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High Gain Control System Design
with Gain Plots

by

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High Gain Control System Design with Gain Plots

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This paper quantifies and summarizes in a general theorem the asymptotic high gain behavior of closed-loop eigenvalues. The theorem is founded on fundamental principles of classical control theory, and its implications are demonstrated conclusively via specialized plots, the Gain Plots (Kurfess and Nagurka, 1991), that offer a new and revealing graphical perspective. The theorem defines the rate (with respect to change in gain) at which closed-hop eigenvalues migrate towards both finite and infinite transmission zeros. This sensitivity information is essential for complete design of control systems in terms of gain selection and its relationship to stability, performance, and robustness. The general theory is graphically related to the classical control definition of sensitivity.

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Introduction

This report investigates high gain behavior of eigenvalues of linear time-invariant (LTI) single-input, single-output (SISO) systems. It is shown that several generic properties associated with the behavior of high gain eigenvalues can be developed leading to a more complete understanding of the concept of high gain asymptotes. The results can be confirmed readily by the use of special eigenvalue magnitude and eigenvalue angle plots, called Gain Plots (GPs) (Kurfess and Nagurka, 1991), that show explicit variations of these quantities with respect to gain. The GPs are particularly well suited for investigation of high gain root sensitivity and offer considerably more insight than the Evans root locus plot

An important problem of classical control system design is that of characterizing the effect of gain on the location of the closed-loop poles (Cannon, 1967; Kwakernaak and Sivan, 1972; Palm, 1986; D'Souza, 1988; Sinha, 1988; Dorf, 1989; Hostetter, *et* a/., 1989; Ogata, 1990; Van de Vegte, 1990; Franklin, *etaU* 1991; Kuo, 1991). Evans root locus plot can provide this information by displaying graphically the variation of the system poles in the complex plane as a scalar control weighting is adjusted from zero to infinity. In particular, from the root locus plot it is well known that some of the closed-loop poles asymptotically approach finite locations at the zeros of the system (the end points of asymptotically finite branches), while the remainder tend to infinite zeros in special configurations. It is, however, not apparent from the root locus how gain variations effect the speed of the poles *(i.e.,* gain rate of location change). Of special interest is the system sensitivity at high gain, where the closedr-loop poles behave asymptotically.

It is shown that high gain asymptotes are characterized by slopes in GPs that give information about the rate at which a locus branch tends to infinity. The asymptotes are indicative of branches tending to infinite zeros at "arrival rates" that can be detected by inspection from slope information.

The LTI SISO system can be represented in the standard state-space form as

$$\dot{x}(t) = Ax(t)+bu(t) \tag{1}$$

$$y(t) = Cx(t) +du(t) \tag{2}$$

The input-output transfer function is the Laplace domain ratio of output y(s) to input u(s). The system is embedded in the closed-loop configuration shown in Figure 1.

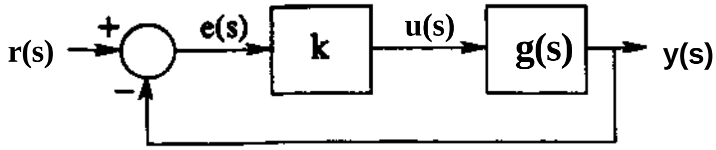

Figure 1. Closed-Loop SISO Negative Feedback Configuration.

The closed-loop poles at zero gain are the open-loop eigenvalues. As the gain increases from zero to infinity, the closed-loop eigenvalues trace out "root loci" in the complex plane. At infinite gain some of these eigenvalues approach finite transmission zeros, defined to be those values of s that satisfy the generalized eigenvalue equation

$$\begin{bmatrix} sI - A & -b \\ C & d \end{bmatrix} \begin{bmatrix} x(0) \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (3)$$

where x(0) represents the initial state and where u is a constant (more generally representing input direction in the multi-input case). Algorithms have been developed for efficient and accurate computation of transmission zeros (Davison and Wang, 1974; Westreich, 1991).

The high gain behavior of the root loci can be viewed another way (Friedland, 1986). The eigenvalues can be considered as always migrating from the open-loop poles to their matching transmission zeros. However, those eigenvalues that do not have matching zeros in the finite part of the s plane are considered to have matching zeros at infinity. In this global SISO perspective, whenever the excess of poles over zeros is greater than two, the eigenvalues migrate towards a real value of +°° and therefore as k—**> some of the closed-loop eigenvalues must migrate to the right half s-plane, indicating an unstable system. Since most practical systems have an excess of two or more poles, the implication is that there is always a finite upper limit to the loop gain. (The ratio of the loop gain at which a system is designed to operate to the gain at which it becomes unstable, expressed logarithmically, is known as the gain margin (Nagurka and Kurfess, 1991).)

After summarizing the basic concept of GPs, this report analyzes high gain eigenvalue behavior of different types of systems. Addressed first are systems possessing only poles at the origin. A subsequent section generalizes the theory to systems with poles located freely on the s-plane. Then the influence of unique transmission zeros is incorporated into the analysis. A general theory explaining high gain sensitivity that is applicable to all LTI SISO systems is presented. Finally, the general theory of high gain eigenvalue behavior is related to eigenvalue sensitivity as defined in classical control theory-

Gain Plots - An Overview

GPs have been proposed by (Kurfess and Nagurka, 1991) as a powerful graphical design tool to represent the behavior of system eigenvalues as an explicit function of gain. Although the GPs are applicable to multi-input, multi-output (MIMO) control systems, the focus of this report is restricted to classical SISO systems.

Given a plant transfer function, g(s), embedded in the standard closed-loop negative feedback system of Figure 1, closed-loop stability can be determined by examining the real parts of the eigenvalues. The eigenvalues are the solutions of the closed-loop characteristic equation

$$kg(s) = -1 \tag{4}$$

The root locus is the solution set of equation (4) as the gain k varies from zero to infinity. Equation (4) is equivalent to two conditions: the angle criterion,

$$\angle kg(s) = \pm 180^{\circ}(2m+1), \quad m = 0, 1, 2, \ldots \tag{5}$$

and the magnitude criterion,

$$|kg(s)| = 1 \tag{6}$$

The shape of the root locus plot is determined entirely by the angle criterion. Then, for any eigenvalue, s, on the root locus, the magnitude criterion is invoked to solve for the corresponding value of k.

For example, Figure 2 is the root locus plot of the open-loop transfer function, g(s),

$$g(s) = \frac{(s+3)}{(s+1)(s+2)} \tag{7}$$

Each branch of the root locus starts at k=0 corresponding to a system open-loop pole (s=-1,-2), and asymptotically approaches either a finite (s=-3) or infinite ($s \rightarrow -\infty$) transmission zero.

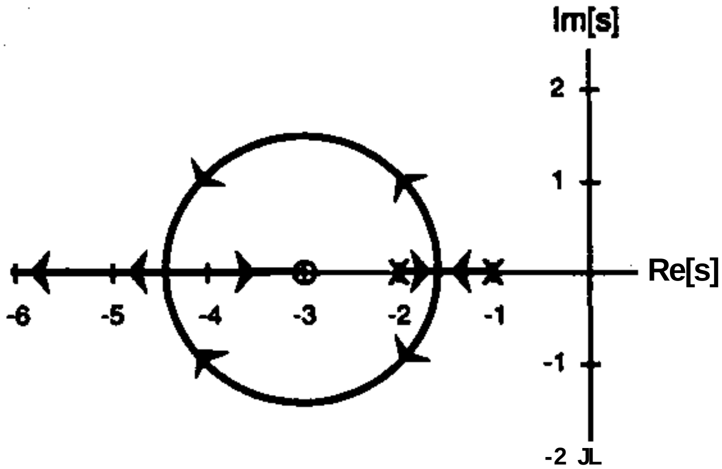

Figure 2. Evans Root Locus Diagram of Equation (7).

An alternative visualization of the root locus plot can be obtained by explicitly graphing the eigenvalue magnitude *vs.* gain and the eigenvalue angle *vs.* gain. These plots are known as the Magnitude Gain Plot (MGP) and Angle Gain Plot (AGP), respectively. It is beneficial to plot the MGP using a log-log scale and the AGP using a semi-log scale (Kurfess and Nagurka, 1991).

Figures 3a,b are the GPs for the system of equation (7). Notice that real eigenvalues are represented by a single line at 180* (= -180*) on the AGP corresponding to the negative real axis. Conversely, complex conjugate eigenvalues have magnitudes that are equal corresponding to a single segment on the MGP.

The AGP reflects the basic construction rule of the root locus, *i.e.*, the angle criterion of equation (5). As a result, the AGP is symmetric about the 180* line. Furthermore, the angle criterion dictates that the eigenvalues must lie on the real axis or be complex conjugates. Thus, a pair of complex conjugate eigenvalues is shown as a single curve in the MGP with corresponding angles symmetrically configured about the 180* line shown in the AGP. As k varies, complex conjugate eigenvalues may become distinct real eigenvalues, causing their angles to become equal (at a multiple of 180*) and permitting their magnitudes to differ.

The MGP shows the presence of two open-loop poles with magnitudes 1 and 2 at k = 0. As k —> oo it shows a single finite transmission zero with magnitude 3 and an asymptote tending toward an infinite transmission zero. The AGP indicates that the two open-loop poles and finite and infinite transmission zeros are located in the left-half plane, on the real axis, since they all have angles of 180*.

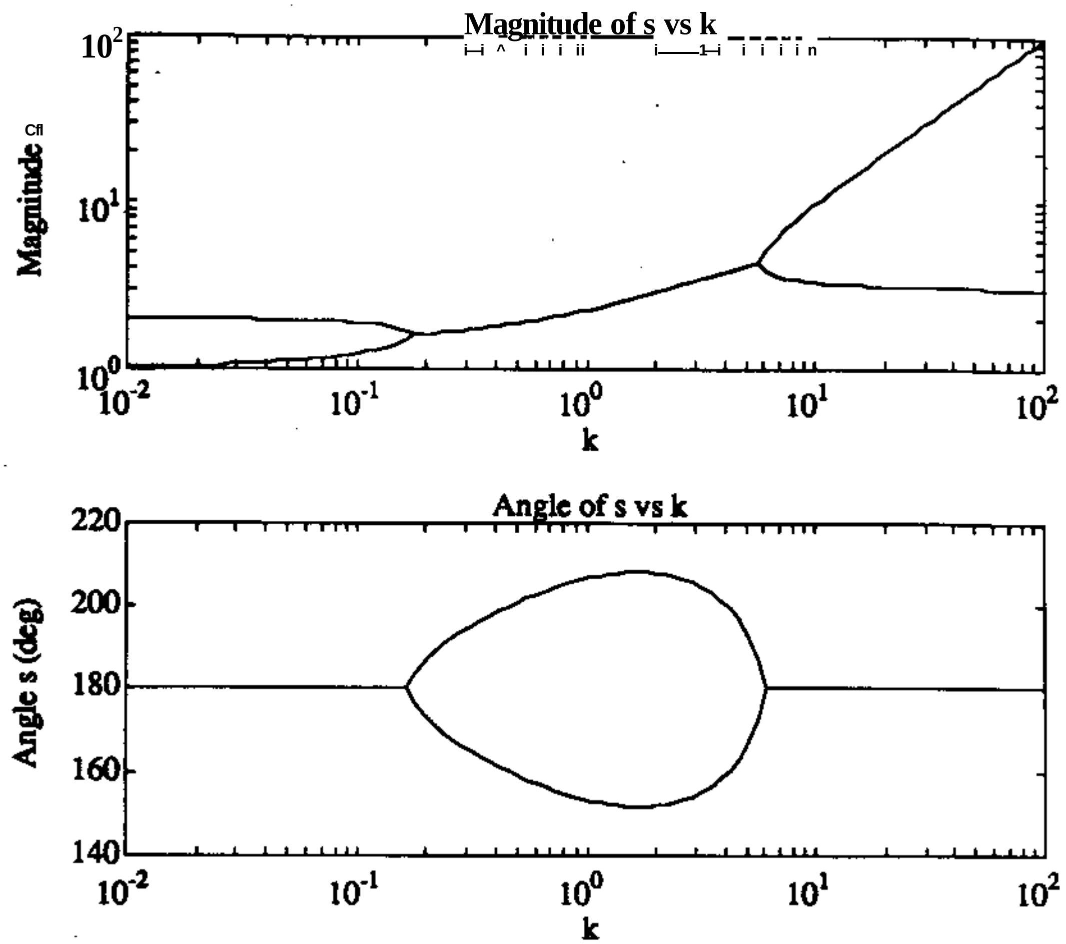

Figure 3a,b. The Magnitude and Angle Gain Plots of Equation (7).

The GPs highlight the break points corresponding to points where branches leave or enter the real axis of the root locus. For this example, these break points occur at k « 0.17 and at k » 5.83. Between these break points the AGP indicates that the loci of the two branch points are not on the real axis and the corresponding single curve of the MGP confirms that the trajectories are those of a complex conjugate pair.

By exposing gain as an independent variable, the GPs are well suited for determining system stability and performance (Kurfess and Nagurka, 1991). In addition, the GPs provide a measure of eigenvalue sensitivity. From the slopes of the GPs, the change in magnitude and angle of each eigenvalue per change in gain can be ascertained. The ability to directly determine these slopes by inspection is crucial in the design of control systems, and is related to the classical controls definition of eigenvalue sensitivity. This report addresses the sensitivity of the eigenvalues with respect to gain, k. Eigenvalue robustness is important since in the design of control systems it is desirable to achieve a system **that possesses the** desired performance characteristics and is insensitive to parameter variations. For example, the magnitude of the closed-loop eigenvalues is relatively robust (i.e., insensitive) to gain variations where the slopes in Figure 3a are relatively small. Furthermore, the angles of the eigenvalues shown in Figure 3b are quite sensitive to gain variations near the break-points.

Eigenvalue sensitivity with respect to gain is defined in control theory as

$$U \, ' \quad dk \;\; s \tag{8}$$

The classical definition of sensitivity normalizes ds/dk by dividing by s/k. The sensitivity of the eigenvalue magnitude can be calculated numerically as a function of k, as depicted in Figure 4 for the system of equation (7). In the calculation the slope of the MGP is used to give the derivative dlsl/dk. Figure 4 shows that the sensitivity tends to infinity for gains associated with eigenvalue break-in and break-out points. Furthermore, it suggests that at high gain values the eigenvalue migrating to the finite transmission zero has zero gain sensitivity, whereas the eigenvalue tending to the infinite zero has constant sensitivity of unity. This sensitivity magnitude is one of several important features of eigenvalue trajectories at high gain.

This report presents a theory for the general behavior of the magnitude and rate of the eigenvalues for high gain LTI SISO systems. The asymptotic angle behavior is well known, and presented in the classical controls literature (Ogata, 1990; Kuo, 1991). The magnitude behavior is not generally addressed, although it offers interesting information that can be linked to the classical definition of sensitivity. After deriving the asymptotic magnitude behavior, it will be graphically related to the traditional definition of sensitivity, depicted in Figure 4, at high gain.

High Gain Magnitude Behavior - Open Loop Poles at Origin

This section investigates the behavior of closed-loop poles as a function of gain for systems whose open-loop poles are located at the origin. Such open-loop poles represent integrators and are commonplace in control theory. For example Newton's second law relating force to acceleration may be viewed as

$$f(t) = ma(t) = m\ddot{x}(t) \tag{9}$$

with a Laplace transform of

$$F(s) = ms^2X(s) \tag{10}$$

Thus, the double integrator is the link between force and position.

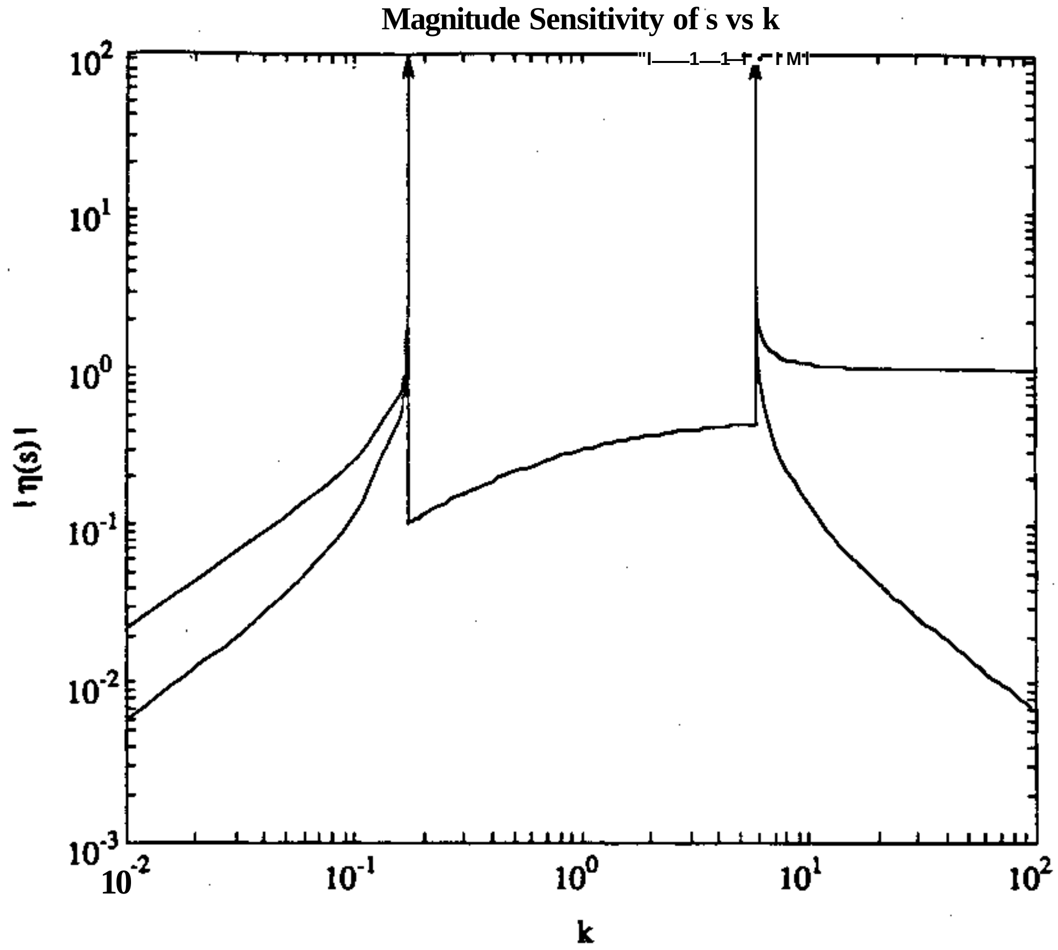

Figure 4. Classical Sensitivity Magnitude Plot of Equation (7).

An analysis of poles at the origin provides an excellent starting point from which to derive a general theorem for high gain sensitivity of SISO systems. The analysis begins with a first order system, and is subsequently extended to a second order system and then a generalization. Finally, the restriction of poles at the origin is removed and general asymptotic rules are derived

Integrator Asymptotic Relations

The transfer function of a plant described by a single integrator, g<sH (ID

has a corresponding closed-loop transfer function,

ga_(5) = l i k (12)

The closed-loop system eigenvalue is given by solving for s when the denominator polynomial of equation (12) is equated to zero. Its magnitude increases proportionally with gain k, and the eigenvalue has sensitivity magnitude \r\(s)\ = 1.

If the plant is described by a double integrator, the transfer function is

* > " £ (13)

yielding a closed-loop transfer function of

$$g_{cL}(s) = \frac{1}{s^2 + k} \quad (14)$$

The two system eigenvalues have magnitudes that increase proportionally to k*/², and have sensitivity magnitudes fn(s)l = 1/2.

In general, for a system with n poles at the origin (n pure integrators) having an open-loop transfer function

g(s) = ^T (15)

the roots of the closed-loop system characteristic equation are

$$s = k^{1/n} \quad (16)$$

The n system eigenvalues have magnitudes that increase proportionally to k^n, and have sensitivity magnitudes hl(s)l = 1/n. This relationship motivates the use of a log-log plot from which the power (n) can be determined by inspection of the slope. Figure 5 summarizes the governing power law relationship, and shows the eigenvalue magnitude behavior with respect to gain as the number of poles increases.

The n eigenvalue angles for plants consisting of pure integrators are dispersed in a Butterworth configuration having angles 8i (Ogata, 1990)

$$\theta_i = \frac{\pm 180°(2k + "}{n} \quad . \quad k - 0 . 1 . 2 \ . \ . \ . \ .i = 0.1 \ . \ . \ . \ .n \qquad 07,$$

The AGP (not shown) corresponding to Figure 5 depicts a set of n horizontal lines. Equation (17) yields the standard root locus asymptote angles for poles migrating toward infinite transmission zeros. In general, if there are n poles and m zeros, the Butterworth pattern is given by

$$\theta_i = \frac{\pm 180^\circ(2k+1)}{n-m}, \quad i = 1, 2, \ldots, \quad k = 0, 1, \ldots, n-m \tag{18}$$

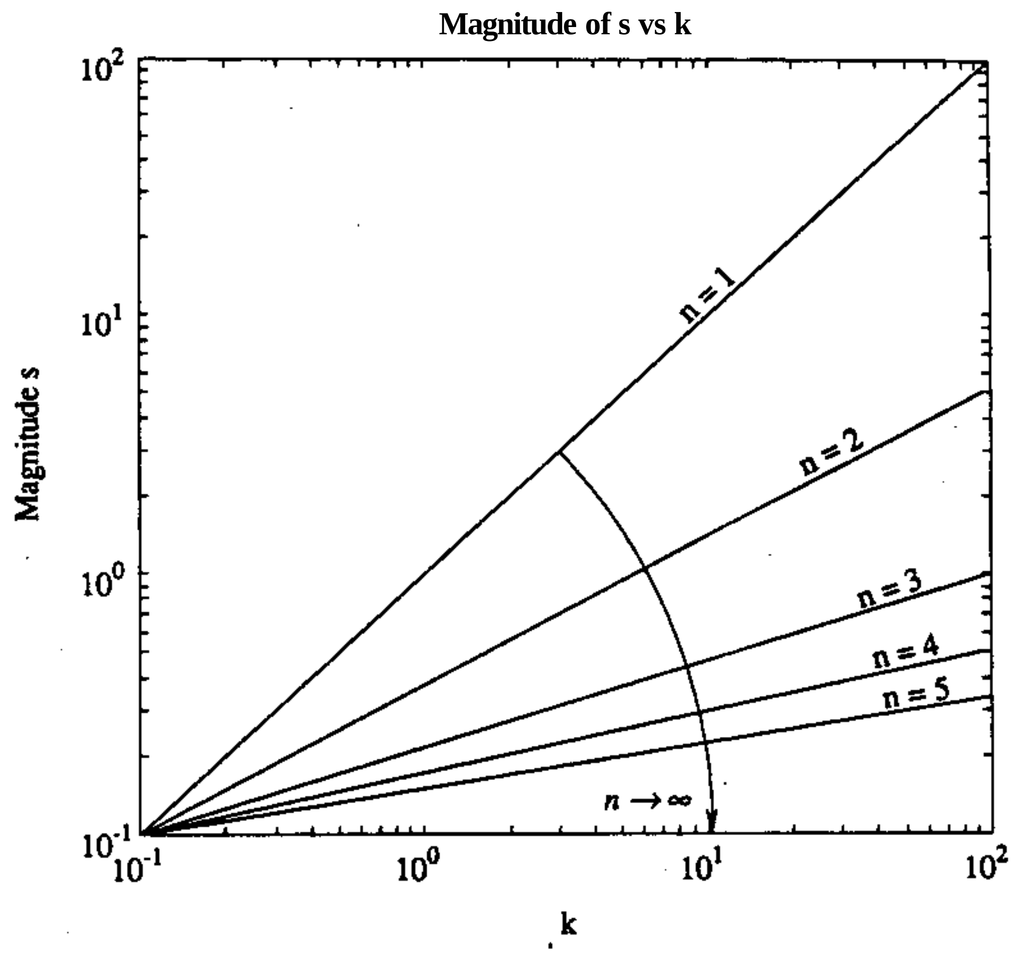

Figure 5. Family of MGPs for Plants of n Integrators.

High Gain Magnitude Behavior - Open Loop Poles Not at Origin

This section investigates high gain behavior of closed-loop eigenvalues for systems with open-loop poles not necessarily at the origin. Several types of plants are considered, including *(i)* a plant having a single pole, *(ii)* a plant with two poles, and *(Hi)* a plant having two poles and a finite transmission zero. Following these examples, a general theory is developed for high gain asymptotic behavior of LTI SISO system eigenvalues.

First Order System

The open-loop transfer function of the plant is given by

$$* \mathrm{w} \quad \frac{\mathbf{bn}}{s + ao} \tag{19}$$

The closed-loop characteristic equation is

$$s + ao + kbo = 0 \tag{20}$$

with a root at

$$s = -(kbo + ao) \tag{21}$$

Equation (21) is a straight line in the Isl-k plane. At high gain, the eigenvalue magnitude is kbo, and it has sensitivity magnitude lr|(s)| = 1.

Second Order System

<u>Two Poles and No Zero</u>

The plant is described by the open-loop transfer function

$$g(s) = \frac{T - \wedge Q}{s^2 + ais + ao} \tag{22}$$

with the corresponding closed-loop characteristic equation

$$s^2 + a_1 s + (a_0 + b_0 k) = 0 \tag{23}$$

The roots of equation (23) can be determined from the quadratic formula and are given by

$$s = \frac{1}{2}\left(-a_1 \pm \sqrt{a_1 - 4(a_0 + b_0 k)}\right) \tag{24}$$

As the gain increases, the solution of equation (24) is a complex conjugate pair $\pm\sqrt{\mathrm{bok}}\, j$. Thus, as $k \to \infty$, the eigenvalue magnitude is proportional to $k^{1/2}$ yielding a sensitivity magnitude lT|(s)l = 1/2.

<u>Two Poles and a Zero</u>

The plant is represented by the open-loop transfer function

$$g(s) = \frac{b \quad s + b_o}{s^2 + ais + ao} \tag{25}$$

The closed-loop characteristic equation is

$$s^2 + (ai + bik)s + (ao + bok) = 0 \tag{26}$$

By adding a zero to the plant transfer function, the gain appears in the coefficients of both the s° and s^1 terms. As k -»«> equation (26) can be approximated by

$$s^2 + biks + bok = O \tag{27}$$

since the high gains mask the relatively smaller terms. The roots of equation (27) can be determined via the quadratic formula as

$$s = \frac{1}{2}\left(-b_1 k \pm b_1 k \sqrt{1 - \frac{4b_0}{b_1^2 k}}\right) \tag{28}$$

Using a square root approximation, equation (28) can be simplified further as

$$s = \frac{1}{2}\left(-b_1 k \pm b_1 k \left(1 - \frac{2b_0}{b_1^2 k}\right)\right) \tag{29}$$

There are two solutions to equation (29), s = {-~, -bo/bi), corresponding respectively to an infinite transmission zero and a finite transmission zero located at s = -bo/bi. The infinite transmission zero is approached at a rate proportional to k, and the finite transmission zero is approached at a rate proportional to If^1, as will be demonstrated in later examples. The sensitivity magnitudes of the infinite and finite eigenvalues are 1 and 0, respectively.

Generalization

In general, a LTISISO system plant may be characterized by the transfer function

$$g(s) = \frac{n(s)}{d(s)} = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{j=0}^{n} a_j s^j} \tag{30}$$

where n(s) is the transfer function numerator of order m in s with coefficients, bi (i=0, 1, ... m), and d(s) is the transfer function denominator of order n in s with coefficients, a_n (j=0,1,.... n) with an=l. Equation (30) represents an n* order system. The characteristic equation, <t>cL(s), for the general system embedded in the closed-loop configuration in Figure 1 is

$$\phi_{cL}(s) = d(s) + kn(s) = \sum_{j=0}^{n} a_j s^j + k \sum_{i=0}^{m} b_i s^i \tag{31}$$

The roots of equation (31) are the system eigenvalues.

Equation (31) can be written as

$$k = -\frac{d(s)}{n(s)} = -\frac{\sum_{j=0}^{n} a_j s^j}{\sum_{i=0}^{m} b_i s^i} \tag{32}$$

From equation (32), as k-*», the ratio of n(s) to d(s) must also tend towards infinity at the same rate. The root locus angle criterion dictates that the eigenvalue trajectories proceed towards infinity at angles given by the Butterworth configuration with n-m infinite trajectories (Kwakernaak and Sivan, 1972).

The magnitude of equation (32) is given by

$$k = \frac{|d(s)|}{|n(s)|} = \frac{\left|\sum_{j=1}^{n} a_j s^j\right|}{\left|\sum_{i=1}^{m} b_i s^i\right|}, \quad a_n = 1 \tag{33}$$

As k->«o, there are two possibilities for the magnitude ratio of d(s) to n(s), *i.e.*,

$$n(s) \to 0 \tag{34}$$

and/or

$$d(s) \to \infty \tag{35}$$

In the literature, it is shown that both of these criteria are satisfied as k->°°. For large gain, the highest order s term for both d(s) and n(s) dominates, since lsl->°°. Thus, in the limit as k—>» equation (33) may be approximated by

$$k \cong \left|\frac{s^n}{b_m s^m}\right| = \left|\frac{s^{n-m}}{K}\right| \tag{36}$$

From equation (36), the asymptotic behavior of the eigenvalues may be realized as

$$|s| \cong (b_m k)^{\left(\frac{1}{n-m}\right)} \tag{37}$$

Thus, high gain eigenvalues migrating toward infinite transmission zeros increase in magnitude at a rate inversely proportional to k^{n-m}, each having a sensitivity magnitude $|T(s)| = 1/(n-m)$. This generalization holds true for LTI SISO systems (minimum and non-minimum phase) possessing n poles and m zeros.

Each of the n-m poles that satisfies equation (37) represents an individual eigenvalue migrating towards an infinite transmission zero. If the first n-m eigenvalues are considered to be those with infinite magnitude at infinite gain, then each individual *infinite* eigenvalue, $s=\lambda_i$ (i=1, 2, ..., n-m), satisfies equation (37). However, there remain m *finite* eigenvalues that also may be used to satisfy equation (37). These m *finite* eigenvalues approach the finite transmission zeros at a rate which may be determined The derivation is non-trivial and yields results that are quite sensible, but not intuitively obvious at first glance.

Eigenvalue Approach Rate to Unique Transmission Zeros

The derivation is slightly different from that developed for the infinite magnitude asymptotes. Initially, unique zero locations are assumed; subsequently, the theory is extended to any LTI SISO system. The derivation begins with the magnitude criterion, and the limit given by equation (34) is invoked. Clearly, for $s=\lambda_i$ (i=n-m, n-m+1, ..., n), $|d(s)|$ must be finite and nonzero if no pole-zero cancellations occur. Thus, $|n(s)|$ evaluated at any specific finite eigenvalue must approach zero as $k \to \infty$

$$\lim_{k \to \infty} \frac{|d(s)|}{|n(s)|} = \lim_{n(s) \to 0} \frac{|d(s)|}{|n(s)|} \quad (38)$$

Furthermore, since the finite eigenvalues asymptotically approach a finite value, $|d(s)|$ may be approximated as a nonzero constant, d^*. Thus as $k \to \infty$, the magnitude criterion requires that $|n(s)|$ evaluated at each finite eigenvalue must approach zero at a rate proportional to k

$$\lim_{k \to \infty} k = \lim_{n(s) \to 0} \left(\frac{d^*}{|n(s)|}\right) \quad (39)$$

Equation (39) summarizes an important property of high gain finite eigenvalue behavior. At high gains, the finite eigenvalues approach finite transmission zeros at a rate inversely proportional to the rate of gain change. In other words, the distance of a high gain eigenvalue from its matching finite transmission zero on the s-plane always changes at a rate inversely proportional to k. The sensitivity magnitude at high gain is zero for this finite eigenvalue.

An alternate perspective of the development above is to consider n(s) as the product of terms given by

$$n(s)=\prod_{i*1}^{m}(s+z_i) \qquad (40)$$

where z\ is the i* zero location. The numerator n(s) evaluated at an admissible value of s tends to zero because one of its terms, S+ZJ, vanishes as s -> z*. For that particular s, the other numerator terms are finite non-zero valued and do not cause n(s) -> 0. Since n(s) and k are inversely related in equation (33), n(s) approaches zero at a rate inversely proportional to the rate that k approaches infinity.

The logarithmic scale of the MGP highlights the power law relation between the magnitude of the asymptotic poles and the gain. Figure 6 is the MGP for the open-loop system given by the transfer function

$$g(s)=\frac{(s+1)}{s(s-1)(s^2+4s+16)} \qquad (41)$$

(studied in Example A-5-3, Ogata, 1990). This system consists of four poles and a single zero. Therefore, it is expected that three asymptotes at 180" and ±60* will occur when k->*>. Furthermore, from equation (37) the slope of the magnitude curves at high values of k is predicted to be 1/3, as seen in Figure 6. In Figure 6, the magnitude of each of the three eigenvalues converges to the line with slope 1/3. This slope corresponds to lt|(s)l = 1/3. This example is studied in depth in Kurfess and Naguika (1991).

As the magnitude of the fourth eigenvalue asymptotically approaches unity, the slope of the MGP at high gain tends toward zero. Thus, the corresponding sensitivity magnitude lr|(s)l also tends to zero. Clearly, the high gain sensitivity magnitude is the slope of the MGP at high gain.

In addition to the ability to determine high gain sensitivity magnitude from the slope of the MGP, it is possible to evaluate the rate at which the eigenvalue approaches the finite transmission zero. The asymptotic behavior of the finite eigenvalue is dominated by unity constant magnitude, since the eigenvalue is approaching the finite zero (at s=l). To more clearly view the asymptotic magnitude behavior of this finite eigenvalue, the distance between it and the finite transmission zero is plotted *vs.* k on a logarithmic scale in Figure 7. By inspection, the slope of the high gain asymptote in Figure 7 is -1 verifying the inverse relationship (k~l) for eigenvalues approaching unique transmission zeros, which was previously derived.

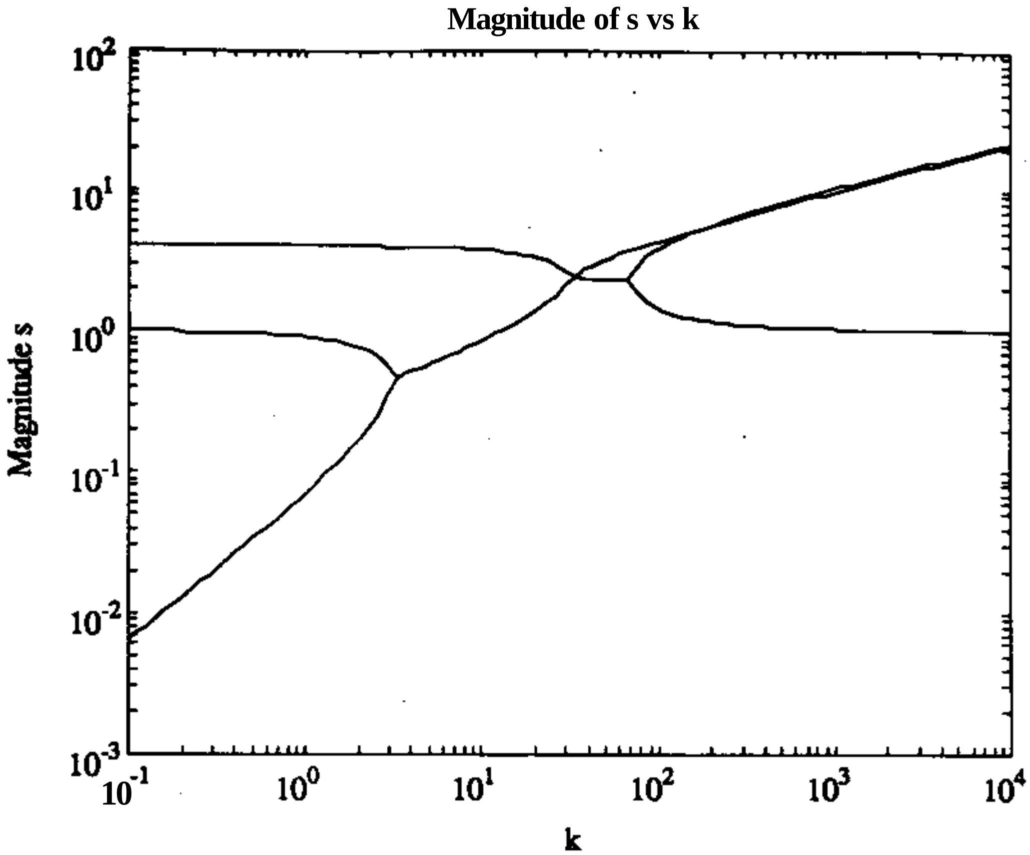

Figure 6. MGP for System Given by Equation (41).

Figure 7 is reminiscent of a Bode plot in that it depicts both high and low independent variable' asymptotes. That is, in addition to the high gain asymptote discussed above, a low gain asymptote that approaches the two complex-conjugate open-loop poles shows a distance (magnitude) of VIT * 3.87 relative to the finite zero. In addition, Figure 7 shows interesting behavior at a gain of k * 70 where a "cusp" is evident. The distance from the finite zero levels off near and at the break-in point of the two complex conjugate poles. The cusp occurs when the two complex conjugate poles meet at the real axis. The pole shown then heads toward the finite transmission zero at s = 1.

At the cusp, the MGP depicts a line with zero slope. This relates directly to the break-in point in the root locus. Poles either leave or arrive at the break points at angles of ±90\ Thus, near the break points, the eigenvalue trajectories *sweep out* a near circular path. This behavior generates a constant magnitude, and a large change in angle. Based on this behavior about break points, the eigenvalue magnitude is not sensitive to changes in k, whereas the eigenvalue angle is highly sensitive to variations in k. Figure 8 is an

enlargement of die MGP shown in Figure 7 that highlights the desensitized eigenvalue TTfflgpifliH* **characteristic** about the break point The abrupt change in Figure 8, for which **the derivative dW/dk** tends to infinity, is indicative of a break-point at k ~ 70.

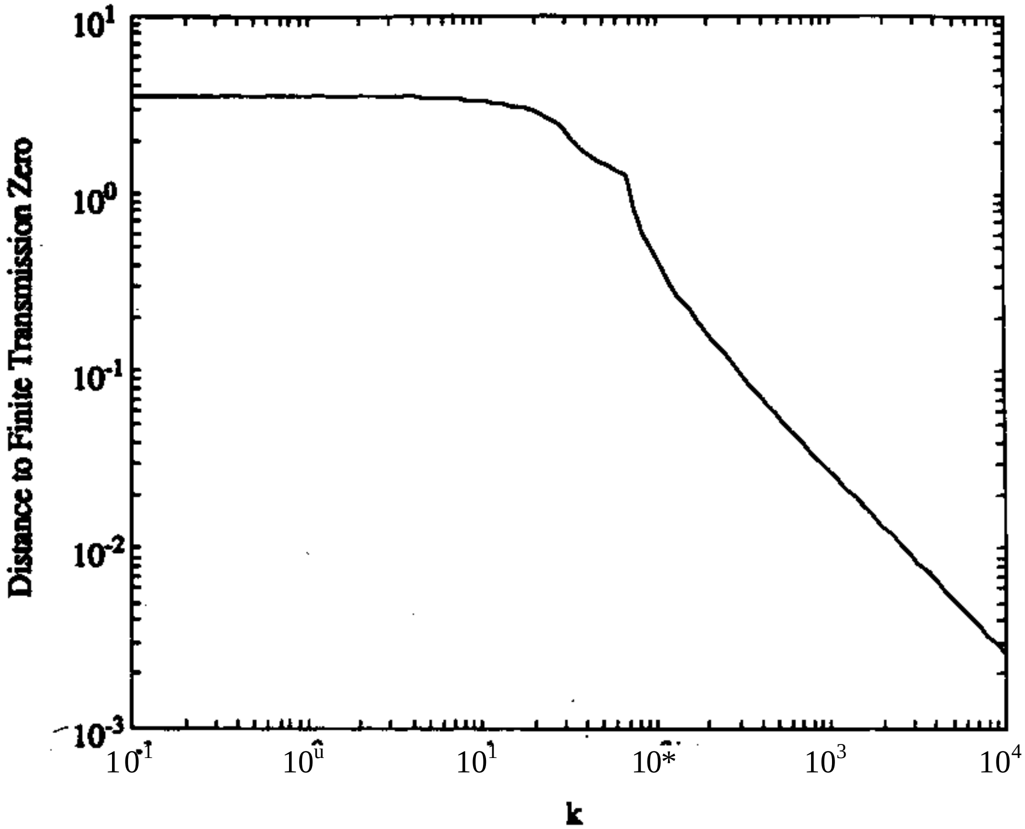

Figure 7. MGP for Asymptotic Approach of Equation (41) to Finite Eigenvalue.

Eigenvalue Approach Rate to Transmission Zeros in General

In **the** previous derivation, unique transmission zeros were assumed This section extends **the** theory to multiple zeros identically located. Again, equation (33) must be satisfied as **k** —> <*>. Therefore, n(s) -» 0 at the same rate that k -» <*>. If w transmission zeros exist at the same location, zi, then as k -»°o

$$s \rightarrow -z_i \tag{42}$$

and

$$n(s) \rightarrow 0 \tag{43}$$

at a rate proportional to T)d(s), where Tjd(s) is the distance sensitivity given by

$$\eta_d(s) = \frac{d(\log|s\text{-}Zil)}{d(\log k)} \qquad (44)$$

T|d(s) may be considered the sensitivity of the eigenvalue distance to the finite transmission zero with respect to gain. Equation (44) employs logarithms to clearly depict the power law relationship governing high gain behavior. Since there is a multiplicity of w non-unique finite transmission zeros, n(s) must approach zero at a rate proportional to $(T|d(s))^w$. In order for equation (34) to hold, T|d(s) must approach zero (or s -* zi) at a rate proportional to $k^{-1/w}$.

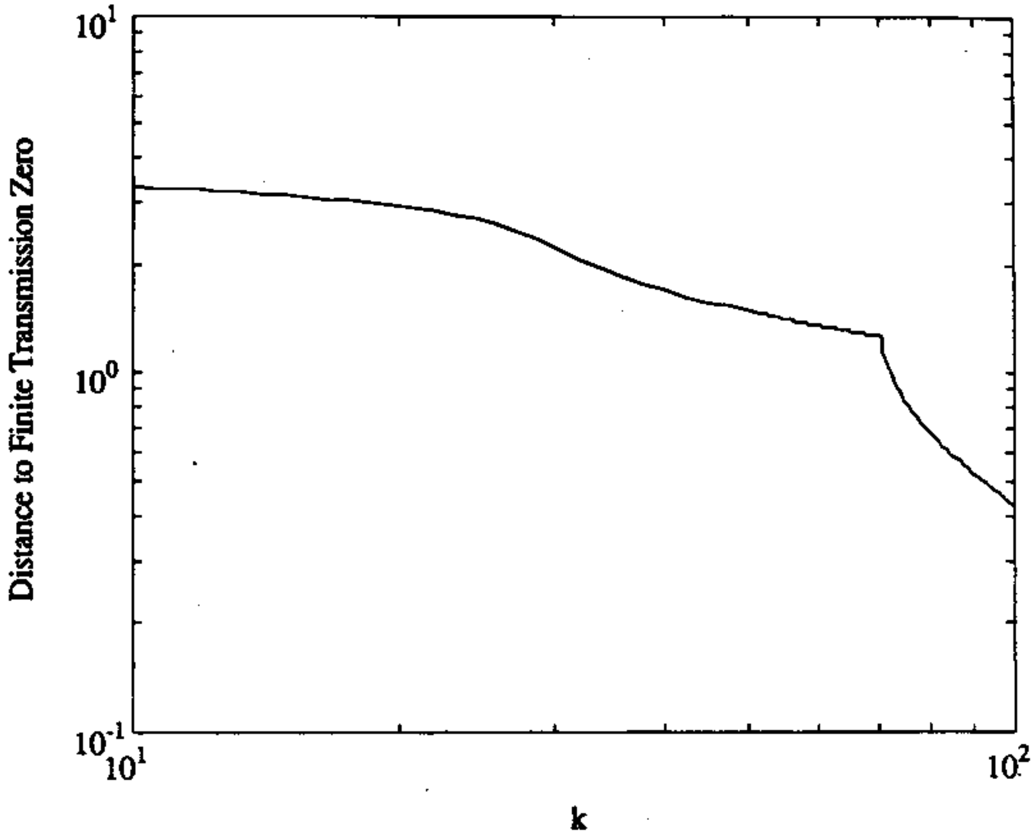

Figure 8. MGP about a Break Point of Equation (41).

Figure 9a,b are the GPs for the open-loop system given by the transfer function

$$g(s) = \frac{(s+D^2}{s(s\text{-}l)(s^2 + 4s+16)} \qquad (45)$$

This is the transfer function given by equation (41) with the addition of a second (identical) transmission zero located at s = -1. As expected from classical control theory, two of the eigenvalues show high gain magnitudes of ~ along asymptotes of ±90*. By inspection, the rate at which their magnitudes approach infinity is proportional to $k^{1}{}^{\wedge 2}$. This is verified by equation (37) where n = 4 and m = 2. The other two eigenvalues approach the finite transmission zero.

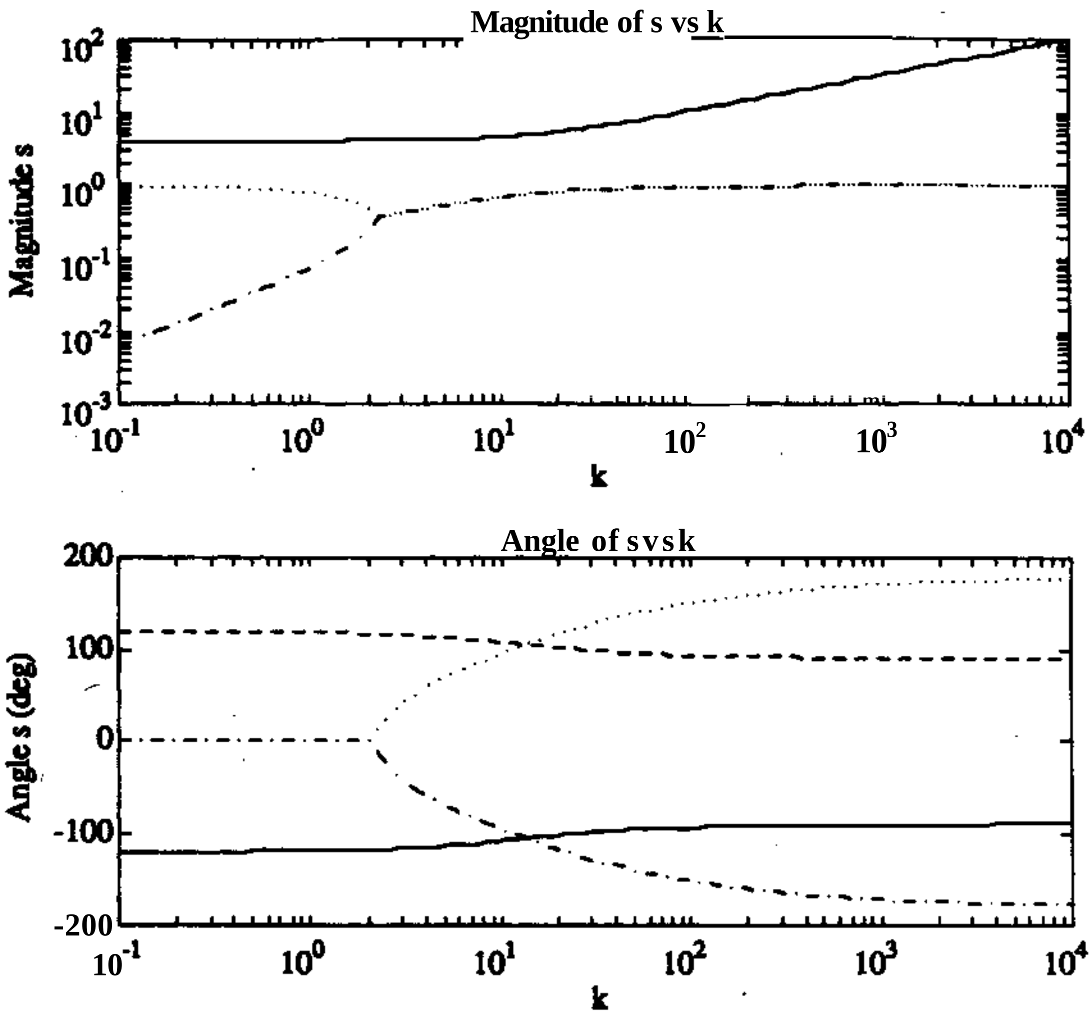

Figure 9a,b. GPs for System Given by Equation (45).

Figure 10 shows the distance from these two transmission zeros to their matching closed-loop eigenvalues. As k -> «>, the slope of the distance asymptote becomes -1/2 indicating that the eigenvalues approach the transmission zeros at a rate proportional to lr$^{\wedge 2}$. This corresponds to the result that T|d(s) must approach zero at a rate proportional to lc-1/w $_w$here w = 2 for this example.

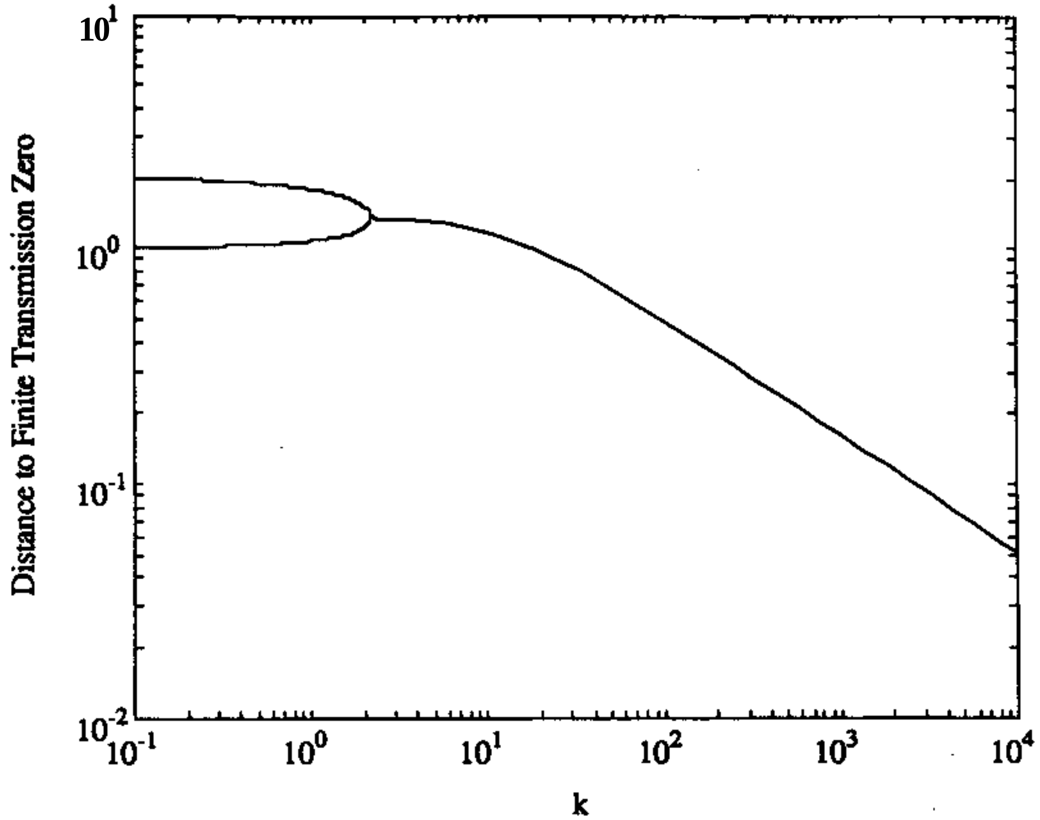

Figure 10. MGP for Asymptotic Approach of Equation (45) to Finite Eigenvalue.

Conclusions

The focus of this report has been the examination of high gain behavior of closed-loop eigenvalues for LTISISO systems in general. It has been shown that the eigenvalues migrating toward infinite transmission zeros proceed at a rate proportional to $k^{l\wedge n\,''m}\backslash$ where n is the number of eigenvalues and m is the number of finite transmission zeros. Furthermore, eigenvalues approach finite transmission zeros at a rate proportional to $k^{\sim w}$, where w is the number of multiple finite zeros at a specific location. In particular, for a system possessing unique finite transmission zeros, w=l, indicating an approach rate proportional to 1/k.

This report has also related the asymptote slopes of the MGP to the sensitivity magnitudes of the closed-loop system eigenvalues at high gain. It can be shown that the high gain sensitivities defined in classical control theory are related to the derivative of log Isl with respect to log k. This motivates the use of the GPs for the purpose of stability, performance, as well as robust system design.

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