Optimal Peremptory Challenges in Trials by Juries: A Bilateral Sequential Process

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This paper explores optimal strategies for the use of peremptory challenges in jury trials where the prospective jurors are examined and then either challenged or seated, one by one. We assume that the lawyers for each side do not necessarily agree about the probability that each prospective juror will vote for conviction, but that the assessment of each side is available to the other. The strategies we develop are optimal noncooperative sequential strategies in the sense that each side maximizes its expected utility at each stage under the assumption that both sides will continue to use these optimal strategies in all future decisions. Under certain regularity conditions we show that it is optimal to be the first side to decide whether to challenge any prospective juror. Necessary and sufficient conditions are given for an optimal strategy to be reversible, which means that it does not matter to either side whether it decides first or second. Specifically, the optimal strategy is reversible if the two sides always agree about the probability that each prospective juror will vote for conviction. We give an algorithm based on backward induction for finding the optimal strategies and discuss simple examples.

The application of relatively sophisticated methods such as polls and market research surveys to the choice of which potential jurors to challenge peremptorily has raised fears for the future of the jury system as we now know it. Some of the cases in which these methods have been used include the Harrisburg Seven trial (Schulman et al. [10]); the Camden, New Jersey, draft board raid trial; the Mitchell-Stans conspiracy trial (Arnold [1], Zeisel [12], and Zeisel and Diamond [13]); the Gainesville, Florida, veterans trial; the Wounded Knee, South Dakota, trials of militant Indians and the Cedar Rapids murder trial arising from the Wounded Knee disturbance; the Buffalo Creek, West Virginia, dam disaster civil damage suit; the Ellsburg-Russo trial; the Joan Little murder trial; and the Attica trials (Shapley [11]). All of these trials involved highly publicized cases of defendants who had taken political positions likely to be very popular with some and very unpopular with others. Furthermore, the nature of the evidence in at least some of these trials was such as to confirm the prejudices of the jurors.
To date these methods have been used more extensively by the defense than by the prosecution (Kairys [9] and Ginger [8]). It can be argued that this use is close to the intent of the jury system, to protect a defendant unpopular with his government by having a group more politically diverse than the government decide his innocence or guilt.

The fear for the jury system arises from the possibility that, now that the defense has blazed the trail, an overzealous prosecution, with the full financial resources of the government, may follow. If this occurs, one might foresee "hanging juries" carefully chosen by sociological methods to have the most negative view of the defendant, and the defense, except in rare instances such as those discussed above, unable to match the resources of the government. "District Attorneys or U.S. Attorneys cannot be expected to stand by doing nothing while defendants in the most serious cases buy themselves a significant edge in trial after trial. The champions of the technique will have to realize that the days when it could be reserved for their favorite defendants will soon be over" (Etzioni [7]). Conceivably, this response by the government could cause a threat to our civil liberties.

To examine whether this possible threat is to be taken seriously, one should first ask what the defense and prosecution would do with information from relevant polls and surveys if they had it. In this paper we present a simplified model of the jury selection process and explore some of its implications. One of our difficulties in undertaking this work is that, while the law of most states is clear about the number of challenges allowed to the defense and prosecution in varying circumstances, the procedure for exercising these challenges is typically left to the trial judge. Usually the judge first examines prospective jurors to be sure that they are qualified and asks questions that might result in dismissal for cause, questions that vary depending on the nature of the trial. In our model each side then has the opportunity to peremptorily challenge the next prospective juror, and, if neither side challenges, the juror is then sworn in. The question of which side is to challenge first is left arbitrary in our mathematics, although in our model it cannot depend on previous uses of the peremptory challenge by either side. Furthermore, we assume that the prosecution and the defense each have an opinion about the likelihood that the prospective juror under consideration would vote for conviction, and that these opinions are known to each other.

This structure leads to a bilateral sequential process, in which decisions are made by each side one by one, without a simultaneous decision by the other side. Bilateral sequential processes may be a better model for many social phenomena, such as arms races and duopoly (Cyert and DeGroot [4, 5]), than traditional game theory that requires simultaneous moves by the players.

Both the information available to each side and the particular se-
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quence we have chosen to study limit the applicability of this paper, and both assumptions need to be relaxed in further work. Nevertheless, the particular structure we have chosen, although somewhat oversimplified, does represent a starting place for examining how effective these new methods are likely to be.

After this paper was submitted for publication, we learned of the earlier related work of Brams and Davis [2, 3]. The relationship between their work and ours will be discussed at the end of Section 1 and immediately after the corollary to Theorem 2.

1. STATEMENT OF THE PROBLEM AND MAIN RESULTS

Prosecution and defense lawyers are about to select a jury of $J$ people. Each prospective juror is interviewed, and each lawyer must then decide whether to accept or challenge (i.e., reject) that person before interviewing the next prospective juror; this decision cannot later be changed. The prosecution is allowed at most $A$ challenges, while the defense has at most $B$ challenges. After questioning each prospective juror $i$, the two sides have possibly different opinions about the probability that he will vote to convict the defendant, giving rise to a vector $(p_{1i}, p_{2i})$ of the opinions of the prosecution and the defense, respectively. These probabilities will be based on all of the information available to the legal teams on each side, including surveys of community attitudes, investigation of the pool of prospective jurors, or experience with similar trials analyzed by age, occupation, sex, race, etc. Thus, as each prospective juror is questioned, the legal teams on each side will identify a profile of the person comprising various demographic and psychological characteristics. From this profile, each side will then use the information just described to assign the probabilities $p_{1i}$ and $p_{2i}$ to that person. For example, if a poll of the community carried out by the defense revealed that older women in the community tended to be more sympathetic than others to the defendant, the defense lawyers would assign a relatively low value of $p_{2i}$ to an older woman.

We shall assume that for each possible profile that might be identified for a given prospective juror, each side knows the probability that the other side will assign to that person as well as the probability that its own side assigns. This would be the case, for example, if the lawyers for each side were familiar with the jury selection tactics of the other side. Furthermore, we shall assume that both sides are sufficiently familiar with the pool of prospective jurors that they know the distribution of profiles in that pool. Hence both sides will know the joint distribution of the bivariate random variable $(P_1, P_2)$ in the pool of prospective jurors after dismissals for cause. This known joint c.d.f. will be noted $F(p_1, p_2)$.

Thus, we assume that the observed vectors $(p_{1i}, p_{2i})$ are independent and identically distributed, each with c.d.f. $F(p_{1i}, p_{2i})$, and are observed
one at a time; i.e., they form a sequential random sample. It is implicit in this process that neither side learns about or changes the c.d.f. $F(p_1, p_2)$ as prospective jurors are questioned. The rule that specifies at each stage which side must declare first whether it wishes to challenge that juror is assumed fixed at the outset and does not depend on the previous decisions of the participants.

From the point of view of the prosecution at any stage in the selection process, the outcome of the entire process will be a random vector $(P_{11}, \cdots, P_{1J})$ of the $p_{1i}$ values of the members of the final jury. We assume that there is no interaction between jurors, so that the overall (random) probability of conviction in the opinion of the prosecution is $P^{(1)} = \prod_{i=1}^{J} P_{1i}$, where the product is taken over the $J$ people on the final jury. (This assumption is probably valid only on the first post-trial ballot taken by the jury prior to any discussion.) Similarly, we define the overall probability of conviction in the opinion of the defense to be $P^{(2)} = \prod_{i=1}^{J} P_{2i}$.

Let $c_1$ denote the utility to the prosecution if the jury votes for conviction and let $n_1$ denote the utility to the prosecution if the jury does not vote for conviction. Then the expected utility of the jury selection process to the prosecution is

$$E[c_1P^{(1)} + n_1(1-P^{(1)})] = (c_1-n_1)E[P^{(1)}]+n_1, \quad (1.1)$$

where the expectation is taken over the distribution of possible values of $P^{(1)}$. Since the prosecution prefers that the jury vote for conviction, it follows that $c_1>n_1$. Hence, by (1.1), the prosecution will maximize its expected utility by using a strategy for selecting jurors that will maximize the value of $E[P^{(1)}]$.

Similarly, let $c_2$ denote the utility to the defense if the jury votes for conviction and let $n_2$ denote the utility if the jury does not vote for conviction. Then the expected utility to the defense is

$$E[c_2P^{(2)} + n_2(1-P^{(2)})] = (c_2-n_2)E[P^{(2)}]+n_2. \quad (1.2)$$

Since the defense prefers that the jury not vote for conviction, then $c_2<n_2$. Hence, by (1.2), the defense will maximize its expected utility by using a strategy for selecting jurors that will minimize the value of $E[P^{(2)}]$.

We shall show that optimal strategies exist for both sides. We define our problem to be reversible for our particular values of $A$, $B$, and $J$ if, under the optimal strategies, it will never matter at any stage which side is required to decide first whether or not to use a challenge. The problem is called universally reversible if it is reversible for all possible values of $A$, $B$, and $J$. Both of these concepts depend on the joint c.d.f. $F(p_1, p_2)$. At any stage of the selection process, after some number of prospective jurors have been acted upon and either mutually accepted or challenged
by one side or the other, the optimal strategy for the prosecution will be
to attempt to maximize the product of the values of $p_1$, for the jurors that
remain to be selected and the optimal strategy for the defense will be to
attempt to minimize the product of the values of $p_2$, for those same jurors.
Thus, the problem is effectively beginning again with new values for $A$, $B$,
and $J$. For any integers $a \leq A$, $b \leq B$, and $j \leq J$, we say that $a$, $b$, and $j$
are reachable if there is positive probability when both sides use optimal
strategies that $a$, $b$, and $j$ are ever these new values. It follows from its
definition that reversibility for $A$, $B$, $J$ implies reversibility for any reach­
able values of $a$, $b$, $j$.

Our main results can be described briefly as follows. In the next three
sections we develop our notation, describe an important regularity as­
sumption, and derive some of the basic relations that characterize the
optimal procedure. Then in Section 5 we show that each side can always
do at least as well by making the first decision regarding any juror as it
can when the opposition decides first. In Theorem 1 we give necessary and
sufficient conditions under which the problem is reversible; it is univer­
sally reversible if and only if these conditions hold for all $A$, $B$, $J$. In
particular, in Theorem 2 and its corollary, universal reversibility is shown
to hold whenever both sides always agree on the $p$-values of prospective
jurors. In Section 6 we specify an algorithm for finding the optimal strate­
gies for both sides. In Section 7 we give examples of problems that are not
universally reversible. Finally, in Section 8 we present some simple nu­
merical examples involving uniform distributions.

Brams and Davis [2] consider a similar problem but restrict themselves
to the case in which both sides always agree on the $p$-values of prospective
jurors. They present some interesting numerical results for this case and
consider other models in which peremptory challenges can be exercised
after groups of prospective jurors have been questioned.

2. DEFINITION AND PROPERTIES OF THE OPTIMAL PROCEDURE

Before investigating reversibility or finding the form of the optimal
procedure, we must define this procedure and describe the sense in which
it is optimal. We observe that the jury must be selected after at most
$A+B+J$ people have been interviewed. Thus, the number of decisions in
the selection process is bounded. Clearly, the lawyer who makes the last
possible decision, when one juror remains to be selected and this lawyer
has one challenge remaining while his opponent has none, has an optimal
choice. Under the assumption that this last possible choice will be made
optimally, the consequences of the next-to-last possible decision are
known. Hence it can also be made optimally. Proceeding by backward
induction, each decision can be made optimally if the side making that
decision is willing to assume that both sides will act optimally on all sub-
sequent decisions. The optimal procedure is taken to be the one resulting from all these optimal choices by both sides. Since this procedure completely defines the actions of both sides, it determines a pair of values \((EP(1), EP(2))\), which represents the best the prosecution and the defense, respectively, can expect to do under the assumption that the other side will proceed optimally according to its own opinions about the prospective jurors.

It is important to keep in mind that the procedure just described is optimal only in the sense of the assumptions that we have made. It is implicit in these assumptions that the two sides do not cooperate or collude in any way. If \(P(1)\) and \(P(2)\) are not necessarily equal, then the jury selection process is a non-zero sum, two-person game, and it is possible that both sides could increase their expected utilities by cooperating in their use of challenges. However, such cooperation or collusion is presumably not permitted in our jury system, and we shall not consider it further in this paper.

3. NOTATION AND A BASIC REGULARITY ASSUMPTION

Let \(w = (w_1, \ldots, w_{A+B+J})\) be a vector such that \(w_i = 1\) or \(2\) for \(i = 1, \ldots, A+B+J\); \(w_i = 1\) means that the prosecution has to decide first about the \(i\)th prospective juror to be questioned, while \(w_i = 2\) means that the defense must decide first. For any vector \(y\) with at least two elements, let \(\emptyset(y)\) denote the vector that is obtained by deleting the first element of \(y\). Suppose now that at some stage the prosecution has \(a\) challenges remaining, the defense has \(b\) challenges remaining, there are \(j\) jurors still to be selected, and the vector \((p_1, p_2)\) associated with the present candidate is \((p_1, p_2)\). In this situation, for \(j = 1\) or \(2\), let \(P(j')\) be the product of the \(p_i\)'s yet to be added to the jury, including the present candidate if he is accepted, and then let \(E[P(1)]\) and \(E[P(2)]\) denote the expected values of these quantities under the optimal procedure described in Section 2.

Let \(v = (v_1, \ldots, v_{a+b+j})\) be the vector consisting of the last \(a+b+j\) elements of \(w\), so that \(v\) specifies who decides first for each remaining potential juror. We write \(M^*(a, b, j, p_1, p_2, v) = E[P(1)]\) and \(M^*(a, b, j, p_1, p_2, v) = E[P(2)]\) to show the explicit dependence of these quantities on the relevant parameters. Next, let \(\mu^*(a, b, j, v) = E[M^*(a, b, j, p_1, P_2, v)]\), where the joint distribution of \((P_1, P_2)\) over the unit square has the c.d.f. \(F\), and let \(\mu^*(a, b, j, v)\) be defined analogously. The quantities \(\mu^*(a, b, j, v)\) and \(\mu^*(a, b, j, v)\) represent the expected products of the \(p_i\)'s and the \(p_2\)'s for the remainder of the process prior to the interviewing of the present candidate. Whenever \(a, b, j,\) and \(v\) are not ambiguous, we shall conserve space by letting \(\alpha = [a-1, b, j]\), \(\beta = [a, b-1, j]\), \(\phi(v)\), and \(\gamma = [a, b, j-1, \phi(v)]\). Thus, \(\alpha, \beta,\) and \(\gamma\) each represent possible stages that might be reached during the jury selection process.
We can think of the stage $\alpha$ being transformed into the stage $\beta$ by the defense giving one of its available challenges to the prosecution for its own use. In this way the number of challenges available to the prosecution increases from $a - 1$ to $a$, while the number available to the defense decreases from $b$ to $b - 1$. It is, therefore, reasonable to suppose that for any fixed history of the process, the prosecution finds it at least as preferable to be at the stage $\beta$ as at the stage $\alpha$. In symbols, this relation can be expressed by the inequality

$$\mu^*(\alpha) \leq \mu^*(\beta). \quad (3.1)$$

For the same reason, it is reasonable to suppose that for any fixed history of the process, the defense finds it at least as preferable to be at the stage $\alpha$ as at the stage $\beta$, i.e., that

$$\mu^*(\alpha) \leq \mu^*(\beta). \quad (3.2)$$

Surprisingly, we have not been able to prove that the inequalities (3.1) and (3.2) hold in full generality for every c.d.f. $F(p_1, p_2)$ and all values of $a$, $b$, $j$, and $v$. Nor, on the other hand, have we succeeded in constructing a counterexample in which at least one of these inequalities does not hold.

We can show, however, that (3.1) and (3.2) must always hold in any problem in which both sides always agree on the $p$-values of prospective jurors. In a problem of this type, $P^{(1)}$ and $P^{(2)}$ must be equal at each stage and the process is actually a zero-sum two-person game. The argument that establishes (3.1) and (3.2) in this case is somewhat roundabout and not directly related to the other developments in this paper, and is omitted.

If one of the inequalities (3.1) or (3.2) does not hold at some stage of the selection process, it means that the lawyers for one side would actually prefer to give one of their available challenges to the other side. It does not seem reasonable that such a situation could arise in any practical problem of jury selection. Hence, we shall herewith assume that in every problem considered in this paper, (3.1) and (3.2) are satisfied for all stages that might arise during the jury selection process.

4. THE FORM OF THE OPTIMAL STRATEGY WHEN BOTH SIDES HAVE AT LEAST ONE CHALLENGE REMAINING

Case 1: Prosecution Makes the First Decision on the Next Candidate

When the prosecution makes the first decision on the next candidate, $v_1 = 1$; i.e., $v$ is of the form $v = 1\emptyset(v)$. By considering the consequences of the two possible decisions, first for the prosecution and then, if the prosecution accepts the juror, for the defense, we can write (for $a \geq 1$, $b \geq 1$, $j \geq 1$)
\( M^*[a, b, j, p_1, p_2, \theta(v)] \)

\[
= \begin{cases} 
\max [\mu^*(\alpha), p_2\mu^*(\gamma)] & \text{if } p_2\mu^*(\gamma) < \mu^*(\beta); \\
\max [\mu^*(\alpha), \mu^*(\beta)] & \text{if } p_2\mu^*(\gamma) > \mu^*(\beta). 
\end{cases} \tag{4.1}
\]

It follows from (3.1) that (4.1) can be rewritten (for \( a \geq 1, b \geq 1, j \geq 1 \)) as

\[
M^*[a, b, j, p_1, p_2, \theta(v)] = \begin{cases} 
\mu^*(\alpha) & \text{if } p_1 < [\mu^*(\alpha)/\mu^*(\gamma)] \text{ and } p_2 < [\mu^*(\beta)/\mu^*(\gamma)], \\
p_1\mu^*(\gamma) & \text{if } p_1 > [\mu^*(\alpha)/\mu^*(\gamma)] \text{ and } p_2 < [\mu^*(\beta)/\mu^*(\gamma)], \\
\mu^*(\beta) & \text{if } p_2 > [\mu^*(\beta)/\mu^*(\gamma)]. 
\end{cases} \tag{4.2}
\]

**TABLE I**

**OPTIMAL DECISIONS**

<table>
<thead>
<tr>
<th>FIRST DECISION (Opponent may challenge if you accept)</th>
<th>FINAL DECISION (Opponent has already accepted)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha ) Small</td>
<td>( \alpha ) Large</td>
</tr>
<tr>
<td>( p_2 )</td>
<td></td>
</tr>
<tr>
<td>Defense: Accept</td>
<td></td>
</tr>
<tr>
<td>Prosecution: Challenge</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>( p_1 ) Small</td>
<td>( p_2 ) Small</td>
</tr>
<tr>
<td>Defense: Accept</td>
<td></td>
</tr>
<tr>
<td>Prosecution: Challenge</td>
<td></td>
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<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>( p_1 ) Large</td>
<td>( p_2 ) Large</td>
</tr>
<tr>
<td>Defense: Accept</td>
<td></td>
</tr>
<tr>
<td>Prosecution: Challenge</td>
<td></td>
</tr>
</tbody>
</table>

* Hypothetical case, opponent has already challenged.

The optimal decisions for both sides regarding the present juror can be deduced from (4.2) and are summarized in Table I. Using (4.2), we define \( p_1 \) to be *large* if \( p_1 > [\mu^*(\alpha)/\mu^*(\gamma)] \) and *small* if \( p_1 < [\mu^*(\alpha)/\mu^*(\gamma)] \). Similarly, we define \( p_2 \) to be *large* if \( p_2 > [\mu^*(\beta)/\mu^*(\gamma)] \) and *small* if \( p_2 < [\mu^*(\beta)/\mu^*(\gamma)] \). These definitions depend on \( a, b, j, \) and \( v \). If the marginal distributions of \( P_1 \) and \( P_2 \) are both continuous, then \( p_1 \) and \( p_2 \) are each either large or small with probability one and Table I completely describes the form of the optimal decisions. For present purposes, we assume that the marginal distributions are continuous, so that the case \( p_1 = [\mu^*(\alpha)/\mu^*(\gamma)] \) or \( p_2 = [\mu^*(\beta)/\mu^*(\gamma)] \), which will be discussed at the end of Section 5, need not be considered here.

By examining Table I and considering the process from the standpoint of the defense, we can see that (for \( a \geq 1, b \geq 1, j \geq 1 \))
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\[ M_*(a, b, j, p_1, p_2, \emptyset(v)) = \begin{cases} 
\mu_*(\beta) & \text{if } p_2 \text{ is large;} \\
\mu_*(\alpha) & \text{if } p_1 \text{ is small and } p_2 \text{ is small;} \\
p_2\mu_*(\gamma) & \text{if } p_1 \text{ is large and } p_2 \text{ is small.}
\end{cases} \quad (4.3) \]

Case II: Defense Makes the First Decision on the Next Candidate

When the defense decides first on the next candidate, \( v_1 = 2 \); i.e., \( v \) is of the form \( v = 2\emptyset(v) \). By almost identical arguments to those used in Case I we can write (for \( a \geq 1, b \geq 1, j \geq 1 \))

\[ M_*[a, b, j, p_1, p_2, 2\emptyset(v)] = \begin{cases} 
\min [\mu_*(\beta), p_2\mu_*(\gamma)] & \text{if } p_1 \text{ is large;} \\
\min [\mu_*(\beta), \mu_*(\alpha)] & \text{if } p_1 \text{ is small,} \\
\mu_*(\beta) & \text{if } p_1 \text{ is large and } p_2 \text{ is large;} \\
p_2\mu_*(\gamma) & \text{if } p_1 \text{ is large and } p_2 \text{ is small;} \\
\mu_*(\alpha) & \text{if } p_1 \text{ is small,}
\end{cases} \quad (4.4) \]

since it follows from (3.2) that \( \mu_*(\alpha) \leq \mu_*(\beta) \). The optimal strategies now follow from (4.4), and we summarize these strategies for both Case I and Case II in Table I. Note that the strategies of both sides do not depend on whether they are making the first or final decision except when \( p_1 \) is small and \( p_2 \) is large, i.e., when both sides find the same juror undesirable. In that case, whoever decides first will accept the juror, forcing his opponent to be the one to use a challenge. Table I gives the complete form of the optimal strategy when \( a \geq 1, b \geq 1, j \geq 1 \). It does not, however, tell us exactly what this strategy is because the categories “large” and “small” depend on \( a, b, j \) and \( v \) through the functions \( \mu_*(\alpha) \) and \( \mu_*(\beta) \) (with various sets of arguments), which can be evaluated (see Section 6) only through the backward induction algorithm.

It follows from Table I that (for \( a \geq 1, b \geq 1, j \geq 1 \))

\[ M^*[a, b, j, p_1, p_2, 2\emptyset(v)] = \begin{cases} 
\mu^*(\beta) & \text{if } p_1 \text{ is large and } p_2 \text{ is large;} \\
p_1\mu^*(\gamma) & \text{if } p_1 \text{ is large and } p_2 \text{ is small;} \\
\mu^*(\alpha) & \text{if } p_1 \text{ is small.}
\end{cases} \quad (4.5) \]

5. THE ADVANTAGE OF MAKING THE FIRST DECISION AND A CHARACTERIZATION OF REVERSIBILITY

We shall now compare the expected value \( M^*[a, b, j, p_1, p_2, 1\emptyset(v)] \) that the prosecution can attain if it makes the first decision about the next prospective juror, with the expected value \( M^*[a, b, j, p_1, p_2, 2\emptyset(v)] \) that it can attain if the defense goes first for that prospective juror, and there is an arbitrary fixed rule \( \emptyset(v) \) for decisions about all future jurors.
From (4.2) and (4.5) we see that (for $a \geq 1$, $b \geq 1$, $j \geq 1$)
\[
M^*[a, b, j, \mu_1, \mu_2, l\emptyset(v)] - M^*[a, b, j, \mu_1, \mu_2, 2\emptyset(v)] = \begin{cases} 
\mu^*(\beta) - \mu^*(\alpha) & \text{if } \mu_1 \text{ small, } \mu_2 \text{ large}; \\
0 & \text{otherwise.}
\end{cases}
\]

Since the prosecution wishes to maximize $M^*$, it follows from (3.1) that the prosecution is at least as well off making the first decision on the next candidate as it is going second.

Similarly, we can compare the expected value $M^*[a, b, j, \mu_1, \mu_2, l\emptyset(v)]$ that the defense can attain when it makes the first decision with the expected value $M^*[a, b, j, \mu_1, \mu_2, 2\emptyset(v)]$ that it can attain when the prosecution makes the first decision. From (4.3) and (4.4) we see that (for $a > 1$, $b > 1$, $j > 1$)
\[
M^*[a, b, j, \mu_1, \mu_2, l\emptyset(v)] - M^*[a, b, j, \mu_1, \mu_2, 2\emptyset(v)] = \begin{cases} 
\mu^*(\beta) - \mu^*(\alpha) & \text{if } \mu_1 \text{ small, } \mu_2 \text{ large}; \\
0 & \text{otherwise.}
\end{cases}
\]

Since the defense wishes to minimize $M^*$, it follows from (3.2) that the defense is also at least as well off as going second; there is no difference unless $\mu_1$ is small and $\mu_2$ is large. (In fact, we have seen from Table I that the strategies are independent of order except in this case.) This argument can be extended by induction to other elements of the vector $v$, and we can conclude that it is always desirable to decide first for any prospective juror. Since reversibility must necessarily hold if either $a = 0$ or $b = 0$ (i.e., only one side has any choices remaining), (5.1) and (5.2) suggest the following result.

**Theorem 1.** The optimal strategy is reversible if and only if for any reachable $a$, $b$, $j$, $v$ either (i) the probability is zero that $\mu_1$ is small and $\mu_2$ is large, or (ii) the probability is zero that $\mu_1$ and $\mu_2$ are either both small or both large, not only for the present values of $a$, $b$, $j$, $v$ but also for any $a$, $b$, $j$, $v$ that are reachable from these present values.

**Proof.** The theorem would follow immediately from (5.1), (5.2), (3.1), and (3.2) if we could show that condition (ii) is equivalent to
\[
(i'') \quad \mu^*(\alpha) = \mu^*(\beta) \quad \text{and} \quad \mu^*(\alpha) = \mu^*(\beta).
\]

But (ii'') means that either side could give the other side one of its challenges without affecting its expected value. Since the $\mu^*$ and $\mu^*$-functions represent expectations over the entire future of the selection process,
(ii') is equivalent to the condition that (with probability one) it is not presently, and will never in the future be, the case that one side wants to challenge a candidate that the other side wants to accept. But this is precisely condition (ii).

Theorem 1, unfortunately, is a characterization of reversibility that is as hard to verify as the original condition itself. Hence, the strength of the theorem lies in its theoretical insight rather than in its practical use. In the usual problems where the defense and the prosecution have essentially opposite goals, \( \mu^*(\alpha) < \mu^*(\beta) \) for all \( a, b, j, v \), and condition (ii) fails. If condition (ii) is ignored, then reversibility is equivalent to the property that there is zero probability that both sides will find the same prospective juror unacceptable.

An important special case of Theorem 1 occurs when both sides always agree in their assessments of jurors, i.e., \( P_1 = P_2 \). A slightly more general problem is considered in the next theorem.

**Theorem 2.** Suppose \( P_2 = kP_1 \) for some \( k > 0 \), i.e., the joint distribution of \( (P_1, P_2) \) lies entirely on a line through the origin. Then universal reversibility holds.

**Proof.** We can ignore the degenerate case where \( P_1 \) and \( P_2 \) are constants and assume that \( P_2 \) is a nondegenerate strictly increasing function of \( P_1 \). Hence, condition (ii) of Theorem 1 cannot possibly be met, i.e., \( \mu^*(\alpha) < \mu^*(\beta) \) for all \( a, b, j, v \). For any values of the arguments, \( M_*(a, b, j, p_1, p_2, v) = E[P^{(a_2)}] = k^{j}E[P^{(a_2)}] = k^{j}M^*(a, b, j, p_1, p_2, v) \). Taking expectations with respect to \( P_1 \) and \( P_2 \), we obtain \( \mu_*(a, b, j, v) = k^{j}\mu^*(a, b, j, v) \). Now suppose that \( p_1 \) is not large and \( p_2 \) is not small for some \( a, b, j, v \). (In the discrete case this may be a weaker assumption than \( p_1 \) small, \( p_2 \) large.) Then

\[
\begin{align*}
(i) & \quad p_1 \leq \mu^*(\alpha)/\mu^*(\gamma), \text{ i.e., } p_2 = kp_1 \leq k\mu^*(\alpha)/\mu^*(\gamma), \text{ and} \\
(ii) & \quad p_2 \leq \mu^*(\beta)/\mu^*(\gamma) = k^{j}\mu^*(\beta)/k^{j-1}\mu^*(\gamma) \\
& \quad = k\mu^*(\beta)/\mu^*(\gamma) > k\mu^*(\alpha)/\mu^*(\gamma),
\end{align*}
\]

which is a contradiction. The result now follows from Theorem 1.

**Corollary.** If \( P_1 = P_2 \), then universal reversibility holds.

This corollary is equivalent to Theorem 1 of Brams and Davis [2]. In effect, they assume that decisions are made simultaneously by the two sides and show that they could just as well be made alternately; we assume that decisions are made alternately and show that they could just as well be made simultaneously.

Theorem 2 and its corollary are true even when the marginal distribution
of $P_1$ or $P_2$ is discrete. The fact that Theorem 1 also holds in this case is a direct consequence of the following argument.

If $p_1 = \mu^*(\alpha)/\mu^*(\gamma)$ or $p_2 = \mu^*(\beta)/\mu^*(\gamma)$, then at least one of these $p$-values is neither "large" nor "small." If the marginal distribution of $P_1$ or $P_2$ is not continuous, this may occur with positive probability. When it occurs, the lawyer whose $p$-value is neither large nor small will be indifferent between his two possible decisions. In that case, the decision that his opponent would prefer that he make is called the \textit{benevolent decision} and the other decision is called the \textit{malevolent decision}. A lawyer who always makes the benevolent decision when he is indifferent, regardless of whether he is deciding first or whether his opponent has already accepted the juror, is said to adopt the \textit{benevolent strategy}. The \textit{malevolent strategy} is defined analogously. Of course, a lawyer may make some benevolent decisions and some malevolent ones.

A lawyer will choose a different strategy depending on whether his opponent has adopted a benevolent or a malevolent strategy. It can be shown, however, that reversibility is not affected by the types of strategies adopted by the two sides, provided that each side knows whether its opponent is benevolent or malevolent. Thus, reversibility will be present when one side uses a benevolent strategy if and only if it is present when that side uses a malevolent strategy. Similarly, the absence of reversibility is not affected by the types of strategies adopted by the two sides. The proofs of these statements are omitted.

6. AN ALGORITHM FOR DETERMINING THE OPTIMAL PROCEDURE

The form of the optimal procedure (as long as $a \geq 1$, $b \geq 1$) was found in Section 4. To completely specify the procedure, it remains only to evaluate the functions $\mu^*$ and $\mu^*$ in order to quantify the notions of "large" and "small" values for $p_1$ and $p_2$. For any two real numbers $s$ and $t$, define the set

$$S(s, t) = \{(p_1, p_2) : p_1 > s \text{ and } p_2 < t\} \quad (6.1)$$

and define the two transformations

$$U_F(s, t) = \int \int_{S(s, t)} p_1 dF(p_1, p_2)$$

and

$$V_F(s, t) = \int \int_{S(s, t)} p_2 dF(p_1, p_2). \quad (6.2)$$

Note that $U_F(s, t) = V_F(s, t) = 0$ if either $s > 1$ or $t < 0$; $U_F(s, t) = U_F(0, t)$ and $V_F(s, t) = V_F(0, t)$ if $s < 0$; $U_F(s, t) = U_F(s, 1)$ and $V_F(s, t) = V_F(s, 1)$ if $t > 1$.

We shall let $F_1$ and $F_2$ denote the marginal c.d.f.'s of $P_1$ and $P_2$, respec-
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Taking expectations on both sides of (4.2)–(4.5), we now obtain (for $a \geq 1, b \geq 1, j \geq 1$) the following relationships:

\[
\begin{align*}
\mu^*[a, b, j, 1\emptyset(v)] &= \mu^*(\alpha) F[\mu^*(\alpha)/\mu^*(\gamma), \mu^*(\beta)/\mu^*(\gamma)] \\
&+ \mu^*(\beta)[1-F_2[\mu^*(\beta)/\mu^*(\gamma)]] \\
&+ \mu^*(\gamma) U_F[\mu^*(\alpha)/\mu^*(\gamma), \mu^*(\beta)/\mu^*(\gamma)], \\
\mu^*[a, b, j, 1\emptyset(v)] &= \mu^*(\alpha) F[\mu^*(\alpha)/\mu^*(\gamma), \mu^*(\beta)/\mu^*(\gamma)] \\
&+ \mu^*(\beta)[1-F_2[\mu^*(\beta)/\mu^*(\gamma)]] \\
&+ \mu^*(\gamma) V_F[\mu^*(\alpha)/\mu^*(\gamma), \mu^*(\beta)/\mu^*(\gamma)], \\
\mu^*[a, b, j, 2\emptyset(v)] &= \mu^*(\alpha) F[\mu^*(\alpha)/\mu^*(\gamma)] \\
&+ \mu^*(\beta) F[\mu^*(\alpha)/\mu^*(\gamma), \mu^*(\beta)/\mu^*(\gamma)] \\
&+ \mu^*(\gamma) U_F[\mu^*(\alpha)/\mu^*(\gamma), \mu^*(\beta)/\mu^*(\gamma)], \\
\mu^*[a, b, j, 2\emptyset(v)] &= \mu^*(\alpha) F[\mu^*(\alpha)/\mu^*(\gamma)] \\
&+ \mu^*(\beta) F[\mu^*(\alpha)/\mu^*(\gamma), \mu^*(\beta)/\mu^*(\gamma)] \\
&+ \mu^*(\gamma) U_F[\mu^*(\alpha)/\mu^*(\gamma), \mu^*(\beta)/\mu^*(\gamma)].
\end{align*}
\]

These equations define recursive relations for determining the values of $\mu^*$ and $\mu^*$ at $(a, b, j)$ from the values of $\mu^*$ and $\mu^*$ at $(a-1, b, j)$, $(a, b-1, j)$, and $(a, b, j-1)$. Hence, the algorithm defined by (6.4) to (6.7) merely requires a set of boundary conditions to completely determine $\mu^*$ and $\mu^*$ for all possible arguments. The boundary conditions, for arbitrary $v$, are

\[
\begin{align*}
\mu^*(a, b, 0, v) &= \mu^*(a, b, 0, v) = 1 \text{ for any } a, b, \quad (6.8) \\
\mu^*(a, 0, j, v) &= u^*(a, j) \text{ for } a \geq 1, j \geq 1, \quad (6.9) \\
\mu^*(a, 0, j, v) &= u^*(a, j) \text{ for } a \geq 1, j \geq 1, \quad (6.10) \\
\mu^*(0, b, j, v) &= v^*(b, j) \text{ for } b \geq 1, j \geq 1, \quad (6.11) \\
\mu^*(0, b, j, v) &= v^*(b, j) \text{ for } b \geq 1, j \geq 1, \quad (6.12)
\end{align*}
\]

where $u^*$ and $v^*$ represent the expected values that can be attained in the one-sided versions of this problem when only one side has any challenges left, and where $u^*$ and $v^*$ are the expected values of the other side in these one-sided versions as they helplessly watch their opponents carry out their strategy. Separate algorithms for evaluating these four functions are given below.

Before generating the algorithms for $u^*$, $u^*$, $v^*$, and $v^*$, we note that $u^*$ depends only on $F_1(p_1)$ and $v^*$ depends only on $F_2(p_2)$, but $u^*$ and $v^*$ depend on the joint c.d.f. $F(p_1, p_2)$. In preparation for dealing with the two
marginal univariate c.d.f.'s, we define for any univariate c.d.f. $\Lambda$ on $[0, 1]$ the transformation

$$T_\Lambda(s) = \int_s^1 (x-s)d\Lambda(x).$$  

(6.13)

The properties of this transformation are given in Section 11.8 of DeGroot [6]. Suppose that $X$ is a random variable with c.d.f. $G$, on $[0, 1]$, that $Y = DX$ for some constant $D > 0$, and that $H$ is the c.d.f. of $Y$. Then for any constant $K$, it is easily shown that

$$E[\max(K, X)] = K + T_\sigma(K),$$  

(6.14)

$$E[\min(K, X)] = E(X) - T_\sigma(K),$$  

(6.15)

$$T_H(s) = DT_\sigma(s/D).$$  

(6.16)

Algorithms for $u^*$ and $v^*$ are now easily obtained using (6.14)–(6.16):

$$u^*(a, j) = E[\max[u^*(a-1, j), P_1u^*(a, j-1)]]$$  

$$= u^*(a-1, j) + u^*(a, j-1)T_{P_1}[u^*(a-1, j)/u^*(a, j-1)],$$  

(6.17)

$$v^*(b, j) = E[\min[v^*(b-1, j), P_2v^*(b, j-1)]]$$  

$$= E(P_2)v^*(b, j-1)$$  

$$- v^*(b, j-1)T_{P_2}[v^*(b-1, j)/v^*(b, j-1)].$$  

(6.18)

The appropriate boundary conditions for (6.17) and (6.18), as well as for the functions $u^*$ and $v^*$, are

$$u^*(a, 0) = u^*(a, 0) = v^*(b, 0) = v^*(b, 0) = 1$$  

for any $a$ or $b$,  

(6.19)

$$u^*(0, j) = v^*(0, j) = [E(P_1)]^j$$  

for any $j \geq 1$,  

(6.20)

$$u^*(0, j) = v^*(0, j) = [E(P_2)]^j$$  

for any $j \geq 1$.  

(6.21)

Algorithms for $u^*$ and $v^*$ are more difficult to obtain. To compute $u^*(a, j)$, for example, we note from (6.17) that the prosecution will challenge the next prospective juror if and only if $P_1 < u^*(a-1, j)/u^*(a, j-1) = Q$, say. Then $u^*(a, j) = E(W)$, where

$$W = \begin{cases} 
    u^*(a-1, j) & \text{if } P_1 < Q, \\
    P_2u^*(a, j-1) & \text{if } P_1 > Q.
\end{cases}$$  

(6.22)

We now obtain

$$u^*(a, j) = u^*(a-1, j)F_1(Q)$$  

$$+ u^*(a, j-1) \int_Q^1 E(P_2|P_1 = p_1)dF_1(p_1)$$  

(6.23)

$$= u^*(a-1, j)F_1(Q) + u^*(a, j-1)V_F(Q, 1).$$

By similar methods, we can obtain

$$v^*(b, j) = v^*(b-1, j)[1 - F_2(R)] + v^*(b, j-1)U_F(0, R)$$  

(6.24)

where $R = v^*(b-1, j)/v^*(b, j-1)$. 


Equations (6.17), (6.18), (6.23), and (6.24), together with the boundary conditions (6.19)–(6.21), form complete algorithms for evaluating the original boundary conditions (6.9)–(6.12). The functions $\mu^*$ and $\mu_*$ can now be computed for any arguments, and the optimal procedure is completely specified for $a \geq 1$ and $b \geq 1$. When $a = 0$ and $b \geq 1$, the defense is playing a one-sided game and we see from (6.18) that the optimal strategy is to challenge the next prospective juror if and only if $P_2 > R$. Similarly, when $b = 0$ and $a \geq 1$, we have seen from (6.17) that the best strategy for the prosecution is to challenge the next prospective juror if and only if $P_1 < Q$. When $a = b = 0$, no strategy at all is involved. The entire optimal strategy has now been specified.

7. EXAMPLES THAT ARE NOT UNIVERSALLY REVERSIBLE

Example 1. It follows directly from Theorem 1 that reversibility cannot hold for any $a$, $b$, $j$ at all if $P_1$ and $P_2$ are independent and neither is a constant.

Example 2. Since the problem is universally reversible if $P_2 = kP_1$, we might suspect that this is also the case when the distribution of $(P_1, P_2)$ lies on the union of two such straight lines, i.e., either $P_2 = k_1 P_1$ (denote this line $L_1$) or $P_2 = k_2 P_1$ (denote this line $L_2$). However, we show that some $F$'s that are not universally reversible satisfy this condition. Without loss of generality, assume $0 < k_1 < k_2$. The three possible cases depending on whether or not $k_1$ and $k_2$ are larger than 1 are illustrated in Figure 1.

Let $T(k_1, k_2)$ be a subset of the unit square with the following property: If $(x_1, y_1) \in L_1 \cap T(k_1, k_2)$ and $(x_2, y_2) \in L_2 \cap T(k_1, k_2)$, then $x_1 \geq x_2$. Such a set can be found for any $0 < k_1 < k_2$—see, e.g., the shaded areas in Figure 1. Suppose that the distribution of $(P_1, P_2)$ lies entirely on $T(k_1, k_2) \cap (L_1 \cup L_2)$. Suppose, furthermore, that very little of the probability lies on $L_2$ and, hence, most of the probability lies on $L_1$. Then for relatively close values of $a$ and $b$, it is clear that any prospective juror for whom $(p_1, p_2)$ lies on $L_2$ will be unsatisfactory to both sides. Lack of universal reversibility follows from Theorem 1.

Comment. We cannot “fix” Example 2 by requiring that the support of the distribution of $(P_1, P_2)$ be all of $L_1 \cup L_2$. In that case we can construct essentially the same example by putting arbitrarily little probability on those parts of the lines outside $T(k_1, k_2)$. We conjecture that if the joint distribution of $(P_1, P_2)$ is absolutely continuous with respect to the Lebesgue measure on the plane, then the optimal strategy is not universally reversible.

8. TWO NUMERICAL EXAMPLES WITH THE SAME MARGINAL DISTRIBUTIONS

Suppose that $P_1$ and $P_2$ each have a marginal uniform distribution on $[0, 1]$ and that $A = B = J = 1$; i.e., one juror is to be selected and each side
has one challenge. We shall determine the optimal strategies for two special cases where (i) \( P_1 = P_2 \) and (ii) \( P_1 \) and \( P_2 \) are independent.

**Example 3:** \( P_1 = P_2 \). By Theorem 2 or its corollary we have universal reversibility. From the proof of Theorem 2 it follows that \( M^* = M_* \) and \( \mu^* = \mu_* \) for any possible common arguments. Furthermore, we can write \( M^*(a, b, j, p, v) \) since \( p_1 \) and \( p_2 \) will always be the same. Thus

Case 1: \( k_2 \leq 1 \)

Case 2: \( k_1 \leq 1 < k_2 \)

Case 3: \( k_1 > 1 \)

Figure 1

\[
M^*[1, 1, 1, p, 1\Theta(v)] = M_*[1, 1, 1, p, 1\Theta(v)] = M^*[1, 1, 1, p, 2\Theta(v)] = M_*[1, 1, 1, p, 2\Theta(v)]. \quad (8.1)
\]

If \( F_0 \) denotes the c.d.f. of \( P_1 \), then since \( P_1 = P_2 \) with probability one,

\[
U_\rho(s, t) = V_\rho(s, t) = \left\{ \begin{array}{ll}
0 & \text{if } s \geq t, \\
\int_s^t w dF_0(w) & \text{if } s < t.
\end{array} \right.
\]

We can now obtain the following results:

\[
\mu^*[1, 0, 1, \Theta(v)] = u^*(1, 1) = u^*(0, 1)
\]

\[
+ u^*(1, 0) T_{\rho_1} [u^*(0, 1) / u^*(1, 0)] = \frac{3}{2} + \int_{1/2}^{1} (x - \frac{1}{2}) dx = \frac{5}{8}. \quad (8.2)
\]
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\[ \mu^*[0, 1, 1, \emptyset(v)] = v^*(1, 1) = v^*(0, 1)\{1 - F_2[v_*(0, 1)/v_*(1, 0)]\} \]
\[ + v^*(1, 0)U_\varphi[0, v_*(0, 1)/v_*(1, 0)] \]
\[ = \frac{1}{2}[1 - F_2(\frac{1}{2})] + U_\varphi(0, \frac{1}{2}) = \frac{1}{4} + \int_0^{1/2} wdw = \frac{72}{8}. \]

Hence, from (4.2) we obtain

\[ M^*(1, 1, 1, p, v) = \begin{cases} \frac{3}{8} & \text{if } p < \frac{3}{8}; \\ p & \text{if } \frac{3}{8} < p < \frac{5}{8}; \\ \frac{5}{8} & \text{if } p > \frac{5}{8}. \end{cases} \] (8.4)

It is found from (8.4) that

\[ \mu^*(1, 1, 1, v) = \frac{1}{2}, \] (8.5)

which merely verifies the fact that both sides start with expectation \( \frac{1}{2} \).

The optimal strategy is as follows. The first candidate will be accepted if his \( p \) value is between \( \frac{3}{8} \) and \( \frac{5}{8} \); otherwise, he will be challenged by one side or the other. If either side challenges the first prospective juror, his opponent will challenge the second prospective juror if and only if he finds \( \frac{1}{2} \) preferable to this second \( p \) value. If the defense uses the first challenge, the expectation for both sides becomes \( \frac{3}{8} \); if the prosecution uses the first challenge, this common expectation becomes \( \frac{5}{8} \). The mutual expectation returns to \( \frac{1}{2} \) if both sides use their challenges.

**Example 4:** \( P_1 \) and \( P_2 \) Independent. By Example 1 there is no reversibility. Therefore, we must determine four different values of \( M^* \) or \( M_* \) and four different values of \( \mu^* \) or \( \mu_* \). It can be shown that (for \( 0 < x < 1, 0 < y < 1 \))

\[ U_\varphi(x, y) = \frac{1}{2}y(1 - x^2) \text{ and } V_\varphi(x, y) = \frac{1}{2}y^2(1 - x). \] (8.6)

We can now obtain the following relations:

\[ \mu^*[0, 1, 1, \emptyset(v)] = v^*(1, 1) = v^*(0, 1)\{1 - F_2[v_*(0, 1)/v_*(1, 0)]\} \]
\[ + v^*(1, 0)U_\varphi[0, v_*(0, 1)/v_*(1, 0)] \]
\[ = \frac{1}{2}[1 - F_2(\frac{1}{2})] + U_\varphi(0, \frac{1}{2}) = \frac{1}{2}, \] (8.7)

\[ \mu^*[1, 0, 1, \emptyset(v)] = u^*(1, 1) = u^*(0, 1) \]
\[ + u^*(1, 0)T_{P_1}[u^*(0, 1)/u^*(1, 0)] = \frac{1}{2} + T_{P_1}(\frac{1}{2}) = \frac{5}{8}, \] (8.8)

\[ \mu_*[0, 1, 1, \emptyset(v)] = v_*(1, 1) = \frac{1}{2}v_*(1, 0) \]
\[ - v_*(1, 0)T_{P_2}[v_*(0, 1)/v_*(1, 0)] = \frac{1}{2} - T_{P_2}(\frac{1}{2}) = \frac{3}{8}, \] (8.9)

\[ \mu_*[1, 0, 1, \emptyset(v)] = u_*(1, 1) = u_*(0, 1)F_1[u^*(0, 1)/u^*(1, 0)] \]
\[ + u_*(1, 0)V_\varphi[u^*(0, 1)/u^*(1, 0), 1] \]
\[ = \frac{1}{2}F_1(\frac{1}{2}) + V_\varphi(\frac{1}{2}, 1) = \frac{1}{2}. \] (8.10)

These equations allow us to obtain the optimal strategy and the \( M^* \).
and $M_*$ values regarding the first prospective juror as presented in Table II. Note that each side has expectation $\frac{1}{2}$ after it uses its challenge since the opponent's strategy is independent of its own perception of the $p$ values. The optimal strategy after the first prospective juror is the same as in Example 3 since each side's strategy when the opponent is out of challenges depends only on the appropriate marginal distribution.

To see how much is gained by making the first decision on the first prospective juror, we use (6.3)-(6.6) to find

$$
\begin{align*}
\mu[1, 1, 1, 10(v)] &= \frac{1}{2}F(\frac{1}{2}, \frac{1}{2}) + \frac{5}{8}[1 - F_2(\frac{1}{2})] \\
&\quad + U_F(\frac{1}{2}, \frac{1}{2}) = \frac{5}{8}, \\
\mu[1, 1, 1, 20(v)] &= \frac{1}{2}F_1(\frac{1}{2}) + \frac{5}{8}\bar{F}(\frac{1}{2}, \frac{1}{2}) \\
&\quad + U_F(\frac{1}{2}, \frac{1}{2}) = 1.62, \\
\mu[1, 1, 1, 10(v)] &= \frac{3}{8}F(\frac{1}{2}, \frac{1}{2}) + \frac{1}{2}[1 - F_2(\frac{1}{2})] \\
&\quad + V_F(\frac{1}{2}, \frac{1}{2}) = 1.62, \\
\mu[1, 1, 1, 20(v)] &= \frac{3}{8}F_1(\frac{1}{2}) + \bar{F}(\frac{1}{2}, \frac{1}{2}) \\
&\quad + V_F(\frac{1}{2}, \frac{1}{2}) = 2.32.
\end{align*}
$$

Thus, each side can improve its expectation by $\frac{1}{3}2$ by going first rather than going second.

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