \sin(\pi y)
\]

The mixed data and Dirichlet problems were approximated using an evenly spaced grid. In addition, computations for the mixed data problem were carried out using a variable grid whose spacing is determined by choosing an even spacing in a \((\xi, \eta)\) coordinate system, and then letting

\[
x = \xi^3 \quad \text{and} \quad y = \eta^3.
\]

This stretching has the effect of accumulating grid points near \( x = 0 \) and \( y = 0 \). For all cases, the computed rate of convergence using criss-cross grids, is of second order. The results shown in Figure 4, especially when compared with those of Figure 3 for the directional grid, are lucid evidence of the necessity of the GDP to the achievement of optimal orders of accuracy.

Also shown in Figure 4 are the values of \( \| h B^n \| \) for the problems described above, confirming the result (26) of section 3.
V. Proof that the Crisa-Cross Grid Satisfies the GDP

For simplicity consider the Dirichlet problem for the uniform grid shown in Figure 1a with \( V_0 \cdot I_T^h \) being the space of \( 1^2 \) - valued piecewise linear functions. No assumptions on \( Q \) will be required. To verify that this grid satisfies the GDP we must show that there is a positive number

\[ 0 < C_G < \infty \]

independent of \( h \) for which the following holds. Given any

\[ f_h \in S^h = \text{div}(U^h) \]

there is a \( y_h \) in \( \hat{I}_T^h \) for which

\[ \text{div}(y_h) = f_h \]

and

\[ \| y_h \|_0 \leq C_G \| f_h \|_{-1} \]

Since \( \hat{I}_T^h \) consists of piecewise linear functions on the grid in Figure 1a, observe that (2) implies that each \( f_h \) in \( S^h \) is a piecewise constant function. What is Interesting is that \( S^h \) is a \( \delta \)-Lcat subspace of the space \( \hat{S}^h \) of all piecewise constant functions on the criss-cross grid in Figure 1a. Indeed, the following gives a rule for determining when a function \( f_h \) in \( \hat{S}^h \) is actually in \( S^h \).
Lemma 3: Let \( f \) be in \( \mathcal{S}^h \). Then \( \ell \ U \) In \( S^k \) if and only if \( f \) is a function satisfying any finite number of conditions. (R. Use Figure 5)

where \( f \) is a function in \( \mathcal{S} \).

Proof. We must construct continuous piecewise linear functions \( u, v \) such that

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = f
\]

in each triangle. To do this we close the system with

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = g
\]

where the piecewise constant function \( g \) is to be determined.

Observe that (6) - (7) is hyperbolic, and we shall solve it by the method of characteristics. The characteristic coordinates are

\[
X - x - 7 = n - x + y
\]

and letting

\[
2U - (u - v), \quad 2V = (u + v)
\]
we obtain

\[
\begin{align*}
\frac{\partial u}{\partial \xi} &= f - g, \\
\frac{\partial v}{\partial \eta} &= f + g.
\end{align*}
\]

Let the arbitrary rectangle \( R \) in Figure 5 be given. We first construct \( U, V, g \) in \( R \). Following this we show that they can be globally extended such that \( \{ u, v \} \) defined by (9) is in \( \mathcal{V}^h \), i.e., it is continuous in \( Q \) as well as linear in each triangle.

Since \( f \) and \( g \) are constants in each triangle \( T_j \) \((j = 1, \ldots, 4)\), then any function \( U \) satisfying the first equation in (10) will be continuous in \( R \) if and only if

\[
\begin{align*}
(11) \quad f_1 - g_1 &= f_4 - g_4, \\
        f_3 - g_3 &= f_2 - g_2,
\end{align*}
\]

where \( f_j, g_j \) are the values of \( f, g \) in the triangle \( T_j \). Similarly, continuity of \( V \) requires

\[
\begin{align*}
(12) \quad f_3 + g_3 &= f_4 + g_4, \\
        f_1 + g_1 &= f_2 + g_2.
\end{align*}
\]

It follows immediately from (11) - (12) that (5) is a necessary condition for (11) - (12) to have a solution \( g_4 \); moreover, it is also sufficient. Indeed, let

\[
(13) \quad g_4 = \text{arbitrary},
\]

then

\[
(14) \quad g_1 - g_4 = f_1 - f_4, \quad g_2 - g_4 = f_1 - f_3, \quad g_3 - g_4 = f_4 - f_3,
\]

is a solution provided (5) holds.
To define $U, V$ globally we first of all define the piecewise constant function $g$ in each rectangle so that (11) - (12) holds. To construct $U$ and $V$ we simply integrate (10) along the characteristics working from rectangle to rectangle. In particular, consider Figure 6 where for simplicity the region $\Omega$ is shown as a rectangle. Along the left and top sides $U$ can be taken as an arbitrary linear function. To determine its values in a given rectangle $R$ we simply integrate (10) from points $A$ to $B$ as shown in Figure 6. The conditions (11) - (12) insure $U$ is a continuous function in $R$, and using the value of $U$ so obtained at $B$ to start the integration in the next box, interelement continuity is assured.

Since $f$ and $g$ are constants in each triangle, $U$ and $V$ will be linear functions of $\xi, \eta$ in each triangle. Hence $u$ and $v$ defined by (9) will be continuous piecewise linear functions (i.e., $\{u, v\}$ in $V^h$).

Note that since the dimension $\dim(S^h)$ of $S^h$ is equal to the number $m$ of triangles in the grid, it follows from (5) that

$$\dim S^h = \frac{3m}{4}.$$ 

Moreover, a locally defined basis can be constructed as follows. For each rectangle $R$ (see Figure 5) we associate three functions $\psi_1, \psi_2, \psi_3$ which vanish outside $R$. The piecewise constant function $\psi_1$ is uniquely determined in $R$ by the requirement that it is identically 1 in $T_i \cup T_{i+1}$ and zero in the other two triangles in $R$. As $R$ varies over all rectangles this process defines $3m/4$ independent functions in $S^h$ and hence the set of such functions is a basis for $S^h$. Interestingly, this shows that $S^h$ is the linear hull of the union of the piecewise constant spaces associated with the directional grids shown in Figures 1b and 1c. Therefore the approximation property [(8), section 2] is certainly valid for the above choice of $S^h$. 

-23-
We now return to the proof of (4), which is contained in the following result.

**Lemma 4.** Therefore - a rumbe/L 0 < C < • independent of h Audi that $f_0 k$ each $f_n$ in $S^h$ we have.

\[
(15) \quad f_h = \text{div}(v_h),
\]

subject to $v^h$ satisfies

\[
(16) \quad \|v_h\|_0 \leq 2c \|f_h\|_{-1}.
\]

**Proof.** To simplify notation we drop all subscripts involving $h$ since all functions that will be encountered will be in $T$ or $\tilde{T}$. As in Lemma 3 we work in the rotated coordinates $(c,r)$ defined by (8). In addition we order the vertices in a sequential manner starting at the bottom of the region and moving left to right as in Figure 7. Observe that, the center of each (rotated) rectangle has an index $(a,0)$, where $a+0$ is an integer, while $Y+6+J$ is integral for the indices $(Y>5)$ of the corner points. We denote the rectangle whose centroid has Index $(a,6)$ by $R^{a,6}$ and let $T_k^{a-b} \ (k=1,2,3,4)$ denote the four enclosed triangles.

Given $f$ in $S^h$ we must construct continuous piecewise linear functions $u$ and $v$ such that

\[
(17) \quad f = \text{div} \left( \begin{array}{c} u \\ v \end{array} \right) = \frac{3u}{\partial c} + \frac{3v}{\partial \eta}.
\]

We let $u^{a,b}$, $v^{a,b}$ denote the values of $u,v$ at the vertices, and let $f^{a,b}$ denote the value of $f$ in the triangle $T_k^{a,b}$. Then a direct calculation gives

\[
(18) \quad fg^{a,b} = D^u + D^v,
\]
where

\[
D_1^+ u = \frac{u^{\alpha+\frac{1}{2}, \beta} - u^{\alpha, \beta}}{h}, \quad D_2^+ v = \frac{v^{\alpha, \beta+\frac{1}{2}} - v^{\alpha, \beta}}{h}.
\]

Similarly,

\[
f_1^{\alpha, \beta} = D_1^+ u + D_2^+ v,
\]

\[
f_3^{\alpha, \beta} = D_1^+ u + D_2^+ v,
\]

\[
f_4^{\alpha, \beta} = D_1^- u + D_2^- v,
\]

where the difference operators \( D_1^-, D_2^- \) are defined by

\[
D_1^- u = \frac{u^{\alpha, \beta} - u^{\alpha-\frac{1}{2}, \beta}}{h}, \quad D_2^- v = \frac{v^{\alpha, \beta} - v^{\alpha, \beta-\frac{1}{2}}}{h}.
\]

Observe that (18) and (20) can be combined into

\[
\sum_{i=1}^{3} \hat{\gamma} = \sum_{i=1}^{3} 2 \gamma_{i}^{\alpha, \beta} + 2 \gamma_{i}^{\alpha', \beta'} = \frac{1}{2} (f_2^{\alpha, \beta} + f_4^{\alpha, \beta}),
\]

a relation which reconfirms the necessity of the condition (5). We rewrite (22) as

\[
\text{div}_{\text{h}} (u) = f_1^{\alpha, \beta},
\]

where \( \text{div}_{\text{h}} (u) \) denotes the difference operator on the left hand side and

\[
f_1^{\alpha, \beta}
\]

denotes the average of \( f \) on the right.

Observe that (22) (or (23)) involves values of \( u \) and \( v \) only at the corner points of the rectangles (i.e., vertices \((Y, \delta)\) where \( Y \cdot 5' i \) is integral). Once these have been determined the values at the centroids of
the rectangles (i.e., vertices \((a, b)\) where \(a + \ell\) is integral) are then given by

\[
\begin{align*}
\alpha^a, b = \frac{1}{2}(u^{a+\frac{1}{2}, b} + u^{a-\frac{1}{2}, b}) + \frac{h}{2}(e_1^{a, b} - e_2^{a, b}), \\
v^{a, b} = \frac{1}{2}(v^{a, b+\frac{1}{2}} + v^{a, b-\frac{1}{2}}) + \frac{h}{2}(e_3^{a, b} - e_2^{a, b}).
\end{align*}
\]

That is, (23) - (25) are three independent relations among the four dependent equations (18), (20).

To solve (23) we introduce a discrete potential \(\phi\) satisfying

\[
\frac{\partial \phi^{a, b}}{\partial x} = \frac{\phi^{a+1, b} - \phi^{a, b}}{2h}, \quad \frac{\partial \phi^{a, b}}{\partial y} = \frac{\phi^{a, b+1} - \phi^{a, b}}{2h}.
\]

Then (23) is equivalent to

\[
(\phi_{2h}^2)(E^{a, b} + E^{a-1, b} + E^{a, b+1} - 2E^{a, b}) - E^{a, b} = 0.
\]

Observe that this equation has a "red-black" decoupling. Indeed, only values of \(\phi\) at the centroids of rectangles (i.e., vertices \((a, b)\) where \(a + 3\) is integral) are involved. Moreover, there are two types of such points. The first are "red" rectangles \(R^{a, b}\) where \(a\) and \(b\) are both integers \((a, b)\). The second are "black" rectangles where \(a\) \(\ell\) and \(b\) \(\ell\).

Since all boundary conditions are natural we can extend the grid to cover \(\ell\) and let \(\phi = 0\) outside \(\ell\). Then (27) becomes a standard five point star on the red rectangular grid, and a standard five point star on the black rectangular grid. Moreover, defining \(u, v\) by (26) we get the standard estimate...
(28) \[ h^2 \sum_{\alpha, \beta} \left\{ |u^{\alpha+\frac{1}{2}, \beta}|^2 + |v^{\alpha, \beta+\frac{1}{2}}|^2 \right\} \leq C \| f \|_{-1}^2 , \]

for some absolute constant \( 0 < C < \infty \). In addition, defining \( u, v \) at the centroids of rectangles by (24) - (25), we get

(29) \[ h^2 \sum_{\alpha, \beta} \left( \int_{\alpha}^{\alpha+1} \int_{\beta}^{\beta+1} \right) \]

where the sum is over all vertices \((a, 0)\). Letting \( u, v \) be the continuous piecewise linear functions whose values at the vertex \((a, \delta)\) is \( u_{a+\frac{1}{2}}, v_{\beta+\frac{1}{2}} \), we get

(30) \[ \| u \|_0^2 + \| v \|_0^2 \leq c (\| f \|_{-1}^2 + a^2, \| f \|_0^2 ) \]

finally for the uniform grid being considered we have the inverse inequality—for function \( f \in S^h \) [3]

(31) \[ \| f \|_0 \leq (C/h) \| f \|_{-1} ; \]

hence (15) - (16) hold with

\[ \nu_h \rightarrow 0 \]
References


Figure 1: Grids.

a) Criss-cross triangles
b-c) Directional triangles
d) Bilinear quadrilaterals
Figure 2. Boundary value specifications used in numerical examples, (a) Mixed data; (b) Dirichlet data.
Figure 3.- $L_2$ error in the Kelvin approximation to $\xi$
$u = 8^\alpha/8x$, $v = 3^\beta/9\beta$, and $\text{div}(u,v) = 8u/8x + 3v/8y$
using the directional grid for the mixed data problem.
Figure 4.- $L_2$ error in the Kelvin approximation to $u = \partial \phi / \partial x$ and $v = \partial \phi / \partial y$ and the $L_2$ norm of the difference in the Kelvin approximation to $\phi$ and the best $L_2$ approximation to $\phi$, using criss-cross grid. (a,d,f) displays $u$; (b,d,h) displays $v$; (c,e,g) displays $\phi$. (a,b,c) for the mixed data problem using a variable grid; (d,e) for the Dirichlet data problem using a regular grid; (f,g,h) for the mixed data problem on a regular grid.
Figure 5.- Generic rectangle R.
Figure 6.- Characteristic directions for U and V.
Figure 7. Ordering of vertices and triangles for Lemma 4.