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FUNCTIONAL INTERPRETATION AND INDUCTIVE DEFINITIONS

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Abstract. Extending Gödel's *Dialectica* interpretation, we provide a functional interpretation of classical theories of positive arithmetic inductive definitions, reducing them to theories of finite-type functionals defined using transfinite recursion on well-founded trees.

§1. Introduction.

Let X be a set, and let Γ be a monotone operator from the power set of X to itself, so that $A \subseteq B$ implies $\Gamma(A) \subseteq \Gamma(B)$. Then the set

$$I = \bigcap \{A \mid \Gamma(A) \subseteq A\}$$

is a least fixed point of Γ; that is, $\Gamma(I) = I$, and I is a subset of any other set with this property. I can also be characterized as the limit of a sequence indexed by a sufficiently long segment of the ordinals, defined by $I_0 = \emptyset$, $I_{\alpha+1} = \Gamma(I_\alpha)$, and $I_\lambda = \bigcup_{\gamma<\lambda} I_\gamma$ for limit ordinals γ. Such inductive definitions are common in mathematics; they can be used, for example, to define substructures generated by sets of elements, the collection of Borel subsets of the real line, or the set of well-founded trees on the natural numbers.

From the point of view of proof theory and descriptive set theory, one is often interested in structures that are countably based, that is, can be coded so that X is a countable set. In that case, the sequence I_α stabilizes before the least uncountable ordinal. In many interesting situations, the operator Γ is given by a positive arithmetic formula $\varphi(x, P)$, in the sense that $\Gamma(A) = \{x \mid \varphi(x, A)\}$ and φ is an arithmetic formula in which the predicate P occurs only positively. (The positivity requirement can be expressed by saying that no occurrence of P is negated when φ is written in negation-normal form.)

The considerations above show that the least fixed point of a positive arithmetic inductive definition can be defined by a Π^1_1 formula. An analysis due to Stephen Kleene [20, 21] shows that, conversely, a positive arithmetic inductive definition can be used to define a complete Π^1_1 set. In the 1960's, Georg Kreisel presented axiomatic theories of such inductive definitions [25, 9]. In particular, the theory ID_1 consists of first-order arithmetic augmented by additional predicates intended to

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denote least fixed-points of positive arithmetic operators. ID_1 is known to have the same strength as the subsystem $\Pi^1_1\text{–}CA^-$ of second order arithmetic, which has a comprehension axiom asserting the existence of sets of numbers defined by Π^1_1 formulas without set parameters. It also has the same strength as Kripke Platek admissible set theory, $KP\omega$, with an axiom asserting the existence of an infinite set. (See [9, 19] for details.)

A Π_2 sentence is one of the form $\forall \bar{x}\, \exists \bar{y}\, R(\bar{x}, \bar{y})$, where $\bar{x}$ and $\bar{y}$ are tuples of variables ranging over the natural numbers, and R is a primitive recursive relation. Here we are concerned with the project of characterizing the Π_2 consequences of the theories ID_1 in constructive or computational terms. This can be done in a number of ways. For example, every Π_2 theorem of ID_1 is witnessed by a function that can be defined in a language of higher-type functionals allowing primitive recursion on the natural numbers as well as a schema of recursion along well-founded trees, as described in Section 2 below. We are particularly interested in obtaining a translation from ID_1 to a constructive theory of such functions that makes it possible to "read off" a description of the witnessing function from the proof of a Π_2 sentence in ID_1.

There are currently two ways of obtaining this information. The first involves using ordinal analysis to reduce ID_1 to a constructive analogue [8, 26, 27], such as the theory $ID_1^{i,sp}$ discussed below, and then using either a realizability argument or a Dialectica interpretation of the latter [18, 7]. One can, alternatively, use a forcing interpretation due to Buchholz [7, 2] to reduce ID_1 to $ID_1^{i,sp}$.

Here we present a new method of carrying out this first step, based on a functional interpretation along the lines of Gödel's "Dialectica" interpretation of first-order arithmetic. Such functional interpretations have proved remarkably effective in "unwinding" computational and otherwise explicit information from classical arguments (see, for example, [22, 23, 24]). Howard [18] has provided a functional interpretation for a restricted version of the constructive theory $ID_1^{i,sp}$, but the problem of obtaining such an interpretation for classical theories of inductive definitions is more difficult, and was posed as an outstanding problem in [6, Section 9.8]. Feferman [12] used a Dialectica interpretation to obtain ordinal bounds on the strength of ID_1 (the details are sketched in [6, Section 9]), and Zucker [36] used a similar interpretation to bound the ordinal strength of ID_2. But these interpretations do not yield Π_2 reductions to constructive theories, and hence do not provide computational information; nor do the methods seem to extend to the theories beyond ID_2. Our interpretation bears similarities to those of Burr [10] and Ferreira and Oliva [13], but is not subsumed by either; some of the differences between the various approaches are indicated in Section 4.

The outline of the paper is as follows. In Section 2, we describe the relevant theories and provide an overview of our results. Our interpretation of ID_1 is presented in three steps. In Section 3, we embed ID_1 in an intermediate theory, OID_1, which makes the transfinite construction of the fixed-point explicit. In Section 4, we present a functional interpretation that reduces OID_1 to a second intermediate theory, $Q_0T_\Omega + (I)$. Finally, the latter theory is interpreted in a constructive theory, QT^i_Ω, using a cut-elimination argument in Section 5. In Section 6, we show that our interpretation extends straightforwardly to cover theories of iterated inductive definitions as well.

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§2. **Background.** In this paper, we interpret classical theories of inductively defined sets in constructive theories of transfinite recursion on well-founded trees. In this section, we describe the relevant theories, and provide an overview of our results.

Take classical first-order Peano arithmetic, *PA*, to be formulated in a language with symbols for each primitive recursive function and relation. The axioms of *PA* consist of basic axioms defining these functions and relations, and the schema of induction,

$$\varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x\, \varphi(x),$$

where φ is any formula in the language, possibly with free variables other than x. ID_1 is an extension of *PA* with additional predicates I_ψ intended to denote the least fixed point of the positive arithmetic operator given by ψ. Specifically, let $\psi(x, P)$ be an arithmetic formula with at most the free variable x, in which the predicate symbol P occurs only positively. We adopt the practice of writing $x \in I_\psi$ instead of $I_\psi(x)$. ID_1 then includes the following axioms:

- $\forall x\, (\psi(x, I_\psi) \rightarrow x \in I_\psi)$,
- $\forall x\, (\psi(x, \theta/P) \rightarrow \theta(x)) \rightarrow \forall x \in I_\psi\, \theta(x)$, for each formula $\theta(x)$.

Here, the notation $\psi(\theta/P)$ denotes the result of replacing each atomic formula $P(t)$ with $\theta(t)$, renaming bound variables to prevent collisions. The first axiom asserts that I_ψ is closed with respect to Γ_ψ, while the second axiom schema expresses that I_ψ is the smallest such set, among those sets that can be defined in the language. Below we will use the fact that this schema, as well as the schema of induction, can be expressed as rules. For example, I_ψ-leastness is equivalent to the rule "from $\forall x\, (\psi(x, \theta'/P) \rightarrow \theta'(x))$ conclude $\forall x \in I_\psi\, \theta'(x)$." To see this, note that the rule is easily justified using the corresponding axiom; conversely, one obtains the axiom for $\theta(x)$ by taking $\theta'(x)$ to be the formula $(\forall z\, (\psi(z, \theta/P) \rightarrow \theta(z))) \rightarrow \theta(x)$ in the rule.

One can also design theories of inductive definitions based on intuitionistic logic. In order for these theories to be given a reasonable constructive interpretation, however, one needs to be more careful in specifying the positivity requirement on ψ. One option is to insist that P does not occur in the antecedent of any implication, where $\neg\eta$ is taken to abbreviate $\eta \rightarrow \perp$. Such a definition is said to be *strictly positive*, and we denote the corresponding axiomatic theory $ID_1^{i,sp}$. An even more restrictive requirement is to insist that $\psi(x)$ is of the form $\forall y \prec x\, P(y)$, where $\prec$ is a primitive recursive relation. These are called *accessibility* inductive definitions, and serve to pick out the well-founded part of the relation. In the case where $\prec$ is the "child-of" relation on a tree, the inductive definition picks out the well-founded part of that tree. We will denote the corresponding theory $ID_1^{i,acc}$.

The following conservation theorem can be obtained via an ordinal analysis [9] or the methods of Buchholz [7]:

THEOREM 2.1. *Every Π_2 sentence provable in ID_1 is provable in $ID_1^{i,acc}$.*

The methods we introduce here provide another route to this result.

Using a primitive recursive coding of pairs and writing $x \in I_y$ for $(x, y) \in I$ allows us to code any finite or infinite sequence of sets as a single set. One can show that in any of the theories just described, any number of inductively defined sets can coded into a single one, and so, for expository convenience, we will assume that each theory uses only a single inductively defined set.

We now turn to theories of transfinite induction and recursion on well-founded trees. The starting point is a quantifier-free theory, T_Ω, of computable functionals over the natural numbers and the set of well-founded trees on the natural numbers. In particular, T_Ω extends Gödel's theory T of computable functionals over the natural numbers. We begin by reviewing the theory T. The set of *finite types* is defined inductively, as follows:

- N is a finite type; and
- assuming σ and τ are finite types, so are $\sigma \times \tau$ and $\sigma \to \tau$.

In the "full" set-theoretic interpretation, N denotes the set of natural numbers, $\sigma \times \tau$ denotes the set of ordered pairs consisting of an element of σ and an element of τ, and $\sigma \to \tau$ denotes the set of functions from σ to τ. But we can also view the finite types as nothing more than datatype specifications of computational objects. The set of *primitive recursive functionals of finite type* is a set of computable functionals obtained from the use of explicit definition, application, pairing, and projections, and a scheme allowing the definition of a new functional F by primitive recursion:

$$\begin{aligned} F(0) &= a, \\ F(x+1) &= G(x, F(x)). \end{aligned}$$

Here, the range of F may be any finite type. The theory T includes defining equations for all the primitive recursive functionals, and a rule providing induction for quantifier-free formulas φ:

$$\frac{\varphi(0) \qquad \varphi(x) \to \varphi(S(x))}{\varphi(t)}$$

Gödel's *Dialectica* interpretation shows:

THEOREM 2.2. *If PA proves a Π_2 theorem $\forall \bar{x}\, \exists \bar{y}\, R(\bar{x}, \bar{y})$, there is a sequence of function symbols $\bar{f}$ such that T proves $R(\bar{x}, \bar{f}(\bar{x}))$. In particular, every Π_2 theorem of PA is witnessed by sequence of primitive recursive functionals of type $N^k \to N$.*

See [15, 6, 35] for details. If (st) is used to denote the result of applying s to t, we adopt the usual conventions of writing, for example, $stuv$ for $(((st)u)v)$. To improve readability, however, we will also sometimes adopt conventional function notation, and write $s(t, u, v)$ for the same term.

In order to capture the Π_2 theorems of ID_1, we use an extension of T that is essentially due to Howard [18], and described in [6, Section 9.1]. Extend the finite types by adding a new base type, Ω, which is intended to denote the set of well-founded (full) trees on N. We add a constant, e, which denote the tree with just one node, and two new operations: sup, of type $(N \to \Omega) \to \Omega$, which forms a new tree from a sequence of subtrees, and $\sup^{-1}$, of type $\Omega \to (N \to \Omega)$, which returns the immediate subtrees of a nontrivial tree. We extend the schema of primitive recursion on N in T to the larger system, and add a principle of primitive recursion on Ω:

$$F(e) = a,$$
$$F(\sup\ h) = G(\lambda n\, F(h(n))),$$

where the range of F can be any of the new types. We call the resulting theory T_Ω, and the resulting set of functionals the *primitive recursive tree functionals*. Below we will adopt the notation $\alpha[n]$ instead of $\sup^{-1}(\alpha, n)$ to denote the nth subtree of α. In that case definition by transfinite recursion can be expressed as follows:[1]

$$F(\alpha) = \begin{cases} a & \text{if } \alpha = e, \\ G(\lambda n\, F(\alpha[n])) & \text{otherwise.} \end{cases}$$

A trick due to Kreisel (see [17, 18]) allows us to derive a quantifier-free rule of transfinite induction on Ω in T_Ω, using induction on N and transfinite recursion.

PROPOSITION 2.3. *The following is a derived rule of* T_Ω:

$$\frac{\varphi(e, x) \qquad \alpha \neq e \wedge \varphi(\alpha[g(\alpha, x)], h(\alpha, x)) \to \varphi(\alpha, x)}{\varphi(s, t)}$$

for quantifier-free formulas φ.

For the sake of completeness, we sketch a proof in the Appendix.

We define QT_Ω to be the extension of T_Ω which allows quantifiers over all the types of the latter theory, strengthening the previous transfinite induction rule with a full transfinite induction axiom schema,

$$\varphi(e) \wedge \forall \alpha\, (\alpha \neq e \wedge \forall n\, \varphi(\alpha[n]) \to \varphi(\alpha)) \to \forall \alpha\, \varphi(\alpha)$$

where φ is any formula in the expanded language. Let QT^i_Ω denote the version of this theory based on intuitionistic logic.

We can also add to QT^i_Ω an ω-*bounding* axiom schema,

$$\forall x\, \exists \alpha\, \psi(x, \alpha) \to \exists \beta\, \forall x\, \exists i\, \psi(x, \beta[i]),$$

where x is of type N, α is of type Ω, and ψ only has quantifiers over N. The following theorem shows that all of the intuitionistic theories described in this section are "morally equivalent," and reducible to T_Ω.

THEOREM 2.4. *The following theories all prove the same* Π_2 *sentences*:

(1) $ID^{i,sp}_1$,
(2) $ID^{i,acc}_1$,
(3) $QT^i_\Omega + (\omega\text{–}bounding)$,
(4) QT^i_Ω,
(5) T_Ω.

[1] We are glossing over issues involving the treatment of equality in our descriptions of both T and T_Ω. All of the ways of dealing with equality in T described in [6, Section 2.5] carry over to T_Ω, and our interpretations work with even the most minimal version of equality axioms associated with the theory denoted T_0 there. In particular, our interpretations to not rely on extensionality, or the assumption $\forall n\, (\alpha[n] = \beta[n]) \to \alpha = \beta$.

Our theory T_Ω is essentially the theory V of Howard [18]. Our theory QT^i_Ω is essentially a finite-type version of the theory U of [18], and contained in the theory V^* described there. One minor difference is that Howard takes the nodes of his trees to be labeled, with end-nodes labeled by a positive natural number, and internal nodes labeled 0.

PROOF. Buchholz [7] presents a realizability interpretation of $ID_1^{i,sp}$ in the theory $ID_1^{i,acc}$. Howard [18] presents an embedding of $ID_1^{i,acc}$ in QT_Ω^i + (ω*–bounding*). Howard [18] also presents a functional interpretation of QT_Ω^i + (ω*–bounding*) in T_Ω, which is included in QT_Ω^i; Proposition 2.3 is used to interpret the transfinite induction axioms of the source theory. Interpreting T_Ω in $ID_1^{i,sp}$ is straightforward, using the set O of Church–Kleene ordinal notations to interpret the type Ω, and interpreting the constants of T_Ω as hereditarily recursive operations over O (see [6, Sections 4.1, 9.5, and 9.6]). ⊣

In fact, Howard's work [18] shows that Theorem 2.4 still holds as stated if one allows arbitrary formulas $\psi(x, \alpha)$ of QT_Ω^i in the ω-bounding axiom schema. In the classical theories considered below, however, the restriction to arithmetic quantifiers is necessary. We have therefore chosen to use the name (ω*–bounding*) for the restricted version.

We can now describe our main results. In Sections 3 to 5, we present the interpretation outlined in the introduction, which yields:

THEOREM 2.5. *Every* Π_2 *sentence provable in* ID_1 *is provable in* QT_Ω^i.

In fact, if ID_1 proves a Π_2 theorem $\forall \bar{x}\, \exists \bar{y}\, R(\bar{x}, \bar{y})$, our proof yields a sequence of function symbols $\bar{f}$ such that QT_Ω^i proves $R(\bar{x}, \bar{f}(\bar{x}))$. By Theorem 2.4, this last assertion can even be proved in T_Ω. Thus we have:

THEOREM 2.6. *If* ID_1 *proves a* Π_2 *theorem* $\forall \bar{x}\, \exists \bar{y}\, R(\bar{x}, \bar{y})$, *there is a sequence of function symbols* $\bar{f}$ *such that* T_Ω *proves* $R(\bar{x}, \bar{f}(\bar{x}))$. *In particular, every* Π_2 *theorem of* ID_1 *is witnessed by sequence of primitive recursive tree functionals of type* $N^k \to N$.

The reduction described by Sections 3 to 5 is thus analogous to the reduction of ID_1 given by Buchholz [7], but relies on a functional interpretation instead of forcing.

§3. Embedding ID_1 in OID_1.

In this section, we introduce a theory OID_1, which makes the transfinite construction of the fixed points of ID_1 explicit. We then show that ID_1 is easily interpreted in OID_1. The theory OID_1 is closely related to Feferman's theory OR_1^ω, as described in [12] and [6, Section 9], and the embedding is similar to the one described there.

Fix any instance of ID_1 with inductively defined predicate I given by the positive arithmetic formula $\psi(x, P)$. The corresponding instance of OID_1 is two-sorted, with variables $\alpha, \beta, \gamma, \ldots$ ranging over type Ω, and variables $i, j, k, n, x, \ldots$ ranging over N. We include symbols for the primitive recursive functions on N, and a function symbol $\sup^{-1}(\alpha, n)$ which returns an element of type Ω. As above, we write $\alpha[n]$ for $\sup^{-1}(\alpha, n)$. Recall that $\alpha[n]$ is intended to denote the nth subtree of α, or e if $\alpha = e$. The language includes an equality symbol for terms of type N, but *not* for terms of type Ω. We include, however, a unary predicate "$\alpha = e$," which holds when α is the tree with just one node. Finally, there is a binary predicate $I(\alpha, x)$, where α ranges over Ω and x ranges over N. We will write $x \in I_\alpha$ instead of $I(\alpha, x)$, and write $x \in I_{\prec\alpha}$ for $\exists i\, (x \in I_{\alpha[i]})$. The axioms of OID_1 are as follows:

(1) defining axioms for the primitive recursive functions,
(2) induction on N,
(3) transfinite induction on Ω,
(4) $\alpha = e \to \alpha[i] = e$,

(5) the schema of *ω–bounding*:

$$\forall x\, \exists \alpha\, \varphi(x, \alpha) \to \exists \beta\, \forall x\, \exists i\, \varphi(x, \beta[i]),$$

where φ has no quantifiers over type $\mathbf{\Omega}$,

(6) $\forall x\, (x \notin I_e)$,

(7) $\forall \alpha\, (\alpha \neq e \to \forall x\, (x \in I_\alpha \leftrightarrow \psi(x, I_{\prec\alpha})))$.

The last two axioms assert that I_α is the hierarchy of sets satisfying $I_e = \emptyset$ and $I_\alpha = \Gamma_\psi(I_{\prec\alpha})$ when $\alpha \neq e$. For any formula φ of ID_1, let $\hat{\varphi}$ be the formula obtained by interpreting $t \in I$ as $\exists \alpha\, (t \in I_\alpha)$.

Theorem 3.1. *If ID_1 proves φ, then OID_1 proves $\hat{\varphi}$.*

We need two lemmas. In the first, let $I_{\prec\alpha} \subseteq I_{\prec\beta}$ abbreviate the formula $\forall x\, (x \in I_{\prec\alpha} \to x \in I_{\prec\beta})$.

Lemma 3.2. *OID_1 proves that for every α and i, $I_{\prec\alpha[i]} \subseteq I_{\prec\alpha}$.*

Proof. Use transfinite induction on α. If α is equal to e, the conclusion is immediate from the fourth axiom. In the inductive step, suppose x is in $I_{\prec\alpha[i]}$. Then for some j, x is in $I_{\alpha[i][j]}$. By the last axiom, we have $\psi(x, I_{\prec\alpha[i][j]})$. By the inductive hypothesis, we have $I_{\prec\alpha[i][j]} \subseteq I_{\prec\alpha[i]}$, and so, by the positivity of ψ, we have $\psi(x, I_{\prec\alpha[i]})$. By the last axiom again, we have $x \in I_{\alpha[i]}$, and hence $x \in I_{\prec\alpha}$, as required. ⊣

Note that if $\eta(x, P)$ is any arithmetic formula involving a new predicate symbol P and $\theta(y)$ is any formula, applying the $\hat{\ }$-translation to $\eta(x, \theta/P)$ changes only the instances of θ. In particular, $\widehat{\eta(x, I)}$ is $\eta(x, \exists \alpha\, (y \in I_\alpha)/P)$.

Lemma 3.3. *Let $\eta(x, P)$ be a positive arithmetic formula. Then OID_1 proves that $\widehat{\eta(x, I)}$ implies $\exists \alpha\, \eta(x, I_{\prec\alpha})$.*

Proof. Use induction on positive arithmetic formulas, expressed in negation-normal form. To handle the base case where $\eta(x, I)$ is $x \in I$, suppose we have $x \in I_\beta$. By a trivial instance of ω-bounding, there are an α and an i such that x is in $I_{\alpha[i]}$. But this means that x is in $I_{\prec\alpha}$, as required.

All the other cases are easy, except when the outermost connective is a universal quantifier. In that case, suppose OID_1 proves that $\widehat{\varphi(x, y, I)}$ implies $\exists \beta\, \varphi(x, y, I_{\prec\beta})$. Using ω bounding, $\forall y\, \widehat{\varphi(x, y, I)}$ then implies $\exists \alpha\, \forall y\, \exists i\, \varphi(x, y, I_{\prec\alpha[i]})$. By Lemma 3.2 and positivity, OID_1 proves $\exists \alpha\, \forall y\, \varphi(x, y, I_{\prec\alpha})$, as required. ⊣

Proof of Theorem 3.1. The defining axioms for the primitive recursive functions and induction axioms of ID_1 are again axioms of OID_1 under the translation, so we only have to deal with the defining axioms for I.

To verify the translation of the closure axiom in OID_1, suppose $\widehat{\psi(x, I)}$. By Lemma 3.3, we have $\exists \alpha\, \psi(x, I_{\prec\alpha})$, which implies $\exists \alpha\, (x \in I_\alpha)$, as required.

This leaves only the leastness property of I, which can be expressed as a rule, "from $\forall x\, (\psi(x, \theta/P) \to \theta(x))$, conclude $\forall x \in I\, \theta(x)$." To verify the translation in OID_1, suppose $\forall x\, (\psi(x, \hat{\theta}/P) \to \hat{\theta}(x))$. It suffices to show that for every α, we have $\forall x \in I_\alpha\, \hat{\theta}(x)$. We use transfinite induction on α. In the base case, when $\alpha = e$, this is immediate from the defining axiom for I_e. In the inductive step, suppose we have $\forall i\, \forall x \in I_{\alpha[i]}\, \hat{\theta}(x)$. This is equivalent to $\forall x \in I_{\prec\alpha}\, \hat{\theta}(x)$. Using the positivity of P, we have $\forall x\,(\psi(x, I_{\prec\alpha}) \to \psi(x, \hat{\theta}/P))$. Using the definition of I_α, we then have $\forall x \in I_\alpha\, \hat{\theta}(x)$, as required. $\dashv$

§4. **A functional interpretation of** $\boldsymbol{OID_1}$**.** Our next step is to interpret the theory OID_1 in a second intermediate theory, $Q_0T_\Omega + (I)$. First, we describe a fragment Q_0T_Ω of QT_Ω, which is obtained by restricting the language of QT_Ω to allow quantification over the natural numbers only, though we continue to allow free variables and constants of all types. We also restrict the language so that the only atomic formulas are equalities $s = t$ between terms of type N. The axioms of Q_0T_Ω are as follows:

(1) any equality between terms of type N that can be derived in T_Ω,
(2) the schema of induction on N,
(3) the schema of transfinite induction, given as a rule:

$$\frac{\theta(e) \qquad \alpha \neq e \wedge \forall n\, \theta(\alpha[n]) \to \theta(\alpha)}{\theta(t)}$$

for any formula θ and term t of type Ω.

In the transfinite induction schema, the formula $\alpha = e$ is should be understood as the formula $f(\alpha) = 0$, where f is the function from Ω to N defined recursively by $f(e) = 0$, $f(\sup\ g) = 1$. Substitution is a derived rule in Q_0T_Ω, which is to say, if the theory proves $\varphi(x)$ where x is a variable of any type, it proves $\varphi(s)$ for any term s of that type. One can show this by a straightforward induction on proofs, using the fact that any substitution instance of one of the axioms or rules of inference above is again an axiom or rule of inference. Similarly, by induction on formulas $\varphi(x)$, one can show that if T_Ω proves $s = t$ for any terms s and t of the appropriate type, then Q_0T_Ω proves $\varphi(s) \leftrightarrow \varphi(t)$.

The following proposition shows that in Q_0T_Ω we can use instances of induction in which higher-type parameters are allowed to vary. For example, the first rule states that in order to prove $\theta(\alpha, x)$ for arbitrary α and x, it suffices to prove $\theta(e, x)$ for an arbitrary x, and then, in the induction step, prove that $\theta(\alpha, x)$ follows from $\theta(\alpha[n], a)$, as n ranges over the natural numbers and a ranges over a countable sequence of parameters depending on n and x.

PROPOSITION 4.1. *The following are derived rules of* Q_0T_Ω:

$$\frac{\theta(e, x) \qquad \alpha \neq e \wedge \forall i\, \forall j\, \theta(\alpha[i], f(\alpha, x, i, j)) \to \theta(\alpha, x)}{\theta(\alpha, x)}$$

and

$$\frac{\psi(0, x) \qquad \forall j\, \psi(n, f(x, n, j)) \to \psi(n+1, x)}{\psi(n, x)}$$

For a fixed instance of ID_1, we now define the theory $Q_0T_\Omega + (I)$ by adding a new binary predicate $I(\alpha, x)$, which is allowed to occur in the induction axioms and the transfinite induction rules, and the following axioms:

(4) $\forall x\,(x \notin I_e)$,
(5) $\forall \alpha\,(\alpha \neq e \to \forall x\,(x \in I_\alpha \leftrightarrow \psi(x, I_{\prec\alpha})))$,
(6) $s \in I_\alpha \leftrightarrow t \in I_\beta$ whenever s, α, t, and β are terms such that T_Ω proves $s = t$ and $\alpha = \beta$.

Proposition 4.1 extends to this new theory, as does the substitution rule. Thanks to axiom (6), if $\varphi(x)$ is any formula of $Q_0 T_\Omega + (I)$ and s and t are any terms such that T_Ω proves $s = t$, then $Q_0 T_\Omega + (I)$ proves $\varphi(s) \leftrightarrow \varphi(t)$.

The goal of this section is to use a functional interpretation to interpret OID_1 in $Q_0 T_\Omega + (I)$. As in Burr [10], we use a variant of Shoenfield's interpretation [29] which incorporates an idea due to Diller and Nahm [11]. The Shoenfield interpretation works for classical logic, based on the connectives $\forall$, $\vee$, and $\neg$. This has the virtue of cutting down on the number of axioms and rules that need to be verified, and keeping complexity down. Alternatively, we could have used a Diller–Nahm variant of the ordinary Gödel interpretation, combined with a double-negation interpretation. The relationship between the latter approach and the Shoenfield interpretation is now well understood (see [33, 3]).

First, we need to introduce some notation. We will often think of an element $\alpha \neq e$ of Ω as denoting a countable set $\{\alpha[i] \mid i \in \mathbb{N}\}$ of elements of Ω. Within the language of QT_Ω, we therefore define $\alpha \sqsubseteq \beta$ by

$$\alpha \sqsubseteq \beta \equiv \forall i\, \exists j\, (\alpha[i] = \beta[j]),$$

expressing inclusion between the corresponding sets. Let $t(i)$ be any term of type Ω, where i is of type N. Then we can define the union of the sets $t(0), t(1), t(2), \ldots$ by

$$\sqcup_i t(i) \equiv \sup_j t(j_0)[j_1],$$

where j_0 and j_1 denote the projections of j under a primitive recursive coding of pairs. In other words, $\sqcup_i t(i)$ represents the set $\{t(i)[k] \mid i \in \mathbb{N}, k \in \mathbb{N}\}$. In particular, we have that for every i, $t(i) \sqsubseteq \sqcup_i t(i)$, since for every k we have $t(i)[k] = (\sqcup_i t(i))[(i,k)]$. Binary unions, $s \sqcup t$, can be defined in a similar way.

We can extend these notions to higher types. Define the set of *pure* Ω*-types* to be the smallest set of types containing Ω and closed under the operation taking σ and τ to $\sigma \to \tau$. Note that every pure Ω-type τ has the form $\sigma_1 \to \sigma_2 \to \ldots \sigma_k \to \Omega$. We can therefore lift the notions above to a, b, and t of arbitrary pure type, by defining them to hold pointwise, as follows:

$$\begin{aligned} a[i] &\equiv \lambda x\, ((ax)[i]), \\ a \sqsubseteq b &\equiv \forall i\, \exists j\, \forall x\, ((ax)[i] = (bx)[j]), \\ \sqcup_i t &\equiv \lambda x\, (\sqcup_i (tx)), \\ s \sqcup t &\equiv \lambda x\, ((sx) \sqcup (tx)), \end{aligned}$$

where in each case x is a tuple of variables chosen so that the resulting term has type Ω. Thus, if a is of any pure type, we can think of a as representing the countable set $\{a[i] \mid i \in \mathbb{N}\}$, in which case $\sqsubseteq$ and $\sqcup$ have the expected behavior.

Below we will be interested in the situation where T_Ω can prove $a \sqsubseteq b$ in the sense that there is an explicit term $f(i)$, not involving x, such that T_Ω proves $(ax)[i] = (bx)[f(i)]$. Notice that when T_Ω proves $a \sqsubseteq b$ in this sense, $Q_0 T_\Omega + (I)$ proves $\varphi(a[i]) \to \varphi(b[f(i)])$ for any formula φ, and hence $\exists i\, \varphi(a[i]) \to \exists j\, \varphi(b[j])$. Notice also that T_Ω proves $t(i) \sqsubseteq \sqcup_i t(i)$, $s \sqsubseteq s \sqcup t$, and $t \sqsubseteq s \sqcup t$ in this sense.

To each formula φ in the language of OID_1, we associate a formula φ^S of the form $\forall a\, \exists b\, \varphi_S(a, b)$, where a and b are tuples of variables of certain pure Ω-types (which are implicit in the definitions below), and φ_S is a formula in the language of $Q_0 T_\Omega + (I)$. The interpretation is defined, inductively, in such a way that the following monotonicity property is preserved: whenever T_Ω proves $b \sqsubseteq b'$ in the sense above, $Q_0 T_\Omega + (I)$ proves $\varphi_S(a, b) \to \varphi_S(a, b')$. In the base case, we define

$$I(\alpha, t)^S \equiv I(\alpha, t),$$
$$(s = t)^S \equiv s = t.$$

In the inductive step, suppose φ^S is $\forall a\, \exists b\, \varphi_S(a, b)$ and ψ^S is $\forall c\, \exists d\, \psi_S(c, d)$. Then we define

$$\begin{aligned}
(\varphi \vee \psi)^S &\equiv \forall a, c\, \exists b, d\, (\varphi_S(a, b) \vee \psi_S(c, d)),\\
(\forall x\, \varphi)^S &\equiv \forall a\, \exists b\, (\forall x\, \varphi_S(a, b)),\\
(\forall \alpha\, \varphi)^S &\equiv \forall \alpha, a\, \exists b\, \varphi_S(a, b),\\
(\neg \varphi)^S &\equiv \forall B\, \exists a\, (\exists i\, \neg \varphi_S(a[i], B(a[i]))).
\end{aligned}$$

Verifying the monotonicity claim above is straightforward; the inner existential quantifier in the clause for negation takes care of the only case that would otherwise have given us trouble. Note in particular the clause for universal quantification over the natural numbers. Our functional interpretation is concerned with bounds; because we can compute "countable unions" using the operator $\sqcup$, we can view quantification over the natural numbers as "small" and insist that the bound provided by b is independent of x. Note also that if φ is a purely arithmetic formula, φ^S is just φ.

The rest of this section is devoted to proving the following:

THEOREM 4.2. *Suppose OID_1 proves φ, and φ^S is the formula $\forall a\, \exists b\, \varphi_S(a, b)$. Then there are terms b of T_Ω involving at most the variables a and the free variables of φ of type Ω such that $Q_0 T_\Omega + (I)$ proves $\varphi_S(a, b)$.*

Importantly, the terms b in the statement of the theorem do not depend on the free variables of φ of type N.

As usual, the proof is by induction on derivations. The details are similar to those in Burr [10]. As in Shoenfield [29], we can take the logical axioms and rules to be the following:

(1) excluded middle: $\neg\varphi \vee \varphi$,
(2) substitution: $\forall x\, \varphi(x) \to \varphi(t)$, and $\forall \alpha\, \varphi(\alpha) \to \varphi(t)$,
(3) expansion: from φ conclude $\varphi \vee \psi$,
(4) contraction: from $\varphi \vee \varphi$ conclude φ,
(5) cut: from $\varphi \vee \psi$ and $\neg\varphi \vee \theta$, conclude $\psi \vee \theta$,
(6) $\forall$-introduction: from $\varphi \vee \psi$ conclude $\forall x\, \varphi \vee \psi$, assuming x is not free in ψ; and similarly for variables of type Ω,
(7) equality axioms.

The translation of excluded middle is

$$\forall B, a'\, \exists a, b'\, (\exists i\, \neg\varphi_S(a[i], B(a[i])) \vee \varphi_S(a', b')).$$

Given B and a', let $a = \sup_i a'$, so that $a[i] = a'$ for every i; in other words, a represents the singleton set $\{a'\}$. Let $b' = B(a')$. Then the matrix of the formula holds with $i = 0$.

The translation of substitution for the natural numbers is equivalent to

$$\forall B, a'\, \exists a, b'\, (\forall i, x\, \varphi_S(x, a[i], B(a[i])) \to \varphi_S(t, a', b')).$$

(In this context, "equivalent to" means that $Q_0T_\Omega + (I)$ proves that the $\cdot_S$ part of the translation is equivalent to the expression in parentheses.) Once again, given B and a', letting $a = \sup_i a'$ and $b' = B(a')$ works.

Handling substitution for Ω and expansion is straightforward. Consider the contraction rule. By the inductive hypothesis we have terms $b = b(a,c)$ and $d = d(a,c)$ satisfying

$$\varphi_S(a, b(a,c)) \vee \varphi_S(c, d(a,c)).$$

Define $f(e)$ to be $b(e,e) \sqcup d(e,e)$. Then T_Ω proves $b(e,e) \sqsubseteq f(e)$ and $d(e,e) \sqsubseteq f(e)$. By substitution and monotonicity we have $\varphi_S(e, f(e)) \vee \varphi_S(e, f(e))$, and hence $\varphi_S(e, f(e))$, as required.

Consider cut. By the inductive hypothesis we have terms $b = b(a,c)$ and $d = d(a,c)$ satisfying

$$\varphi_S(a, b(a,c)) \vee \psi_S(c, d(a,c)), \tag{1}$$

and terms $a' = a'(B,e)$ and $f = f(B,e)$ satisfying

$$\exists i \, \neg\varphi_S(a'(B,e)[i], B(a'(B,e))[i]) \vee \theta_S(e, f(B,e)). \tag{2}$$

We need terms $d' = d'(c',e')$ and $f' = f'(c',e')$ satisfying

$$\psi_S(c', d'(c',e')) \vee \theta_S(e', f'(c',e')).$$

Given c' and e', and the terms $b(a,c)$, $d(a,c)$, $a'(B,e)$, and $f(B,e)$, define $B' = \lambda a \, b(a,c')$, define $a'' = \sup_i a'(B',e')$, and then define $d' = d(a'',c')$ and $f' = f(B',e')$. Since T_Ω proves $B'(a'') = b(a'',c')$, from (1) we have

$$\varphi_S(a'', B'(a'')) \vee \psi_S(c', d').$$

Since $a''[i] = a'(B',e')$ for every i, from (2) we have

$$\neg\varphi_S(a'', B'(a'')) \vee \theta_S(e', f').$$

Applying cut in $Q_0T_\Omega + (I)$, we have $\psi_S(c',d') \vee \theta_S(e',f')$, as required.

The treatment of $\forall$-introduction over N and Ω is straightforward. We can take the equality axioms to be reflexivity, symmetry, transitivity, and congruence with respect to the basic function and relation symbols in the language. These, as well as the defining equations for primitive recursive function symbols in the language and the defining axioms for I, are verified by the fact that for formulas whose quantifiers ranging only over N, $\varphi^S = \varphi$.

Thus we only have to deal with the other axioms of OID_1, namely, ω bounding, induction on N, and transfinite induction on Ω. Note that if φ has quantifiers ranging only over N, the definition of $\exists$ in terms of $\forall$ implies that $(\exists \alpha \, \varphi(\alpha))^S$ is equivalent to $\exists \alpha \, \exists i \, \varphi(\alpha[i])$. To interpret the translation of ω-bounding, we therefore need to define a term $\beta = \beta(\alpha)$ satisfying

$$\forall x \, \exists i \, \varphi_S(x, \alpha[i]) \rightarrow \exists j \, \forall x \, \exists k \, \varphi_S(x, (\beta[j])[k]).$$

Setting $\beta = \sup_j \alpha$ means that for every j we have $\beta[j] = \alpha$, so this β works.

We can take induction on the natural numbers to be given by the rule "from $\varphi(0)$ and $\varphi(x) \rightarrow \varphi(x+1)$ conclude $\varphi(t)$ for any term t." From a proof of the first hypothesis, we obtain a term $b = b(a)$ satisfying

$$\varphi_S(0, a, b). \tag{3}$$

From a proof of the second hypothesis, we obtain terms $a' = a'(B', a'')$ and $b'' = b''(B', a'')$ satisfying

$$\forall i\, \varphi_S(x, a'[i], B'(a'[i])) \to \varphi_S(x+1, a'', b''). \tag{4}$$

If suffices to define a function $f(x, \hat{a})$ and show that we can prove

$$\varphi_S(x, \hat{a}, f(x, \hat{a})), \tag{5}$$

since if we then define $\hat{b}(\hat{a}) = \sqcup_x f(x, \hat{a})$, we have $\varphi_S(x, \hat{a}, \hat{b})$ by the monotonicity property of our translation. Define f by

$$f(0, \hat{a}) = b(\hat{a}),$$
$$f(x+1, \hat{a}) = b''(\lambda a\, f(x, a), \hat{a}).$$

Let B' denote $\lambda a\, f(x, a)$, so $f(x+1, \hat{a}) = b''(B', \hat{a})$. Let $A(x, \hat{a})$ denote the formula (5). From (3), we have $A(0, \hat{a})$, and from (4) we have $\forall i\, A(x, a'(B', \hat{a})[i]) \to A(x+1, \hat{a})$. Using Proposition 4.1, we obtain $A(x, \hat{a})$, as required.

Transfinite induction, expressed as the rule "from $\varphi(e)$ and $\forall n\, \varphi(\alpha[n]) \to \varphi(\alpha)$ conclude $\varphi(\alpha)$," is handled in a similar way. From a proof of the first hypothesis we obtain a term $b = b(a)$ satisfying

$$\varphi_S(e, a, b). \tag{6}$$

From a proof of the second hypothesis we obtain terms $a' = a'(\alpha, B', a'')$ and $b'' = b''(\alpha, B', a'')$ satisfying

$$\forall i\, \forall n\, \varphi_S(\alpha[n], a'[i], B'(a'[i])) \to \varphi_S(\alpha, a'', b''). \tag{7}$$

It suffices to define a function f satisfying

$$\varphi_S(\alpha, \hat{a}, f(\alpha, \hat{a}))$$

for every α and $\hat{a}$, since then $\hat{b} = f(\alpha, \hat{a})$ is the desired term. Let $A(\alpha, \hat{a})$ be this last formula, and define f by recursion on α:

$$f(\alpha, \hat{a}) = \begin{cases} b(a) & \text{if } \alpha = e, \\ b''(\alpha, \lambda a\, (\sqcup_j f(\alpha[j], a)), \hat{a}) & \text{otherwise.} \end{cases}$$

Write B' for the expression $\lambda a\, (\sqcup_j f(\alpha[j], a))$, so we have $f(\alpha, \hat{a}) = b''(\alpha, B', \hat{a})$ when $\alpha \neq e$. We will use the transfinite induction rule given by Proposition 4.1 to show that $A(\alpha, \hat{a})$ holds for every α and $\hat{a}$. From (6), we have $A(e, \hat{a})$, so it suffices to show

$$\alpha \neq e \wedge \forall n, i\, A(\alpha[n], a'[i]) \to A(\alpha, \hat{a}),$$

where a' is the term $a'(\alpha, B', \hat{a})$. Arguing in $Q_0 T_\Omega + (I)$, assume $\alpha \neq e$ and $\forall n, i\, A(\alpha[n], a'[i])$, that is,

$$\forall n, i\, \varphi_S(\alpha[n], a'[i], f(\alpha[n], a'[i])).$$

By monotonicity, we have

$$\forall n, i\, \varphi_S(\alpha[n], a'[i], \sqcup_j f(\alpha[j], a'[i])).$$

By the definition of B', this is just

$$\forall n, i\, \varphi_S(\alpha[n], a'[i], B'(a'[i])).$$

By (7), this implies

$$\varphi_S(\alpha, \hat{a}, f(\alpha, \hat{a})),$$

which is $A(\alpha, \hat{a})$ as required. This concludes the proof of Theorem 4.2.

Our theory $Q_0T_\Omega + (I)$ is inspired by Feferman [12], and, in particular, the theory denoted $T_\Omega + (\mu)$ in [6, Section 9]. That theory, like $Q_0T_\Omega + (I)$, combines a classical treatment of quantification over the natural numbers with a constructive treatment of the finite types over Ω.

The principal novelty of our interpretation, however, is the use of the Diller–Nahm method in the clause for negation, and the resulting monotonicity property. This played a crucial rule in the interpretation of transfinite induction. The usual Dialectica interpretation would require us to choose a single candidate for the failure of an inductive hypothesis, something that cannot be done constructively. Instead, using the Diller–Nahm trick, we recursively "collect up" a countable sequence of possible counterexamples. (The original Diller–Nahm trick involved using only finite sequences of counterexamples; we are grateful to Paulo Oliva for pointing out to us that the extension of the method to more general sequences of counterexamples seems to have been first used by Stein [32].)

Similar uses of monotonicity can be found in functional interpretations developed by Kohlenbach [23, 24] and Ferreira and Oliva [13], as well as in the forcing interpretations described in Avigad [5]. The functional interpretations of Avigad [4], Burr [10], and Ferreira and Oliva [13] also make use of the Diller–Nahm trick. But Kohlenbach, Ferreira, and Oliva rely on majorizability relations, which cannot be represented in Q_0T_Ω, due to the restricted uses of quantification in that theory. Our interpretation is perhaps closest to the one found in Burr [10], but a key difference is in our interpretation of universal quantification over the natural numbers; as noted above, because we are computing bounds and our functionals are closed under countable sequences, the universal quantifier is absorbed by the witnessing functional.

§5. Interpreting $Q_0T_\Omega + (I)$ in QT^i_Ω. The hard part of the interpretation is now behind us. It is by now well known that one can embed infinitary proof systems for classical logic in the various constructive theories listed in Theorem 2.4. This idea was used by Tait [34], to provide a constructive consistency proof for the subsystem Σ^1_1–CA of second-order arithmetic. It was later used by Sieg [30, 31] to provide a direct reduction of the classical theory ID_1 to the constructive theory $ID^{i,sp}_2$, as well as the corresponding reductions for theories of transfinitely iterated inductive definitions (see Section 6). Here we show that, in particular, one can define an infinitary proof system in QT^i_Ω, and use it to interpret $Q_0T_\Omega + (I)$ in a way that preserves Π_2 formulas. The methods are essentially those of Sieg [30, 31], adapted to the theories at hand. In fact, our interpretation yields particular witnessing functions in T_Ω, yielding Theorem 2.5.

Let us define the set of infinitary *constant* propositional formulas, inductively, as follows:

- $\top$ and $\bot$ are formulas.
- If $\varphi_0, \varphi_1, \varphi_2, \ldots$ are formulas, so are $\bigvee_{i \in N} \varphi_i$ and $\bigwedge_{i \in N} \varphi_i$.

Take a *sequent* Γ to be a finite set of such formulas. As usual, we write Γ, Δ for $\Gamma \cup \Delta$ and Γ, φ for $\Gamma \cup \{\varphi\}$. We define a cut-free infinitary proof system for such formulas with the following rules:

- $\Gamma, \top$ is an axiom for each sequent Γ.
- From Γ, φ_i for some i conclude $\Gamma, \bigvee_{i \in N} \varphi_i$.
- From Γ, φ_i for every i conclude $\Gamma, \bigwedge_{i \in N} \varphi_i$.

We also define a mapping $\varphi \mapsto \neg\varphi$ recursively, as follows:

- $\neg\top = \bot$.
- $\neg\bot = \top$.
- $\neg \bigvee_{i \in N} \varphi_i = \bigwedge_{i \in N} \neg\varphi_i$.
- $\neg \bigwedge_{i \in N} \varphi_i = \bigvee_{i \in N} \neg\varphi_i$.

Note that the proof system does not include the cut rule, namely, "from Γ, φ and $\Gamma, \neg\varphi$ include Γ." In this section we will show that it is possible to represent propositional formulas and infinitary proofs in the language of QT^i_Ω in such a way that QT^i_Ω proves that the set of provable sequents is closed under cut. We will then show that this infinitary proof system makes it possible to interpret $Q_0T_\Omega + (I)$ in a way that preserves Π_2 sentences. This will yield Theorem 2.5. In fact, our interpretation will yield explicit functions witnessing the truth of the Π_2 from the proof in $Q_0T_\Omega + (I)$.

We can represent formulas in QT^i_Ω as well-founded trees whose end nodes are labeled either $\top$ or $\bot$ and whose internal nodes are labeled either $\bigvee$ or $\bigwedge$. A well-founded tree is simply an element of Ω. As in the Appendix, if α is an element of Ω, then one can assign to each node of α a unique "address," σ, where σ is a finite sequence of natural numbers. Since these can be coded as natural numbers, a labeling of α from the set $\{\top, \bot, \bigvee, \bigwedge\}$ is a function l from N to N. The assertion that α, l is a formula, i.e., that the labeling has the requisite properties, is given by a universal formula in QT^i_Ω. Using λ-abstraction we can define functions F with recursion of the following form:

$$F(\alpha, l) = \begin{cases} G(l(\emptyset)) & \text{if } \alpha = e, \\ H(\lambda n\, F(\alpha[n], \lambda\sigma\, l((n)\hat{}\sigma))) & \text{otherwise,} \end{cases}$$

where $\emptyset$ denotes the sequence of length 0. This yields a principle of recursive definition on formulas, which can be used, for example, to define the map $\varphi \to \neg\varphi$. (This particular function can be defined more simply by just switching $\top$ with $\bot$ and $\bigwedge$ with $\bigvee$ in the labeling.) A principle of induction on formulas is obtained in a similar way. We can now represent proofs as well-founded trees labeled by finite sets of formulas and rules of inference, yielding principles of induction and recursion on proofs as well.

We will write $\vdash \Gamma$ for the assertion that Γ has an infinitary proof, and we will write $\vdash \varphi$ instead of $\vdash \{\varphi\}$. The proofs of the following in QT^i_Ω are now standard and straightforward (see, for example, [28, 31]).

Lemma 5.1 (Weakening). *If* $\vdash \Gamma$ *and* $\Gamma' \supseteq \Gamma$ *then* $\vdash \Gamma'$.

Lemma 5.2 (Excluded middle). *For every formula* φ, $\vdash \{\varphi, \neg\varphi\}$.

Lemma 5.3 (Inversion).

- *If* $\vdash \Gamma, \bot$, *then* $\vdash \Gamma$.
- *If* $\vdash \Gamma, \bigwedge_{i \in N} \varphi_i$, *then* $\vdash \Gamma, \varphi_i$ *for every* i.

The first and third of these is proved using induction on proofs in QT^i_Ω. The second is proved using induction on formulas.

Lemma 5.4 (Admissibility of cut). *If* $\vdash \Gamma, \varphi$ *and* $\vdash \Gamma, \neg\varphi$, *then* $\vdash \Gamma$.

Proof. We show how to cast the usual proof as a proof by induction on formulas, with a secondary induction on proofs. For any formula φ, define

$$\varphi^\vee = \begin{cases} \varphi & \text{if } \varphi \text{ is } \top \text{ or of the form } \bigvee_{i\in N} \psi_i, \\ \neg\varphi & \text{otherwise.} \end{cases}$$

We express the claim to be proved as follows:

> For every formula φ, for every proof d, the following holds: if d is a proof of a sequent of the form $\Gamma, \varphi^\vee$, then $\vdash \Gamma, \neg(\varphi^\vee)$ implies $\vdash \Gamma$.

The most interesting case occurs when $\varphi = \varphi^\vee$ is of the form $\bigvee_{i\in N} \psi_i$, and the last inference of d is of the form

$$\frac{\Gamma, \bigvee_{i\in N} \psi_i, \psi_j}{\Gamma, \bigvee_{i\in N} \psi_i}$$

Given a proof of $\Gamma, \bigwedge_{i\in N} \neg\psi_i$, apply weakening and the inner inductive hypothesis for the immediate subproof of d to obtain a proof of Γ, ψ_j, apply inversion to obtain a proof of $\Gamma, \neg\psi_j$, and then apply the outer inductive hypothesis to the subformula $\neg\psi_j$ of φ. ⊣

We now assign, to each formula $\varphi(\bar{x})$ in the language of $Q_0T^i_\Omega + (I)$, an infinitary formula $\hat{\varphi}(\bar{x})$. More precisely, to each formula $\varphi(\bar{x})$ we assign a function $F_\varphi(\bar{x})$ of T_Ω, in such a way that QT^i_Ω proves "for every $\bar{x}$, $F_\varphi(\bar{x})$ is an infinitary propositional formula." We may as well take $\vee$, $\neg$, and $\forall$ to be the logical connectives of $Q_0T^i_\Omega + (I)$, and use the Shoenfield axiomatization of predicate logic given in the last section. For formulas not involving I_α, the assignment is defined inductively as follows:

- $\widehat{s=t}$ is equal to $\top$ if $s = t$, and $\bot$ otherwise.
- $\widehat{\varphi \vee \psi}$ is equal to $\bigvee_j \widehat{\theta}_j$, where $\theta_0 = \varphi$ and $\theta_j = \psi$ for $j > 0$.
- $\widehat{\forall x\, \varphi(x)}$ is $\bigwedge_j \widehat{\varphi}(j)$.
- $\widehat{\neg\varphi}$ is $\neg\widehat{\varphi}$.

If I corresponds to the inductive definition $\psi(x, P)$, the interpretation of $x \in I_\alpha$ is defined recursively:

$$x \in I_\alpha = \begin{cases} \bot & \text{if } \alpha = e, \\ \widehat{\psi(x, I_{\prec\alpha})} & \text{otherwise.} \end{cases}$$

The following lemma asserts that this interpretation is sound.

Lemma 5.5. *If* $Q_0T_\Omega + (I)$ *proves* $\varphi(\bar{x})$, *then* QT^i_Ω *proves that for every* $\bar{x}, \vdash \hat{\varphi}(\bar{x})$.

Proof. We simply run through the axioms and rules of inference in $Q_0T_\Omega + (I)$. If $s = t$ is a theorem of T_Ω, it is also a theorem of QT^i_Ω. Hence QT^i_Ω proves $\widehat{s=t} = \top$, and so $\vdash \widehat{s=t}$.

The interpretation of the logical axioms and rules are easily validated in the infinitary propositional calculus augmented with the cut rule, and the interpretation of the defining axioms for I_α are trivially verified given the translation of $\widehat{t \in I_\alpha}$.

This leaves only induction on N and transfinite induction on Ω. We will consider transfinite induction on Ω; the treatment of induction on N is similar.

We take transfinite induction to be given by the rule "from $\varphi(e)$ and $\alpha \neq e \wedge \forall n\, \varphi(\alpha[n]) \rightarrow \varphi(\alpha)$ conclude $\varphi(\alpha)$." Arguing in $Q_0 T_\Omega + (I)$, suppose for every instantiation of α and the parameters of φ there is an infinitary derivation of the $\widehat{\cdot}$ translation of these hypothesis. Use transfinite induction to show that for every α there is an infinitary proof of $\widehat{\varphi}(\alpha)$. When $\alpha = e$, this is immediate. In the inductive step we have infinitary proofs of $\widehat{\varphi}(\alpha[n])$ for every n. Applying the $\bigwedge$-rule, we obtain an infinitary proof of $\widehat{\forall n\, \varphi(\alpha[n])}$, and hence, using ordinary logical operations in the calculus with cut, a proof of $\widehat{\varphi}(\alpha)$. $\dashv$

We note that with a little more care, one can obtain cut-free proofs of the induction and transfinite induction axioms; see, for example, [7].

Lemma 5.6. *Let φ be a formula of the form $\forall \bar{x}\, \exists \bar{y}\, R(\bar{x}, \bar{y})$, where R is primitive recursive. Then QT^i_Ω proves that $\vdash \hat{\varphi}$ implies φ.*

Proof. Using a primitive recursive coding of tuples we can assume, without loss of generality, that each of $\bar{x}$ and $\bar{y}$ is a single variable. Using the inversion lemma, it suffices to prove the statement for Σ_1 formulas, which we can take to be of the form $\exists y\, S(y)$ for some primitive recursive S. Use induction on proofs to prove the slightly more general claim that given any proof of either $\{\widehat{\exists y\, S(y)}\}$ or $\{\widehat{\exists y\, S(y)}, \bot\}$ there is a j satisfying $S(j)$. In a proof of either sequent, the last rule rule can only have been a $\bigvee$ rule, applied to a sequent of the form $\{\widehat{S(j)}\}$ or $\{\widehat{\exists y\, S(y)}, \widehat{S(j)}\}$. If $\widehat{S(j)}$ equals $\top$, j is the desired witness; otherwise, apply the inductive hypothesis. $\dashv$

Putting the pieces together, we have shown:

Theorem 5.7. *Every Π_2 theorem of $Q_0 T_\Omega + (I)$ is a theorem of QT^i_Ω.*

Together with Theorems 3.1 and 4.2, this yields Theorem 2.5. Note that every time we used induction on formulas or proofs in the lemmas above, the arguments give explicit constructions that are represented by terms of T_Ω. So we actually obtain, from an ID_1 proof of a Π_2 sentence, a T_Ω term witnessing the conclusion and a proof that this is the case in QT^i_Ω. By Theorem 2.4, this can be converting to a proof in T_Ω, if desired.

Our reduction of ID_1 to a constructive theory has been carried out in three steps, amounting, essentially, to a functional interpretation on top of a straightforward cut elimination argument. A similar setup is implicit in the interpretation of ID_1 due to Buchholz [7], where a forcing interpretation is used in conjunction with an infinitary calculus akin to the one we have used here. We have also considered alternative reductions of $Q_0 T_\Omega + (I)$ that involve either a transfinite version of the Friedman A-translation [14] or a transfinite version of the Dialectica interpretation. These yield interpretations of $Q_0 T_\Omega + (I)$ not in QT^i_Ω, however, but in a Martin–Löf type theory $ML_1 V$ with a universe and a type of well-founded sets [1]. $ML_1 V$ is known to have the same strength as ID_1, but although many consider $ML_1 V$ to be a legitimate constructive theory in its own right, we do not know of any reduction of $ML_1 V$ to one of the other constructive theories listed in Theorem 2.4 that does not subsume a reduction of ID_1. Thus the methods described in this section seem to provide an easier route to a stronger result.

§6. Iterating the interpretation. In this section, we consider theories ID_n of finitely iterated inductive definitions. These are defined in the expected way: ID_{n+1} bears the same relationship to ID_n that ID_1 bears to PA. In other words, in ID_{n+1} one can introduce a inductive definitions given by formulas $\psi(x, P)$, where ψ is a formula in the language of ID_n together with the new predicate P, in which P occurs only positively.

Writing Ω_0 for N and Ω_1 for Ω, we can now define a sequence of theories T_{Ω_n}. For each $n \geq 1$ take $T_{\Omega_{n+1}}$ to add to T_{Ω_n} a type Ω_{n+1} of trees branching over Ω_n, with corresponding constant e and functionals $\sup\colon (\Omega_n \to \Omega_{n+1}) \to \Omega_{n+1}$ and $\sup^{-1}\colon \Omega_{n+1} \to (\Omega_n \to \Omega_{n+1})$. Once again, we extend primitive recursion in T_{Ω_n} to the larger system and add a principle of primitive recursion on Ω_{n+1}. The theories $QT^i_{\Omega_{n+1}}$ are defined analogously. It is convenient to act as though for each $i < j$, Ω_j is closed under unions indexed by Ω_i; this can arranged by fixing injections of each Ω_i into Ω_{j-1}.

In this section, we show that our interpretation extends to ID_n, to yield the following generalization of Theorem 2.5:

THEOREM 6.1. *Every Π_2 sentence provable in ID_n is provable in $QT^i_{\Omega_n}$.*

As with Theorem 2.5, the proof yields a particular term witnessing the Π_2 assertion, and the correctness of that witnessing term can be established in T_{Ω_n}, by a generalization of Theorem 2.4. The interpretation can be further extended to theories of transfinitely iterated inductive definitions, as described in [9]. We do not, however, know of any ordinary mathematical arguments that are naturally represented in such theories.

To extend the theories OID_1 to theories OID_n, we first have to generalize the schema of ω-bounding. For each $i < j$, define the schema of Ω_i–Ω_j-bounding as follows:

$$\forall \alpha^{\Omega_i}\, \exists \beta^{\Omega_j}\, \varphi(\alpha, \beta) \to \exists \beta^{\Omega_j}\, \forall \alpha^{\Omega_i}\, \exists \gamma^{\Omega_i}\, \varphi(\alpha, \beta[\gamma]).$$

for every formula φ with quantifiers ranging over the types $\Omega_0, \dots, \Omega_i$. With this notation, the ω-bounding schema is now corresponds to Ω_0–Ω_1-bounding.

We extend the theories OID_1 to theories OID_n in the expected way, where now OID_n includes the schema of Ω_i–Ω_j-bounding for each $i < j \leq n$. The fixed points $I_1, \dots, I_n$ of ID_n are interpreted iteratively according to the recipe in Section 3. In particular, if $\psi_j(x, P)$ is gives the definition of the jth inductively defined predicate I_j, the translation of ψ_j has quantifiers ranging over at most Ω_{j-1}, and $t \in I_j$ is interpreted as $\exists \alpha^{\Omega_j}\, (t \in I_{j,\alpha})$, where the predicates $I_j(\alpha, x)$ are defined in analogy to $I(\alpha, x)$. This yields:

THEOREM 6.2. *If ID_n proves φ, then OID_n proves $\hat{\varphi}$.*

Next, we define theories $Q_{n-1}T_{\Omega_n} + (I)$ in analogy to the theory $Q_0 T_\Omega + (I)$ of Section 4, except that we include the Ω_i–Ω_j-bounding axioms for $i < j < n$ in $Q_{n-1}T_{\Omega_n} + (I)$. Now it is quantification over the types $\Omega_0, \dots, \Omega_{n-1}$ that is considered "small," and absorbed into the target theory. In particular, for $i < j < n$, the Ω_i–Ω_j-bounding axioms of OID_n are unchanged by the functional interpretation. The Ω_i–Ω_n bounding axioms for $i < n$, induction on N, and transfinite induction on Ω_n are interpreted as in Section 4. With the corresponding modifications to φ^S, we then have the analogue to Theorem 4.2:

THEOREM 6.3. *Suppose OID_n proves φ, and φ^S is the formula $\forall a\, \exists b\, \varphi_S(a, b)$. Then there are terms b of T_{Ω_n} involving at most the variables a and the free variables of φ of type Ω_n such that $Q_{n-1}T_{\Omega_n} + (I)$ proves $\varphi_S(a, b)$.*

In the last step, we have to embed $Q_{n-1}T_{\Omega_n} + (I)$ into an infinitary proof system in $QT^i_{\Omega_n}$. The method of doing this is once again found in [30, 31], and an extension of the argument described in Section 5. We extend the definition of the infinitary propositional formulas so that when, for each $\alpha \in \Omega_j$ with $j < n$, φ_α is a formula, so are $\bigvee_{\alpha\in\Omega_j} \varphi_\alpha$ and $\bigwedge_{\alpha\in\Omega_j} \varphi_\alpha$. The proof of cut elimination, and the verification of transfinite induction and the defining axioms for the predicates $I_j(\alpha, x)$, are essentially unchanged. The only additional work that is required is to handle the bounding axioms; this is taken care of using a style of bounding argument that is fundamental to the ordinal analysis of such infinitary systems (see [26, 27, 30, 31]).

LEMMA 6.4. *For every $i < j < n$, $QT^i_{\Omega_n}$ proves the translation of the Ω_i–Ω_j bounding axioms.*

PROOF (SKETCH). Since $QT^i_{\Omega_n}$ establishes the provability of the law of the excluded middle in the infinitary language, it suffices to show that for every sequent Γ with quantifiers ranging over at most Ω_i, if $\vdash \Gamma, \forall\alpha^{\Omega_i}\, \exists\beta^{\Omega_j}\, \varphi(\alpha, \beta)$, then there is a β in Ω^j such that for every α in Ω_i, $\vdash \exists\gamma^{\Omega_i}\, \varphi(\alpha, \beta[\gamma])$. But this is essentially a consequence of the "Boundedness lemma for Σ" in Sieg [31, page 162]; the requisite β is defined by an explicit recursion on the derivation. ⊣

This gives us the proper analogue of Theorem 5.7, and hence Theorem 6.1.

THEOREM 6.5. *Every Π_2 theorem of $Q_{n-1}T_{\Omega_n} + (I)$ is a theorem of $QT^i_{\Omega_n}$.*

Appendix: Kreisel's trick and induction with parameters. For completeness, we sketch a proof of Proposition 2.3. Full details can be found in [17, 18].

PROPOSITION 2.3. The following is a derived rule of T_Ω:

$$\frac{\varphi(e, x) \qquad \alpha \neq e \wedge \varphi(\alpha[g(\alpha, x)], h(\alpha, x)) \rightarrow \varphi(\alpha, x)}{\varphi(s, t)}$$

for quantifier-free formulas φ.

PROOF. We associate to each node of an element of Ω a finite sequence σ of natural numbers, where the ith child of the node corresponding to σ is assigned $\sigma\hat{\ }(i)$. Then the subtree α_σ of α rooted at σ (or e if σ is not a node of α) can be defined by recursion on Ω as follows:

$$e_\sigma = e,$$
$$(\sup f)_\sigma = \begin{cases} \sup f & \text{if } \sigma = \emptyset, \\ (f(i))_\tau & \text{if } \sigma = (i)\hat{\ }\tau \end{cases}$$

Here $\emptyset$ denotes the sequence of length 0.

Now, given φ, g, and h as in the statement of the lemma, we define a function $k(\alpha, x, n)$ by primitive recursion on n. The function k uses the the second clause of the rule to compute a sequence of pairs (σ, y) with the property that $\varphi(\alpha_\sigma, y)$ implies $\varphi(\alpha, x)$. For readability, we fix α and x and write $k(n)$ instead of $k(\alpha, x, n)$.

We also write $k_0(n)$ for $(k(n))_0$ and $k_1(n)$ for $(k(n))_1$.

$$k(0) = (\emptyset, x),$$
$$k(n+1) = \begin{cases} (k_0(n)\hat{\ })g(\alpha_{k_0(n)}, k_1(n))), \\ \quad h(\alpha_{k_0(n)}, k_1(n))) & \text{if } \alpha_{k_0(n)} \neq e, \\ k(n) & \text{otherwise.} \end{cases}$$

Ordinary induction on the natural numbers shows that for every n, $\varphi(\alpha_{k_0(n)}, k_1(n))$ implies $\varphi(\alpha, x)$. So, it suffices to show that for some n, $\alpha_{k_0(n)} = e$.

Since $k_0(0) \subseteq k_0(1) \subseteq k_0(2) \subseteq \cdots$ is an increasing sequence of sequences, it suffices to establish the more general claim that for every α and every function f from N to N, there is an n such that $\alpha_{(f(0),\dots,f(n-1))} = e$. To that end, by recursion on Ω, define

$$g(\alpha, f) = \begin{cases} 1 + g(\alpha[f(0)], \lambda n\, f(n+1)) & \text{if } \alpha \neq e, \\ 0 & \text{otherwise} \end{cases}$$

Let $h(m) = g(\alpha_{(f(0),\dots,f(m-1))}, \lambda n\, f(n+m))$. By induction on m we have $h(0) = m + h(m)$ as long as $\alpha_{(f(0),\dots,f(m-1))} \neq e$. In particular, setting $m = h(0)$, we have $h(h(0)) = 0$, which implies $\alpha_{(f(0),\dots,f(h(0)-1))} = e$, as required. ⊣

The following principles of induction and recursion were used in Section 4.

PROPOSITION 4.1. The following are derived rules of Q_0T_Ω:

$$\frac{\theta(e, x) \qquad \alpha \neq e \wedge \forall i\, \forall j\, \theta(\alpha[i], f(\alpha, x, i, j)) \rightarrow \theta(\alpha, x)}{\theta(\alpha, x)}$$

and

$$\frac{\psi(0, x) \qquad \forall j\, \psi(n, f(x, n, j)) \rightarrow \psi(n+1, x)}{\psi(n, x)}$$

PROOF. Consider the first rule. For any element α of Ω and finite sequence of natural numbers σ (coded as a natural number), once again we let α_σ denote the subtree of α rooted at σ. Let τ be the type of x. We will define a function $h(\alpha, g, \sigma)$ by recursion on α, which returns a function of type $N \rightarrow \tau$, with the property that $h(\alpha, g, \emptyset) = g$, and for every σ, $\theta(\alpha_\sigma, x)$ holds for every x in the range of $h(\alpha, g, \sigma)$. Applying the conclusion to $h(\alpha, \lambda i\, x, \emptyset)$ will yield the desired result.

The function h is defined as follows:

$$h(\alpha, g, \sigma) = \begin{cases} g & \text{if } \alpha = e \text{ or } \sigma = \emptyset, \\ h(\alpha[i], \lambda l\, f(\alpha, g(l_0), i, l_1), \sigma') & \text{if } \alpha \neq e \text{ and } \sigma = \sigma'\hat{\ }(i). \end{cases}$$

Using transfinite induction on α, we have

$$\forall \sigma\, \forall v\, \theta(\alpha_\sigma, h(\alpha, g, \sigma)(v))$$

for every α, and hence and hence $\theta(\alpha, h(\alpha, \lambda i\, x, \emptyset)(0))$. Since $h(\alpha, \lambda i\, x, \emptyset)(0) = (\lambda i\, x)(0) = x$, we have the desired conclusion.

The second rule is handled in a similar way. ⊣

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