A Model of Moderation: Finding Skiba Points on a Slippery Slope *

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Abstract

A simple model is considered that rewards "moderation" - finding the right balance between sliding down either of two "slippery slopes". Optimal solutions are computed as a function of two key parameters: (1) the cost of resisting the underlying uncontrolled dynamics and (2) the discount rate. Analytical expressions are derived for bifurcation lines separating regions where it is optimal to fight to stay balanced, to give in to the attraction of the "left" or the "right", or to decide based on one's initial state. The latter case includes situations both with and without so-called Dechert-Nishimura-Skiba (DNS) points defining optimal solution strategies. The model is unusual for having two DNS points in a one-state model, having a single DNS point that bifurcates into two DNS points, and for the ability to explicitly graph regions within which DNS points occur in the 2-D parameter space. The latter helps give intuition and insight concerning conditions under which these interesting points occur.

Running Head: Skiba Points in a Model of Moderation
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INTRODUCTION

It is often both difficult and advantageous to maintain a position of moderation between two opposing factions, both of which seek to win the allegiance of those who remain uncommitted. Such situations present an interesting challenge: When and how should one remain unaligned if that is indeed the best option? More colloquially, how does one stay on top of a slippery slope? The solution to a simple model of this problem proves insightful both for the original context and as a window into quite interesting behavior concerning multiple so-called Dechert-Nishimura-Skiba (DNS) points within a one-dimensional model. We motivate the problem by sketching its application in three disparate contexts.

"Swing Voters"

Consider a legislature debating an issue for which there will be an up or down vote and for which the vote will be close. Both the "yes" and "no" lobbies court "uncommitted" representatives, and these uncommitted "swing voters" can trade (log roll) their cooperation for considerable benefit (cf. (Ordeshook, 1986; Schickler and Rich, 1997)). Politicians who are already firmly in the "yes" or "no" camp - e.g., because they have made public commitments to their constituency about their vote - do not have similar leverage, and those who begin to lean one way or the other may quickly come to be known ("pigeon-holed") as belonging to that camp and not the other. In terms of conventional power indices (e.g., the Shapley-Shubik or Banzhaf index) one can think of this as having most others "vote first" and cancel (balance out) each others' votes, leaving those few in the middle to determine the winning coalitions. Similar arguments apply to Supreme Court Justices (Blasecki, 1990) and blocks of voters in general elections. Of course, independents do not always reap rewards; in "machine politics" parties dole out favors to loyalists and independents are disadvantaged. The claim here is only that there are some situations in which unaligned moderates have leverage, not that it is always the case. (Dixit and Londregan, 1996) characterize when each situation pertains.

"Arbitration" - both Formal and Less Formal

Formal arbitrators wield considerable power, regularly resolving multi-million dollar commercial disputes, particularly in labor-management relations. Arbitrators jeopardize their ability to obtain that power if they become tainted by bias (Kaufman and Duncan, 1992). E.g., the neutrality of the arbitrator is a non-negotiable precondition mentioned explicitly in the National Arbitration Forum's "Arbitration Bill of Rights". Less formally, a neutral third party can command respect in a partisan dispute. Statements from either side of the dispute are given little credence; they are perceived as mere posturing. In contrast, the opinions of an unaligned third party who has protected a reputation for impartiality are taken seriously. Consultants are sometimes hired to obtain such an independent perspective, and however poor the methodology (Clarke, 2002), US News and World Report rankings of universities are taken more seriously than individual school's own self-serving claims to offer the best education. At a macro level, India sought leadership among third world nations by remaining unaligned in the Cold War. In highly charged contexts such that "he who is not my friend is my enemy", third parties may find it hard to resist becoming aligned with one side or the other, but may preserve a unique power if they can do so.

"Preserving a Balanced World View"

A more abstract example is an individual seeking to maximize intellectual honesty by remaining open-minded. Psychologists have shown that people often suffer from a "confirmation bias" (Gilovich, 1991). Once we begin to believe one thing, we are most attentive to evidence that reinforces that prior belief and are more skeptical toward contrary evidence. So with respect to any issue - e.g., "public schools are [or are not] doing a good job" - if one does not exert effort to avoid it, and bias in one direction or another may tend to be amplified over time.

What these situations have in common is that maintaining a "middle ground" is valuable but difficult. I.e., it can require effort, which presumably is costly. Visually, one can imagine this as standing on top of a narrow hilltop, with "slippery slopes" leading off to the left and to the right. Remaining atop the hill confers benefit, and at the peak the pulls from either side are balanced. However, if one moves a bit to the left or to the right, one has to fight (gravity) to keep from slipping further. If one falls, one does not fall forever. That is, the cross-section of the hill looks more like a bell-curve than an upside down U. Once one has moved from the middle all the way to one side or the other, there is little pressure to move beyond that point. E.g., there is not in general a force that would tend to push a Congressperson to be more conservative [or liberal] than the conservative [liberal] party whip wants that Congressperson to be.

A stylized model of this has a state variable x denoting the decision maker's position, with -1 and 1 representing the positions of the two opposing sides and $x = 0$ being the sought-for middle ground. If the state is currently between -1 and 1 there is tendency to be pulled out to the end of that range that is closest. If for some reason the decision maker moved beyond -1 or 1 (i.e.,$|x| > 1$), there would be a tendency to drift back to the nearest pole. Those tendencies could, however, be moderated by exerting effort (control variable ν) to adjust the state, but at a (convex) cost. Various functional forms might be suitable. One that is analytically convenient is

$$\min_{\nu} \int_0^{\infty} e^{-rt} \left(x^2 + c\nu^2\right) dt \tag{1}$$

$$\text{s.t} \qquad \dot{x} = x - x^3 + \nu, \tag{2}$$

where $r > 0$ is a discount rate and parameter c governs the cost of adjusting one's position.

ANALYSIS OF THE MODEL

This section consists of five subsections. In the first subsection the optimality conditions are stated. After that the steady state properties of the system are summarized. In the next subsection the dynamic behavior of the model near the bifurcation lines is analyzed. Thereafter the model is explored in respect to optimal strategies. In the last subsection the optimal strategies are analyzed as parameters vary.

Optimality Conditions

The optimal control problem persuted above is given by

$$\begin{aligned} \min_{\nu} \int_0^{\infty} e^{-rt} \left(x^2 + c\nu^2\right) dt \\ \text{s. t.} \qquad \dot{x} = x - x^3 + \nu, \end{aligned} \tag{3}$$

with state variable x, control variable ν, discount rate r and costs c. To solve this problem by applying Pontryagin's minimum principle (see, e.g., (Feichtinger and Hartl, 1986)) we consider the current value Hamiltonian

$$H = x^2 + c\nu^2 + \lambda(x - x^3 + \nu), \tag{4}$$

where λ denotes the co-state variable in current value terms.

Following the standard methods of optimal control theory we derive the necessary optimality condition

$$\nu^{\star} = \arg\min_{\nu} H,$$

which implies

$$\lambda = -2c\nu \Leftrightarrow \nu = -\frac{\lambda}{2c}, \tag{5}$$

by setting $H_\nu = 0$.

Since $H_{\nu\nu} = 2c > 0$ the Legendre-Clebsch condition is satisfied. Furthermore, the co-state equation is given by

$$\dot{\lambda} = r\lambda - H_x = \lambda\left[r + (3x^2 - 1)\right] - 2x. \tag{6}$$

Considering Eq. 5 we derive the differential equation for the control variable ν

$$\dot{\nu} = \nu\left(r + (3x^2 - 1)\right) + \frac{x}{c}. \tag{7}$$

The state equation Eq. 4 and the adjoint equation Eq. 6, where ν is given by the Hamiltonian minimizing condition Eq. 5, yield the canonical system as necessary optimality conditions for our control problem. Since the Hamiltonian is not convex with respect to the state variable, the usual (Mangasarian) sufficiency conditions are not satisfied.

Regions of Stability

We next examine the steady states of the canonical system as functions of its two parameters r and c and determine their stability properties. As we can see in Fig. 1, the parameter space is divided into four regions, with different numbers of steady states and different stability properties. (See also Tab. 1.) The exact computations of the steady states and their properties for these regions can be found in Appendices A and B, respectively. The origin is always a steady state. There are in addition up to two pairs of steady states on either side of the origin. The two pairs are referred to as the inner and outer steady states based on their distance from the origin.

Insert Figure 1 About Here

Insert Table 1 About Here

Region I

For parameters lying in Region I the only steady state is at the origin. This equilibrium is a saddle, and the region is delimited by the positive r and c axes and the curve (labeled α) defined by $c = 12/(r+2)^2$, for $r \leq 4$ and by $c = 1/(r-1)$ for $r \geq 4$.

Region II

In Region II there are five equilibria. The origin and outer equilibria are saddles, and the two inner equilibria are unstable foci. The curve labeled β forms the boundary of Region II. It is defined by two rational functions whose exact formulae are found in Appendix B. As will be seen shortly, Region II is split into two subregions with different behavior of optimality.

Region III

Region III also has five equilibria with saddles at the origin and two outer equilibria, but the inner equilibria have mutated into unstable nodes. Region III lies between the curves α, β and γ_1, where the latter is defined by $c = 1/(r-1)$

for $1 < r < 4$.

Region IV

The big difference between Region IV and the other regions is that the origin is an unstable node, not a saddle. No inner equilibria exist, but the outer equilibria remain saddles. Region IV lies above the curve γ, defined by $c = 1/(r-1)$ for $r > 1$.

Bifurcation lines

These four regions are divided by bifurcation curves. (See Fig. 1.) Crossing these curves can mean a dramatic change in the system's behavior. New equilibria can emerge while others disappear or change their stability properties. As the possibility of such limiting cases is zero, they are of no vital importance for applications, but they nevertheless give insight into the mathematical formulations of radical changes in the model behavior as parameters vary. At these lines, catastrophes can take place, and even models near such bifurcation lines can show some strange behavior.

Bifurcation line α

For models with parameters lying on this line, the origin is a saddle, and the inner and outer equilibria coincide at non-isolated fixed points. When moving from Region I onto this line, the saddle at the origin in Region I trifurcates into one saddle and two non-isolated fixed points. Continuing by moving on into

Region II, each non-isolated fixed point bifurcates further into a saddle and an unstable node.

Bifurcation line β

This line separates region II, where the inner equilibria are nodes, from region III, where the inner equilibria are unstable foci. Lying on β they take on an intermediate state as degenerated nodes. Curve β lies above line α and they intersect at $r = 0$ and $c = 3$.

Bifurcation line γ

Bifurcation line γ is split into γ_1 for $1 < r < 4$ and γ_2 for $r \geq 4$. On γ_1 the origin changes from a saddle to a non-isolated fixed point. This is the intermediate state between being a saddle in region III and an unstable node in region IV. The outer equilibria are unchanged as saddles, while the inner equilibria cease to exist at γ_1.

On γ_2 the origin is also a non-isolated fixed point, but no other equilibria exist. To be exact the outer equilibria coincide with the origin for $r > 4$, while for $r = 4$ all equilibria coincide at the origin. So for this case no saddle exists and therefore the standard methods to find an optimal (extremal) solution are not applicable. Heuristically this strange behavior can be understood by considering that γ_2 divides region I and IV which have very different optimal behavior, as we will now elaborate.

Optimal Strategies

Having analyzed the dynamic systems in terms of equilibria and their properties, we next explore when various strategies are optimal. It turns out that there are essentially three strategies that may be optimal depending on the values of parameters r and c: (1) always move to the middle (origin), (2) (almost) always fall off to one side or the other, and (3) decide based on one's initial position.

The stability regions and bifurcation lines play an important role in defining when the various strategies are optimal, but the correspondence is not one for one. In particular, stability Region II, which is defined by two rational functions continuously connected at $r = 2(\sqrt{3} - 1)$, is divided into two subregions (Region IIa and Region IIb) by the bifurcation line δ. (See Appendix B for details.) So different strategies are optimal in different parts of a single stability region (namely Region II), and the same strategy may be optimal for different stability regions (e.g., Regions I and IIa).

Strategy A: Always Move to the Middle

In stability Regions I and IIa, it is always optimal to move to the origin, regardless of the starting position ($x(0)$). This makes intuitive sense because in these regions r and c are small. Hence the cost of exerting effort is modest (c small) and the low discount rate (r small) implies that the decision maker weighs the long-run future benefits of being at the origin heavily relative to the short-run costs of exerting effort to get there.

Strategy B: (Almost) Always Fall Off to One Side or the Other

Stability Region IV represents the opposite case. If the starting position is exactly at the origin, it is optimal to stay there. Otherwise, parameters r and c are so large that if the decision maker ever deviates even slightly from the origin, the decision maker is so short-sighted and the costs of control so big that the benefits of returning to the origin are not worth the effort. That does not mean that the optimal strategy is to be utterly passive. It is still optimal to exert some effort to slow the slide, but not enough to alter the end result.

Strategy C: Move to the Middle if and only if One Starts Nearby

Regions IIb and III present an intermediate case. If the decision maker's initial position is not too far from the origin, it is optimal to move back to the middle. But if the initial position is too far from the origin, moving to the origin is not worth the effort, although again some effort should be exerted to slow the slide. In between there are points of indifference, one on either side of the origin, from which the decision maker is equally happy moving left or right.

The character of the indifference points differs, however, in Regions IIb and III. In Region III where the inner equilibria are unstable nodes, the indifference points occur at the inner (unstable) equilibria, and the initial level of effort ($\nu^{\star}(0)$) is only infinitesimally different whether one (arbitrarily) chooses to move left or right from that point.

In Region IIb the indifference points do not necessarily correspond to the inner equilibria. Furthermore, from the indifference point, if one chooses (arbitrarily) to move back to the origin the initial level of effort ($\nu^\star$) is noticeably greater (in absolute value) than is optimal if one chooses instead to move out to the outer equilibria. In both cases, the indifference points are so-called DNS points.

Change in Optimal Strategy As Parameters Vary

This section examines in more detail how the optimal strategy varies as one of the two parameters in turn is increased.

Increasing c for Fixed Values of the Discount Rate r

Figure 2 shows phase portraits when $r = 1.5$ and the cost parameter c is $0.5, 12/3.5^2, 1.75$, and 2.5, respectively. When costs are low ($c = 0.5$; Fig. 2a) it is always optimal to move to the origin. As c increases further toward the bifurcation line α, the $\dot{x} = 0$ and $\dot{v} = 0$ isoclines begin to converge creating a bottleneck through which trajectories originating further outside have a hard time passing. This visually corresponds to the increasing costs of reaching the origin when one starts far away.

Insert Figure 2 About Here

When c reaches the bifurcation line ($c = 12/3.5^2$; Fig. 2b) the isoclines touch creating non-isolated fixed points. This is the critical case, where the origin is optimal for states starting inside the inner equilibria and standard methods cannot be used to analyze cases outside this interval.

Increasing c beyond this bifurcation line splits the non-isolated fixed points into a saddle (outside equilibrium) and an unstable node (inside equilibrium), and the origin is only the optimal endpoint for initial conditions inside the indifference points, i. e. the inner equilibria (Fig. 2c). Outside that range, the extreme positions (outer equilibria) are optimal. As c increases further, the range of initial conditions for which it is optimal to converge to the origin shrinks until at bifurcation line γ, the inner equilibria (thresholds) coincide with the origin, and the origin becomes a non-isolated fixed point, and the optimal strategy is to move out to the extremes unless one begins precisely on the origin.

Increasing c with r fixed at 0.5 gives slightly different results (figures not shown). The cost parameter can become very large before it is no longer optimal to always converge to the origin, and there is no critical change when crossing bifurcation curve α. From a heuristic point of view, this means the decision maker is far-sighted enough to want to reach the origin even when there are strong alternatives (the outer equilibria exist). In mathematical terms the difference can be found in the relationship between the stable manifold and the emerging outer equilibria. While for $r \geq 1$ these equilibria lie on the stable manifold, that is not the case for $r < 1$. So driving to the origin remains always optimal until c reaches the DNS-bifurcation line δ. Numerical calculations show that there is no r small enough to make going to the origin universally optimal, without regard to both the cost parameter c and the initial position. Rather δ

crosses the y axis. Extrapolation of the bifurcationline δ and direct calculations suggests this crossing point in about $c \approx 55$. (I.e., for high enough costs, the origin is never universally optimal.)

Increasing c with $r > 4$ gives a simple but quite different pattern. Strategy A (always going to the origin) is optimal for $c < \gamma$, and Strategy B (almost always not going to the origin) is optimal for $c \geq \gamma$. There is never an intermediate case (Strategy C) when $r > 4$.

Increasing r for Fixed Values of the Cost Parameter c

We next consider how the optimal solution changes for a given cost parameter c as the discount rate parameter r increases, i.e., as the decision maker gets more and more myopic. In particular, Figures 3a and 3b show phase portraits when $c = 2.5$ and the discount rate parameter r is 0.32 and 0.8, respectively.

Insert Figure 3 About Here

When $c = 2.5$ and r is small it is always optimal to converge to the origin. That remains true until r reaches bifurcation line δ even though between lines α and δ two other sets of equilibria emerge, because those equilibria are bypassed by the optimal solution. (See Fig. 3a.) Between δ and β there are two DNS points, one on either side of the origin. For initial conditions between the DNS points it is optimal to converge to the origin; for initial conditions outside the DNS points, it is optimal to converge to the nearer of the two outer equilibria. (See Fig. 3b.) Two things happen as r increases within this DNS region (Region IIa). First, the DNS points move inward. Second, the gap shrinks between the optimal initial control level when one moves right and the optimal initial control level when one moves left. That gap shrinks to zero when r reaches bifurcation line β that separates Region IIb, where the inner equilibria are unstable foci, from Region III, where they are unstable nodes.

As r increases further in Region III those inner equilibria (unstable nodes) continue to converge until they collapse into the origin at bifurcation line γ_1. I.e., the decision maker becomes so present-oriented, that it is never worth working to get back to the origin.

When c is somewhat smaller (e.g., $c = 1$) the sequence as r increases is similar except that crossing bifurcation line α takes one directly to the condition of two unstable foci (inner equilbria) separating regions where it is optimal to converge to the origin from those where it is optimal to approach the outer equilibria.

When c is very small such an intermediate region does not exist. To the left of γ it is always optimal to converge to the origin; to the right, for any initial position other than the origin, it is optimal to slide out to the outer equilibria. Note that when control costs are extremely small, the γ curve occurs for very large values of the discount rate ($r > 4$) and so may not be practically relevant.

CONCLUSION

Even though this "model of moderation" is very simple (one state, one co-state, and just two parameters), from a mathematical point of view very interesting features emerge. Despite the model's simplicity, the existence of multiple DNS points has been shown. Moreover, we found a region in parameter space split by a DNS bifurcation line. Even though the number of equilibria and their properties are the same throughout this region, DNS points exist on one side of this bifurcation line but not on the other. Furthermore, precisely because of the model's simplicity, analytical expressions can be written for this and the model's other various bifurcation lines and associated regions.

The model's optimal solutions have consistent and sensible interpretations, even in the critical extreme cases, and a variety of intriguing extensions can be envisioned. To begin with, other functional forms for the state dynamics could be investigated representing different "curvatures" of the "slippery slope". Also, the model is essentially separable at the origin because trajectories involving positive state values never become negative and vice versa. Hence, there is no reason why investigations need to be restricted to cases that are symmetric about the origin. Furthermore, in this paper the one-dimensional state space can be thought of as reflecting the cross-section of a hill. That is appropriate when the "issue space" in question is one-dimensional with two opposing camps. Sometimes, however, the neutral position is not intermediate between just two alternatives but rather is central relative to a large number of alternatives that can not neatly be arrayed along a line. Hence, two-dimensional versions of the model could be of interest, and their investigation might yield closed two-dimensional DNS thresholds, which to the best of our knowledge have never before been discovered in applied models.

A EXISTENCE OF EQUILIBRIA

Considering the dynamical system

$$\begin{aligned} \dot{x} &= x - x^3 + \nu \\ \dot{\nu} &= \nu(r + 3x^2 - 1) + \frac{x}{c} \end{aligned} \tag{8}$$

the equilibria must satisfy

$$\begin{aligned} \nu &= x^3 - x \\ \nu &= -\frac{x}{c(r + 3x^2 - 1)}. \end{aligned}$$

Setting these expressions for ν equal to each other and assuming $x \neq 0$, we get

$$\begin{aligned} -1 &= c\left(r + \left(3x^2 - 1\right)\right)\left(x^2 - 1\right) \\ 0 &= c\left(3x^4 + (r-4)\,x^2 + 1 - r\right) + 1 \qquad \text{setting } y = x^2 \\ 0 &= c\left(3y^2 + (r-4)\,y + 1 - r\right) + 1, \end{aligned}$$

a quadratic in $y = x^2$, whose solutions are

$$y = \frac{4 - r \pm \sqrt{(r+2)^2 - 12/c}}{6}. \tag{9}$$

Abbreviating $w = \sqrt{(r+2)^2 - 12/c}$ the formal solutions of Eq. 9 together with $x = 0$ are

Comparing the x values of E_2-E_5 it is obvious that in absolute values the x coordinates of E_2 and E_3 are always larger then E_4 and E_5. This justifies

	$\hat{x}$	$\hat{\nu}$
E_1	0	0
E_2	$\sqrt{\frac{4-r+w}{6}}$	$-\sqrt{\frac{4-r+w}{6}}\left(\frac{r+2-w}{6}\right)$
E_3	$-\sqrt{\frac{4-r+w}{6}}$	$\sqrt{\frac{4-r+w}{6}}\left(\frac{r+2-w}{6}\right)$
E_4	$\sqrt{\frac{4-r-w}{6}}$	$-\sqrt{\frac{4-r-w}{6}}\left(\frac{r+2+w}{6}\right)$
E_5	$-\sqrt{\frac{4-r-w}{6}}$	$\sqrt{\frac{4-r-w}{6}}\left(\frac{r+2+w}{6}\right).$

referring to E_2 respectivly E_3 as outer equilibria, wheras we refer to E_4 and E_5 as inner equilibria.

Now we determine the regions of existence for the equilibria. Considering Eq. 9 and noticing that $y = x^2$ the existence of equilibria is determined by the two conditions $y \geq 0$ and $w \in \mathbb{R}$. These conditions can be written as

$$(r+2)^2 - 12/c \geq 0 \tag{10}$$

$$4 - r \pm w \geq 0. \tag{11}$$

Setting these conditions to zero, we get two boundary curves delimiting the regions to be considered. After a short calculation we get $c = 12/(c+2)^2$ and $c = 1/(r-1)$ from Eqs. 10-11. These two curves are tangent to each other at $r = 4$. Now every different case given by these solutions has to be analysed to prove the existence of the equilibria. To cover every region with a different number of equilibria we have to distinguish eight cases. The solution is summarized in Tab. 1. In the following derivations we only consider the inner and outer equilibria.

Case 1: $c < 12/(c+2)^2 \Rightarrow w \leq 0$ and therefore w is not real, and no equilibria exist apart from the origin.

Case 2: From $c = 12/(c+2)^2$ and $r < 4$ we get $w = 0$. This implies that inner and outer equilibria coincide.

Case 3: For $r = 4$ and $c = 1/3$ all equilibria coincide with the origin.

Case 4: Considering $r > 4$ and $c < 1/(r-1)$ we see that $4 - r - w < 0$ and therefore we only have to find the sign of $4 - r + w$. But for that case the following equivalence relation holds

$$\begin{aligned} c < \frac{1}{r-1} \quad &\Leftrightarrow \quad 12r - 12 - \frac{12}{c} < 0 \\ &\Leftrightarrow \quad (r+2)^2 - \frac{12}{c} < r^2 - 8r + 16 \\ &\Leftrightarrow \quad w^2 < (r-4)^2 \qquad r > 4!! \\ &\Leftrightarrow \quad w < r - 4 \\ &\Leftrightarrow \quad 4 - r + w = y < 0, \end{aligned}$$

and so no real solutions for $y = x^2$ exist and therefore we have no equilibria apart from the origin in this case.

Case 5: For $c > 12/(r+2)^2$ and $r \leq 1$ we get $w > 0$ and $w \leq \sqrt{3(3-4/c)} < 3 \leq 4 - r$ implying $4 - r \pm w > 0$, therefore all five equilibria exist and are distinct.

Case 6: Assuming $c > 12/(r+2)^2$, $c < 1/(r-1)$ for $1 < r < 4$, yields $3 - 1/c > w > 0$ and $4 - r > 3 - 1/c$. Therefore $4 - r \pm w > 0$, and as in the latter case all five equilibria exist and are distinct.

Case 7: For $r < 4$ and $c = 1/(r-1)$ the following equation holds

$$\begin{aligned} w &= \sqrt{(r+2)^2 - \frac{12}{c}} \\ &= \sqrt{r^2 - 8r + 16} \\ &= |r-4| \\ &= 4 - r, \end{aligned} \tag{12}$$

this yields $y = 4 - r - w = 0$ and $y = 4 - r + w > 0$. Therefore the inner equilibria coincide with the origin, while the outer equilibria exist and differ from the origin.

Case 8: For $r > 4$ and $c = 1/(r-1)$ considering Eq. 12, $y = 4 - r + w = 0$ and $y = 4 - r - w < 0$ holds. This means the outer equilibria coincide with the origin but no inner equilibria exist.

Case 9: $c > 1/(r-1)$ implies $w > |r-4|$ which in turn implies $4 - r + w > 0$ and $4 - r - w < 0$. Therefore only the outer equilibira exist.

B STABILITY PROPERTIES

Knowing the number of equilibria for the different regions, we analyse now their stability properties. The characterization of the equilibrium behavior ensues from calculating the determinante, trace and discriminant of the Jacobi matrix J.

We get the common form of J, by linearizing the system of differential equations Eq. 8

$$J(x,\nu) = \begin{pmatrix} 1-3x^2 & 1 \\ 6x\nu + 1/c & r+3x^2-1 \end{pmatrix} \tag{13}$$

calculating Δ, τ and D gives

$$\begin{aligned} \tau &= r \\ \Delta &= -(3x^2-1)(r+3x^2-1) - (6x\nu + 1/c) \\ D &= r^2 - 4((3x^2-1)(r+3x^2-1) - (6x\nu + 1/c)) \end{aligned}$$

with

$$\begin{aligned} \tau &\ \ldots\ \mathrm{tr}(J) \\ \Delta &\ \ldots\ \det(J) \\ D &\ \ldots\ \tau^2 - 4\Delta. \end{aligned}$$

In the following subsections these formal results will be analyzed for the different equilibria.

Origin

At the origin the Jacobi matrix Eq. 13 simplifies to

$$J(0,0) = \begin{pmatrix} 1 & 1 \\ 1/c & r-1 \end{pmatrix}, \tag{14}$$

and so we get

$$\tau = r \tag{15}$$

$$\Delta = r - 1 - 1/c \tag{16}$$

$$D = r^2 - 4r + 4 + 4/c \tag{17}$$

The stability properties are completely determined by the signs of the three parameters Δ, τ and D. As $\tau = r > 0$ always holds we only have to consider the occurrence of $\Delta = 0$ and $D = 0$. Furthermore it can be seen immediatly that $D > 0 \quad \forall r, c > 0$. Therefore only equation $\Delta = 0$ has to be analyzed.

We distinguish three cases for $\mathrm{sgn}(\Delta)$ and therefore three regions in parameter space.

Case 1: $\Delta < 0 \Leftrightarrow r < 1 + 1/c$ together with $\tau > 0$ and $D > 0$ characterizes a saddle point.

Case 2: $\Delta = 0 \Leftrightarrow r = 1 + 1/c$ gives the critical case of a non-isolated fixpoint at the origin.

Case 3: $\Delta > 0 \Leftrightarrow r > 1 + 1/c$ is associated with an unstable node.

Considering the division of the parameter space given in Fig. 1 we see that the origin is a saddle in regions I, II and III, a non-isolated fixpoint at γ, and an unstable node in region IV.

Inner Equilibria

In case of an inner equilibria the Jacobi matrix Eq. 13 becomes

$$J = \frac{1}{6} \begin{pmatrix} 3(r + w - 2) & 6 \\ (w^2 + 2(1-r)w - 2r - 8) + 6/c & 3(r - w + 2) \end{pmatrix}$$

and

$$
\begin{aligned}
\tau &= r \\
\Delta &= \frac{1}{12}(-5w^2 + 4(4-r)w + (r+2)^2) - \frac{1}{c} \\
D &= \frac{1}{3}(5w^2 - 4\,(4-r)\,w + 2r^2 - 2r - 2) + \frac{12}{c}.
\end{aligned} \tag{18}
$$

Finding the regions where $\mathrm{sgn}(\Delta)$ differs we have to solve $\Delta = 0$, which implies

$$
\begin{aligned}
0 &= \frac{1}{12}(-5w^2 + 4(4-r)w + (r+2)^2) - \frac{1}{c} \\
0 &= \frac{1}{12}(-4w^2 + 4(4-r)w) \\
0 &= w^2 - (4-r)w \\
0 &= w(w - 4 + r)
\end{aligned}
$$

and therefore the following equations must hold

$$
w = 4 - r \quad \text{or} \quad w = 0.
$$

Since inner equilibria only exist for $r \leq 4$ the inequality $w \geq 0$ holds and we get

$$
\Delta = 0 \Leftrightarrow \begin{cases} c = \dfrac{1}{r-1} & r \leq 4 \\[2em] c = \dfrac{12}{(r+2)^2} & \forall r. \end{cases}
$$

Next we analyze the case $D = 0$. Starting from equation 18 we get

$$
\begin{aligned}
D &= \frac{1}{3}\left(5w^2 + 4(r-4)w + 2r^2 - 4r - 4 + \frac{12}{c}\right) \\
&= \frac{1}{3}\left(5w^2 + 4(r-4)w + 3r^2 - r^2 - 4r - 4 + \frac{12}{c}\right) \\
&= \frac{1}{3}\left(4w^2 + 4(r-4)w + 3r^2\right).
\end{aligned}
$$

This is a quadratic in w with solutions

$$w_{1,2} = \frac{1}{2}\left(4 - r \pm \sqrt{-2(r^2 + 4r - 8)}\right). \tag{19}$$

As $w \geq 0$ is demanded we have to make sure that the right side of Eq. 19 is real and positive. Finding the zeros of $r^2 + 4r - 8$

$$r_{1,2} = 2(\pm\sqrt{3} - 1),$$

it is obvious that the solutions of Eq. 19 for $0 < r \leq 2(\sqrt{3} - 1)$ are real. Next we have to show the positivity of 19. Since $r \leq 2(\sqrt{3} - 1)$ we get $r < 4$ and so positivity is given for the positive root. Therefore only the case of the negative root has to be analyzed. But in this case we get the following equivalence

$$\begin{aligned}
4 - r - \sqrt{-2(r^2 + 4r - 8)} \geq 0 \quad &\Leftrightarrow \quad 4 - r \geq \sqrt{-2(r^2 + 4r - 8)} \\
&\Leftrightarrow \quad 16 - 8r + r^2 \geq -2r^2 - 8r + 16 \\
&\Leftrightarrow \quad r^2 \geq 0,
\end{aligned}$$

which holds since $4 - r > 0$.

As we have found now the domain where equation Eq. 19 holds, we can start finding the solution curves explicitly. Replacing w by its definition yields to equation

$$\begin{aligned}
\sqrt{(r+2)^2 - \frac{12}{c}} &= \frac{1}{2}\left(4 - r \pm \sqrt{-2(r^2 + 4r - 8)}\right) \\
4\left((r+2)^2 - \frac{12}{c}\right) &= \left(4 - r \pm \sqrt{-2(r^2 + 4r - 8)}\right)^2 \\
5r^2 + 32r - 16 - \frac{48}{c} &= \pm 2(4 - r)\sqrt{-2(r^2 + 4r - 8)}.
\end{aligned}$$

Expressing c explicitly we get

$$
\begin{aligned}
c_{1,2} &= \frac{48}{5r^2+32r-16\pm 2(4-r)\sqrt{-2(r^2+4r-8)}} \\
c_{1,2} &= \frac{48\left(5r^2+32r-16\mp 2(4-r)\sqrt{-2(r^2+4r-8)}\right)}{33r^4+288r^3+672r^2-768} \\
c_{1,2} &= \frac{16\left(5r^2+32r-16\mp 2(4-r)\sqrt{-2(r^2+4r-8)}\right)}{(r+4)^2(11r^2+8r-16)}.
\end{aligned}
$$

These are rational functions continuously connected at $r = 2(\sqrt{3}-1)$ and can therefore be treated as one curve δ. As this curve is given by rational functions we have to find the singularities and the behavior at these singularities. Setting $11r^2 + 8r - 16 = 0$ gives the solutions

$$-\frac{4}{11}\left(1 \pm 2\sqrt{3}\right),$$

as we are only interested in positive values, the only interesting solution is

$$-\frac{4}{11}\left(1 - 2\sqrt{3}\right). \tag{20}$$

Readily it can be seen that Eq. 20 is the only candidate for a singularity, because $(r+4)^2$ has no positive solution. Considering the two curves at this singularity it can easily be proven that in case of c_1 it is a real singularity, while in case c_2 the singularity can be lifted.

We have proved now, that the domain where δ exists is part of the domain where the inner equilibira exist. Therefore δ is delimiting regions with different $\mathrm{sgn}(D)$ and therefore different stability properties, for the inner equilibria. Evaluating D at points for the two different regions, we get $D > 0$ inside region II and $D < 0$ inside region III. Combining this result with Eq. 20 the inner equilibria are unstable saddles in region III, unstable nodes in region II, and degenerate nodes on the curves δ. (See Tab. 1.)

Outer Equilibria

Similar to the case of the inner equilibria the Jacobi matrix Eq. 8 becomes

$$J = \frac{1}{6} \begin{pmatrix} 3(r+w-2) & 6 \\ (w^2 + 2(1-r)w - 2r - 8) + 6/c & 3(r+2+w) \end{pmatrix}$$

and

$$\begin{aligned} \tau &= r \\ \Delta &= \frac{1}{12}\left(-5w^2 - 4(4-r)w + (r+2)^2\right) - \frac{1}{c} \\ D &= \frac{1}{3}\left(5w^2 + 4(4-r)w + 2r^2 - 2r - 2)\right) + \frac{1}{c} \end{aligned}$$

Solving $\Delta = 0$ leads to

$$0 = \frac{1}{12}\left(-4w^2 + 4(r-4)w\right).$$

Therefore the following equations must hold

$$w = r - 4 \quad \text{or} \quad w = 0.$$

Since $w \geq 0$ has to be guaranteed we get

$$\Delta = 0 \Leftrightarrow \begin{cases} c = \dfrac{1}{r-1} & r \geq 4 \\ c = \dfrac{12}{(r+2)^2} & \forall r \end{cases}$$

Analogous to the case of the inner equilibria the following equation for $D = 0$ holds.

$$\frac{1}{3}\left(4w^2 + 4(4-r)w + 3r^2\right) = 0$$

This is a quadratic in w with solutions

$$w_{1,2} = \frac{1}{2}\left(r - 4 \pm \sqrt{-2(r^2 + 4r - 8)}\right).$$

We have the same constraints on r as for the inner equilibria. Therefore the right side is always negative, and the quadratic has no solution. This means D does not change sign for the outer equilibira. Evaluating D for an arbitrary point shows $D > 0$. Summarizing the results of this subsection, the outer equilibria are saddles in regions II, III and IV as well as on the curves β and γ_1, and they are non isolated fixpoints at curves α and γ_2. (See Tab. 1.)

References

Blasecki, J. L. (1990). Justice Lewis F. Powell: Swing voter or staunch conservative? *The Journal of Politics*, 52(2):530–547.

Clarke, M. (2002). Quantifying quality: What can the us news and world report rankings tell us about the quality of higher education? *Education Policy Analysis Archives*, 10(16).

Dixit, A. and Londregan, J. (1996). The determinants of success of special interests in redistributive politics. *The Journal of Politics*, 58(4):1132–1155.

Feichtinger, G. and Hartl, R. F. (1986). *Optimale Kontrolle ökonomischer Prozesse*. Walter de Gruyter.

Gilovich, T. (1991). *How We Know What Isn't So: The Fallibility of Human Reason in Everyday Life*. Free Press, New York, N.Y.

Kaufman, S. and Duncan, G. T. (1992). A formal framework for mediator mechanisms and motivations. *Journal of Conflict Resolution*, 36(4):688–708.

Ordeshook, P. (1986). *Game Theory and Political Science*. Cambridge University Press.

Schickler, E. and Rich, A. (1997). Controlling the floor: Parties as procedural coalitions in the house. *American Journal of Political Science*, 41(4):1340–1375.

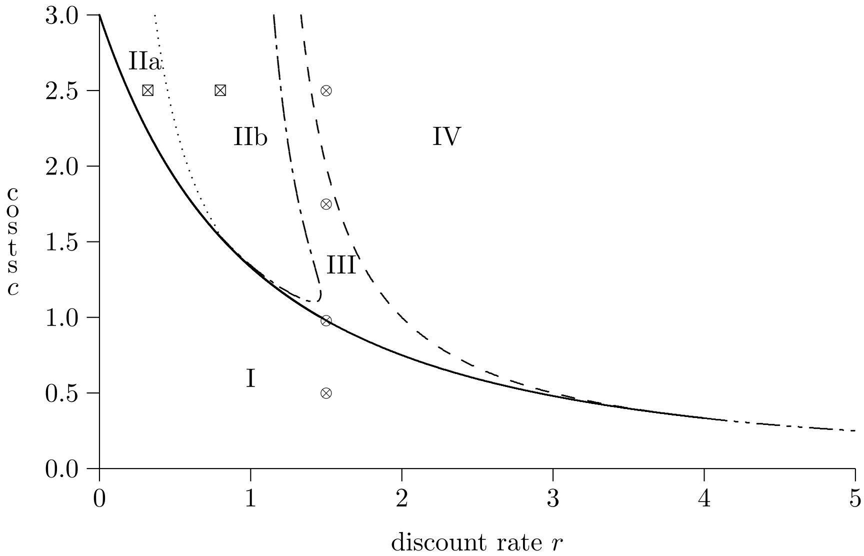

Figure 1: Regions of different stability and optimality divided by bifurcation lines α, β, γ and DNS bifurcation line δ.
$\otimes$ and $\boxtimes$ mark different positions of models depicted in Fig. 2 and Fig. 3.

Caption

	α
	β
	γ_1
	γ_2
	δ
$\otimes$	$r = 1.5$
$\boxtimes$	$c = 2.5$

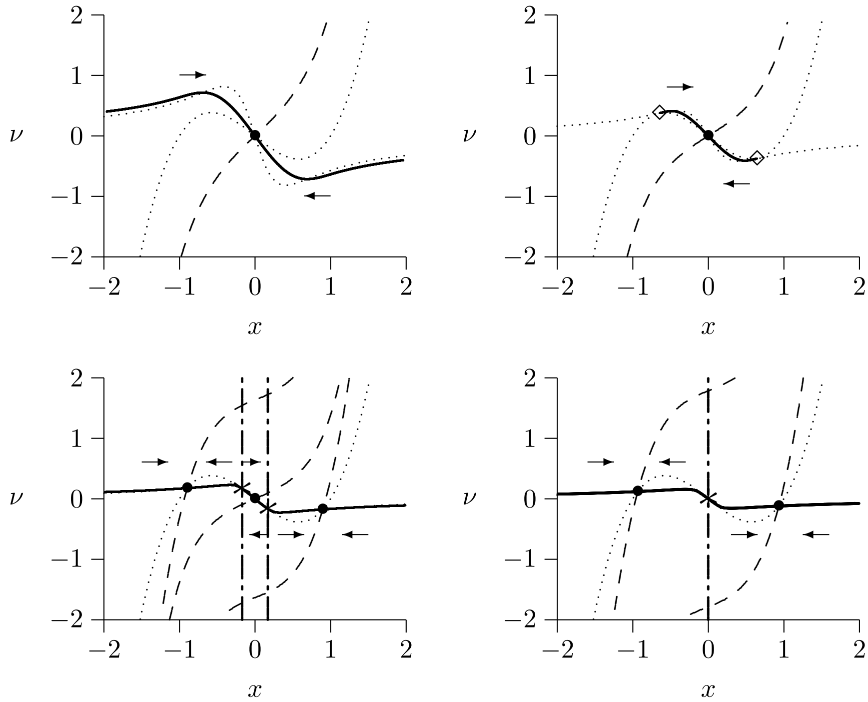

Figure 2: For constant discount rate $r = 1.5$ and different costs c the system dynamics is shown together with its optimal behavior and direction, strating in the upper left and moving clockwise the four cost parameters are
a) $c = 0.5$ b) $c = 12/3.5^2$ c) $c = 1.75$ d) $c = 2.5$

Caption

—— stable manifold

—— optimal path

– – unstable manifold

······· isoclines

—·— DNS threshold

• saddle

○ unstable focus

∗ unstable node

◇ non-isolated fixed point

→ direction of optimal path

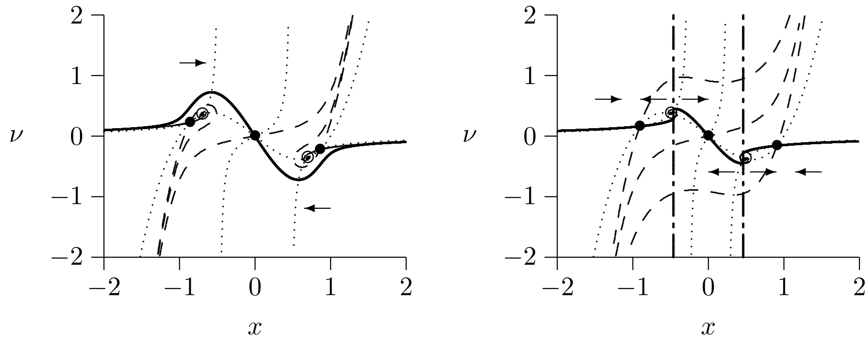

Figure 3: For constant cost $c = 2.5$ and different discount rates r the system dynamics is shown together with its optimal behavior and direction. On the left (case a) $r = 0.32$ on the right (case b) $r = 0.8$.

	Character of Equilibria at:			
Region/Curve	Origin	Inner Equilibria	Outer Equilibria	# of DNS points
I	saddle	–	–	–
II a	saddle	unstable focus	saddle	–
II b	saddle	unstable focus	saddle	2
III	saddle	unstable node	saddle	2
IV	unstable node	–	saddle	1
α	saddle	non-isolated e.*	non-isolated e.*	–
β	saddle	degenerated node	saddle	2
γ_1	non-isolated e.*	non-isolated e.*	saddle	1
γ_2	non-isolated e.*	–	non-isolated e.*	–
P_0	non-isolated e.*	non-isolated e.*	non-isolated e.*	–

Table 1: Number and properties of equilibria. See Fig. 1 for definitons of regions and bifurcation curves.
* equilibria of this row coincide