ON INJECTIVE MODULES AND COGENERATORS

by

F. Kasch, H.-J. Schneider
and H. J. Stolberg

Report 69-23

May, 1969
1. Introduction

In this paper all rings are assumed with identity and all modules are unitary. For a ring \( R \) denote by \( M_R \) respectively \( nN \) a right respectively left \( R \)-module.

A module \( M_R \) is a generator iff for every right \( R \)-module \( A_R \)

\[
A = \sum_{\varphi \in \text{Hom}_R(M, A)} \text{ima}(\varphi),
\]

where \( \text{ima}(\varphi) \) denotes the image of the homomorphism \( \varphi \). Dually \( M_n \) is a cogenerator iff for every right \( R \)-module \( A_R \)

\[
0 = \sum_{\varphi \in \text{Hom}_R(A, M)} \text{ker}(\varphi),
\]

where \( \text{ker}(\varphi) \) denotes the kernel of the homomorphism \( \varphi \).

For a ring \( R \) the modules \( R_R \) and \( R_n \) are projective and generators. In general these modules are neither injective nor cogenerators.

The representation theory of finite groups is to a large extent based on the fact that the group ring of a finite group with coefficients in a field is on both sides injective and a cogenerator.

In this connection there exists the following well known theorem (see [9],[4]): If \( R \) is Noetherian or Artinian on one
side and if \( R \) is injective or a cogenerator on one side, then \( R \) is Artinian, injective and a cogenerator on both sides.

A ring with these properties is called a Quasi-Frobenius-ring (\( = \text{QF ring} \)).

Several authors ([1],[6],[8],[9]) have considered the hypotheses of the above theorem dropping the assumptions on chain conditions.

There exist examples which show that being either injective or a cogenerator on one side does not imply all these properties for the ring on both sides.

The question arises as to which combination of injective and cogenerator properties have to be assumed to ensure that \( R \) is an injective cogenerator on both sides.

A still open (well known) conjecture in this direction states:

\[ R \text{ is injective and a cogenerator iff } R^e \text{ is injective and a cogenerator.} \]

The following results have been established.

**Theorem 1** (T. Onodera [8], T. Kato [6]): The following statements are equivalent:

1. \( R \) and \( R^e \) are cogenerators.
2. \( R \) and \( R^e \) are injective and \( R \) is a left and right S-ring.*

**Theorem 2** (F. Sandomierski, unpublished): The following statements are equivalent:

1. \( R \) is injective and \( R^e \) is a cogenerator.
2. \( R \) is a cogenerator and \( R^e \) is injective.

*For the definition of left respectively right S-ring see 3.3.
In this paper we generalize some of the results in the literature from rings to modules. Some of our proofs when restricted to the case of rings are simpler than those in the literature. Our setting is the following: Let \( R \) be a ring and \( M_\_R \), a right \( R \)-module. Set \( r = \text{End}(M_\_R) \) with \( r \) operating on the left side of \( M_\_R \) in the usual manner. Thus \( M = J_\_R \) is a \( F-R \) bimodule. If \( M = R \), we have the special case \( R = R \), which is the one dealt with in the literature. We remark that there exists a ring homomorphism \( R \rightarrow r \rightarrow \text{End}(M_\_R) \). If this mapping is an isomorphism then we say that \( R \) is naturally ring isomorphic to \( \text{End}(M_\_R) \).

In what follows we use a theorem of P. Pahl. Since this result is unpublished we include a proof of

**Theorem 3 (P. Pahl [10]):** Let \( M_\_R \) be a generator. Then the following statements are equivalent:

1. \( M_\_R \) is selfinjective.
2. \( M_\_R \) is injective.
3. \( \text{ry}_R \) is injective.

Generalizing theorem 1 and 2 we obtain

**Theorem 4:** The following statements are equivalent:

1. \( M_\_R \) and \( M_\_R \) are projective cogenerators and \( R \) is naturally ring isomorphic to \( \text{End}(M_\_R) \).
2. \( M_\_R \) and \( J_\_R \) are injective generators, \( R \) is a right \( S \)-ring and \( T \) is a left \( S \)-ring.
Theorem 5: The following statements are equivalent:

(1) $M_R$ is an injective generator and $\mathfrak{I}$ is a cogenerator.

(2) $M_R$ is a finitely generated, projective cogenerator, $J_\mathfrak{I}$ is injective and $R$ is naturally ring isomorphic to $\text{End}(\mathfrak{I})$.

(3) $M_\mathfrak{r}, jji, R_\mathfrak{r}, R_\mathfrak{r}, r, ^T$ are injective, projective, finitely generated, generators, cogenerators and semiperfect.

We intend to make this paper as self-contained as possible. Thus we include several proofs and facts which are already in the literature. Since these are spread over several papers and some are not even available we think that this may prove useful to the reader.
2. Changing sides.

The setting described in section 1 holds throughout this paper.

2.1 Lemma: Let \( x \in M, xR \) simple and \( xR \) contained in an injective submodule of \( M \). Then \( Tx \) is simple.

Proof: Let \( y_T x \neq 0 \) for some \( y \in T \). Then \( y_T xR \neq 0 \) and since \( xR \) is simple it follows that

\[
xR \ni y_T x \rightarrow y_T xR \ni y_T xR
\]

is an isomorphism. Since \( xR \) is contained in an injective submodule of \( M \), there exists \( y_1 \in T \) such that \( y_1 y_T x = x \). Hence \( Ty_1 x = Tx \) and \( Tx \) is simple.

2.2 Corollary: If \( M \) is injective then the socle of \( M \) is contained in the socle of \( T \).

2.3 Lemma: Let \( x,y \in M, yR \cong xR \) and \( xR \) contained in an injective submodule of \( M \). Then \( Tx \) is isomorphic to a submodule of \( TV \).

Proof: By assumption there exists \( y \in T \) such that

\[
yR \ni y \mapsto Ay \in xR
\]

is the given isomorphism \( yR \cong xR \). Hence there exist \( f_1 \circ f_0 \in R \) such that \( Ay = xR \), \( Ay_2 = x \). Consider the \( T \)-homomorphism

\[
Tx \ni \gamma x \mapsto \gamma xR = yAy \in TV.
\]

Since \( yxR_1 = 0 \) implies \( \gamma xR_1 = yx = 0 \), this is a monomorphism.
2.4 Corollary; Let $M$, be injective, $x, y \in M$, $yR \sim xR$ and $xR$ simple. Then $Fx$ is simple and isomorphic to $Iy$.

Proof: By 2.1 and 2.3.

2.5 Corollary; Let $x, y \in M$, $yR \sim xR$ and $yR$ and $xR$ contained in injective submodules of $M$. Then the injective hulls of $IV$ and $Fx$ are isomorphic.

Proof: By 2.3 $Fx$ is isomorphic to a submodule of $Fy$ and $TV$ is isomorphic to a submodule of $Fx$. Hence an injective hull of $Fx$ is isomorphic to a submodule of an injective hull of $Fy$ and vice versa. By [3] two injective modules which are isomorphic to submodules of each other are isomorphic. This implies 2.5.
3. Annihilators.

Let $U$ be a submodule of $M_R$. The left annihilator of $U$ in $F$ is given by

$$l_p(u) = \{ ye^{ly}U = 0 \}.$$  

Similarly for a right ideal $J_f$ of $T$ we have

$$l_{\Gamma}(\Omega) = \{ y\in\Gamma | y\Omega = 0 \}.$$  

Then $l_{\Gamma}(U)$ and $l_{\Gamma}(J_f)$ are left ideals in $I$. For a left ideal $A$ of $F$ we have two right annihilators

$$r_r(A) = \{ ye^{rl}A y = 0 \},$$  

$$r_M(A) = \{ me^{M}A n = 0 \}.$$  

Clearly $r_r(A)$ is a right ideal of $T$ and $r_M(A)$ is a submodule of $M$. Obviously these definitions do not depend on the assumption that $r$ is the endomorphism ring of $M_R$.

3.1 Lemma: Let $jJ_P$ be a cogenerator and $A$ a left ideal of $T$. Then

$$l_{\Gamma}r_{\Gamma}(A) = l_{\Gamma}r_{\Gamma}(A) = A.$$  

Proof (see for example [8], [12]):

i) $l_{\Gamma}r_{\Gamma}(A) = A$: This is well known. We give a proof for completeness. By definition $A^c l_{\Gamma}r_{\Gamma}(A)$. Assume $\gamma \in l_{\Gamma}r_{\Gamma}(A)$ and $A. \gamma$. Since $T$ is a cogenerator, there exists a $T$-homomorphism $f: r/A \rightarrow r$ such that $f(\gamma)^A 0$, where $I\gamma$ is the coset of $\gamma$ in $I/A$. Let
\( \mathbf{v} \ast \mathbf{T} \rightarrow \mathbf{T} / \mathbf{A} \)

be the natural epimorphism. Then \( \mathbf{fv}(\mathbf{e}) / 0 \) and \( \mathbf{fv} \in \text{Hom}_{\mathbf{A}}(\mathbf{T}, \mathbf{T}/\mathbf{A}) \). Hence \( \mathbf{fv} \) is given by multiplication on the right with an element \( y_0 = \mathbf{fv}(\mathbf{e}) \in \mathbf{T} \). Since \( \mathbf{fv}(\mathbf{A}) = 0 \) it follows that \( \mathbf{Ay}_0 = 0 \), that is \( y_0 \in \mathbf{r}_r(\mathbf{A}) \). Since \( \mathbf{fi}(\mathbf{e}) \neq 0 \) we have \( \mathbf{fy}_0 \neq 0 \) which contradicts \( \mathbf{e} \in \mathbf{l}_r \mathbf{r}_r(\mathbf{A}) \).

ii) \( l_r^\mathbf{M}(\mathbf{A}) = \mathbf{A} \): From \( \mathbf{r}_r(\mathbf{A}) \mathbf{M} \subseteq \mathbf{r}_r(\mathbf{M})(\mathbf{A}) \) it follows that \( l_p \mathbf{M}(\mathbf{A}) \subseteq l_r(\mathbf{r}_r(\mathbf{A}) \mathbf{M}) = l_r \mathbf{r}_r(\mathbf{A}) = \mathbf{A} \). Since \( \mathbf{A} \subseteq l_p \mathbf{M} \) the proof is complete.

3.2 Lemma ([9]): \( \mathbf{M} \) is a cogenerator iff for every simple right \( \mathbf{R} \)-module there exists a monomorphism into an injective submodule of \( \mathbf{M}_R \).

Proof: \( ^{\wedge} \): Let \( \mathbf{U}_R \) be a simple module and let \( \mathbf{U} : \mathbf{R} \rightarrow \mathbf{Q} \) be an injective hull of \( \mathbf{U}_R \). Since \( \mathbf{M}_R \) is a cogenerator there exists \( \mathbf{q_0} : \mathbf{Q} \rightarrow \mathbf{M} \) such that \( \mathbf{pr} \neq 0 \). Assume \( \ker(\mathbf{q_0}) \neq 0 \). Since \( \mathbf{T}(\mathbf{U}) \) is large in \( \mathbf{Q} \), \( \mathbf{T}(\mathbf{U}) \cap \ker(\mathbf{q_0}) \neq 0 \) and by simplicity of \( \mathbf{U} \), \( \mathbf{T}(\mathbf{U}) \subseteq \ker(\mathbf{q}) \) which implies \( \#(\mathbf{T}) = 0 \). Therefore \( \mathbf{q} \) is a monomorphism and \( \mathbf{qT} \) is a monomorphism of \( \mathbf{U} \) into the injective submodule \( \mathbf{ima}(\mathbf{q}) = \mathbf{q}(\mathbf{Q}) \) of \( \mathbf{M} \).

\( ^{\wedge_0} \): Let \( \mathbf{A}_R \neq 0 \) be an arbitrary module and let \( a \in \mathbf{A} \), \( a \neq 0 \). Let \( \mathbf{B} \) be a maximal submodule of the cyclic submodule \( a\mathbf{R} \). By assumption there exists a monomorphism \( \mathbf{T} : a\mathbf{R}/\mathbf{B} \rightarrow \mathbf{Q} \) where \( \mathbf{Q} \) is an injective submodule of \( \mathbf{M}_R \). If \( i : a\mathbf{R} \rightarrow a\mathbf{R}/\mathbf{B} \) is the natural epimorphism, then \( \mathbf{Tt}/(a) \neq 0 \). Consider the diagram
The existence of $\delta$ is assured since $Q$ is injective. But then $15 \in \text{Hom}_R(AR, M)$ and $*\delta(a) = TV(a) \wedge 0$.

Remark: By Lemma 3.2 $M_R$ is a cogenerator iff for every simple right $R$-module $U$ there exists a monomorphism $U \rightarrow M$ which can be factorized through an injective module. Dually $M_R$ is a generator iff for every simple right $R$-module $U$ there exists an epimorphism $M \twoheadrightarrow U$ which can be factorized through a projective module.

3.3 A ring $R$ is called a left $S$-ring iff for every proper left ideal $A$ of $R$, $r_A(A)^\wedge 0$. Right $S$-rings are defined similarly.

Lemma ([7]): $R$ is a left $S$-ring iff for every simple left $R$-module there exists an isomorphic left ideal in $R$.

Proof: Let $A$ be a left ideal of $R$. One verifies easily that $r_R(A) \wedge H. (R/A?ri- rxeR) \in \text{Hom}_R(R/A, R)$ is a right $R$-isomorphism.

^: Every simple left $R$-module is isomorphic to $R/A$ for a certain maximal left ideal $A$. Since $r_R(A)^\wedge 0$ this implies $\text{Hom}_R(R/A, R)^\wedge 0$ and using the simplicity of $R/A$ we get the desired isomorphism.
".: If \( R \) contains a copy of \( R/A \), where \( A \) is maximal, then
\( \text{Hom}_R(R/A, R) \neq 0 \), hence \( r_o(A) \neq 0 \). Since every proper left ideal
is contained in a maximal one \( R \) is a left \( S \)-ring.

3.4 Corollary: If \( _R R \) is a cogenerator, then \( R \) is a left \( S \)-ring. If \( _R R \) is injective and \( R \) is a left \( S \)-ring then \( R \) is a cogenerator.

Remark: This corollary gives a proof of the easy part of theorem
1. Note that the assumptions are only needed on one side in this
case.

3.5 Proposition: Let \( _K M \) be a cogenerator and \( T \) a left \( S \)-ring.
Then \( M \) is injective.

R.

Proof (Special case in [8], page 406):
Let \( T : M \rightarrow Q \) be a monomorphism of \( M \) into an injective module
\( Q \). Then \( A := \text{Hom}_R(Q, M)T \) is a left ideal in \( I \). Assume \( A \neq 0 \). Then there exists \( \phi \in T \), \( (p \neq 0) \) such that \( A\phi = 0 \). Furthermore
\( T \) a monomorphism implies \( r\phi \neq 0 \). But since \( V \) is a cogenerator,
there exists \( \psi \in \text{Hom}_R(Q, M) \) such that \( r\psi \neq 0 \). This is a
contradiction. Hence \( A = T \) and there exists \( T' \in \text{Hom}_R(Q, M) \)
such that \( L = T'T \in Jl \).
Thus \( T(M) \) is a direct summand of \( Q \), hence \( T(M) \) and as a
consequence \( M \) are also injective.

3.6 Corollary: Let \( _K M \) be a cogenerator and \( R \) a left \( S \)-ring.
Then \( R \) is injective.
Remarks:

1) Corollary 3.6 finishes the proof of theorem 1.

2) Note that in the proof of proposition 3.5 we only need that $M_\mathbb{R}$ is a $Q$ cogenerator and $Q$ can be taken to be an injective hull of $M_\mathbb{R}$.

3) By dualizing the proof we get: Let $M_\mathbb{R}$ be a generator and $r$ a right $S$-ring. Then $M_\mathbb{R}$ is projective.

3.7 Lemma ([10]): Let $M_\mathbb{R}$ be injective and $U, V$ submodules of $I_\mathbb{R}^*$. Then

$$i_r(unv) = i_r(u) + i_r(v).$$

Proof: It is easily verified that $1_{I_\mathbb{R}}(U) + l_r(V) \subseteq 1_{I_\mathbb{R}}(UDV)$.

Let $a \in l_{I_\mathbb{R}}(unv)$ and define

$$\varphi_1: U^*u \mapsto u \in M,$$

$$\varphi_2: V^*v \mapsto (1+a)v \in M.$$

Then for $w \in U \cap V$: $\varphi_1(w) = w = (1+a)w = \varphi_2(w)$. Thus we have a well-defined $R$ homomorphism

$$\varphi: U + V \ni u + v \mapsto u + (1+a)v' \in M.$$

By assumption $tp$ can be extended to an $R$ homomorphism of $M_\mathbb{R}$ into $M_\mathbb{R}$, that is there exists $p \in T$ such that $pj(U+V) = \varphi$. Hence $pu = u$ for all $u \in U$ and $pv = (1+a)v$ for all $v \in V$. Thus $a = (P-1_M) + (1_M+a-p) 6 l_p(U) + l_{I_\mathbb{R}}(V)$ and the proof is complete.

3.8 Proposition: Let $M_\mathbb{R}$ be injective and $T$ a cogenerator. Then $T$ is a semiperfect ring.
Proof: By [5] we need only show that $JT$ is complemented with respect to addition. Let $\mathcal{C} \subset j^1$ and let $B_T \subset M_T$ be maximal with respect to $r_M(A) \cap B = 0$. It follows from 3.1 and 3.7 that

$$\mathcal{I} = l_r(0) = l_r(r_M(A) n B) = l^U + l_r(B) = A + l_r(B).$$

Assume $pjl \subset l_p(B)$ and $T = A + Q$. Then $r_M(A) \cap r_M(Q) = r_M(A + J) = r_M(D = 0$. Since $B \subset r_M(B) \subset r_M(J)$, the maximality of $B$ implies $B = r_M(B) = r_M(J)$. Hence by 3.1 $SI = l_r(0) = 1(B)$ and $l_p(B)$ is minimal with respect to $F = A + 1(B)$.

3.9 Corollary: Assume $R_\sim$ is injective and $A_R$ is a cogenerator. Then $R$ is semiperfect and $R_{R_\sim}$ is a cogenerator.

Proof: That $R$ is semiperfect is immediate from 3.8. Thus there exists only a finite number, say $n$, of isomorphism classes of simple left $R$-modules and the same number for the isomorphism classes of simple right $R$-modules. Let $U_1, \ldots, U_n$ be a set of representatives for the isomorphism classes of simple right $R$-modules. By 3.2 there exist monomorphisms

$$T_i: U_i \mapsto Q_i, i = 1, \ldots, n$$

with $\mathcal{Q} \subset R$ and $Q_i$ injective. Let $U_i' = T_i(U_i)$. Then $U_i' = U_i$ and $U_i \subset Q_i$. We may then consider $U, \ldots, U_n$ as a set of representatives of the $n$ isomorphism classes and write $U^\sim = R \setminus$. It follows then from 2.1 that $x_T^\sim$ is also simple. Since $R_T$ is injective, $x_{iR}^\sim x^{\sim j}$ for $i \sim j$ by 2.3 and then by 3.2 $R_{i\sim^\sim}$ is a cogenerator.
Remarks:

1) Theorem 2 follows from corollary 3.6 and corollary 3.9.

2) Note that in the above proof we need not use the fact that \( R \) is semiperfect but only that the number of isomorphism classes of simple \( R \)-modules is finite. This follows from the fact that \( R/\text{rad}(R) \) is semisimple and this in turn is an easy consequence of the fact that \( _{-}R \) is complemented with respect to addition. More generally we prove: If \( _{\mathcal{R}}M \) is complemented with respect to addition, then \( \overline{M} = M/\text{rad}(M) \) is a semisimple \( R \)-module. To see this let \( v: M \to \overline{M} = M/\text{rad}(M) \) be the natural epimorphism and \( A \) a submodule of \( M \). Set \( A = v^{-1}(\overline{A}) \). Let \( A' \) be a complement of \( A \) such that \( M = A + A' \) and \( A' \) minimal in this equation. Then it is easy to see that \( AHA' \) is a small submodule of \( M \) and hence \( A \cap A' \subseteq \text{rad}(M) \). It follows then that \( \overline{M} = A \oplus \overline{A'} \) and \( \overline{M} \) is semisimple.
4. Generators.

In this section we consider certain results on generators which appear in an unpublished paper of P. Pahl [10]. For the convenience of the reader we include here the proofs which are to some extent related to the Morita theorem (see [2], Chapter II or [11], 4.11).

4.1 Consider the \( R \)-bihomomorphism

\[
\phi: \text{Hom}_R(M, \cdot) \otimes M \rightarrow (\cdot \otimes \text{Hom}_R(M, \cdot)), \quad \phi(m, \rho) = m \otimes \rho
\]

and the \( R \)-bihomomorphism

\[
\tau: M \otimes \text{Hom}_R(M, \cdot) \otimes \text{Hom}_R(M, \cdot) \rightarrow (M \otimes \text{Hom}_R(M, \cdot)), \quad \tau(m, \rho, \sigma) = (m \otimes \rho)(\sigma)
\]

The following relations are easily verified:

\[
M_{\phi} m \otimes \rho \otimes m = \rho(m) \otimes m = \rho(m) \otimes m, \quad \phi(\rho \otimes m),
\]

\[
\phi(\rho \otimes m) = \phi(\rho \otimes m) \phi \quad \text{for}
\]

\[
\phi_i, \phi_i', \quad \phi_i, \phi_i' \in \text{Hom}_R(M \otimes R, M) \quad \text{and} \quad m, i, m \in M
\]

We will use these relations throughout this section.

4.2 Lemma: (1) if \( M_i \) is a generator then \( I_i \) is finitely generated and projective.

(2) If \( M_i \) is finitely generated and projective then \( J_i \) is a generator.

(3) If \( Mg \) is a generator then \( R \) is naturally ringisomorphic to \( \text{End}(I_i) \).
Proof: (1) If $M$ is a generator then it follows that $\Psi$ is an epimorphism. Hence there exist $T^\ldots,^n \in \text{Hom}_R(M_R, R_R)$ and $e_1, \ldots, e_n \in M$ such that
\[
\Psi(\xi_j \circ e_j) = \sum_j \Psi(m_j \circ e_j) e_j = 1 \in R.
\]
Then for arbitrary $m \in M$
\[
m - m^* 1 = m \in \sum_j \Psi(m_j \circ e_j) e_j.
\]
Since $\Psi(\xi_j \circ e_j) \in \text{Hom}_R(I, T)$ we get by the dual basis lemma that $I^M$ is finitely generated and projective.

(2) If $M_\mathfrak{m}$ is finitely generated and projective then there exist $<p_1, \ldots, p_k > \in \text{Hom}_R(M_R, R_R)$ and $m_1, \ldots, m_k \in M$ such that for all $m \in M$
\[
m = \sum_j \Psi(m_j \circ e_j) m.
\]
Hence $1_\mathfrak{m} = \Psi(m. \circ p.)$ and since $<j(f(\circ t_i)) > \in \text{Hom}_R(J_\mathfrak{m}, T)$ we get that $I^m_\mathfrak{m}$ is a generator.

(3) Let $^\mathfrak{m}$ be a generator. Consider the ring homomorphism
\[
p: R \rightarrow (\text{Max } t \rightarrow x_t \in M) \in \text{Hom}_R(I, J_B, X).
\]
Let $f \in \text{Hom}_R(I, J_B, Y)$ and $m \in M$. Since $M_R$ is a generator there exist $y_1, \ldots, y_n \in \text{Hom}_R(M_R, R^\mathfrak{m})$ and $e_1, \ldots, e_n \in M$ such that
\[ mf = (f \circ (m \circ r_j) \circ e_i) f = f \circ \left( f \circ (m \circ i \circ e_i) f \right) = \]
\[ = m \circ \left( f \circ (r_j \circ e_i) f \right) = m \circ r_j \quad \text{where} \quad r_j \in R. \]

Hence \( p \) is an epimorphism.

Let \( r \in R \) and \( Mr = 0 \). Then \( e_j r = 0 \) and
\[ r = lr = f \circ \left( \sum_i \eta_i \circ \psi(e_j \circ \varphi_i) m_i \right) = 0. \]

Therefore \( p \) is a monomorphism.

4.3 Lemma: \( M_R \) is a generator iff \( \$ \) is an isomorphism.

Proof: \( ^\ast \): We mentioned already in the proof of lemma 4.2 that \( \ast \) is an epimorphism. Assume that \( \$ (f \circ \varphi \circ \varphi^\prime) = 0 \). Then
\[ E p. \circ m_1 \circ f = \sum_i \sum_j \eta_j \circ \psi(e_j \circ \varphi_i) m_i = \]
\[ = T_j \circ \left( e_j \circ \varphi_i \right) \circ m_1 = \]
\[ = \sum_j \sum_i \eta_j \circ \psi(e_j \circ \varphi_i) m_i = \]
\[ = f \circ f \circ (e_j \circ \varphi_i) \circ m_1 = \sum_j \sum_i \eta_j \circ \psi(e_j \circ \varphi_i) m_i = 0, \]

where \( T_j \) and \( e_j \) are chosen as in the proof of lemma 4.2. Thus \( \$ \) is a monomorphism.

\( \circ \): \( M_R \) is a generator since \( \$ \) is an epimorphism and \( R \) is a generator.

4.4 Lemma: Let \( M_R \) be a generator and \( A \) a right ideal of \( R \). Denote by \( t : A_R \rightarrow R \) the inclusion map. Then
\[ \text{Hom}(M_R, A_R) \otimes M_R \rightarrow \text{Hom}(M_R, R_R) \otimes M \]
\[ \text{Hom}(M_R, C) \otimes M \text{ is exact} \]
(2) \(-1(A) = \text{Hom}_R(M, A) \otimes M\) (where we are identifying \(\text{Hom}_R(M, A) \otimes M\) with a submodule of \(\text{Hom}_R(M, R) \otimes M\) via (1)).

Proof: Since \(\text{Hom}_R\) is left exact and by lemma 4.2 (1) \(\mathfrak{J}l\) is projective hence flat, we obtain (1). Since \(M\) is a generator \(<\text{Hom}_R(M, A) \otimes M\>) = A\) and since \(\$\) is an isomorphism by lemma 4.3, (2) follows.

4.5 Lemma: If \(M\) is a generator and \(T-p\) is injective, then \(M\) is injective.

Proof: Let \(A\) be a right ideal of \(R\) and \(f \in \text{Hom}_R(A, M)\).

We must find \(g \in \text{Hom}_R(R^+, M)\) such that \(g|A = f\). Consider the following diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}_R(M, A) \\
\text{Hom}(1, f) & \rightarrow & \text{Hom}_R(M, R) \\
\Gamma & \rightarrow & \text{Hom}_R(M, R)
\end{array}
\]

where \(\Gamma: A \rightarrow R\) is the inclusion map. In this diagram the row is exact and all mappings are right \(T\)-homomorphisms. Since \(T-p\) is injective \(g\) exists making the diagram commutative. From this we get a commutative diagram with exact row (see lemma 4.4):

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}_R(M, A) \\
\text{Hom}(1, f) & \rightarrow & \text{Hom}_R(M, R) \\
\Gamma & \rightarrow & \text{Hom}_R(M, R)
\end{array}
\]
Using the isomorphism

$$6: r \circ M \cong z i \otimes m \cong M \oplus y, m \in M$$

and the fact that \( \$ \) is an isomorphism by lemma 4.3 we set

\[ g = efS \left( \sum \right) \]$^1$.

If \( a \in A \) then \( \#^{-1}(a) = \ell \wedge K \wedge H \wedge e \wedge M \) (by lemma 4.4). Hence

\[ g(a) = 6^{vl_i_m}(T, (p^*O_m^*) = 6^{p^*I_{Sim}^*} = T f(<p,m.) = f(-f<p,m.) = f(a) \] and \( g \) is the desired homomorphism.

A module \( M_0 \) is called selfinjective iff for every exact \( R \) sequence \( 0 \to A \to H M_n \) and every \( R \)-homomorphism \( f: A_0 \to M \) there exists \( g: M_0 \to M \) such that \( f = get. \)

**4.6 Proposition (P. Pahl):** Let \( M_R \) be selfinjective and \( -f_l \) flat. Then \( r_f \) is injective.

**Proof:** Let \( A \) be a right ideal of \( T, \) \( T \) \( \) an arbitrary homomorphism. We have to show that there exists \( y_o \in V \) such that \( f(A) = y_o A \) for all \( A \in A. \)

Since \( M \) is flat \( 0 \to A \otimes M \to M \) is exact. Thus \( A \oplus M \) will be considered as a submodule of \( r \otimes M. \) Let \( 6: r \otimes M \to M \) be the isomorphism described above. Its restriction

\[ 6^0: A \otimes M \cong z i \oplus A_i \otimes m \in M \]

is an isomorphism. Set \( \hat{g} = 6(f \circ l_m) \) \( 6^{-1}. \) Then \( \hat{g} \) is a right \( R \)-homomorphism of \( AM \) into \( M. \) Since \( M \) is selfinjective there exists \( y_0 \in T \) such that \( y_0 | AM = \$. \) Then \( \hat{g}(Am) = f(A)m = yAm \) where \( m \in M, \) \( A \in A \) and thus \( f(A) = yA \) for all \( A \in A. \)
4.7 Proof of theorem 3 (as stated in the introduction):

(2) \Rightarrow (1): by definition.

(1) \Rightarrow (3): by proposition 4.6.

(3) \Rightarrow (2): by lemma 4.5.

Remark: (1) \Rightarrow (2) is a special case of the fact that if a module is injective with respect to a generator, then it is injective.
5. Proofs of theorems 4 and 5.

We again refer the reader to the introduction for the statements of theorems 4 and 5.

5.1 Lemma: Let $\mathcal{M}_R$ be a projective cogenerator. Then $R^\mathcal{M}$ is a cogenerator.

Proof: Since $\mathcal{M}_R$ is projective there exists a monomorphism $r?: \mathcal{M}_R \rightarrow \mathcal{M}_R$ where $\mathcal{M}_R = \bigoplus_{i \in I} \mathcal{M}_{R_i}$ for all $i \in I$. Let $A$ be an arbitrary module and $0 \neq a \in A$. We have to find $g: \mathcal{M}_R \rightarrow R^\mathcal{M}$ such that $g(a) \neq 0$. Since $\mathcal{M}_R$ is a cogenerator there exists $f: \mathcal{M}_R \rightarrow \mathcal{M}_R$ such that $f(a) \neq 0$. Since $r?$ is a monomorphism we get $r?f(a) \neq 0$. Hence there exists a projection $w: \mathcal{M}_R \rightarrow \bigoplus_{i \in I} \mathcal{M}_{R_i}$ such that $\sum_{i \in I} w_i(a) \neq 0$. Thus we can set $g = \sum_{i \in I} w_i f$.

5.2 Lemma: Let $R^\mathcal{M}$ be a cogenerator and $R$ a left $S$-ring. If $\mathcal{M}$ is faithful then $\mathcal{M}_R$ is a generator.

Proof: ([8], page 407): Consider the left ideal $A$ of $R$ defined by $A = \bigoplus_{i \in I} \text{ima}(p)$. Assume $A \subseteq R$. Since $R$ is a left $\text{peHom}_R(M_R, R_R)$ $S$-ring there exists $r \in R$, $r \neq 0$ such that $Ar = 0$. It follows then that $p(M_R) = 0$ for all $p \in \text{Hom}_R(M_R, R_R)$. Since $R^\mathcal{M}$ is a cogenerator this implies $M_R^0 = 0$ and since $\mathcal{M}_R^K$ is faithful we get $r^0 = 0$. Hence $A = R$ and $\mathcal{M}^\mathcal{M}$ is a generator.

5.3 Proof of theorem 4:

$(1) \Rightarrow (2)$: Since $\mathcal{M}$ and $t^\mathcal{M}$ are projective cogenerators it follows by lemma 5.1 that $R^\mathcal{M}$ and $J_{\mathcal{M}_R}$ are cogenerators. This implies by corollary 3.4 that $R$ is a right $S$-ring and $T$ is a
left $S$-ring. By proposition 3.5 we get that $M_R$ is injective and since $\mathfrak{J}$ is flat it follows by proposition 4.6 that $T\mathfrak{J}$ is injective. Using the assumption that $R$ is naturally ring-isomorphic to $\text{End}(\mathfrak{J}I)$ we get similarly that $\mathfrak{I}#$ and $\mathfrak{I}R$ are injective. Then it follows by corollary 3.9 that $R$ and $\mathfrak{I}U$ are cogenerators. Hence $\mathfrak{I}$ and $M_R$ are generators by lemma 5.2.

$(2) \Rightarrow (1)$: Let $U_0$ be simple. Then there exists $x \in R$ such that $U_0 \cong xR$ since $R$ is a right $S$-ring. Since $M_\mathfrak{J}$ is a generator $M_\mathfrak{J}$ is faithful and there exists $m \in M$ such that $mx \neq 0$. Then $U_0 \cong xR \cong mxR \subseteq M$. Using the assumption that $M_R$ is injective we get by lemma 3.2 that $M_L$ is a cogenerator. By lemma 4.2 $R$ is naturally ring-isomorphic to $\text{End}(\mathfrak{J}^\mathfrak{I}M)$ and $M_\mathfrak{I}$ is projective. Similarly it follows that $\mathfrak{I}J_\mathfrak{I}$ is a projective cogenerator.

5.4 Proof of theorem 5:

$(1) \Rightarrow (3)$: By lemma 4.2 $\mathfrak{J}\mathfrak{I}$ is projective. Hence by lemma 5.1 it follows that $\mathfrak{I}T$ is a cogenerator. Since $M_R$ is an injective generator we get by theorem 3 that $\mathfrak{I}R$ is injective. Hence by corollary 3.9 $T$ is semiperfect, $\mathfrak{I}R$ is a cogenerator and by corollary 3.6 $\mathfrak{I}F$ is injective. Using the fact that $\mathfrak{I}J_\mathfrak{I}$ is finitely generated and projective it follows that $\mathfrak{I}N$ is injective (since $\mathfrak{I}J^\mathfrak{I}$ is injective) and $\mathfrak{I}M$ is semiperfect (since $T$ is semiperfect). Obviously $\mathfrak{I}J_\mathfrak{I}$ is faithful. Hence $\mathfrak{I}J_\mathfrak{I}$ is a generator by lemma 5.2. Then it follows by lemma 4.2 that $M$ is a finitely generated projective generator.
By the Morita theorem it follows that the functor \( \mathcal{R} \) is an equivalence between module categories. Since \( \mathcal{R} \mathcal{M} = \mathcal{M} \) and \( \mathcal{IL} \) is a cogenerator we then get that \( \mathcal{M} \) is a cogenerator.

We now apply the above results under the assumptions \( \mathcal{M} \) is an injective generator and \( \mathcal{M} \) is a cogenerator, where \( \mathcal{R} \) is viewed as the endomorphism ring of \( \mathcal{M} \) via lemma 4.2. This allows us to obtain the rest of the statements in (3). This

(3) \( \Rightarrow \) (1): trivial.

(2) \( \Rightarrow \) (3): Since \( \mathcal{M} \) is finitely generated and projective we get by lemma 4.2 that \( \mathcal{M} \) is a generator. By assumption \( \mathcal{R} \) can be identified with the endomorphism ring of \( \mathcal{M} \). Switching sides we then copy the proof of (1) \( \Rightarrow \) (3) to obtain (3).

(3) \( \Rightarrow \) (2): Since \( \mathcal{L} \) is a generator it follows by lemma 4.2 that \( \mathcal{R} \) is naturally ringisomorphic to \( \text{End}(\mathcal{L}) \). The other statements in (2) follow trivially.

Carnegie-Mellon University
Pittsburgh, Pennsylvania
May, 1969
References


