Design of Linear Quadratic Regulators: The Low Error Weighting Case

S. Wang, T. Kurfess, M. Nagurka .

EDRC 24-72-91

Design of
Linear Quadratic Regulators:
The Low Error Weighting Case

by

Ssu-Kuei Wang
Thomas R. Kurfess
Mark L. Nagurka

October 1991

Department of Mechanical Engineering
Carnegie Mellon University
Pittsburgh, PA 15213

Abstract

The branches of optimal root loci that approach infinity as the error weighting is decreased can be characterized by a combination of several Butterwonh patterns existing on **separate Riemann** sheets. Algorithmic approaches have been reported to find the order of **these Butterwonh patterns.** This report presents a geometric technique, involving **eigenvalue polar** plots, that provides direct realization of the directions and radii of the asymptotic eigenvalue patterns. A graphically-based systematic procedure is proposed and employed in **a** sample problem for analyzing Butterworth patterns.

This research was supported by the Engineering Design Research Center at Carnegie Mellon University, a National Science Foundation Engineering ResearchCXSwHSprj- ';> ^h.

1. Introduction

The optimal root loci are used to find the closed-loop eigenvalues of the linear **quadratic regulator** as the input weight of the performance index is varied. For multivariable regulators, it has been shown that the asymptotic poles which approach optimal infinite zeros (OIZ) can be grouped into several Butterworth patterns of different orders and with different radii [1]. An iterative algorithm to determine the **OIZ** angle and radius for all orders has been proposed by Kouvaritakis [2] and modified by Keenhi and Fannin [3]. The present report outlines a graphical procedure for obtaining the information of the **OIZ** from a set of eigenvalue polar plots. To illustrate the method, an example problem is solved.

2. Linear Quadratic Regulator Problem

Consider the linear time-invariant state space system

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx \tag{2}$$

where x is an nxl vector and **u** and y are mxl vectors. Assume that (A,B) is stabilizable and (C,A) is detectable. The objective is to find an optimal control which minimizes the cost function

$$J = \int_0^\infty (y^T Q y + \rho u^T R u)dt \tag{3}$$

where Q and R are weighting matrices and p is a scalar. As p varies from ©o to 0 (corresponding to "expensive" to "cheap" control, respectively), the closed-loop poles of the system trace out the optimal root locus consisting of m Butterwonh patterns on m Riemann sheets [4].

3. Methodology

From [2] the asymptotic approach to OIZ of order 2n can be expressed as

$$s^{2n} = \lambda_i\left(\frac{1}{\rho}\right) \tag{4}$$

or

-- / 1 \ 1/2n :n

S = K i ^ | CJ-i (5)

where ***X{*** **is a complex** number with magnitude K* and angle 6i. From equation (5), the **radius of the OIZ of order 2n** is

$$\omega_i = K_i \left(\frac{1}{\rho}\right)^{1/2n} \tag{6}$$

The optimal root locus may be viewed via an alternate graphical representation that exposes the relationship between the control weighting scalar, p, and the eigenvalue locations in polar coordinates. The proposed plots, called eigenvalue polar plots, portray the magnitude and angle of each closed-loop system eigenvalue in the complex plane as a function of p. The visualization is based on the adjustment of p in the same fashion **employed** in constructing the optimal root locus plot.

Eigenvalue polar plots recast, and in so doing enrich, the information presented in the optimal root locus plot, and offer advantages for control system analysis and design. **For** example, by exposing the correspondence of weighting values to specific eigenvalue locations, the plots are a useful pole-placement tool for achieving closed-loop designs meeting stability and performance specifications.

In the eigenvalue magnitude plot, equation (6) implies a straight line of slope l/2n **when 1/p** approaches infinity. Because an n^{th} order Butterworth pattern corresponds to a In^{111} order OIZ, according to equation (6), the order of the Butterworth pattern can be determined from the slope of the eigenvalue magnitude plot. The corresponding direction 0j can be found from the eigenvalue angle plot.

To determine the radius of the OIZ, the constant $K_{(}$ in equation (6) should be determined first. Choosing a point (1/po, Isol) in the eigenvalue magnitude plot with po **large,** the constant $K_{\dot{x}}$ can be expressed in terms of po and Isol. Thus, the approach to OIZ **of order** 2n **can be** obtained as

$$s_i \cong \left[|s_0| \frac{\rho_0^{J/2n}}{\rho^{1/2n}} \right] e^{j\theta_i} \tag{7}$$

Equation (7) may be used to approximate the closed-loop optimal eigenvalues for low values of p, provided that certain criteria are met. The following steps may be used to graphically compute a valid asymptotic approximation to the closed-loop eigenvalue:

1. Inspect the eigenvalue magnitude plot to determine n, the order of the Butterworth pattern, from theslope=l/2n.

2. For the corresponding eigenvalue, determine the angle, θ_i, from the eigenvalue angle plot.

3. **Choose** a test point $(1/\rho_0, |s_0|)$ from the eigenvalue magnitude plot such **that** the corresponding curve segment exhibits straight line asymptotic behavior.

4. Choose a value of ρ and verify that it is in the range of asymptotic behavior on the eigenvalue magnitude plot. (If it is not, the approximation given by equation (7) may not be valid.)

5. Substitute the values of ρ_0, $|s_0|$, θ_i and ρ into equation (7) to approximate the eigenvalue.

An example problem demonstrates the procedure.

4. Example

This example, adapted from [1], considers the longitudinal motion of an aircraft. The system is described by equations (1) and (2) with

$$A = \begin{bmatrix} -0.158 & 0.02633 & -9.81 & 0 \\ -0.1571 & -1.03 & 0 & 120.5 \\ 0 & 0 & 0 & 1 \\ 0.0005274 & -0.01652 & 0 & -1.466 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0006056 & 0 \\ 0 & -9.496 \\ 0 & 0 \\ 0 & -5.565 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We wish to control the system such that a criterion of the form of equation (3) is minimized where

$$Q = \begin{bmatrix} 0.02 & 0 \\ 0 & 50 \end{bmatrix} \qquad R = \begin{bmatrix} 0.0004 & 0 \\ 0 & 2500 \end{bmatrix}$$

Applying the proposed technique, first order and second order Butterworth patterns were found from the eigenvalue magnitude plot of Figure 1 by examining the slope of 1/2 and 1/4, respectively. The corresponding angles were obtained from the eigenvalue angle plot of Figure 2. From Figure 1, test points were chosen as $(1/\rho_0, |s_0|) = (10^7, 13)$ for the first order pattern and $(10^7, 50)$ for the second order pattern, and the asymptotic behavior was determined from equation (7) as follows:

First order Butterworth pattern: $s = \left[\dfrac{0.0041}{\rho^{1/2}}\right] e^{j(-180^\circ)}$

$$\textbf{Second order} \text{ Butterworth pattern: } s = \left[\frac{k\&\&2.}{p}\right] e^{j(\pm 135^\circ)}$$

These **are** in **close agreement** with the results determined in [1]:

$$\text{First order Butterworth pattern: } s = \left[\frac{0.004283}{p^{1/2}}\right] e^{j(-180^\circ)}$$

$$\text{Second order Butterworth pattern: } s = \left[\frac{0.8871}{p^{1/4}}\right] e^{j(\pm 135^\circ)}$$

It should be emphasized that the data points used to determine the eigenvalue locations in the eigenvalue polar plots were read directly from the figures. By using the actual numerical values employed in plotting the graph, the relations match (to the accuracy shown) those calculated from [1].

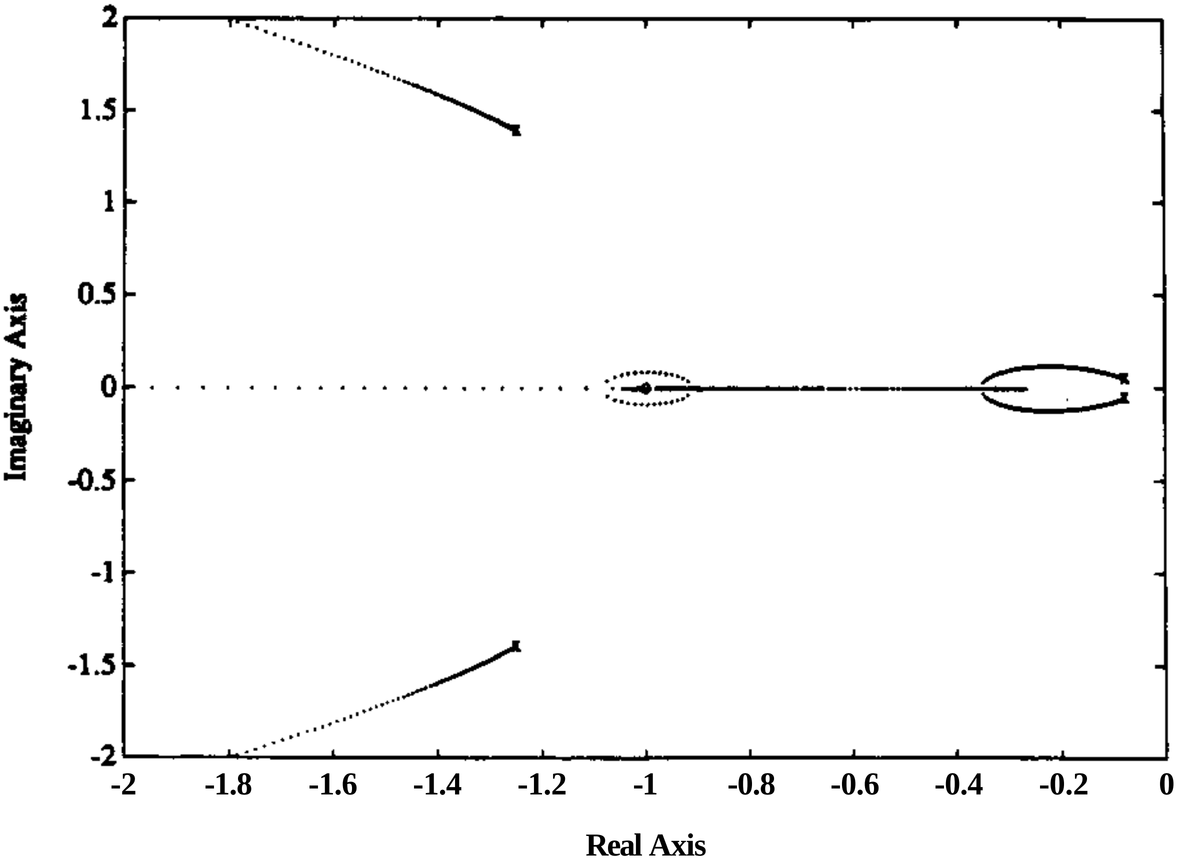

Figure 1. Optimal Root Locus Plot (p Implicit).

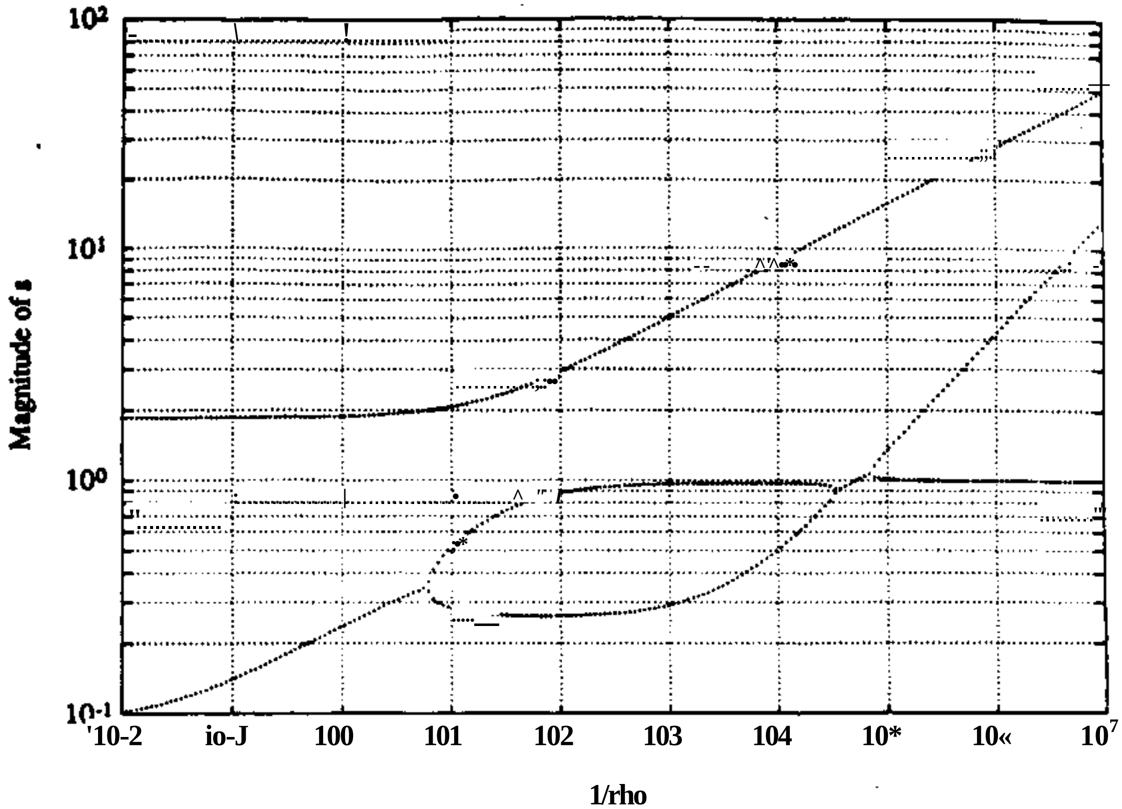

Figure 2. Eigenvalue Magnitude Plot (p Explicit).

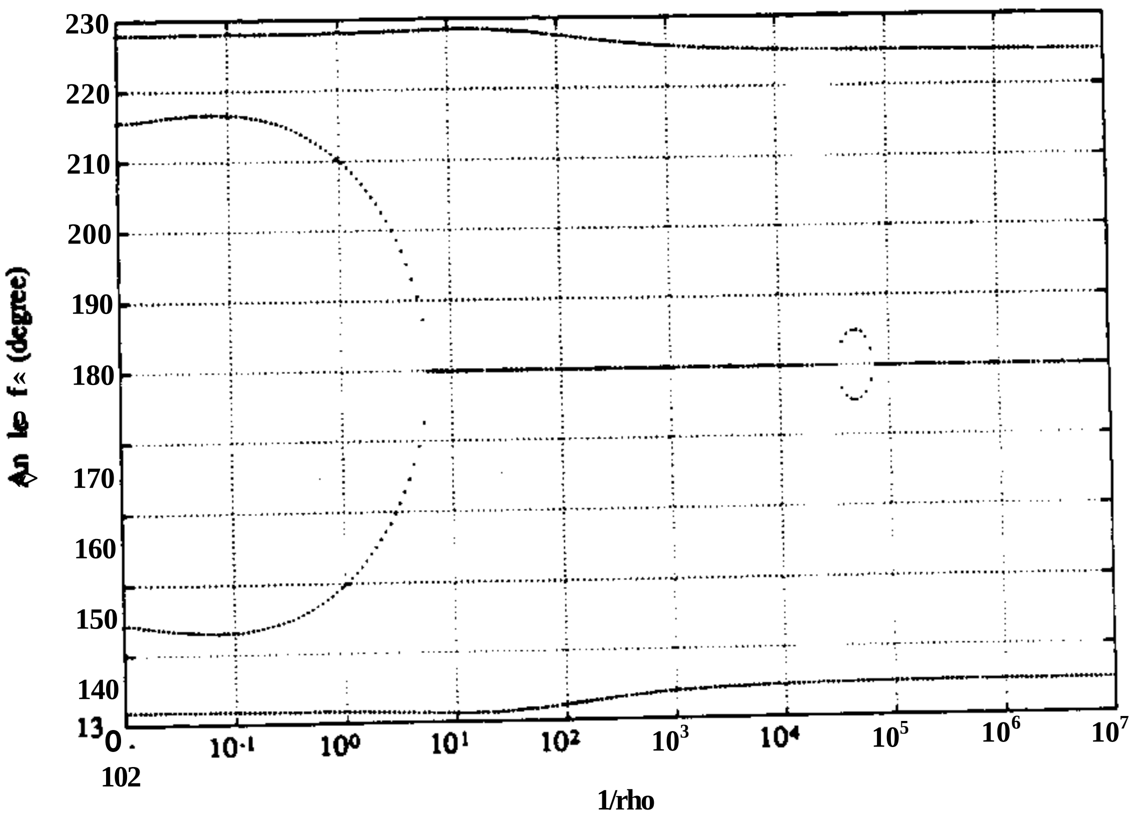

Figure 3. Eigenvalue Angle Plot (p Explicit).

5. Conclusions

This report has presented a graphical technique for approximating the behavior of **closed-loop eigenvalues in the** linear quadratic optimal control problem with low control **weighting. The** proposed geometric tools are eigenvalue polar plots that are built from the same data used in creating the optimal root locus plot. Hence, if the optimal root locus is generated, it is a minor effort to draw the eigenvalue polar plots and geometrically determine the closed-loop eigenvalues at low values of control weighting. In an example problem, the geometric analysis yields exceptionally accurate results that have been verified by comparison to documented techniques.

6. References

[1] H. Kwakernaak, "Asymptotic Root Loci of Multivariable Linear Optimal Regulators," ***IEEE Transactions of Automatic Controls,*** **vol. AC-21, pp. 378-382, 1976.**

[2] B. Kouvaritakis, "On the Asymptotic Behaviour of Optimal Root Loci," *International Journal of Control,* vol. 33, No.6, 1165-1170, 1981.

[3] S.S. Keerthi and D.R. Fannin, "Comments on and Modifications to 'On the Asymptotic Behaviour of Optimal Root Loci/' *International Journal of Control,* vol.37, No. 1, pp. 215-220, 1983

[4] B. Kouvaritakis, "The Optimal Root Loci of Linear Multivariable Systems," ***International Journal of Control,*** **vol. 28, No.l, 33-62, 1978.**