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Geometry of Differentiable Manifolds

Walter Noll
Carnegie Mellon University, wn0g@andrew.cmu.edu

Sea-Mean Chiou

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Chapter 1

Preliminaries

11. Multilinearity

Let $(\mathcal{V}_i \mid i \in I)$ be a family of linear spaces, we define (see (04.24) of [FDS]), for each $j \in I$ and each $\mathbf{v} \in \times_{i\in I} \mathcal{V}_i$, the mapping $(\mathbf{v}.j) : \mathcal{V}_j \to \times_{i\in I} \mathcal{V}_i$ by the rule

$$((\mathbf{v}.j)(\mathbf{u}))_i := \left\{ \begin{array}{lll} \mathbf{v}_i & \text{if} & i \in I\backslash\{j\} \\ \\ \mathbf{u} & \text{if} & i = j \end{array} \right\} \qquad \text{for all} \quad \mathbf{u} \in \mathcal{V}_j. \tag{11.1}$$

<u>Definition</u> : *Let the family $(\mathcal{V}_i \mid i \in I)$ and $\mathcal{W}$ be linear spaces. We say that the mapping $\mathbf{M} : \times_{i\in I} \mathcal{V}_i \to \mathcal{W}$ is* **multilinear** *if, for every $\mathbf{v} \in \times_{i\in I} \mathcal{V}_i$ and every $j \in I$ the mapping $\mathbf{M} \circ (\mathbf{v}.j) : \mathcal{V}_j \to \mathcal{W}$ is linear, so that $\mathbf{M} \circ (\mathbf{v}.j) \in \mathrm{Lin}(\mathcal{V}_j, \mathcal{W})$. The set of all multilinear mappings from $\times_{i\in I} \mathcal{V}_i$ to $\mathcal{W}$ is denoted by*

$$\mathrm{Lin}_I(\times_{i\in I} \mathcal{V}_i \,,\, \mathcal{W}). \tag{11.2}$$

Let linear spaces $\mathcal{V}$ and $\mathcal{W}$ and a set I be given.

Let $\mathrm{Perm}\, I$ be the permutation group, which consists of all invertible mappings from I to itself. For every permutation $\sigma \in \mathrm{Perm}\, I$ we define a mapping $\mathrm{T}_\sigma : \mathcal{V}^I \to \mathcal{V}^I$ by

$$\mathrm{T}_\sigma(\mathbf{v}) = \mathbf{v} \circ \sigma \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}^I, \tag{11.3}$$

that is $(\mathrm{T}_\sigma(\mathbf{v}))_i := \mathbf{v}_{\sigma(i)}$ for all $i \in I$. In view of $\mathbf{v} \circ (\sigma \circ \rho) = (\mathbf{v} \circ \sigma) \circ \rho$, we have $\mathrm{T}_{\sigma\circ\rho} = \mathrm{T}_\rho \circ \mathrm{T}_\sigma$ for all $\sigma, \rho \in \mathrm{Perm}\, I$. It is not hard to see that, for every multilinear mapping $\mathbf{M} : \mathcal{V}^I \to \mathcal{W}$ and every permutation σ, the composition $\mathbf{M}\circ\mathrm{T}_\sigma$ is again a multilinear mapping from $\mathcal{V}^I$ to $\mathcal{W}$, i.e. $\mathbf{M}\circ\mathrm{T}_\sigma \in \mathrm{Lin}_I(\mathcal{V}^I, \mathcal{W})$.

<u>Definition</u> : *A multilinear mapping $\mathbf{M} : \mathcal{V}^I \to \mathcal{W}$ is said to be (completely)* **symmetric** *if*

$$\mathbf{M} \circ \mathrm{T}_\sigma = \mathbf{M} \qquad \text{for all} \quad \sigma \in \mathrm{Perm}\, I\,,$$

and is said to be (completely) **skew** *if*

$$\mathbf{M} \circ \mathrm{T}_\sigma = \mathrm{sgn}\,(\sigma)\, \mathbf{M} \qquad \text{for all} \quad \sigma \in \mathrm{Perm}\, I\,.$$

The set of all (completely) symmetric multilinear mappings and the set of all (completely) skew multilinear mappings from $\mathcal{V}^I$ to $\mathcal{W}$ will be denoted by $\mathrm{Sym}_I(\mathcal{V}^I, \mathcal{W})$ and by $\mathrm{Skew}_I(\mathcal{V}^I, \mathcal{W})$; respectively.

Both $\mathrm{Sym}_I(\mathcal{V}^I,\ \mathcal{W})$ and $\mathrm{Skew}_I(\mathcal{V}^I,\ \mathcal{W})$ are subspaces of the linear space $\mathrm{Lin}_I(\mathcal{V}^I,\ \mathcal{W})$ with dimensions

$$\dim \mathrm{Sym}_I(\mathcal{V}^I,\ \mathcal{W}) = \binom{\dim \mathcal{V} + \#I - 1}{\#I} \dim \mathcal{W} \tag{11.4}$$

and

$$\dim \mathrm{Skew}_I(\mathcal{V}^I,\ \mathcal{W}) = \binom{\dim \mathcal{V}}{\#I} \dim \mathcal{W}. \tag{11.5}$$

For every $k \in$, we write $\mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W})$, $\mathrm{Sym}_k(\mathcal{V}^k, \mathcal{W})$ and $\mathrm{Skew}_k(\mathcal{V}^k, \mathcal{W})$ for $\mathrm{Lin}_{k^]}(\mathcal{V}^{k^]}, \mathcal{W})$, $\mathrm{Sym}_{k^]}(\mathcal{V}^{k^]}, \mathcal{W})$ and $\mathrm{Skew}_{k^]}(\mathcal{V}^{k^]}, \mathcal{W})$; respectively.

In applicatins, we often use the following identifications

$$\begin{aligned} \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W}) &\cong \mathrm{Lin}_{k-1}(\mathcal{V}^{k-1},\ \mathrm{Lin}\,(\mathcal{V}, \mathcal{W})) \\ &\cong \mathrm{Lin}(\mathcal{V},\ \mathrm{Lin}_{k-1}(\mathcal{V}^{k-1}, \mathcal{W})) \end{aligned}$$

and inclusions

$$\begin{aligned} \mathrm{Sym}_k(\mathcal{V}^k, \mathcal{W}) &\subset \mathrm{Sym}_{k-1}(\mathcal{V}^{k-1}, \mathrm{Lin}\,(\mathcal{V}, \mathcal{W})), \\ \mathrm{Skew}_k(\mathcal{V}^k, \mathcal{W}) &\subset \mathrm{Skew}_{k-1}(\mathcal{V}^{k-1}, \mathrm{Lin}\,(\mathcal{V}, \mathcal{W})). \end{aligned}$$

In particular, we shall use $\mathrm{Sym}_2(\mathcal{V}^2,) \cong \mathrm{Sym}\,(\mathcal{V}, \mathcal{V}^*) := \mathrm{Sym}\,(\mathcal{V}, \mathrm{Lin}\,(\mathcal{V},))$ and $\mathrm{Skew}_2(\mathcal{V}^2,) \cong \mathrm{Skew}\,(\mathcal{V}, \mathcal{V}^*) := \mathrm{Skew}\,(\mathcal{V}, \mathrm{Lin}\,(\mathcal{V},))$. It can be shown that $\mathrm{Skew}\,(\mathcal{V}, \mathcal{V}^*)$ has invertiable mapping if and only if $\dim \mathcal{V}$ is even. (See Prop.3 of Sect.87, [FDS].)

Given a number $k \in$ and a multilinear mapping $\mathbf{A} \in \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W})$, the mapping $\sum_{\sigma \in \mathrm{Perm}\, k^]} (\mathrm{sgn}\,\sigma)\, \mathbf{A} \circ \mathrm{T}_\sigma : \mathcal{V}^k \to \mathcal{W}$ is a completely skew multilinear mapping. Moreover, it can be easily shown that

$$\frac{1}{k!} \sum_{\sigma \in \mathrm{Perm}\, k^]} (\mathrm{sgn}\,\sigma)\ \mathbf{W} \circ \mathrm{T}_\sigma = \mathbf{W}$$

for all skew multilinear mapping $\mathbf{W} \in \mathrm{Skew}_k(\mathcal{V}^k, \mathcal{W})$.

<u>Definition</u> : *Given a number $k \in$, we define the* **alternating assignment** $\mathrm{Alt} : \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W}) \to \mathrm{Skew}_k(\mathcal{V}^k, \mathcal{W})$ *by*

$$\mathrm{Alt}\ \mathbf{A} := \frac{1}{k!} \sum_{\sigma \in \mathrm{Perm}\, k^]} (\mathrm{sgn}\,\sigma)\, \mathbf{A} \circ \mathrm{T}_\sigma \tag{11.6}$$

for all linear spaces $\mathcal{V}$ and $\mathcal{W}$ and all $\mathbf{A} \in \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W})$.

Given $p \in$. We define, for each $i \in (p+1)^]$, a mapping $\mathrm{del}_i : \mathcal{V}^{p+1} \to \mathcal{V}^p$ by

$$(\mathrm{del}_i(\mathbf{v}))_j := \left\{ \begin{array}{lll} \mathbf{v}_j & \text{if} & 1 \le i \le j-1 \\ & & \\ \mathbf{v}_{i+1} & \text{if} & j \le i \le p \end{array} \right\} \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}^{p+1}. \tag{11.7}$$

Intuitively, $\mathrm{del}_i(\mathbf{v})$ is obtained from $\mathbf{v}$ by deleting the i-th term.

When the alternating assignment Alt restricted to the subspace $\mathrm{Lin}\,(\mathcal{V},\, \mathrm{Skew}_p(\mathcal{V}^p, \mathcal{W}))$ of $\mathrm{Lin}\,(\mathcal{V},\, \mathrm{Lin}_p(\mathcal{V}^p, \mathcal{W})) \cong \mathrm{Lin}_{p+1}(\,\mathcal{V}^{p+1}, \mathcal{W})$, we have

$$(p+1)\,(\mathrm{Alt}\,\mathbf{A})\mathbf{v} = \sum_{i\in(p+1)^]} (-1)^{i-1}\mathbf{A}(\mathbf{v}_i, \mathrm{del}_i\mathbf{v}) \tag{11.8}$$

for all $\mathbf{v} \in \mathcal{V}^{p+1}$ and all $\mathbf{A} \in \mathrm{Lin}\,(\mathcal{V},\, \mathrm{Skew}_p(\,\mathcal{V}^p, \mathcal{W}))$. Similarly, when the alternating assignment Alt restricted to the subspace $\mathrm{Skew}_p(\,\mathcal{V}^p, \mathrm{Lin}(\mathcal{V}, \mathcal{W}))$ of $\mathrm{Lin}\,(\mathcal{V},\, \mathrm{Lin}_p(\,\mathcal{V}^p, \mathcal{W})) \cong \mathrm{Lin}_{p+1}(\,\mathcal{V}^{p+1}, \mathcal{W})$, we have

$$(p+1)\,(\mathrm{Alt}\,\mathbf{B})\mathbf{v} = \sum_{i\in(p+1)^]} (-1)^{p+1-i}\mathbf{B}(\mathrm{del}_i\mathbf{v}, \mathbf{v}_i) \tag{11.9}$$

for all $\mathbf{v} \in \mathcal{V}^{p+1}$ and all $\mathbf{B} \in \mathrm{Skew}_p(\mathcal{V}^p, \mathrm{Lin}(\mathcal{V}, \mathcal{W}))$.

Definition: *An* **algebra** *is a linear space $\mathcal{V}$ together with a bilinear mapping $\mathbf{B} \in \mathrm{Lin}_2(\mathcal{V}^2, \mathcal{V})$. An algebra $\mathcal{V}$ is called a* **Lie Alegebra** *if the bilinear mapping $\mathbf{B}$ is skew-symmetric, i.e. $\mathbf{B} \in \mathrm{Skew}_2(\mathcal{V}^2, \mathcal{V})$, and satisfies* **Jacobi indetity**

$$\mathbf{B}(\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2), \mathbf{v}_3) + \mathbf{B}(\mathbf{B}(\mathbf{v}_2, \mathbf{v}_3), \mathbf{v}_1) + \mathbf{B}(\mathbf{B}(\mathbf{v}_3, \mathbf{v}_1), \mathbf{v}_2) = \mathbf{0} \tag{11.10}$$

for all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathcal{V}$.

By using the inclusion $\mathrm{Skew}_2(\mathcal{V}^2, \mathcal{V}) \subset \mathrm{Lin}(\mathcal{V},\, \mathrm{Lin}(\mathcal{V}, \mathcal{V}))$ and (11.9), we see taht (11.10) can rewriten as

$$\mathrm{Alt}\,(\mathbf{B} \circ \mathbf{B}) = \mathbf{0} \tag{11.11}$$

where $(\mathbf{B} \circ \mathbf{B})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) := \mathbf{B}(\mathbf{B}(\mathbf{v}_1, \mathbf{v}_2), \mathbf{v}_3)$ for all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathcal{V}$.

Remark 1: In the literature the **alternating assignment** given in (11.6) is often called "*skew-symmetric operator*" ([B-W]), "*complete antisymmetrization*" ([F-C]). The **symmetric assignment**, "*symmetric operator*" or "*complete symmetrization*" $\mathrm{Sym} : \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W}) \to \mathrm{Sym}_k(\mathcal{V}^k, \mathcal{W})$ is given by

$$\mathrm{Sym}\,\mathbf{M} := \frac{1}{k!} \sum_{\sigma\in\mathrm{Perm}\,k^]} \mathbf{M} \circ \mathrm{T}_\sigma \tag{11.12}$$

for all linear spaces $\mathcal{V}$ and $\mathcal{W}$ and all $\mathbf{M} \in \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W})$. ∎

Remark 2: Both assignments given in (11.6) and (11.12) are "*natural linear assignments*" from a functor to another functor (see (13.16) of Sect.13). More precisely, the alternating assignment is a natural linear assgnment from the functor Ln_k to the functor Sk_k and the symmetric assignment is a natural linear assgnment from the functor Ln_k to the functor Sm_k (see Sect. 13). ∎

12. Isocategories, isofunctors and Natural Assignments

An **isocategory**[*] [‡] is given by the specification of a class OBJ whose members are called **objects**, a class ISO whose members are called ISO**morphisms**,

(i) a rule that associates with each $\phi \in$ ISO a pair $(\mathrm{Dom}\,\phi, \mathrm{Cod}\,\phi)$ of objects, called the **domain** and **codomain** of ϕ,

(ii) a rule that associates with each $\mathcal{A} \in OBJ$ a member of ISO denoted by $1_{\mathcal{A}}$ and called the **identity** of $\mathcal{A}$,

(iii) a rule that associates with each pair (ϕ, ψ) in ISO such that $\mathrm{Cod}\,\phi = \mathrm{Dom}\,\psi$ a member of ISO denoted by $\psi \circ \phi$ and called the **composite** of ϕ and ψ, with $\mathrm{Dom}\,(\psi \circ \phi) = \mathrm{Dom}\,\phi$ and $\mathrm{Cod}\,(\psi \circ \phi) = \mathrm{Cod}\,\psi$.

(iv) a rule that associates with each $\phi \in$ ISO a member of ISO denoted by $\phi^{\leftarrow}$ and called the **inverse** of ϕ.

subject to the following three axioms:

(I1) $\phi \circ 1_{\mathrm{Dom}\,\phi} = \phi = 1_{\mathrm{Cod}\,\phi} \circ \phi$ for all $\phi \in$ ISO,

(I2) $\chi \circ (\psi \circ \phi) = (\chi \circ \psi) \circ \phi$ for all $\phi, \psi, \chi \in$ ISO such that $\mathrm{Cod}\,\phi = \mathrm{Dom}\,\psi$ and $\mathrm{Cod}\,\psi = \mathrm{Dom}\,\chi$.

(I3) $\phi^{\leftarrow} \circ \phi = 1_{\mathrm{Dom}\,\phi}$ and $\phi \circ \phi^{\leftarrow} = 1_{\mathrm{Cod}\,\phi}$ for all $\phi \in$ ISO.

Given $\phi \in$ ISO, one writes $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ or $\mathcal{A} \xrightarrow{\phi} \mathcal{B}$ to indicate that $\mathrm{Dom}\,\phi = \mathcal{A}$ and $\mathrm{Cod}\,\phi = \mathcal{B}$.

There is one to one correspondence between an object $\mathcal{A} \in OBJ$ and the corresponding identity $1_{\mathcal{A}} \in$ ISO. For this reason, we will usually name an isocategory by giving the name of its class of ISOmorphisms.

Let isocategories ISO and ISO$'$ with object-classes OBJ and OBJ' be given. We can then form the **product-isocategory** ISO $\times$ ISO$'$ whose object-class $OBJ \times OBJ'$ consists of pairs $(\mathcal{A}, \mathcal{A}')$ with $\mathcal{A} \in OBJ$, $\mathcal{A}' \in OBJ'$ and ISOmorphism-class ISO $\times$ ISO$'$ consists of pairs (ϕ, ϕ') with $\phi \in$ ISO, $\phi' \in$ ISO$'$ and the following

(a) For every $(\phi, \phi') \in$ ISO $\times$ ISO$'$, $\mathrm{Dom}\,(\phi, \phi') := (\mathrm{Dom}\,\phi, \mathrm{Dom}\,\phi')$ and $\mathrm{Cod}\,(\phi, \phi') := (\mathrm{Cod}\,\phi, \mathrm{Cod}\,\phi')$.

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[*] A category, introduced by Eilenberg and MacLane, is defined by (i), (ii) and (iii) with the axioms (I1) and (I2). Roughly speaking, an isocategory is a special category whose "morphisms" are called ISO-morphisms.

[‡] Since isocategories are widely used in differential geometry, we introduced them directly instead of making them as a special category.

(b) Composition in $\mathrm{ISO} \times \mathrm{ISO}'$ is defined by termwise composition, i.e. by $(\psi, \psi') \circ (\phi, \phi') := (\,\psi \circ \phi\,,\, \psi' \circ \phi'\,)$ for all $\phi, \psi \in \mathrm{ISO}$ and $\phi', \psi' \in \mathrm{ISO}'$ such that $\mathrm{Dom}\,(\psi, \psi') = \mathrm{Cod}\,(\phi, \phi')$.

(c) The identity of a given pair $(\mathcal{A}, \mathcal{A}') \in OBJ \times OBJ'$ is defined to be $1_{(\mathcal{A},\mathcal{A}')} = (1_{\mathcal{A}}, 1_{\mathcal{A}'})$.

The product of an arbitary family of isocategories can be defined in a similar manner. In particular, if a isocategory ISO and an index set I are given, one can form the I-**power-isocategory** ISO^I of ISO; its ISOmorphism-class consists of all families in ISO indexed on I. In the case when I is of the form $I := n^{]}$, we write $\mathrm{ISO}^n := \mathrm{ISO}^{n^{]}}$ for short. For example, we write $\mathrm{ISO}^2 := \mathrm{ISO} \times \mathrm{ISO}$. We identify ISO^1 with ISO and ISO^{m+n} with $\mathrm{ISO}^m \times \mathrm{ISO}^n$ for all $m, n \in$ in the obvious manner. The isocategory ISO^0 is the trival one whose only object is $\emptyset$ and whose only ISOmorphism is $1_\emptyset$.

A **functor** Φ is given by the specification of:

(i) a pair $(\mathrm{Dom}\,\Phi, \mathrm{Cod}\,\Phi)$ of categories, called the **domain-category** and **codomain-category** of Φ,

(ii) a rule that associates with every $\phi \in \mathrm{Dom}\,\Phi$ a member of $\mathrm{Cod}\,\Phi$ denoted by $\Phi(\phi)$,

subject to the following conditions:

(F1) We have $\mathrm{Cod}\,\Phi(\phi) = \mathrm{Dom}\,\Phi(\psi)$ and $\Phi(\psi \circ \phi) = \Phi(\psi) \circ \Phi(\phi)$ for all $\phi, \psi \in \mathrm{Dom}\,\Phi$ such that $\mathrm{Cod}\,\phi = \mathrm{Dom}\,\psi$.

(F2) For every identity $1_{\mathcal{A}}$ in $\mathrm{Dom}\,\Phi$, where $\mathcal{A}$ belongs to the object-class of $\mathrm{Dom}\,\Phi$, $\Phi(1_{\mathcal{A}})$ is an identity in $\mathrm{Cod}\,\Phi$.

An **isofunctor** is a functor whose domain-category and codomain-category are isocategories. In this book we only deal with isofunctors.

Let isocategories ISO and ISO' with object-classes OBJ and OBJ' be given. We say that Φ is an **isofunctor from** ISO **to** ISO' and we write $\mathrm{ISO} \xrightarrow{\Phi} \mathrm{ISO}'$ or $\Phi : \mathrm{ISO} \longrightarrow \mathrm{ISO}'$ to indicate that $\mathrm{ISO} = \mathrm{Dom}\,\Phi$ and $\mathrm{ISO}' = \mathrm{Cod}\,\Phi$. By (F2), we can associate with each $\mathcal{A} \in OBJ$ exactly one object in OBJ', denoted by $\Phi(\mathcal{A})$, such that

$$\Phi(1_{\mathcal{A}}) = 1_{\Phi(\mathcal{A})}. \tag{12.1}$$

It easily follows from (I3), (F1) and (F2) that every isofunctor Φ satisfies

$$\Phi(\phi^{\leftarrow}) = \big(\Phi(\phi)\big)^{\leftarrow} \quad \text{for all} \quad \phi \in \mathrm{Dom}\,\Phi. \tag{12.2}$$

One can construct new isofunctors from given isofunctors in the same way as new mappings are constructed from given mappings. (See, for example, Sect. 03

and 04, [FDS].) Thus, if Φ and Ψ are isofunctors such that $\operatorname{Cod}\Phi = \operatorname{Dom}\Psi$, one can define the **composite isofunctor** $\Psi \circ \Phi : \operatorname{Dom}\Phi \to \operatorname{Cod}\Psi$ by

$$(\Psi \circ \Phi)(\phi) := \Psi(\Phi(\phi)) \quad \text{for all} \quad \phi \in \operatorname{Dom}\Phi \tag{12.3}$$

Also, given isofunctors Φ and Ψ, one can define the **product-isofunctor**

$$\Phi \times \Psi : \ \operatorname{Dom}\Phi \times \operatorname{Dom}\Psi \longrightarrow \operatorname{Cod}\Phi \times \operatorname{Cod}\Psi$$

of Φ and Ψ by

$$(\Phi \times \Psi)(\phi, \psi) := (\Phi(\phi), \Psi(\psi)) \tag{12.4}$$

for all $\phi \in \operatorname{Dom}\Phi$ and all $\psi \in \operatorname{Dom}\Psi$.

Product-isofunctors of arbitrary families of isofunctors are defined in a similar way. In particular, if a isofunctor Φ and an index set I are given, we define the I-**power-isofunctor** $\Phi^{\times I} : (\operatorname{Dom}\Phi)^I \to (\operatorname{Cod}\Phi)^I$ of Φ by

$$\Phi^{\times I}(\phi_i \mid i \in I) = (\Phi(\phi_i) \mid i \in I) \tag{12.5}$$

for all families $(\phi_i \mid i \in I)$ in $\operatorname{Dom}\Phi$. We write $\Phi^{\times n} := \Phi^{\times n]}$ when $n \in$.

We now assume that an isocategory ISO with object-class OBJ is given.

The **identity-isofunctor** $\operatorname{Id} : \mathrm{ISO} \to \mathrm{ISO}$ of ISO is defined by

$$\operatorname{Id}(\phi) = \phi \quad \text{for all} \quad \phi \in \mathrm{ISO}. \tag{12.6}$$

We then have

$$\operatorname{Id}(\mathcal{A}) = \mathcal{A} \quad \text{for all} \quad \mathcal{A} \in OBJ. \tag{12.7}$$

If I is an index set, then the identity-isofunctor of ISO^I is $\operatorname{Id}^{\times I}$. In particular, the identity-isofunctor of $\mathrm{ISO} \times \mathrm{ISO}$ is $\operatorname{Id} \times \operatorname{Id}$.

Given an object $\mathcal{C} \in OBJ$. The **trivial-isofunctor** $\operatorname{Tr}_{\mathcal{C}} : \mathrm{ISO} \to \mathrm{ISO}$ **for** $\mathcal{C}$ is defined by

$$\operatorname{Tr}_{\mathcal{C}}(\phi) = 1_{\mathcal{C}} \quad \text{for all} \quad \phi \in \mathrm{ISO}. \tag{12.8}$$

We then have

$$\operatorname{Tr}_{\mathcal{C}}(\mathcal{A}) = \mathcal{C} \quad \text{for all} \quad \mathcal{A} \in OBJ. \tag{12.9}$$

One often needs to consider a variety of "accounting isofunctors" whose domain and codomain isocategories are obtained from ISO by product formation. For example, the **switch-isofunctor** $\operatorname{Sw} : \mathrm{ISO}^2 \to \mathrm{ISO}^2$ is defined by

$$\operatorname{Sw}(\phi, \psi) := (\psi, \phi) \quad \text{for all} \quad \phi, \psi \in \mathrm{ISO}. \tag{12.10}$$

Given any index set I, the **equalization-isofunctor** $\operatorname{Eq}_I : \mathrm{ISO} \to \mathrm{ISO}^I$ is defined by

$$\operatorname{Eq}_I(\phi) := (\phi \mid i \in I) \quad \text{for all} \quad \phi \in \mathrm{ISO}. \tag{12.11}$$

We write $\mathrm{Eq}_n := \mathrm{Eq}_{n]}$ when $n \in$.

Let a index set I and a family $(\Phi_i \mid i \in I)$ of isofunctors, with $\mathrm{Dom}\,\Phi_i = \mathrm{ISO}$ for all $i \in I$, be given. We then identify the family $(\Phi_i \mid i \in I)$ with the **termwise-formation isofunctor**

$$(\Phi_i \mid i \in I) : \mathrm{ISO} \to \bigtimes_{i \in I} \mathrm{Cod}\,\Phi_i$$

defined by

$$(\Phi_i \mid i \in I) := \bigtimes_{i \in I} \Phi_i \circ \mathrm{Eq}_I,$$

so that

$$(\Phi_i \mid i \in I)(\phi) = \bigtimes_{i \in I} \Phi_i(\phi), \quad \text{for all} \quad \phi \in \mathrm{ISO}. \tag{12.12}$$

In particular, if $I = 2^{]}$, we then identify the pair (Φ_1, Φ_2) with the **pair-formation isofunctor** $(\Phi_1, \Phi_2) : \mathrm{ISO} \to \mathrm{Cod}\,\Phi_1 \times \mathrm{Cod}\,\Phi_2$.

Let isofunctors Φ and Ψ, both from ISO to ISO$'$, be given. A **natural assignment** α **form** Φ **to** Ψ is a rule that associates with each object $\mathcal{F}$ of ISO a mapping

$$\alpha_{\mathcal{F}} : \Phi(\mathcal{F}) \to \Psi(\mathcal{F}),$$

such that

$$\Psi(\chi) \circ \alpha_{\mathrm{Dom}\,\chi} = \alpha_{\mathrm{Cod}\,\chi} \circ \Phi(\chi) \qquad \text{for all} \quad \chi \in \mathrm{ISO}; \tag{12.13}$$

i.e. the diagram

$$\begin{tikzcd}
\Phi(\mathrm{Dom}\,\chi) \arrow[r, "\alpha_{\mathrm{Dom}\,\chi}"] \arrow[d, "\Phi(\chi)"'] & \Psi(\mathrm{Dom}\,\chi) \arrow[d, "\Psi(\chi)"] \\
\Phi(\mathrm{Cod}\,\chi) \arrow[r, "\alpha_{\mathrm{Cod}\,\chi}"'] & \Psi(\mathrm{Cod}\,\chi)
\end{tikzcd}$$

is commutative. We write $\alpha : \Phi \longrightarrow \Psi$ to indicate that Φ is the **domain isofunctor**, denoted by Dmf_α, and Ψ is the **codomain isofunctor**, denoted by Cdf_α.

One can construct new natural assignments from given ones in the same way as new mappings from given ones. Let natural assignments $\alpha : \Phi \to \Psi$ and $\beta : \Psi \to \Theta$ be given. We can define the **composite assignment** $\beta \circ \alpha : \Phi \to \Theta$, by assigning to each object $\mathcal{F}$ of $\mathrm{Dom}\,\Phi = \mathrm{Dom}\,\Psi$ the mapping $(\beta \circ \alpha)_{\mathcal{F}} := \beta_{\mathcal{F}} \circ \alpha_{\mathcal{F}}$. If α, β are natural assignment, one can define the **product-assignment** $\alpha \times \beta$ by assigning to each pair $(\mathcal{F}, \mathcal{G})$ of objects the mapping $(\alpha \times \beta)_{(\mathcal{F},\mathcal{G})} := \alpha_{\mathcal{F}} \times \beta_{\mathcal{G}}$.

Given a natural assignment $\alpha : \Phi \to \Psi$ and a isofunctor Θ such that $\mathrm{Cod}\,\Theta = \mathrm{Dom}\,\Phi = \mathrm{Dom}\,\Psi$, one can define the **composite assignment**

$\alpha \circ \Theta : \Phi \circ \Theta \to \Psi \circ \Theta$ by assigning to each object $\mathcal{F}$ of $\operatorname{Dom} \Phi = \operatorname{Dom} \Psi$ the mapping $(\alpha \circ \Theta)_{\mathcal{F}} := \alpha_{\Theta(\mathcal{F})}$.

13. Tensor Functors

We say that an isocategory ISO is **concrete** if ISO consists of mappings, the object-class *OBJ* consists of sets, and if domain and codomain, composition, identity and inverse have the meanning they are usually given for sets and mappings. (See, e.g. Sect. 01 – 04 of [FDS]).

Examples of concrete isocategory

The following are some concrete isocategories to be used in this book:

(A) The category FIS whose object-class *FS* consists of all finite dimensional flat spaces over and whose ISOmorphism-class FIS consists of all flat isomorphism from one such space onto another or itself.

(B) Fix a field and we consider the concrete isocategory whose object-class *LS* consists of all finite dimensional linear spaces over and whose ISOmorphism-class LIS consists of all linear isomorphism from one such space onto another or itself.

(C) Given $s \in$, the category DIF^s whose object-class *DF* consists of all C^s manifolds and whose ISOmorphism-class DIF^s consists of all diffeomorphism from one such manifold onto another or itself.

From now on, *in this section*, we will deal only with LIS and the categories obtained from it by product formation, such as $\mathrm{LIS}^m \times \mathrm{LIS}^n$ when $m, n \in$. We use the term **tensor functor of degree** $n \in$ for functor from LIS^n to LIS. (Under this definition, composition of tensor functors is somewhat strange: the second one of those functors must be of degree 1!!!!!!!!!!!!!)

Examples of tensor functor

Here is a list of important tensor functors used in linear algebra and differential geometry:

(1) The **product-space functor** $\mathrm{Pr} : \mathrm{LIS}^2 \to \mathrm{LIS}$. It is defined by

$$\mathrm{Pr}(\mathbf{A}, \mathbf{B}) := \mathbf{A} \times \mathbf{B} \quad \text{for all} \quad (\mathbf{A}, \mathbf{B}) \in \mathrm{LIS}^2. \tag{13.1}$$

We have $\mathrm{Pr}(\mathcal{V}, \mathcal{W}) := \mathcal{V} \times \mathcal{W}$ (the *product-space* of $\mathcal{V}$ and $\mathcal{W}$) for all $\mathcal{V}, \mathcal{W} \in LS$.

(2) Given $k \in$, the k-**lin-map-functor** $\mathrm{Lin}_k : \mathrm{LIS}^k \times \mathrm{LIS} \to \mathrm{LIS}$. It assigns to each list $(\mathcal{V}_i \,|\, i \in k^{]}\,)$ in LS and each $\mathcal{W} \in LS$ the linear space

$$\mathrm{Lin}_k((\mathcal{V}_i \,|\, i \in k^{]}\,), \mathcal{W}) := \mathrm{Lin}_k\Big(\bigtimes_{i \in k^{]}} \mathcal{V}_i \,,\, \mathcal{W}\Big) \tag{13.2}$$

of all k-multilinear mappings from $\bigtimes_{i \in k^{]}} \mathcal{V}_i$ to $\mathcal{W}$, and it assigns to every list $(\mathbf{A}_i \,|\, i \in k^{]}\,)$ in LIS and each $\mathbf{B} \in \mathrm{LIS}$ the linear mapping

$$\mathrm{Lin}_k((\mathbf{A}_i \,|\, i \in k^{]}) \,,\, \mathbf{B}) \tag{13.3}$$

from $\mathrm{Lin}_k\big(\bigtimes_{i \in k^{]}} \mathrm{Dom}\,\mathbf{A}_i, \mathrm{Dom}\,\mathbf{B}\big)$ to $\mathrm{Lin}_k\big(\bigtimes_{i \in k^{]}} \mathrm{Cod}\,\mathbf{A}_i, \mathrm{Cod}\,\mathbf{B}\big)$ defined by

$$\mathrm{Lin}_k((\mathbf{A}_i \,|\, i \in k^{]}\,), \mathbf{B})\mathbf{T} := \mathbf{BT} \circ \bigtimes_{i \in k^{]}} \mathbf{A}_i^{-1} \tag{13.4}$$

for all $\mathbf{T} \in \mathrm{Lin}\big(\bigtimes_{i \in k^{]}} \mathrm{Dom}\,\mathbf{A}_i, \mathrm{Dom}\,\mathbf{B}\big)$.

When $k = 1$, $\mathrm{Lin}_1 : \mathrm{LIS} \times \mathrm{LIS} \to \mathrm{LIS}$ is called the **lin-map-functor** and abreviated by $\mathrm{Lin} := \mathrm{Lin}_1$.

(3) Given $k \in$, the k-**multilin-functor** $\mathrm{Ln}_k : \mathrm{LIS}^2 \to \mathrm{LIS}$. It is defined by

$$\mathrm{Ln}_k := \mathrm{Lin}_k \circ (\mathrm{Eq}_k \times \mathrm{Id}). \tag{13.5}$$

We have

$$\mathrm{Ln}_k(\mathbf{A}, \mathbf{B})\mathbf{T} := \mathbf{BT} \circ (\mathbf{A}^{-1})^{\times k} \tag{13.6}$$

for all $\mathbf{A}, \mathbf{B} \in \mathrm{LIS}$ and all $\mathbf{T} \in \mathrm{Lin}_k((\mathrm{Dom}\,\mathbf{A})^k, \mathrm{Dom}\,\mathbf{B})$. and

$$\mathrm{Ln}_k(\mathcal{V}, \mathcal{W}) := \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W}) \tag{13.7}$$

for all $\mathcal{V}, \mathcal{W} \in LS$

There are two very important "*subfunctors*" (see [E-M]), Sm_k and Sk_k, given in following. The **symmetric-k-multilin-functor** $\mathrm{Sm}_k : \mathrm{LIS}^2 \to \mathrm{LIS}$ assigns to every pair of linear spaces $(\mathcal{V}, \mathcal{W}) \in LS^2$ the linear sapce

$$\mathrm{Sm}_k(\mathcal{V}, \mathcal{W}) := \mathrm{Sym}_k(\mathcal{V}^k, \mathcal{W}) \tag{13.8}$$

of all *symmetric* k-multilinear mappings from $\mathcal{V}^k$ to $\mathcal{W}$. It is clear that

$$\mathrm{Sm}_k(\mathbf{A}, \mathbf{B})\mathbf{T} := \mathbf{BT} \circ (\mathbf{A}^{-1})^{\times k} \tag{13.9}$$

for all $\mathbf{A}, \mathbf{B} \in \mathrm{LIS}$ and all $\mathbf{T} \in \mathrm{Sym}_k((\mathrm{Dom}\,\mathbf{A})^k, \mathrm{Dom}\,\mathbf{B})$. The **skew-$k$-multilin-functor** $\mathrm{Sk}_k : \mathrm{LIS}^2 \to \mathrm{LIS}$ is defined in the same manner as Sm_k, except that $\mathrm{Sym}_k(\mathcal{V}^k, \mathcal{W})$ in (13.8) is replaced by the linear space $\mathrm{Skew}_k(\mathcal{V}^k, \mathcal{W})$ of all *skew* k-multilinear mappings from $\mathcal{V}^k$ to $\mathcal{W}$.

(4) Given $n \in$, the k-**linform-functor** Lnf_k, the k-**symform-functor** Smf_k, the k-**skewform-functor** Skf_k, all from LIS to LIS. They are defined by

$$\mathrm{Lnf}_k := \mathrm{Ln}_k \circ (\mathrm{Id}, \mathrm{Tr})\ ,\ \mathrm{Smf}_k := \mathrm{Sm}_k \circ (\mathrm{Id}, \mathrm{Tr})\ ,\ \mathrm{Skf}_k := \mathrm{Sk}_k \circ (\mathrm{Id}, \mathrm{Tr}). \quad (13.10)$$

Given $\mathcal{V} \in LS$, we have

$$\mathrm{Lnf}_k(\mathcal{V}) := \mathrm{Lin}_k(\mathcal{V}^k,), \quad (13.11)$$

the space of all k-multilinear forms on $\mathcal{V}^k$. We have

$$\mathrm{Lnf}_k(\mathbf{A})\boldsymbol{\omega} := \boldsymbol{\omega} \circ (\mathbf{A}^{-1})^{\times k} \quad \text{for all} \quad \boldsymbol{\omega} \in \mathrm{Lin}_k((\mathrm{Dom}\,\mathbf{A})^k,) \quad (13.12)$$

and all $\mathbf{A} \in$ LIS. The formulas (13.11) and (13.12) remain valid if Lin is replaced by Sym or Skew and Lnf by Smf or Skf correspondingly.

When $k = 1$, we have $\mathrm{Lnf}_1 = \mathrm{Smf}_1 = \mathrm{Skf}_1$ which is called the **duality-functor** and denoted by $\mathrm{Dl} : \mathrm{LIS} \to \mathrm{LIS}$.

(5) The **lineon-functor** $\mathrm{Ln} : \mathrm{LIS} \to \mathrm{LIS}$. It is defined by

$$\mathrm{Ln} := \mathrm{Lin} \circ \mathrm{Eq}_2. \quad (13.13)$$

We have

$$\mathrm{Ln}(\mathcal{V}) := \mathrm{Lin}(\mathcal{V}, \mathcal{V}) \quad \text{for all} \quad \mathcal{V} \in LS \quad (13.14)$$

and

$$\mathrm{Ln}(\mathbf{A})\mathbf{T} := \mathbf{A}\mathbf{T}\mathbf{A}^{-1} \quad \text{for all} \quad \mathbf{A} \in \mathrm{LIS} \ \text{ and } \ \mathbf{T} \in \mathrm{Ln}(\mathrm{Dom}\,\mathbf{A}). \quad (13.15)$$

$\boxed{\text{It is clear that } \mathrm{Lin}_1 = \mathrm{Ln}_1\text{, however, } \mathrm{Ln}_1 \neq \mathrm{Ln}\text{! Notation?}}$

Remark : In much of the literature (see [K-N], Sect. 2 of Ch.I or [M-T-W], §3.2) the use of the term "tensor" is limited to tensor functors of the form $\mathbf{T}^r_s := \mathrm{Lin} \circ (\mathrm{Lnf}_s, \mathrm{Lnf}_r) : \mathrm{LIS} \to \mathrm{LIS}$ with $r, s \in$, or to tensor functors that are naturally equivalent to one of this form. Given $\mathcal{V} \in LS$ a member of the linear space $\mathbf{T}^r_s(\mathcal{V})$ is called a "tensor of contravariant order r and covariant order s." ∎

Let a family of tensor functors $(\Phi_i \mid i \in k^]})$ and a tensor functor Ψ with $\mathrm{Dom} \times_{i \in k^]} \Phi_k = \mathrm{LIS}^k = \mathrm{Dom}\,\Psi$ be given. We say that a natural assignment $\beta : \times_{i \in k^]} \Phi_k \to \Psi$ is a k-**linear assignment** if, for every $\mathcal{F} \in LS^k$, the mapping

$$\beta_{\mathcal{F}} : \mathop{\times}_{i \in k^]} \Phi_i(\mathcal{F}_i) \to \Psi(\mathcal{F}) \quad (13.16)$$

is k-linear.

The following are examples for bilinear natural assignments.

(6) Given $k \in$, the **alternating assgnment** $\mathrm{Alt} : \mathrm{Ln}_k \to \mathrm{Sk}_k$ it assigns each pair $(\mathcal{V}, \mathcal{W}) \in LS^2$ the mapping

$$\mathrm{Alt}_{(\mathcal{V},\mathcal{W})}\mathbf{A} := \sum_{\sigma \in \mathrm{Perm}\, k^{]}} (\mathrm{sgn}\,\sigma)\mathbf{A} \circ \mathrm{T}_\sigma \tag{13.17}$$

where $\mathrm{Perm}\, k^{]}$ is the permutation group of $k^{]}$ and T_σ is defined as in (11.3), for all $\mathbf{A} \in \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W})$.

(7) The **tensor product** $\mathrm{tpr} : \mathrm{Id} \times \mathrm{Id} \to \mathrm{Lin} \circ (\mathrm{Dl} \times \mathrm{Id}) \circ \mathrm{Sw}$ assigns each pair $(\mathcal{V}, \mathcal{W}) \in LS^2$ the mapping

$$\mathrm{tpr}_{(\mathcal{V},\mathcal{W})} : \mathcal{V} \times \mathcal{W} \to \mathrm{Lin}(\mathcal{W}^*, \mathcal{V}) \tag{13.18}$$

defined by

$$\mathrm{tpr}_{(\mathcal{V},\mathcal{W})}(\mathbf{v}, \mathbf{w}) := \mathbf{v} \otimes \mathbf{w} \quad \text{for all} \quad \mathbf{v} \in \mathcal{V} \text{ and } \mathbf{w} \in \mathcal{W}, \tag{13.19}$$

where $\mathbf{v} \otimes \mathbf{w}$ is the tensor product defined according to Def. 1 of Sect. 25, [FDS], with the identification $\mathcal{W} \cong \mathcal{W}^{**}$.

We use $\mathbf{v} \otimes \mathbf{w} \in \mathrm{Lin}(\mathcal{W}^*, \mathcal{V})$ but others use $\mathbf{v} \otimes \mathbf{w} \in \mathrm{Lin}(\mathcal{V}^*, \mathcal{W})$ (see e.g. [B-W]). Our definition of $\otimes$ bring up the switch functor Sw here!!!!!!!!!!!!!!!!!!!!

The **wedge product** $\mathrm{wpr} : \mathrm{Id} \times \mathrm{Id} \to \mathrm{Lin} \circ (\mathrm{Dl} \times \mathrm{Id}) \circ \mathrm{Sw}$ is defined by

$$\mathrm{wpr}_{(\mathcal{V},\mathcal{W})}(\mathbf{v}, \mathbf{w}) := \mathbf{v} \wedge \mathbf{w} \quad \text{for all} \quad \mathbf{v} \in \mathcal{V} \text{ and } \mathbf{w} \in \mathcal{W}, \tag{13.20}$$

where $\mathbf{v} \wedge \mathbf{w}$ is the wedge product defined according to (12.9) of Sect. 12, [FDS], Vol.2, with the identification $\mathcal{W} \cong \mathcal{W}^{**}$.

We have $\mathrm{wpr} = \frac{1}{2}\,\mathrm{Alt} \circ \mathrm{tpr}$. Need more development!!!!!!!!!!!!!!!!!!!

We now assume that the field relative to which LS and LIS are defined in above is the field of real number. Given $\mathcal{V}, \mathcal{W} \in LS$, the set

$$\mathrm{Lis}(\mathcal{V}, \mathcal{W}) := \left\{ \mathbf{A} \in \mathrm{LIS} \;\middle|\; \mathrm{Dom}\,\mathbf{A} = \mathcal{V}, \mathrm{Cod}\,\mathbf{A} = \mathcal{W} \right\} \tag{13.21}$$

is then an open subset of the linear space $\mathrm{Lin}(\mathcal{V}, \mathcal{W})$. (See, for example, the Differentiation Theorem for Inversion Mappings in Sect.68 of [FDS].).

Let a tensor functor Φ be given. For every pair of objects $(\mathcal{V}, \mathcal{W})$ of $\mathrm{Dom}\,\Phi$, we define the mapping

$$\Phi_{(\mathcal{V},\mathcal{W})} : \mathrm{Lis}(\mathcal{V}, \mathcal{W}) \to \mathrm{Lis}(\Phi(\mathcal{V}), \Phi(\mathcal{W})) \tag{13.22}$$

by

$$\Phi_{(\mathcal{V},\mathcal{W})}(\mathbf{A}) := \Phi(\mathbf{A}) \quad \text{for all} \quad \mathbf{A} \in \mathrm{Lis}(\mathcal{V}, \mathcal{W}). \tag{13.23}$$

Indeed, we can view (13.22) as a bilinear assignment from $\mathrm{Lin} = \mathrm{Ln}_1$ to $\mathrm{Lin} \circ (\Phi \times \Phi)$. The one to be used in (13.27)

$$\Phi_{(\mathcal{V},\mathcal{V})} : \mathrm{Lis}(\mathcal{V}) \to \mathrm{Lis}(\Phi(\mathcal{V}))$$

is a linear assignment from Ln to $\mathrm{Ln} \circ \Phi$ and hence whose gradient is also a linear assignment from Ln to $\mathrm{Ln} \circ \Phi$!!!!!!!!!!!!!!!!

We say that the tensor functor Φ is **analytic** if $\Phi_{(\mathcal{V},\mathcal{W})}$ is an analytic mapping for every pair of objects $(\mathcal{V},\mathcal{W})$ of $\mathrm{Dom}\,\Phi$. We say that a natural assignment $\alpha : \Phi \to \Psi$ is an **analytic** assignment if the mapping $\alpha_{\mathcal{F}} : \Phi(\mathcal{F}) \to \Psi(\mathcal{F})$ is an analytic mapping for every object $\mathcal{F}$ of $\mathrm{Dom}\,\Phi$. All the tensor functors listed in above are in fact analytic. (The fact that they are of class C^∞ can easily be inferred from the results of Ch.6 of [FDS]. Proofs that they are analytic can be inferred, for example, from the results that will be presented in Ch.2 of Vol.2 of [FDS].)

Theorem : *Let an analytic tensor functor Φ be given and associate with each $\mathcal{V} \in \mathrm{Dom}\,\Phi$ the mapping*

$$\Phi^{\bullet}_{\mathcal{V}} : \mathrm{Ln}(\mathcal{V}) \to \mathrm{Ln}(\Phi(\mathcal{V})) \tag{13.24}$$

defined by

$$\Phi^{\bullet}_{\mathcal{V}} := \nabla_{\mathbf{1}_{\mathcal{V}}} \Phi_{(\mathcal{V},\mathcal{V})}. \tag{13.25}$$

(The gradient-notation used here is explained in [FDS], Sect.63.) *Then $\Phi^{\bullet}$ is a linear assignment from* Ln *to* $\mathrm{Ln} \circ \Phi$. *We call $\Phi^{\bullet}$ the* **derivative** *of Φ.*

Proof: Let a pair of objects $(\mathcal{V},\mathcal{W})$ of $\mathrm{Dom}\,\Phi$ and $\mathbf{A} \in \mathrm{Lis}(\mathcal{V},\mathcal{W})$ be given. It follows from (13.23), from axiom (F1), and from (12.2) that

$$\Phi_{(\mathcal{W},\mathcal{W})}(\mathbf{A}\mathbf{L}\mathbf{A}^{-1}) = \Phi(\mathbf{A})\Phi_{(\mathcal{V},\mathcal{V})}(\mathbf{L})\Phi(\mathbf{A})^{-1} \tag{13.26}$$

for all $\mathbf{L} \in \mathrm{Lis}(\mathcal{V},\mathcal{V})$. By (13.15) we may write (13.26) as

$$\big(\Phi_{(\mathcal{W},\mathcal{W})} \circ \mathrm{Ln}(\mathbf{A})\big)(\mathbf{L}) = \big(\mathrm{Ln}(\Phi(\mathbf{A})) \circ \Phi_{(\mathcal{V},\mathcal{V})}\big)(\mathbf{L}) \tag{13.27}$$

for all $\mathbf{L} \in \mathrm{Lis}(\mathcal{V},\mathcal{V})$. Taking the gradient of (13.27) with respect to $\mathbf{L}$ at $\mathbf{L} := \mathbf{1}_{\mathcal{V}}$ yields

$$\Phi^{\bullet}_{\mathcal{W}} \circ \mathrm{Ln}(\mathbf{A}) = (\mathrm{Ln} \circ \Phi)(\mathbf{A}) \circ \Phi^{\bullet}_{\mathcal{V}}. \tag{13.28}$$

In view of (12.13) it follows that $\Phi^{\bullet}$ is a natural assignment from Ln to $\mathrm{Ln} \circ \Phi$. The linearity of $\Phi^{\bullet}$ follows from the definition of gradient. ∎

We now list the derivatives of a few analytic tensor functors. The formulas given are valid for every $\mathcal{V} \in LS$.

(6) $\mathrm{Ln}^{\bullet}_{\mathcal{V}} : \mathrm{Ln}(\mathcal{V}) \to \mathrm{Ln}(\mathrm{Ln}(\mathcal{V}))$ is given by

$$(\mathrm{Ln}^{\bullet}_{\mathcal{V}}\mathbf{L})\mathbf{M} = \mathbf{LM} - \mathbf{ML} \quad \text{for all} \quad \mathbf{L}, \mathbf{M} \in \mathrm{Ln}(\mathcal{V}) \tag{13.29}$$

(This formula is an easy consequence of (13.15) and, [FDS] (68.9).).

(7) Let $k \in$ be given. In order to describe

$$(\mathrm{Lnf}_k)^{\bullet}_{\mathcal{V}} : \mathrm{Ln}(\mathcal{V}) \to \mathrm{Ln}(\mathrm{Lin}_k(\mathcal{V}^k,)), \tag{13.30}$$

we define, for every $\mathbf{L} \in \mathrm{Ln}(\mathcal{V})$ and every $j \in k^{]}$, $D_j(\mathbf{L}) \in (\mathrm{Ln}(\mathcal{V}))^k$ by

$$(D_j(\mathbf{L}))_i := \left\{ \begin{array}{ll} \mathbf{L} & if \quad i = j \\ \\ \mathbf{1}_{\mathcal{V}} & if \quad i \neq j \end{array} \right\} \qquad \text{for all} \quad i \in k^{]}. \tag{13.31}$$

We then have

$$((\mathrm{Lnf}_k)^{\bullet}_{\mathcal{V}}\mathbf{L})\boldsymbol{\omega} = -\sum_{j \in k^{]}} \boldsymbol{\omega} \circ D_j(\mathbf{L}) \quad \text{for all} \quad \boldsymbol{\omega} \in \mathrm{Lin}_k(\mathcal{V}^k,) \tag{13.32}$$

and all $\mathbf{L} \in \mathrm{Ln}(\mathcal{V})$. The formula (13.32) remains valid if Lnf is replaced by Smf or Skf and Lin by Sym or Skew, correspondingly.

The General Chain Rule for gradients (see [FDS], Sect.63) and the definition (13.25) immediately lead to the following

Chain Rule for Analytic Tensor Functors

Let Φ and Ψ be analytic tensor functors. Then the composite functor $\Psi \circ \Phi$ is also an analytic tensor functor and we have

$$(\Psi \circ \Phi)^{\bullet} = (\Psi^{\bullet} \circ \Phi) \circ \Phi^{\bullet}, \tag{13.33}$$

where the composite assignments on the right are explained in the end of Sect.12.

For example, (13.33) shows that, for each $\mathcal{V} \in LS$,

$$(\mathrm{Ln} \circ \mathrm{Ln})^{\bullet}_{\mathcal{V}} : \mathrm{Ln}(\mathcal{V}) \to \mathrm{Ln}(\mathrm{Ln}(\mathrm{Ln}(\mathcal{V})))$$

is given by

$$(\mathrm{Ln} \circ \mathrm{Ln})^{\bullet}_{\mathcal{V}} = \mathrm{Ln}^{\bullet}_{\mathrm{Ln}(\mathcal{V})}\mathrm{Ln}^{\bullet}_{\mathcal{V}}. \tag{13.34}$$

In view of (13.29.) above, (13.34) gives

$$\begin{aligned} \big(((\mathrm{Ln} \circ \mathrm{Ln})^{\bullet}_{\mathcal{V}}\mathbf{L})\mathbf{K}\big)\mathbf{M} &= \big((\mathrm{Ln}^{\bullet}_{\mathcal{V}}\mathbf{L})\mathbf{K} - \mathbf{K}(\mathrm{Ln}^{\bullet}_{\mathcal{V}}\mathbf{L})\big)\mathbf{M} \\ &= \mathbf{L}(\mathbf{KM}) - (\mathbf{KM})\mathbf{L} - \mathbf{K}(\mathbf{LM} - \mathbf{ML}) \end{aligned} \tag{13.35}$$

for all $\mathcal{V} \in LS$, all $\mathbf{K} \in \mathrm{Ln}(\mathrm{Ln}(\mathcal{V}))$, and all $\mathbf{L}, \mathbf{M} \in \mathrm{Ln}(\mathcal{V})$.

If Φ and Ψ are analytic tensor functors so is $\mathrm{Pr} \circ (\Phi, \Psi)$ and we have

$$(\mathrm{Pr} \circ (\Phi, \Psi))^{\bullet}_{\mathcal{V}} = (\Phi^{\bullet}_{\mathcal{V}}\mathbf{L}) \times \mathbf{1}_{\Psi(\mathcal{V})} + \mathbf{1}_{\Psi(\mathcal{V})} \times (\Phi^{\bullet}_{\mathcal{V}}\mathbf{L}) \tag{13.36}$$

for all $\mathcal{V} \in LS$ and all $\mathbf{L} \in \mathrm{Ln}(\mathcal{V})$.

Let α be an analytic assignment of degree $n \in$. If we associate with each $\mathcal{V} \in LS$ the mapping $(\nabla\alpha)_{\mathcal{V}} := \nabla(\alpha_{\mathcal{V}})$, the gradient of the mapping $\alpha_{\mathcal{V}}$, then $\nabla\alpha$ is again an analytic assignment of degree n and we have $\mathrm{Dmf}_{\nabla\alpha} = \mathrm{Dmf}_{\alpha}$ and $\mathrm{Cdf}_{\nabla\alpha} = \mathrm{Lin} \circ (\mathrm{Dmf}_{\alpha}, \mathrm{Cdf}_{\alpha})$. We call $\nabla\alpha$ the **gradient** of α.

Let tensor functors Φ_1, Φ_2, Ψ, all of degree $n \in$ but not necessarily analytic, be given. Each bilinear assignment $\beta : \mathrm{Pr} \circ (\Phi_1, \Phi_2) \to \Psi$ is then analytic and its gradient $\nabla\beta : \mathrm{Pr} \circ (\Phi_1, \Phi_2) \to \mathrm{Lin} \circ (\mathrm{Pr} \circ (\Phi_1, \Phi_2), \Psi)$ is given by

$$\big((\nabla\beta)_{\mathcal{V}}(\mathbf{v}_1, \mathbf{v}_2)\big)(\mathbf{u}_1, \mathbf{u}_2) = \beta_{\mathcal{V}}(\mathbf{v}_1, \mathbf{u}_2) + \beta_{\mathcal{V}}(\mathbf{u}_1, \mathbf{v}_2) \tag{13.37}$$

for all $\mathcal{V} \in LS$, all $\mathbf{v}_1, \mathbf{u}_1 \in \Phi_1(\mathcal{V})$, and all $\mathbf{v}_2, \mathbf{u}_2 \in \Phi_2(\mathcal{V})$.

If α is an analytic assignment of degree $n \in$ and if Φ is any isofunctor from LIS^k to LIS^n with $k \in$, then $\alpha \circ \Phi$ is an analytic assignment of degree k and we have $\nabla(\alpha \circ \Phi) = (\nabla\alpha) \circ \Phi$.

14. Short Exact Sequences

Let a pair $(\mathbf{I}, \mathbf{P})$ of mappings be given such that $\mathrm{Cod}\,\mathbf{I} = \mathrm{Dom}\,\mathbf{P}$. We often write

$$\mathcal{U} \xrightarrow{\mathbf{I}} \mathcal{V} \xrightarrow{\mathbf{P}} \mathcal{W} \quad \text{or} \quad \mathcal{W} \xleftarrow{\mathbf{P}} \mathcal{V} \xleftarrow{\mathbf{I}} \mathcal{U} \tag{14.1}$$

to indicate that $\mathcal{U} = \mathrm{Dom}\,\mathbf{I}$, $\mathcal{V} = \mathrm{Cod}\,\mathbf{I} = \mathrm{Dom}\,\mathbf{P}$ and $\mathrm{Cod}\,\mathbf{P} = \mathcal{W}$. If $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ are linear spaces and if $\mathbf{I}$ is injective linear mapping, $\mathbf{P}$ is surjective linear mapping with

$$\mathrm{Rng}\,\mathbf{I} = \mathrm{Null}\,\mathbf{P},$$

we say that $(\mathbf{I}, \mathbf{P})$, or (14.1), is a **short exact sequence** *. In the literature, a short exact sequence is often expressed as

$$\mathbf{0} \longrightarrow \mathcal{U} \xrightarrow{\mathbf{I}} \mathcal{V} \xrightarrow{\mathbf{P}} \mathcal{W} \longrightarrow \mathbf{0}\,.$$

Let a short exact sequence $\mathcal{U} \xrightarrow{\mathbf{I}} \mathcal{V} \xrightarrow{\mathbf{P}} \mathcal{W}$ be given.

Notation: *The set of all linear right-inverses of* $\mathbf{P}$ *is denoted by*

$$\mathrm{Riv}(\mathbf{P}) := \left\{ \mathbf{K} \in \mathrm{Lin}\,(\mathcal{W}, \mathcal{V}) \;\middle|\; \mathbf{PK} = \mathbf{1}_{\mathcal{W}} \right\}, \tag{14.2}$$

and the set of all linear left-inverses of $\mathbf{I}$ *is denoted by*

$$\mathrm{Liv}(\mathbf{I}) := \left\{ \mathbf{D} \in \mathrm{Lin}\,(\mathcal{V}, \mathcal{U}) \;\middle|\; \mathbf{DI} = \mathbf{1}_{\mathcal{U}} \right\}. \tag{14.3}$$

Proposition 1: *There is a bijection* $\mathbf{\Lambda} : \mathrm{Riv}(\mathbf{P}) \to \mathrm{Liv}(\mathbf{I})$ *such that, for every* $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$

$$\mathcal{U} \xleftarrow[\mathbf{\Lambda}(\mathbf{K})]{} \mathcal{V} \xleftarrow[\mathbf{K}]{} \mathcal{W} \tag{14.4}$$

is again a short exact sequence. We have

$$\mathbf{KP} + \mathbf{I\Lambda}(\mathbf{K}) = \mathbf{1}_{\mathcal{V}} \quad \text{for all} \quad \mathbf{K} \in \mathrm{Riv}(\mathbf{P}). \tag{14.5}$$

Proof: It is easily seen that $(\mathbf{K} \mapsto \mathrm{Rng}\,\mathbf{K})$ is a bijection from $\mathrm{Riv}(\mathbf{P})$ to the set of all supplements of $\mathrm{Null}\,\mathbf{P} = \mathrm{Rng}\,\mathbf{I}$ in $\mathcal{V}$. Also, $(\mathbf{D} \mapsto \mathrm{Null}\,\mathbf{D})$ is a bijection from $\mathrm{Liv}(\mathbf{I})$ to the set of all supplements of $\mathrm{Rng}\,\mathbf{I} = \mathrm{Null}\,\mathbf{P}$ in $\mathcal{V}$. The mapping $\mathbf{\Lambda}$ is the composite of the first of these bijections with the inverse of the second one.

* The term short exact sequence comes from the more general concept of an "exact sequence" which is not needed here.

Let $\mathbf{K} \in \operatorname{Riv}(\mathbf{P})$ be given. Both $\mathbf{KP}$ and $\mathbf{I\Lambda}(\mathbf{K})$ are idempotents with $\operatorname{Rng}\mathbf{KP} = \operatorname{Rng}\mathbf{K}$ and $\operatorname{Rng}\mathbf{I\Lambda}(\mathbf{K}) = \operatorname{Rng}\mathbf{I}$. Since $\operatorname{Rng}\mathbf{K}$ and $\operatorname{Rng}\mathbf{I}$ are supplementary in $\mathcal{V}$, it follows that

$$\mathbf{KP} + \mathbf{I\Lambda}(\mathbf{K}) = \mathbf{1}_{\mathcal{V}}. \tag{14.6}$$

Since $\mathbf{K} \in \operatorname{Riv}(\mathbf{P})$ was arbitrary, the assertion follows. ∎

Proposition 2: $\operatorname{Riv}(\mathbf{P})$ *is a flat in* $\operatorname{Lin}(\mathcal{W},\mathcal{V})$ *whose direction space is*

$$\big\{\, \mathbf{IL} \,\big|\, \mathbf{L} \in \operatorname{Lin}(\mathcal{W},\mathcal{U}) \,\big\},$$

$\operatorname{Liv}(\mathbf{I})$ *is a flat in* $\operatorname{Lin}(\mathcal{V},\mathcal{U})$ *whose direction space is*

$$\big\{ -\mathbf{LP} \,\big|\, \mathbf{L} \in \operatorname{Lin}(\mathcal{W},\mathcal{U}) \,\big\}.$$

Proof: Given $\mathbf{K},\mathbf{K}' \in \operatorname{Riv}(\mathbf{P})$, we have $\mathbf{1}_{\mathcal{W}} = \mathbf{P}\mathbf{K} = \mathbf{P}\mathbf{K}'$ and hence $\mathbf{P}(\mathbf{K}-\mathbf{K}') = \mathbf{0}$. It follows that $\operatorname{Rng}(\mathbf{K}-\mathbf{K}') \subset \operatorname{Null}\mathbf{P} = \operatorname{Rng}\mathbf{I}$ and hence $\mathbf{K}-\mathbf{K}' = \mathbf{IL}$ for some $\mathbf{L} \in \operatorname{Lin}(\mathcal{W},\mathcal{U})$. On the other hand, given $\mathbf{K} \in \operatorname{Riv}(\mathbf{P})$ and $\mathbf{L} \in \operatorname{Lin}(\mathcal{W},\mathcal{U})$, we have $\mathbf{P}(\mathbf{IL}) = \mathbf{0}$ and hence $\mathbf{1}_{\mathcal{W}} = \mathbf{PK} = \mathbf{P}(\mathbf{K}+\mathbf{IL})$, which implies $\mathbf{K}+\mathbf{IL} \in \operatorname{Riv}(\mathbf{P})$. These facts show that $\operatorname{Riv}(\mathbf{P})$ is a flat in $\operatorname{Lin}(\mathcal{W},\mathcal{V})$ with direction space $\big\{\, \mathbf{IL} \,\big|\, \mathbf{L} \in \operatorname{Lin}(\mathcal{W},\mathcal{U}) \,\big\}$.

Similar arguments show that $\operatorname{Liv}(\mathbf{I})$ is a flat in $\operatorname{Lin}(\mathcal{V},\mathcal{U})$ with direction space $\big\{ -\mathbf{LP} \,\big|\, \mathbf{L} \in \operatorname{Lin}(\mathcal{W},\mathcal{U}) \,\big\}$. ∎

Proposition 3: *Let* $\mathbf{K}$ *and* $\mathbf{K}'$ *in* $\operatorname{Riv}(\mathbf{P})$ *be given and determine* $\mathbf{L} \in \operatorname{Lin}(\mathcal{W},\mathcal{U})$ *such that* $\mathbf{K}-\mathbf{K}' = \mathbf{IL}$. *Then*

$$\mathbf{\Lambda}(\mathbf{K}) - \mathbf{\Lambda}(\mathbf{K}') = -\mathbf{LP}. \tag{14.7}$$

Proof: It follows from (14.5) that $\mathbf{KP} + \mathbf{I\Lambda}(\mathbf{K}) = \mathbf{1}_{\mathcal{V}} = \mathbf{K}'\mathbf{P} + \mathbf{I\Lambda}(\mathbf{K}')$ and hence

$$\mathbf{I}(\mathbf{\Lambda}(\mathbf{K}) - \mathbf{\Lambda}(\mathbf{K}')) = -(\mathbf{K}-\mathbf{K}')\mathbf{P}.$$

Since $\mathbf{K}-\mathbf{K}' = \mathbf{IL}$ and $\mathbf{I}$ is injective, we obtain $\mathbf{\Lambda}(\mathbf{K}) - \mathbf{\Lambda}(\mathbf{K}') = -\mathbf{LP}$. ∎

It follows from the injectivity of $\mathbf{I}$ and from the surjectivity of $\mathbf{P}$ that both the direction space $\{\mathbf{I}\}\operatorname{Lin}(\mathcal{W},\mathcal{U})$ of $\operatorname{Riv}(\mathbf{P})$ and the direction space $\operatorname{Lin}(\mathcal{W},\mathcal{U})\{\mathbf{P}\}$ of $\operatorname{Liv}(\mathbf{I})$ are naturally isomorphic to $\operatorname{Lin}(\mathcal{W},\mathcal{U})$. Hence we may and will consider $\operatorname{Lin}(\mathcal{W},\mathcal{U})$ to be the external translation space (see Conventions and Notations) of both $\operatorname{Riv}(\mathbf{P})$ and $\operatorname{Liv}(\mathbf{I})$. We have

$$\dim \operatorname{Riv}(\mathbf{P}) = (\dim \mathcal{W})(\dim \mathcal{U}) = \dim \operatorname{Liv}(\mathbf{I}). \tag{14.8}$$

Proposition 4: *The mapping* $\mathbf{\Lambda} : \mathrm{Riv}(\mathbf{P}) \to \mathrm{Liv}(\mathbf{I})$, *as described in Prop. 1, is a flat isomorphism whose gradient* $\nabla\mathbf{\Lambda} \in \mathrm{Lin}(\mathrm{Lin}(\mathcal{W},\mathcal{U}))$ *is* $-\mathbf{1}_{\mathrm{Lin}(\mathcal{W},\mathcal{U})}$, *so that*

$$\nabla\mathbf{\Lambda}(\mathbf{L}) = -\mathbf{L} \quad \text{for all} \quad \mathbf{L} \in \mathrm{Lin}(\mathcal{W},\mathcal{U}). \tag{14.9}$$

Proof: It follows from Prop. 2 and the identification $\mathrm{Lin}(\mathcal{W},\mathcal{U})\{\mathbf{P}\} \cong \mathrm{Lin}(\mathcal{W},\mathcal{U})$ that $\mathbf{\Lambda} : \mathrm{Riv}(\mathbf{P}) \to \mathrm{Liv}(\mathbf{I})$ is a flat isomorphism with $\nabla\mathbf{\Lambda} = -\mathbf{1}_{\mathrm{Lin}(\mathcal{W},\mathcal{U})}$. ∎

<u>**Notation:**</u> *Let* $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$ *be given. We define the mapping*

$$\mathbf{\Gamma}^{\mathbf{K}} : \mathrm{Riv}(\mathbf{P}) \to \mathrm{Lin}\,(\mathcal{W},\mathcal{U})$$

by

$$\mathbf{\Gamma}^{\mathbf{K}}(\mathbf{K}') := -\mathbf{\Lambda}(\mathbf{K})\mathbf{K}' \qquad \text{for all} \quad \mathbf{K}' \in \mathrm{Riv}(\mathbf{P}). \tag{14.10}$$

Proposition 5: *For every* $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$, *the mapping* $\mathbf{\Gamma}^{\mathbf{K}} : \mathrm{Riv}(\mathbf{P}) \to \mathrm{Lin}\,(\mathcal{W},\mathcal{U})$ *is a flat isomorphism whose gradient* $\nabla\mathbf{\Gamma}^{\mathbf{K}} \in \mathrm{Lin}(\mathrm{Lin}(\mathcal{W},\mathcal{U}))$ *is* $-\mathbf{1}_{\mathrm{Lin}(\mathcal{W},\mathcal{U})}$; *i.e.*

$$\nabla\mathbf{\Gamma}^{\mathbf{K}}(\mathbf{L}) = -\mathbf{L} \quad \text{for all} \quad \mathbf{L} \in \mathrm{Lin}(\mathcal{W},\mathcal{U}).$$

Proof: Let $\mathbf{K}_1, \mathbf{K}_2 \in \mathrm{Riv}(\mathbf{P})$ be given; then we determine $\mathbf{L} \in \mathrm{Lin}(\mathcal{W},\mathcal{U})$ such that $\mathbf{K}_1 - \mathbf{K}_2 = \mathbf{IL}$. It follows from (14.10) and $\mathbf{\Lambda}(\mathbf{K})\mathbf{I} = \mathbf{1}_{\mathcal{U}}$ that

$$\mathbf{\Gamma}^{\mathbf{K}}(\mathbf{K}_1) - \mathbf{\Gamma}^{\mathbf{K}}(\mathbf{K}_2) = -\mathbf{\Lambda}(\mathbf{K})(\mathbf{K}_1 - \mathbf{K}_2) = -\mathbf{\Lambda}(\mathbf{K})(\mathbf{IL}) = -\mathbf{L}.$$

Since $\mathbf{K}_1, \mathbf{K}_2 \in \mathrm{Riv}(\mathbf{P})$ were arbitrary, the assertion follows. ∎

Proposition 6: *We have*

$$\begin{aligned} \mathbf{K} - \mathbf{K}' &= \mathbf{I}\,\mathbf{\Gamma}^{\mathbf{K}}(\mathbf{K}') \\ \mathbf{\Lambda}(\mathbf{K}) - \mathbf{\Lambda}(\mathbf{K}') &= -\mathbf{\Gamma}^{\mathbf{K}}(\mathbf{K}')\mathbf{P} \end{aligned} \tag{14.11}$$

and hence $\mathbf{\Gamma}^{\mathbf{K}'}(\mathbf{K}) = -\mathbf{\Gamma}^{\mathbf{K}}(\mathbf{K}')$ *for all* $\mathbf{K}, \mathbf{K}' \in \mathrm{Riv}(\mathbf{P})$. *Moreover,*

$$\mathbf{\Gamma}^{\mathbf{K}_1}(\mathbf{K}_3) - \mathbf{\Gamma}^{\mathbf{K}_2}(\mathbf{K}_3) = \mathbf{\Gamma}^{\mathbf{K}_1}(\mathbf{K}_2) \tag{14.12}$$

for all $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3 \in \mathrm{Riv}(\mathbf{P})$.

Proof: In view of (14.5) and (14.10), we have

$$\mathbf{K} - \mathbf{K}' = (\mathbf{KP} - \mathbf{1}_{\mathcal{V}})\mathbf{K}' = -(\mathbf{I}\,\mathbf{\Lambda}(\mathbf{K}))\mathbf{K}' = \mathbf{I}\,\mathbf{\Gamma}^{\mathbf{K}}(\mathbf{K}')$$

for all $\mathbf{K}', \mathbf{K} \in \mathrm{Riv}(\mathbf{P})$. The second equation $(14.11)_2$ follows from $(14.11)_1$ and Prop. 2 with $\mathbf{L}$ replaced by $\mathbf{\Gamma}^{\mathbf{K}}(\mathbf{K}')$.

We observe from (14.11) that

$$\begin{aligned}\mathbf{I}\,\mathbf{\Gamma}^{\mathbf{K}_1}(\mathbf{K}_2) = \mathbf{K}_1 - \mathbf{K}_2 &= (\mathbf{K}_1 - \mathbf{K}_3) - (\mathbf{K}_2 - \mathbf{K}_3)\\ &= \mathbf{I}(\mathbf{\Gamma}^{\mathbf{K}_1}(\mathbf{K}_3) - \mathbf{\Gamma}^{\mathbf{K}_2}(\mathbf{K}_3))\end{aligned}$$

for all $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3 \in \mathrm{Riv}(\mathbf{P})$. Since $\mathbf{I}$ is injective, (14.12) follows. ∎

Remark: We consider $\mathrm{Lin}(\mathcal{W},\mathcal{U})$ to be the external translation space of $\mathrm{Riv}(\mathbf{P})$. Given $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$, in view of $(14.11)_1$, we have

$$\mathbf{\Gamma}^{\mathbf{K}}(\mathbf{K}') = \mathbf{K} - \mathbf{K}' \quad \text{for all} \quad \mathbf{K}' \in \mathrm{Riv}(\mathbf{P}).$$

Roughly speaking, the flat isomorphism $\mathbf{\Gamma}^{\mathbf{K}} : \mathrm{Riv}(\mathbf{P}) \to \mathrm{Lin}(\mathcal{W},\mathcal{U})$ identify $\mathrm{Riv}(\mathbf{P})$ with $\mathrm{Lin}(\mathcal{W},\mathcal{U})$ by choosing $\mathbf{K}$ as the "*zero*" (or "*orgin*"). ∎

15. Brackets and Twists

We assume now that linear spaces $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{Z}$ and a short exact sequence

$$\mathrm{Lin}(\mathcal{W},\mathcal{Z}) \xrightarrow{\mathbf{I}} \mathcal{V} \xrightarrow{\mathbf{P}} \mathcal{W} \tag{15.1}$$

are given. Recall from Prop. 1 of Sec. 14 that to every linear right-inverse $\mathbf{K}$ of $\mathbf{P}$ there corresponds exactly one linear left-inverse $\mathbf{\Lambda}(\mathbf{K})$ of $\mathbf{I}$ such that

$$\mathrm{Lin}(\mathcal{W},\mathcal{Z}) \underset{\mathbf{\Lambda}(\mathbf{K})}{\longleftarrow} \mathcal{V} \underset{\mathbf{K}}{\longleftarrow} \mathcal{W} \tag{15.2}$$

is again a short exact sequence. In view of the identification

$$\mathrm{Lin}\big(\mathcal{W}, \mathrm{Lin}\,(\mathcal{W},\mathcal{Z})\big) \cong \mathrm{Lin}_2\,(\mathcal{W}^2,\mathcal{Z}) \tag{15.3}$$

we may identify the external translation space $\mathrm{Lin}\big(\mathcal{W}, \mathrm{Lin}\,(\mathcal{W},\mathcal{Z})\big)$ of $\mathrm{Riv}(\mathbf{P})$ with $\mathrm{Lin}_2\,(\mathcal{W}^2,\mathcal{Z})$.

Assumption : From now on, we assume that in this section, a flat $\mathcal{F}$ in $\mathrm{Riv}(\mathbf{P})$ with direction space $\{\mathbf{I}\}\mathrm{Sym}_2\,(\mathcal{W}^2,\mathcal{Z})$ is given. Here $\mathrm{Sym}_2\,(\mathcal{W}^2,\mathcal{Z})$ is regarded as a subspace of $\mathrm{Lin}_2\,(\mathcal{W}^2,\mathcal{Z}) \cong \mathrm{Lin}\big(\mathcal{W}, \mathrm{Lin}\,(\mathcal{W},\mathcal{Z})\big)$.

Proposition 1: *For every* $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{F}$,

$$(\mathbf{\Lambda}(\mathbf{K}_1)\mathbf{v})(\mathbf{P}\mathbf{v}') - (\mathbf{\Lambda}(\mathbf{K}_1)\mathbf{v}')(\mathbf{P}\mathbf{v}) = (\mathbf{\Lambda}(\mathbf{K}_2)\mathbf{v})(\mathbf{P}\mathbf{v}') - (\mathbf{\Lambda}(\mathbf{K}_2)\mathbf{v}')(\mathbf{P}\mathbf{v}) \tag{15.4}$$

holds for all $\mathbf{v}, \mathbf{v}' \in \mathcal{V}$.

Proof: Let $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{F}$ be given. Then we determine $\mathbf{L} \in \mathrm{Sym}_2(\mathcal{W}^2, \mathcal{Z})$ such that $\mathbf{K}_1 - \mathbf{K}_2 = \mathbf{IL}$. It follows from Prop.3 of Sect.14 that

$$(\mathbf{\Lambda}(\mathbf{K}_1)\mathbf{v})(\mathbf{Pv}') - (\mathbf{\Lambda}(\mathbf{K}_2)\mathbf{v})(\mathbf{Pv}') = -\mathbf{L}(\mathbf{Pv}, \mathbf{Pv}')$$

holds for all $\mathbf{v}, \mathbf{v}' \in \mathcal{V}$. By interchanging $\mathbf{v}$ and $\mathbf{v}'$ and observing that $\mathbf{L}$ is symmetric, we conclude that (15.4) follows. ▮

<u>**Definition**</u>: *In view of Prop. 1, the* $\mathcal{F}$-**bracket** $\mathbf{B}_{\mathcal{F}} \in \mathrm{Skw}_2(\mathcal{V}^2, \mathcal{Z})$ *can be defined such that*

$$\mathbf{B}_{\mathcal{F}}(\mathbf{v}, \mathbf{v}') := (\mathbf{\Lambda}(\mathbf{K})\mathbf{v})(\mathbf{Pv}') - (\mathbf{\Lambda}(\mathbf{K})\mathbf{v}')(\mathbf{Pv}) \quad \text{for all} \quad \mathbf{v}, \mathbf{v}' \in \mathcal{V} \tag{15.5}$$

is valid for all $\mathbf{K} \in \mathcal{F}$. *Using the identification* (15.3) *we also have*

$$\mathbf{B}_{\mathcal{F}} \in \mathrm{Lin}\big(\mathcal{V}, \mathrm{Lin}(\mathcal{V}, \mathcal{Z})\big).$$

Proposition 2: *The* $\mathcal{F}$-**bracket** $\mathbf{B}_{\mathcal{F}} \in \mathrm{Lin}\big(\mathcal{V}, \mathrm{Lin}(\mathcal{V}, \mathcal{Z})\big)$ *satisfies*

$$\begin{aligned} \mathbf{B}_{\mathcal{F}}(\mathbf{I}\,\mathbf{M}) &= \mathbf{M}\,\mathbf{P} \qquad \text{for all} \quad \mathbf{M} \in \mathrm{Lin}(\mathcal{W}, \mathcal{Z}), \\ (\mathbf{B}_{\mathcal{F}}\mathbf{v})\mathbf{K} &= \mathbf{\Lambda}(\mathbf{K})\mathbf{v} \qquad \text{for all} \quad \mathbf{K} \in \mathcal{F} \text{ and all } \mathbf{v} \in \mathcal{V}. \end{aligned} \tag{15.6}$$

If $\dim \mathcal{Z} \neq 0$, *then* $\mathbf{B}_{\mathcal{F}}$ *is injective; i.e.* $\mathrm{Null}\,\mathbf{B}_{\mathcal{F}} = \{\mathbf{0}\}$.

Proof: The equations $(15.6)_1$ and $(15.6)_2$ follow from Definition (15.5) together with $\mathbf{\Lambda}(\mathbf{K})\,\mathbf{I} = \mathbf{1}_{\mathrm{Lin}(\mathcal{W},\mathcal{Z})}$ and $\mathbf{PK} = \mathbf{1}_{\mathcal{W}}$, respectively.

Let $\mathbf{v} \in \mathrm{Null}\,\mathbf{B}_{\mathcal{F}}$ be given, so that $\mathbf{B}_{\mathcal{F}}\,\mathbf{v} = \mathbf{0}$ and hence

$$\mathbf{0} = \big(\mathbf{B}_{\mathcal{F}}\mathbf{v}\big)\,\mathbf{IM} = \mathbf{B}_{\mathcal{F}}\,(\mathbf{v}, \mathbf{IM}) = -\big(\mathbf{B}_{\mathcal{F}}(\mathbf{I}\,\mathbf{M})\big)\mathbf{v}$$

for all $\mathbf{M} \in \mathrm{Lin}(\mathcal{W}, \mathcal{Z})$. Using $(15.6)_1$, it follows that $-\mathbf{MPv} = \mathbf{0}$ for all $\mathbf{M} \in \mathrm{Lin}(\mathcal{W}, \mathcal{Z})$, which can happen, when $\dim \mathcal{Z} \neq 0$, only if $\mathbf{Pv} = \mathbf{0}$ and hence $\mathbf{v} \in \mathrm{Null}\,\mathbf{P} = \mathrm{Rng}\,\mathbf{I}$. Thus we may choose $\mathbf{M}' \in \mathrm{Lin}(\mathcal{W}, \mathcal{Z})$ such that $\mathbf{v} = \mathbf{IM}'$ and hence $\mathbf{B}_{\mathcal{F}}(\mathbf{IM}') = \mathbf{0}$. Using $(15.6)_1$ again, it follows that $\mathbf{M}'\,\mathbf{P} = \mathbf{0}$. Since $\mathbf{P}$ is surjective , we conclude that $\mathbf{M}' = \mathbf{0}$ and hence $\mathbf{v} = \mathbf{0}$. Since $\mathbf{v} \in \mathrm{Null}\,\mathbf{B}_{\mathcal{F}}$ was arbitrary, it follows that $\mathrm{Null}\,\mathbf{B}_{\mathcal{F}} = \{\mathbf{0}\}$. ▮

<u>**Definition**</u>: *The* $\mathcal{F}$-**twist**

$$\mathbf{T}_{\mathcal{F}} : \mathrm{Riv}(\mathbf{P}) \to \mathrm{Skw}_2(\mathcal{W}^2, \mathcal{Z}) \tag{15.7}$$

is defined by

$$\mathbf{T}_{\mathcal{F}}(\mathbf{K}) := -\mathbf{B}_{\mathcal{F}} \circ (\mathbf{K} \times \mathbf{K}) \qquad \text{for all} \quad \mathbf{K} \in \mathrm{Riv}(\mathbf{P}), \tag{15.8}$$

where $\mathbf{B}_{\mathcal{F}}$ *is the* $\mathcal{F}$*-bracket defined by* (15.5).

Proposition 3: *For every* $\mathbf{H} \in \mathcal{F}$*, we have*

$$\mathbf{T}_{\mathcal{F}} = \mathbf{\Gamma}^{\mathbf{H}} - \mathbf{\Gamma}^{\mathbf{H}\sim} \tag{15.9}$$

where $^{\sim}$ *denotes the value-wise switch, so that* $\mathbf{\Gamma}^{\mathbf{H}\sim}(\mathbf{K})(\mathbf{s},\mathbf{t}) = \mathbf{\Gamma}^{\mathbf{H}}(\mathbf{K})(\mathbf{t},\mathbf{s})$ *for all* $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$ *and all* $\mathbf{s},\mathbf{t} \in \mathcal{W}$.

Proof: Let $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$ and $\mathbf{s},\mathbf{t} \in \mathcal{W}$ be given. By (15.8) and (15.5), we see that for every $\mathbf{H} \in \mathcal{F}$ we have

$$\begin{aligned} \mathbf{T}_{\mathcal{F}}(\mathbf{K})(\mathbf{s},\mathbf{t}) &= -\mathbf{B}_{\mathcal{F}}(\mathbf{K}\mathbf{s},\mathbf{K}\mathbf{t}) \\ &= -\mathbf{\Lambda}(\mathbf{H})(\mathbf{K}\mathbf{s})\mathbf{P}(\mathbf{K}\mathbf{t}) + \mathbf{\Lambda}(\mathbf{H})(\mathbf{K}\mathbf{t})\mathbf{P}(\mathbf{K}\mathbf{s}). \end{aligned} \tag{15.10}$$

We conclude from $\mathbf{P}\,\mathbf{K} = \mathbf{1}_{\mathcal{W}}$, (15.10) and (14.10) that

$$\mathbf{T}_{\mathcal{F}}(\mathbf{K})(\mathbf{s},\mathbf{t}) = \mathbf{\Gamma}^{\mathbf{H}}(\mathbf{K})(\mathbf{s},\mathbf{t}) - \mathbf{\Gamma}^{\mathbf{H}}(\mathbf{K})^{\sim}(\mathbf{s},\mathbf{t}).$$

Since $\mathbf{s},\mathbf{t} \in \mathcal{W}$ and $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$ were arbitrary, (15.9) follows. ▮

Remark: It is clear from (15.9) and (11.6) that

$$\mathbf{T}_{\mathcal{F}} = 2\ \mathrm{Alt} \circ \mathbf{\Gamma}^{\mathbf{H}} \qquad \text{for all} \quad \mathbf{H} \in \mathcal{F}.$$

The numerical factor 2 is conventional which reduces numerical factors in calculations. ▮

Proposition 4: *The* $\mathcal{F}$*-torsion* $\mathbf{T}_{\mathcal{F}}$ *is a surjective flat mapping whose gradient*

$$\nabla\mathbf{T}_{\mathcal{F}} \in \mathrm{Lin}\left(\mathrm{Lin}_2\left(\mathcal{W}^2,\mathcal{Z}\right),\mathrm{Skw}_2\left(\mathcal{W}^2,\mathcal{Z}\right)\right)$$

is given by

$$(\nabla\mathbf{T}_{\mathcal{F}})\mathbf{L} = \mathbf{L}^{\sim} - \mathbf{L} \tag{15.11}$$

for all $\mathbf{L} \in \mathrm{Lin}_2\left(\mathcal{W}^2,\mathcal{Z}\right)$.

Proof: Let $\mathbf{H} \in \mathcal{F}$ be given. It follows from (15.8) and (15.5)

$$\mathbf{T}_{\mathcal{F}}\left(\mathbf{H} - \tfrac{1}{2}\mathbf{I}\mathbf{L}\right) = \mathbf{L} \qquad \text{for all} \quad \mathbf{L} \in \mathrm{Skw}_2\left(\mathcal{W}^2,\mathcal{Z}\right)$$

and hence $\mathbf{T}_{\mathcal{F}}$ is surjective.

Prop. 3 together with Prop. 4 in Sec. 14 shows that the $\mathcal{F}$-torsion $\mathbf{T}_{\mathcal{F}}$ is a flat mapping whose gradient is given by (15.11). ▮

In view of definitions (15.8), (15.5) and (15.11), we have $\mathbf{T}_{\mathcal{F}}{}^{<}(\{\mathbf{0}\}) = \mathcal{F}$.

Definition: *We say that* $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$ *is* $\mathcal{F}$**-twist-free** (*or* $\mathcal{F}$**-symmetric**) *if* $\mathbf{T}_{\mathcal{F}}(\mathbf{K}) = \mathbf{0}$, *i.e. if* $\mathbf{K} \in \mathcal{F}$.

$\mathcal{F}$ is a flat in $\mathrm{Riv}(\mathbf{P})$ with the (external) direction space $\mathrm{Sym}_2\,(\mathcal{W}^2, \mathcal{Z})$ and hence

$$\dim \mathbf{T}_{\mathcal{F}}{}^{<}(\{\mathbf{0}\}) = \dim \mathrm{Sym}_2\,(\mathcal{W}^2, \mathcal{Z}) = \frac{n(n+1)}{2} m, \tag{15.12}$$

where $n := \dim \mathcal{W}$ and $m := \dim \mathcal{Z}$. The mapping

$$\mathbf{S}_{\mathcal{F}} := \left(\mathbf{1}_{\mathrm{Riv}(\mathbf{P})} + \tfrac{1}{2}\mathbf{I}\mathbf{T}_{\mathcal{F}}\right)\Big|^{\mathbf{T}_{\mathcal{F}}{}^{<}(\{\mathbf{0}\})} \tag{15.13}$$

is the projection of $\mathrm{Riv}(\mathbf{P})$ onto $\mathbf{T}_{\mathcal{F}}^{<}(\{\mathbf{0}\})$ with $\mathrm{Null}\,\nabla\mathbf{S}_{\mathcal{F}} = \mathrm{Skw}_2\,(\mathcal{W}^2, \mathcal{Z})$. If $\mathbf{K} \in \mathrm{Riv}(\mathbf{P})$, we call

$$\mathbf{S}_{\mathcal{F}}(\mathbf{K}) = \mathbf{K} + \frac{1}{2}\mathbf{I}\big(\mathbf{T}_{\mathcal{F}}(\mathbf{K})\big)$$

the $\mathcal{F}$-**symmetric part** of $\mathbf{K}$.

Chapter 2

Manifolds and Bundles

21. Charts, Atlases and Manifolds

Let a set $\mathcal{M}$ and $r \in \tilde{}$ be given. A **chart** χ for $\mathcal{M}$ is defined to be a bijection whose domain is included in $\mathcal{M}$ and whose codomain is an open subset of a specified flat space, denote by $\mathrm{Pag}\,\chi$ and called the **page** of χ. The translation space of $\mathrm{Pag}\,\chi$ is denoted by

$$\mathcal{V}_\chi := \mathrm{Pag}\,\chi - \mathrm{Pag}\,\chi. \tag{21.1}$$

Let f be a mapping whose domain is a subset of $\mathcal{M}$ and whose codomain is an open subset $\mathcal{D}$ of a specified flat space. We say that f is **C^r-related** to a given chart χ for $\mathcal{M}$ if

(R1) $\chi_>(\mathrm{Dom}\,\chi \cap \mathrm{Dom}\,f)$ is an open subset of $\mathrm{Pag}\,\chi$,

(R2) $f \circ \chi^{\leftarrow} : \chi_>(\mathrm{Dom}\,\chi \cap \mathrm{Dom}\,f) \to \mathcal{D}$ is of class C^r.

We say that two charts χ and γ for $\mathcal{M}$ are **C^r-compatible** if γ is C^r-related to χ and χ is C^r-related to γ.

Pitfall: In general, C^r-compatibility is not an equivalence relation. ▮

A class $\mathfrak{A}$ of charts for $\mathcal{M}$ is called a **C^r-atlas** of $\mathcal{M}$ if

(A1) Any two charts in $\mathfrak{A}$ are C^r-compatible,

(A2) The domain of the charts in $\mathfrak{A}$ cover $\mathcal{M}$, i.e.

$$\mathcal{M} = \bigcup \{\mathrm{Dom}\,\chi \mid \chi \in \mathfrak{A}\}. \tag{21.2}$$

It is clear that a C^r-atlas is also a C^s-atlas for every $s \in 0..r$.

Proposition 1: *Let $\mathfrak{A}$ be a C^r-atlas for $\mathcal{M}$ and let χ be a chart that is C^r-compatible with all charts in $\mathfrak{A}$. If f is a mapping that is C^r-related to every chart in $\mathfrak{A}$ then it is also C^r-related to χ .*

Proof: Let $x \in \mathrm{Dom}\,\chi \cap \mathrm{Dom}\,f$ be given. By (A2) we may may choose $\alpha \in \mathcal{A}$ such that $x \in \mathrm{Dom}\,\alpha$. We put

$$\mathcal{G} := \mathrm{Dom}\,\chi \cap \mathrm{Dom}\,\alpha \cap \mathrm{Dom}\,f. \tag{21.3}$$

Since α is injective we have

$$\alpha_>(\mathcal{G}) = \alpha_>(\mathrm{Dom}\,\chi \cap \mathrm{Dom}\,\alpha) \cap \alpha_>(\mathrm{Dom}\,f \cap \mathrm{Dom}\,\alpha).$$

Since χ and f are both C^r-related to α, it follows from (R1) that both $\alpha_>(\mathrm{Dom}\,\chi \cap \mathrm{Dom}\,\alpha)$ and $\alpha_>(\mathrm{Dom}\,f \cap \mathrm{Dom}\,\alpha)$ are open subsets of $\mathrm{Pag}\alpha$ and hence that $\alpha_>(\mathcal{G})$ is also open in $\mathrm{Pag}\alpha$. Since $\alpha \,\square\, \chi^{\leftarrow}$ is continuous by (R2), it follows that $\chi_>(\mathcal{G}) = (\alpha \,\square\, \chi^{\leftarrow})^<(\alpha_>(\mathcal{G}))$ is an open neighborhood of $\chi(x)$ in $\mathrm{Pag}\,\chi$. Using (0.1) and (0.2) it is easily seen that

$$(f \,\square\, \chi^{\leftarrow})\big|_{\chi_>(\mathcal{G})} = (f \,\square\, \alpha^{\leftarrow})\big|_{\alpha_>(\mathcal{G})} \circ (\alpha \,\square\, \chi^{\leftarrow})\big|_{\chi_>(\mathcal{G})}^{\alpha_>(\mathcal{G})}.$$

Since both $f \,\square\, \alpha^{\leftarrow}$ and $\alpha \,\square\, \chi^{\leftarrow}$ are of class C^r by (R2), it follows from the chain rule that the restriction of $f \,\square\, \alpha^{\leftarrow}$ to a neighborhood $\chi_>(\mathcal{G})$ of $\chi(x)$ in $\mathrm{Pag}\,\chi$ is of class C^r. Since $x \in \mathrm{Dom}\,\chi \cap \mathrm{Dom}\,f$ was arbitrary, it follows that the domain $\chi_>(\mathrm{Dom}\,\chi \cap \mathrm{Dom}\,f)$ of $f \,\square\, \chi^{\leftarrow}$ is open in $\mathrm{Pag}\,\chi$ and that $f \,\square\, \chi^{\leftarrow}$ is of class C^r, *i.e.* that f is C^r-related to χ. ∎

We say that a C^r-atlas $\mathfrak{A}$ for $\mathcal{M}$ is **C^r-saturated** if every chart for $\mathcal{M}$ that is C^r-compatible with all charts in $\mathfrak{A}$ already belongs to $\mathfrak{A}$. The following is an immediate consequence of Prop. 1.

Proposition 2: *Let $\mathfrak{A}$ be a C^r-atlas for $\mathcal{M}$. Then there is exactly one saturated C^r-atlas $\overline{\mathfrak{A}}$ that includes $\mathfrak{A}$. In fact, $\overline{\mathfrak{A}}$ consists of all charts that are C^r-compatible with all charts in $\mathfrak{A}$.*

Definition: *Let $r \in \tilde{}$ be given. A C^r-**manifold** is a set $\mathcal{M}$ endowed with structure by the prescription of a saturated C^r-atlas for $\mathcal{M}$, which is called the* **chart-class** *of $\mathcal{M}$ and is denoted by $\mathrm{Ch}^r\mathcal{M}$, or if no confusion is likely, simply by $\mathrm{Ch}\mathcal{M}$.*

In view of Prop. 2, the structure of a C^r-manifold on $\mathcal{M}$ is uniquely determined by specifying a C^r-atlas included in $\mathrm{Ch}\mathcal{M}$. Of course, two different such atlases may determine one and the same C^r-structure.

Let $\mathcal{M}$ be a C^r-manifold with chart-class $\mathrm{Ch}^r\mathcal{M}$. Then, for every $s \in 0..r$, $\mathcal{M}$ has also the natural structure of a C^s-manifold, determined by $\mathrm{Ch}^r\mathcal{M}$ regarded as a C^s-atlas. Of course, the chart-class $\mathrm{Ch}^s\mathcal{M}$ of the C^s-manifold structure includes $\mathrm{Ch}^r\mathcal{M}$, but we have $\mathrm{Ch}^r\mathcal{M}\ \ \mathrm{Ch}^s\mathcal{M}$ if $s < r$.

Examples of manifold

Example 1: Let $\mathcal{D}$ be an open subset of a flat space. Then the singleton $\{\mathbf{1}_{\mathcal{D}}\}$ is a C^ω-atlas of $\mathcal{D}$. It determines on $\mathcal{D}$ a natural C^ω-structure and hence a natural C^r-structure for every $r \in$.

Example 2: (**Product manifold**) Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds of class C^r, then the product $\mathcal{M} \times \mathcal{N}$ has the natural structure of a C^r manifold. ∎

We now assume that a C^r-manifold $\mathcal{M}$ with chart-class $\mathrm{Ch}\mathcal{M}$ is given. We use the notation

$$\mathrm{Ch}_x\mathcal{M} := \left\{ \chi \in \mathrm{Ch}\mathcal{M} \;\middle|\; x \in \mathrm{Dom}\,\chi \right\} . \tag{21.4}$$

It is easily seen that the spaces $\mathrm{Pag}\,\chi$ and $\mathcal{V}_\chi$, $\chi \in \mathrm{Ch}_x\mathcal{M}$, all have the same dimension. This dimension is called the **dimension of $\mathcal{M}$ at** x, and is denoted by $\dim_x\mathcal{M}$.

The C^r-manifold $\mathcal{M}$ is endowed with a natural topology, namely the coarsest topology that renders all $\chi \in \mathrm{Ch}\mathcal{M}$ continuous. A subset $\mathcal{P}$ of $\mathcal{M}$ is open if and only if, for each $\chi \in \mathrm{Ch}\mathcal{M}$, the image $\chi_>(\mathcal{P} \cap \mathrm{Dom}\,\chi)$ is an open subset of $\mathrm{Pag}\,\chi$. Given $x \in \mathcal{M}$, one can construct a neighborhood-basis $\mathfrak{B}_x$ of x in $\mathcal{M}$ in the following manner: Choose a chart $\chi \in \mathrm{Ch}_x\mathcal{M}$ and a neighborhood-basis $\mathfrak{N}_{\chi(x)}$ of $\chi(x)$ in $\mathrm{Pag}\,\chi$. Then put

$$\mathfrak{B}_x := \left\{ \chi^<(\mathcal{N} \cap \mathrm{Cod}\,\chi) \;\middle|\; \mathcal{N} \in \mathfrak{N}_{\chi(x)} \right\} . \tag{21.5}$$

Pitfall: The natural topology of $\mathcal{M}$ need not be separating.

Let $\mathcal{P}$ be an open subset of $\mathcal{M}$. Then $\mathcal{P}$ has the natural structure of a C^r-manifold whose chart-class $\mathrm{Ch}\,\mathcal{P}$ is

$$\mathrm{Ch}\,\mathcal{P} := \left\{ \chi \in \mathrm{Ch}\mathcal{M} \;\middle|\; \mathrm{Dom}\,\chi \subset \mathcal{P} \right\}. \tag{21.6}$$

The natural topology of $\mathcal{P}$ as a C^r-manifold concides with the topology of $\mathcal{P}$ induced by the topology of $\mathcal{M}$.

Let f be a mapping whose domain is an open subset of $\mathcal{M}$ and whose codomain is an open subset $\mathcal{D}$ of a specified flat space $\mathcal{E}$ with translation space $\mathcal{V} := \mathcal{E} - \mathcal{E}$. We say that f is **of class** C^s, with $s \in 0..r$, if it is C^s-related to every chart $\chi \in \mathrm{Ch}\mathcal{M}$, i.e. if $f \circ \chi^{\leftarrow}$ is of class C^s for all charts $\chi \in \mathrm{Ch}\mathcal{M}$. (Since $\mathrm{Dom}\,f$ is open, $\mathrm{Dom}\,f \circ \chi^{\leftarrow} = \chi_>(\mathrm{Dom}\,\chi \cap \mathrm{Dom}\,f)$ is automatically open in $\mathrm{Pag}\,\chi$ when $\chi \in \mathrm{Ch}\mathcal{M}$.) It follows from Prop. 1 that f is of class C^s if $f \circ \chi^{\leftarrow}$ is of class C^s for every chart χ in some C^r-atlas included in $\mathrm{Ch}\mathcal{M}$. If f is of class C^s with $s \geq 1$ and if $\chi \in \mathrm{Ch}\mathcal{M}$, we define the **gradient**

$$\nabla_\chi f : \mathrm{Dom}\,\chi \cap \mathrm{Dom}\,f \to \mathrm{Lin}(\mathcal{V}_\chi, \mathcal{V})$$

of f in the chart χ by

$$(\nabla_\chi f)(x) := \nabla_{\chi(x)}(f \circ \chi^{\leftarrow}) \quad \text{for all} \quad x \in \mathrm{Dom}\,\chi \cap \mathrm{Dom}\,f. \tag{21.7}$$

More generally, for every $s \in 1..r$, the **gradient of order** s

$$\nabla_\chi^{(s)} f : \mathrm{Dom}\,\chi \cap \mathrm{Dom}\,f \to \mathrm{Sym}_s((\mathcal{V}_\chi)^s, \mathcal{V})$$

of f in the chart χ defined by

$$(\nabla_\chi^{(s)} f)(x) := \nabla_{\chi(x)}^{(s)}(f \circ \chi^{\leftarrow}) \quad \text{for all} \quad x \in \operatorname{Dom}\chi \cap \operatorname{Dom} f. \tag{21.8}$$

The following transformation rules are easy concequences of the rules of calculus.

Proposition 3: *Let f be a mapping of class C^1, $x \in \operatorname{Dom} f$ and $\chi, \gamma \in \mathrm{Ch}_x\mathcal{M}$. Then*

$$(\nabla_\gamma f)(x) = (\nabla_\chi f)(x)(\nabla_\gamma \chi)(x). \tag{21.9}$$

If f is also of class C^2, then

$$(\nabla_\gamma^{(2)} f)(x) = (\nabla_\chi^{(2)} f)(x) \circ (\nabla_\gamma \chi(x) \times \nabla_\gamma \chi(x)) + (\nabla_\chi f)(x) \nabla_\gamma^{(2)} \chi(x). \tag{21.10}$$

In the case when $f := \gamma$ the formulas (21.7) and (21.8) reduce to

$$(\nabla_\gamma \gamma)(x) = \mathbf{1}_{\mathcal{V}_\gamma} \qquad \text{and} \qquad (\nabla_\gamma^{(2)} \gamma)(x) = \mathbf{0}.$$

Hence Prop. 3 has the following consequence:

Proposition 4: *Let $x \in \mathcal{M}$ and $\chi, \gamma \in \mathrm{Ch}_x\mathcal{M}$ be given. If $r \geq 1$, then $(\nabla_\chi \gamma)(x) \in \operatorname{Lin}(\mathcal{V}_\chi, \mathcal{V}_\gamma)$ is invertible and*

$$(\nabla_\chi \gamma)(x)^{-1} = (\nabla_\gamma \chi)(x). \tag{21.11}$$

If $r \geq 2$, we also have

$$(\nabla_\gamma^{(2)} \chi)(x) = -(\nabla_\gamma \chi)(x)\big((\nabla_\chi^{(2)} \gamma)(x) \circ (\nabla_\gamma \chi(x) \times \nabla_\gamma \chi(x))\big). \tag{21.12}$$

If the manifold $\mathcal{M}$ is itself the underlying manifold of an open subset of a flat space (see Example 1 above), then a mapping f is of class C^s as described above if and only if it is of class C^s in the ordinary sence (see Notations).

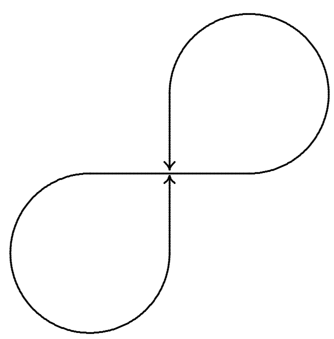
Let f be a mapping whose domain is a neighborhood of a given point $x \in \mathcal{M}$ and whose codomain is an open subset of a specified flat space. We say that f is **differentiable at** x if $f \circ \chi^{\leftarrow}$ is differentiable at $\chi(x)$ for some, and hence all, $\chi \in \mathrm{Ch}_x\mathcal{M}$. If this is the case, (21.7) remains meaningful for the given $x \in \mathcal{M}$ and the transformation formula (21.9) remains valid. The concept of "s times differentiable at x" when $s \in 0..r$ is defined in a similar way.

More generally, let C^r-manifolds $\mathcal{M}$ and $\mathcal{M}'$ be given. Let g be a mapping whose domain and codomain are open subsets of $\mathcal{M}$ and $\mathcal{M}'$, respectively. We say that g is **of class** C^s with $s \in 0..r$ if $\chi' \circ g \circ \chi^{\leftarrow}$ is of class C^s in the ordinary sense for all $\chi \in \mathrm{Ch}\mathcal{M}$ and all $\chi' \in \mathrm{Ch}\mathcal{M}'$.

<u>Definition</u>: *Let $\mathcal{M}$ be a C^r-manifold and let $\mathcal{P}$ be a subset of $\mathcal{M}$. We say that $\mathcal{P}$ is a* **submanifold** *of $\mathcal{M}$ if for each point $x \in \mathcal{P}$ there is a chart $\chi \in \mathrm{Ch}_x\mathcal{M}$ such that $\chi_>(\mathcal{P} \cap \mathrm{Dom}\,\chi)$ is an open subset of a flat $\mathcal{F}_\chi$ of* $\mathrm{Pag}\,\chi$.

Let $\mathcal{P}$ be a C^r submanifold of the manifold $\mathcal{M}$. We left it the readers to show that $\mathcal{P}$ has the natural structure of a C^r manifold. The natural topology of $\mathcal{P}$ as a C^r-manifold concides with the topology of $\mathcal{P}$ induced by the topology of $\mathcal{M}$, i.e. $\mathcal{P}$ a topological subspace of $\mathcal{M}$.

Let $f : \mathcal{S} \to \mathcal{M}$ be a C^s mapping from a manifold $\mathcal{S}$ to another manifold $\mathcal{M}$. The mapping f is called a C^s **immersion at** $x \in \mathcal{S}$ if there exists an open neighborhood $\mathcal{N}_x$ of x (in $\mathcal{S}$) such that the restriction $f|_{\mathcal{N}_x}$ is injective and $f_>(\mathcal{N}_x)$ is a submanifold of $\mathcal{M}$. We say that f is an **immersion** if it is an immersion at every $y \in \mathcal{S}$. If f is an immersion, the domain $\mathcal{S}$ called an **immersed manifold** of $\mathcal{M}$. However, being an immersion is a "local property" and hence the range $\mathrm{Rng}\,f := f_>(\mathcal{S})$ of f may not be a submanifold of $\mathcal{M}$. For example (see [L]):



Figue11.1

An injective immersion f from manifold $\mathcal{A}$ to manifold $\mathcal{B}$ is an **imbedding** if the range $\mathrm{Rng}\,f := f_>(\mathcal{A})$ is a submanifold of $\mathcal{B}$. The domain of an imbedding is called an **imbedded manifold** of its codomain manifold. It is clear that for every submanifold $\mathcal{P}$ of a given manifold $\mathcal{M}$ the inclusion $\mathbf{1}_{\mathcal{P}\subset\mathcal{M}}$ is an imbedding.

Remark: Let $\mathcal{A}$ and $\mathcal{B}$ be topological spaces and $f : \mathcal{A} \to \mathcal{B}$ be an injection. We say that f is an imbedding if the topology of $\mathcal{A}$ is induced by f from the topology of $\mathcal{B}$. ▮

More details on submanifolds

22. Bundles

We assume that $r \in {}^{\sim}$ with $r \geq 2$ and a C^r-manifold $\mathcal{M}$ are given. Let a number $s \in 0..r$ be given and let $\tau : \mathcal{B} \to \mathcal{M}$ be a surjective mapping from a given set $\mathcal{B}$ to the manifold $\mathcal{M}$.

Let a concrete isocategory ISO with object class *OBJ* be given with the following properties:

(i) Each set in *OBJ* has the natural structure of a C^s-manifold.

(ii) Every isomorphism in ISO is a C^s-diffeomorphism.

The most inportant special cases are (1) the isocategory of LIS consisting of all linear isomorphisms, whose object class *LS* consist of all (finite dimensional) linear spaces and (2) the isocategory of FIS consisting of all flat isomorphisms, whose object class *FS* consist of all flat spaces. The object sets in *LS* and *FS* have the natural structure of C^ω-manifolds and the isomorphisms in LIS and FIS are C^ω-diffeomorphisms.

<u>Definition</u>: *An* ISO-**bundle chart** *for $\mathcal{B}$ (for τ) is a bijection*

$$\phi : \tau^{<}(\mathcal{O}_\phi) \to \mathcal{O}_\phi \times \mathcal{V}_\phi,$$

where $\mathcal{O}_\phi$ is an open subset of $\mathcal{M}$ and $\mathcal{V}_\phi$ is a set in OBJ such that the diagram

$$\begin{array}{ccc} \tau^{<}(\mathcal{O}_\phi) & \xrightarrow{\phi} & \mathcal{O}_\phi \times \mathcal{V}_\phi \\ & {\scriptstyle \tau|^{\mathcal{O}_\phi}_{\tau^{<}(\mathcal{O}_\phi)}} \searrow & \big\downarrow {\scriptstyle \mathrm{ev}_1} \\ & & \mathcal{O}_\phi \end{array} \quad . \tag{22.1}$$

is commutative, i.e. $\mathrm{ev}_1 \circ \phi = \tau|^{\mathcal{O}_\phi}_{\tau^{<}(\mathcal{O}_\phi)}$.

<u>Notation</u>: *For every $y \in \mathcal{M}$, we denote $\mathcal{B}_y := \tau^{<}(\{y\})$ and for every* ISO-*bundle chart ϕ we use the following notations*

$$\phi\rfloor_y := \mathrm{ev}_2 \circ \phi \circ \left(\mathbf{1}_{\mathcal{B}_y \subset \tau^{<}(\mathcal{O}_\phi)}\right) : \mathcal{B}_y \to \mathcal{V}_\phi \tag{22.2}$$

for all $y \in \mathcal{O}_\phi$, i.e. we have the following commutative diagram

$$\begin{array}{ccccc} & & & & \mathcal{V}_\phi \\ & & \nearrow^{\phi\rfloor_y} & & \big\uparrow {\scriptstyle \mathrm{ev}_2} \\ \mathcal{B}_y & \hookrightarrow & \tau^{<}(\mathcal{O}_\phi) & \xrightarrow{\phi} & \mathcal{O}_\phi \times \mathcal{V}_\phi \end{array} \quad .$$

Put (22.1) and (22.2) together, we have the following commutative diagram

$$\begin{tikzcd}
 & & \mathcal{V}_\phi \\
\mathcal{B}_y \arrow[r, hook] \arrow[urr, "\phi\rfloor_y"] & \tau^<(\mathcal{O}_\phi) \arrow[r, "\phi"] \arrow[dr, "\tau\big|^{\mathcal{O}_\phi}_{\tau^<(\mathcal{O}_\phi)}"'] & \mathcal{O}_\phi \times \mathcal{V}_\phi \arrow[u, "\mathrm{ev}_2"'] \arrow[d, "\mathrm{ev}_1"] \\
 & & \mathcal{O}_\phi
\end{tikzcd} .$$

Let ϕ and ψ be ISO-bundle charts for $\mathcal{B}$. We say that ϕ and ψ are C^s-**compatible** if

$$\psi \,\square\, \phi^{\leftarrow} : (\mathcal{O}_\phi \cap \mathcal{O}_\psi) \times \mathcal{V}_\phi \to (\mathcal{O}_\phi \cap \mathcal{O}_\psi) \times \mathcal{V}_\psi \tag{22.3}$$

is a C^s-diffeomophism such that, for every $y \in \mathcal{O}_\phi \cap \mathcal{O}_\psi$, the mapping

$$\psi\big\rfloor_y \circ \phi\big\rfloor_y^{\leftarrow} : \mathcal{V}_\phi \to \mathcal{V}_\psi \tag{22.4}$$

belongs to ISO.

A class $\mathfrak{A}$ of ISO-bundle charts for $\mathcal{B}$ is called a C^s ISO-**bundle atlas** for $\mathcal{B}$ if

(BA1) every two ISO-bundle charts in $\mathfrak{A}$ are C^s-compatiable,

(BA2) for every $x \in \mathcal{M}$ there is a bundle chart $\phi \in \mathfrak{A}$ with $x \in \mathcal{O}_\phi$; i.e. we have

$$\mathcal{M} = \bigcup_{\phi \in \mathfrak{A}} \mathcal{O}_\phi \ .$$

Proposition 1: *Let $\mathfrak{A}$ be a* ISO-*bundle atlas for $\mathcal{B}$ and let ϕ be a* ISO-*bundle chart that is C^s-compatible with all* ISO-*bundle charts in $\mathfrak{A}$. If ψ is a* ISO-*bundle chart that is C^s-compatible with every* ISO-*bundle chart in $\mathfrak{A}$ then it is also C^s-compatible with ϕ.*

Proof: Let $x \in \mathcal{O}_\phi \cap \mathcal{O}_\psi$ be given. By (BA2), we may choose a ISO-bundle chart $\theta \in \mathfrak{A}$ such that $x \in \mathcal{O}_\theta$. Put $\mathcal{O} := \mathcal{O}_\phi \cap \mathcal{O}_\psi \cap \mathcal{O}_\theta$. Since both ϕ and ψ are C^s-compatible with θ, we see that the restriction

$$\psi \,\square\, \phi^{\leftarrow}\Big|_{\phi(\tau^<\{\mathcal{O}\})} = (\psi \,\square\, \theta^{\leftarrow})\Big|_{\theta(\tau^<\{\mathcal{O}\})} \circ (\theta \,\square\, \phi^{\leftarrow})\Big|^{\theta(\tau^<\{\mathcal{O}\})}_{\phi(\tau^<\{\mathcal{O}\})}$$

on $\phi(\tau^<\{\mathcal{O}\})$ is a C^s-diffeomorphism and the induced mapping

$$\psi\big\rfloor_x \circ \phi\big\rfloor_x^{\leftarrow} = (\psi\big\rfloor_x \circ \theta\big\rfloor_x^{\leftarrow}) \circ (\theta\big\rfloor_x \circ \phi\big\rfloor_x^{\leftarrow})$$

is a ISO-isomorphism. Since $x \in \mathcal{O}_\phi \cap \mathcal{O}_\psi$ was arbitrary, we conclude that ψ and ϕ are C^s-compatible. ▮

We say that a ISO-bundle atlas $\mathfrak{A}$ of $\mathcal{B}$ is **C^s-saturated** if every ISO-bundle chart for $\mathcal{B}$ that is C^s-compatible with all ISO-bundle charts in $\mathfrak{A}$ already belongs to $\mathfrak{A}$. The following is an immediate consequence of Prop. 1.

Proposition 2: *Let $\mathfrak{A}$ be a C^s ISO-bundle atlas for $\mathcal{B}$. Then there is exactly one C^s-saturated ISO-bundle atlas $\overline{\mathfrak{A}}$ that includes $\mathfrak{A}$. In fact, $\overline{\mathfrak{A}}$ consists of all ISO-bundle charts that are C^s-compatible with all ISO-bundle charts in $\mathcal{B}$.*

Let $\mathfrak{A}$ be a saturated ISO-atlas for $\mathcal{B}$ and let ϕ be a ISO-bundle chart in $\mathfrak{A}$. On each fibre $\mathcal{B}_x$, $x \in \mathcal{O}_\phi$, we can transport the ISO-structure of $\mathcal{V}_\phi$ by means of $\phi\rfloor_x : \mathcal{B}_x \to \mathcal{V}_\phi$. The result is independent of the choice of ϕ, since every pair of bundle charts ϕ and ψ in $\mathfrak{A}$ are compatible and hence $\psi\rfloor_x \circ \phi\rfloor_x^{\leftarrow} : \mathcal{V}_\phi \to \mathcal{V}_\psi$ is a ISO-isomorphism.

<u>**Definition:**</u> *A C^s ISO-**bundle** over $\mathcal{M}$ is a set $\mathcal{B}$ and a mapping $\tau : \mathcal{B} \to \mathcal{M}$ endowed with structure by the prescription of a saturated C^s ISO-bundle atlas for $\mathcal{B}$, which is called the **bundle structure** for $\mathcal{B}$ and is denoted by $\mathrm{Ch}^s(\mathcal{B}, \mathcal{M})$, or if no confusion is likely, simply by $\mathrm{Ch}(\mathcal{B}, \mathcal{M})$. We denote the ISO-bundle by $(\mathcal{B}, \tau, \mathcal{M})$ or simply by $\mathcal{B}$.*

The mapping τ is called the **bundle-projection**. For every $x \in \mathcal{M}$, $\mathcal{B}_x := \tau^<(\{x\})$ is called the **fiber over** x and the inclusion mapping of $\mathcal{B}_x$ in $\mathcal{B}$ is called the **bundle inclusion at** x. Right inverses of τ are called **cross sections of** $\mathcal{B}$. We also use the following notation

$$\mathrm{Ch}_x(\mathcal{B}, \mathcal{M}) := \{\, \phi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M}) \mid x \in \mathcal{O}_\phi \,\}. \tag{22.5}$$

As explained above, for every $x \in \mathcal{M}$, the fiber $\mathcal{B}_x$ is naturally endowed with the structure of a ISO-set in such a way that $\phi\rfloor_x : \mathcal{B}_x \to \mathcal{V}_\phi$ is in ISO (is an isomorphism) for all $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$. Thus the dimension of $\mathcal{B}_x$ can be obtained from all $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$.

Locally (relative to $\mathcal{M}$), the manifold structure of the **bundle manifold** $\mathcal{B}$ is completely determined by the manifold structure of the **base manifold** $\mathcal{M}$ and the manifold structures of $\mathcal{V}_\phi$ for a single $\phi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})$. Every bundle chart ϕ in $\mathrm{Ch}(\mathcal{B}, \mathcal{M})$ transports the manifold structure from $\mathcal{O}_\phi \times \mathcal{V}_\phi$ to $\tau^<(\mathcal{O}_\phi)$, and hence a manifold chart can be easily obtained from ϕ.

Let $\mathbf{b} \in \mathcal{B}$ be given and put $x := \tau(\mathbf{b})$. The dimension of $\mathcal{B}$ at $\mathbf{b}$ can be obtained from the codomain of each bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$. We have

$$\dim_{\mathbf{b}} \mathcal{B} = m + n,$$

where $\dim_x\mathcal{M} = m$ and $\dim_{\mathbf{b}}\mathcal{B}_x = n$.

Let ISO-bundles $(\mathcal{B}', \tau', \mathcal{M}')$ and $(\mathcal{B}, \tau, \mathcal{M})$ be given. We say that $(\mathcal{B}', \tau', \mathcal{M}')$ is a ISO-**subbundle** of $(\mathcal{B}, \tau, \mathcal{M})$ provided $\mathcal{B}'$ is a submanifold of $\mathcal{B}$, $\mathcal{M}'$ is a submanifold of $\mathcal{M}$ and $\tau' = \tau\big|_{\mathcal{B}'}^{\mathcal{M}'}$ such that, for each bundle chart $\varphi \in \mathrm{Ch}(\mathcal{B}', \mathcal{M}')$, we have $\varphi = \phi\big|_{\mathrm{Dom}\,\varphi}^{\mathrm{Cod}\,\varphi}$ for some bundle chart $\phi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})$.

It is easily seen that for every open subset $\mathcal{P}$ of $\mathcal{M}$, $\left(\tau^{<}(\mathcal{P}),\, \tau\big|_{\tau^{<}(\mathcal{P})}^{\mathcal{P}},\, \mathcal{P}\right)$ is an open subbundle of $(\mathcal{B}, \tau, \mathcal{M})$.

<u>**Definition**</u>**:** *A* **cross section on** $\mathcal{O}$ *of* $\mathcal{B}$*, where* $\mathcal{O}$ *is an open submanifold of* $\mathcal{M}$*, is a mapping* $\mathbf{s} : \mathcal{O} \to \mathcal{B}$ *such that* $\tau \circ \mathbf{s} = \mathbf{1}_{\mathcal{O}\subset\mathcal{M}}$*. For every* $p \in 0..s$*, we denote the collection of all* C^p *cross sections of* $\mathcal{B}$ *by* $\mathrm{Sec}^p\mathcal{B}$*.*

If ISO is the category DIF_s that consists of all C^s-diffeomorphisms between C^s manifolds, we call $\mathcal{B}$ a C^s-**bundle**. If ISO = FIS, we call $\mathcal{B}$ a **flat-space bundle**. If ISO = LIS, we call $\mathcal{B}$ a **linear-space bundle**.

Proposition 3: *Let* $\mathcal{D}$ *be an open subset of a flat space* $\mathcal{E}$ *and let* $\mathcal{V}$*,* $\mathcal{W}$ *be linear spaces. Let* $F : \mathcal{D} \to \mathrm{Lin}(\mathcal{V}, \mathcal{W})$ *be given. If* $f : \mathcal{D} \times \mathcal{V} \to \mathcal{W}$ *is defined by*

$$f(x, \mathbf{v}) := F(x)\mathbf{v} \qquad \text{for all} \quad (x, \mathbf{v}) \in \mathcal{D} \times \mathcal{V} \tag{22.6}$$

then f *is of class* C^p*,* $p \in$ *, if and only if* F *is of class* C^p*.*

Proof: The assertion follows from the Partial Gradient Theorem [FDS]. ∎

If $\mathcal{B}$ is a linear-space bundle, then it follows from (22.3), (22.4) and Prop. 3 that for every pair of bundle charts $\phi, \psi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})$, the mapping

$$\psi \diamond \phi : \mathcal{O}_\phi \cap \mathcal{O}_\psi \to \mathrm{Lin}(\mathcal{V}_\phi, \mathcal{V}_\psi)$$

defined by

$$(\psi \diamond \phi)(x) := \psi\big\rfloor_x \circ \phi\big\rfloor_x^{-1} \quad \text{for all} \quad x \in \mathcal{O}_\phi \cap \mathcal{O}_\psi \tag{22.7}$$

is of class C^s.

Before closing this section, we give two examples of constructing a new bundle from given ones. We omit the details.

Examples :

(1) **Trivial bundles :** $\mathcal{M} \times \mathcal{G}$, where $\mathcal{G} \in OBJ$. The fiber $\mathcal{B}_x = \{x\} \times \mathcal{G}$ at $x \in \mathcal{M}$ is $\mathcal{G}$ tagged with x. ∎

(2) **Fiber-product bundles :** Let two bundles $(\mathcal{A},\alpha,\mathcal{M})$ and $(\mathcal{B},\beta,\mathcal{M})$ over the same base manifold $\mathcal{M}$ be given. Put

$$\begin{aligned}\mathcal{A}\times_{\mathcal{M}}\mathcal{B} &:= \bigcup_{x\in\mathcal{M}} \mathcal{A}_x\times\mathcal{B}_x \\ \alpha\times_{\mathcal{M}}\beta &:= \alpha\circ \mathrm{ev}_1 = \beta\circ\mathrm{ev}_2\end{aligned} \quad ; \quad \begin{array}{ccc} \mathcal{A}\times_{\mathcal{M}}\mathcal{B} & \xrightarrow{\mathrm{ev}_2} & \mathcal{B} \\ {\scriptstyle \mathrm{ev}_1}\downarrow & & \downarrow{\scriptstyle \beta} \\ \mathcal{A} & \xrightarrow[\alpha]{} & \mathcal{M} \end{array} . \tag{22.8}$$

The bundle $(\,\mathcal{A}\times_{\mathcal{M}}\mathcal{B}\,,\,\alpha\times_{\mathcal{M}}\beta\,,\,\mathcal{M}\,)$ is called the **fiber-product bundle** of $(\mathcal{A},\alpha,\mathcal{M})$ and $(\mathcal{B},\beta,\mathcal{M})$. The bundle projection $\alpha\times_{\mathcal{M}}\beta : \mathcal{A}\times_{\mathcal{M}}\mathcal{B}\to\mathcal{M}$ is given by

$$\alpha\times_{\mathcal{M}}\beta(\mathbf{v}) :\in \big\{y \,\big|\, \mathbf{v}\in\mathcal{A}_y\times\mathcal{B}_y\big\}. \tag{22.9}$$

Let bundle charts $\phi\in\mathrm{Ch}(\mathcal{A},\mathcal{M})$ and $\psi\in\mathrm{Ch}(\mathcal{B},\mathcal{M})$ be given. The mapping

$$\phi\times_{\mathcal{M}}\psi : (\tau_1\times_{\mathcal{M}}\tau_2)^{<}(\mathcal{O}_\phi\cap\mathcal{O}_\psi)\to(\mathcal{O}_\phi\cap\mathcal{O}_\psi)\times(\mathcal{V}_\phi\times\mathcal{V}_\psi) \tag{22.10}$$

given by

$$\phi\times_{\mathcal{M}}\psi(\mathbf{v}) = \big(\,y\,,\,(\phi\rfloor_y\times\psi\rfloor_y)\mathbf{v}\,\big) \quad \text{for all} \quad \mathbf{v}\in\mathcal{A}\times_{\mathcal{M}}\mathcal{B} \tag{22.11}$$

is a bundle chart for $(\,\mathcal{A}\times_{\mathcal{M}}\mathcal{B}\,,\,\phi\times_{\mathcal{M}}\psi\,,\,\mathcal{M}\,)$. ∎

23. The tangent bundle

Let $r \in^{\times}$, a C^r-manifold $\mathcal{M}$, and a point $x \in \mathcal{M}$ be given.

Definition: *The* **tangent space** *of* $\mathcal{M}$ *at* x *is defined to be*

$$\mathrm{T}_x\mathcal{M} := \left\{ \mathbf{t} \in \bigtimes_{\alpha \in \mathrm{Ch}_x\mathcal{M}} \mathcal{V}_\alpha \;\middle|\; (23.2) \textit{ holds} \right\}, \tag{23.1}$$

where the condition (23.2) *is given by*

$$\mathbf{t}_\gamma = \nabla_\chi \gamma(x)\, \mathbf{t}_\chi \quad \text{for all} \quad \chi, \gamma \in \mathrm{Ch}_x\mathcal{M}. \tag{23.2}$$

$\mathrm{T}_x\mathcal{M}$ *is endowed with the natural structure of a linear space as shown below and* $\dim \mathrm{T}_x\mathcal{M} = \dim_x\mathcal{M}$.

For every $\chi \in \mathrm{Ch}_x\mathcal{M}$, define the evaluation mapping $\mathrm{ev}_\chi : \mathrm{T}_x\mathcal{M} \to \mathcal{V}_\chi$ by

$$\mathrm{ev}_\chi(\mathbf{t}) := \mathbf{t}_\chi \qquad \text{for all} \qquad \mathbf{t} \in \mathrm{T}_x\mathcal{M}.$$

It follows from (21.10) that the evaluation mapping ev_χ is invertible and that its inverse $\mathrm{ev}_\chi^{\leftarrow} : \mathcal{V}_\chi \to \mathrm{T}_x\mathcal{M}$ is given by

$$(\mathrm{ev}_\chi^{\leftarrow})(\mathbf{u}) = \big(\, \nabla_\chi \alpha(x)\mathbf{u} \;\big|\; \alpha \in \mathrm{Ch}_x\mathcal{M} \,\big) \quad \text{for all} \quad \mathbf{u} \in \mathcal{V}_\chi.$$

Hence we have

$$\mathrm{ev}_\chi \circ \mathrm{ev}_\gamma^{\leftarrow} = \nabla_\gamma \chi(x) \;\in\; \mathrm{Lis}\,(\mathcal{V}_\gamma, \mathcal{V}_\chi) \tag{23.3}$$

for all $\gamma, \chi \in \mathrm{Ch}_x\mathcal{M}$. It follows from that the linear-space structure on $\mathrm{T}_x\mathcal{M}$ obtained from that of $\mathcal{V}_\chi$ by ev_χ does not depend on the choice of $\chi \in \mathrm{Ch}_x\mathcal{M}$ and hence is intrinsic to $\mathrm{T}_x\mathcal{M}$. We consider $\mathrm{T}_x\mathcal{M}$ to be endowed with this structure.

Let f be a mapping whose domain $\mathcal{D}$ is a neighborhood of x in $\mathcal{M}$ and whose codomain is an open subset of a flat space with translation space $\mathcal{V}$. It follows from (23.3) and (21.7) that

$$\nabla_\chi f(x) \circ \mathrm{ev}_\chi \;\in\; \mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathcal{V})$$

is the same for all $\chi \in \mathrm{Ch}_x\mathcal{M}$. Hence we may define the **gradient of** f **at** x by

$$\nabla_x f := \nabla_\chi f(x) \circ \mathrm{ev}_\chi \;\in\; \mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathcal{V}) \tag{23.4}$$

for all $\chi \in \mathrm{Ch}_x\mathcal{M}$. In particular, if we put $f := \chi$ we get $\nabla_x \chi = \mathrm{ev}_\chi$ and hence

$$(\nabla_x \chi)\, \mathbf{t} = \mathbf{t}_\chi \qquad \text{for all} \qquad \chi \in \mathrm{Ch}_x\mathcal{M}. \tag{23.5}$$

Also, if f is given as above, we have

$$\nabla_x f = \nabla_\chi f(x) \nabla_x \chi \qquad \text{for all} \qquad \chi \in \mathrm{Ch}_x\mathcal{M}. \tag{23.6}$$

Let $\mathcal{P}$ be an open neighborhood of x in $\mathcal{M}$. By (21.6) we have $\mathrm{Ch}_x\mathcal{P} \subset \mathrm{Ch}_x\mathcal{M}$ and the mapping

$$(\mathbf{t} \mapsto \mathbf{t}|_{\mathrm{Ch}_x\mathcal{P}}) : \mathrm{T}_x\mathcal{M} \to \mathrm{T}_x\mathcal{P}$$

is a natural bijection; we use it to identify

$$\mathrm{T}_x\mathcal{P} \cong \mathrm{T}_x\mathcal{M}. \tag{23.7}$$

<u>Definition</u>: *The* **tangent bundle** $\mathrm{T}\mathcal{M}$ *of* $\mathcal{M}$ *is defined to be the union of all the tangent spaces of* $\mathcal{M}$:

$$\mathrm{T}\mathcal{M} := \bigcup_{x\in\mathcal{M}} \mathrm{T}_x\mathcal{M}. \tag{23.8}$$

It is endowed with the natural structure of a C^{r-1}*-linear-space bundle as shown below.*

In view of the identifications (23.7) we may regard $\mathrm{T}\mathcal{P}$ as a subset of $\mathrm{T}\mathcal{M}$ when $\mathcal{P}$ is an open subset of $\mathcal{M}$.

Let $\mathcal{D}$ be an open subset of a flat space $\mathcal{E}$ with translation space $\mathcal{V} := \mathcal{E} - \mathcal{E}$. Then the singleton $\{\mathbf{1}_\mathcal{D}\}$ is a C^ω-atlas of $\mathcal{D}$. It determines on $\mathcal{D}$ a natural C^ω-manifold structure and hence a natural C^r-manifold structure for every $r \in$. Given $x \in \mathcal{D}$, the linear isomorphism $\mathrm{ev}_{\mathbf{1}_\mathcal{D}} : \mathrm{T}_x\mathcal{D} \to \mathcal{V}$ will be used for the identification

$$\mathrm{T}_x\mathcal{D} \cong \{x\} \times \mathcal{V}. \tag{23.9}$$

Let f be a mapping whose domain is an open neighborhood of x and whose codomain is an open subset of a flat space $\mathcal{E}'$ with translation space $\mathcal{V}'$. If f is differentiable at $x \in \mathcal{D}$ then the gradient $\nabla_x f$ in the ordinary sense of (23.4) belongs to $\mathrm{Lin}(\{x\} \times \mathcal{V}, \mathcal{V}')$ when the identification (23.9) is used. No confusion is likely because we have

$$\nabla_x f(x, \mathbf{v}) = \nabla_x f\, \mathbf{v} \qquad \text{for all} \qquad \mathbf{v} \in \mathcal{V} \tag{23.10}$$

when $\nabla_x f$ is used with both meanings.

If $\mathcal{D}$ is the underlying manifold of an open subset of a flat space, then (23.9) gives rise to the idetification

$$\mathrm{T}\mathcal{D} \cong \mathcal{D} \times \mathcal{V}. \tag{23.11}$$

Note that the family $(\,\mathrm{T}_x\mathcal{M} \mid x \in \mathcal{M}\,)$ is disjoint. The **bundle projection** $\mathrm{pt} : \mathrm{T}\mathcal{M} \to \mathcal{M}$ of the tangent bundle is given by

$$\mathrm{pt}(\mathbf{t}) :\in \left\{\, x \in \mathcal{M} \;\middle|\; \mathbf{t} \in \mathrm{T}_x\mathcal{M} \,\right\}. \tag{23.12}$$

Every manifold chart $\chi \in \mathrm{Ch}\mathcal{M}$ induces a bundle chart for $\mathrm{T}\mathcal{M}$ as shown in the following. We define the **tangent-bundle chart**

$$\mathrm{tgt}_\chi : \mathrm{pt}^<(\mathrm{Dom}\ \chi) \to \mathrm{Dom}\ \chi \times \mathcal{V}_\chi \tag{23.13}$$

by

$$\mathrm{tgt}_\chi(\mathbf{t}) = (\, z\,,\ (\nabla_z \chi)\,\mathbf{t}\,) \qquad \text{where} \quad z := \mathrm{pt}(\mathbf{t}). \tag{23.14}$$

It is easily seen that tgt_χ is invertible and that

$$\mathrm{tgt}_\chi^{\leftarrow}(z, \mathbf{u}) = (\nabla_z \chi)^{-1}\mathbf{u} \tag{23.15}$$

for all $z \in \mathrm{Dom}\chi$ and all $\mathbf{u} \in \mathcal{V}_\chi$. Let $\chi, \gamma \in \mathrm{Ch}\mathcal{M}$ be given. It follows from (21.7) and (23.6) that

$$\nabla_{\chi(z)}(\gamma \,\square\, \chi^{\leftarrow}) = (\nabla_\chi \gamma)(z) = (\nabla_z \gamma)(\nabla_z \chi)^{-1} \tag{23.16}$$

for all $z \in \mathrm{Dom}\ \gamma \cap \mathrm{Dom}\ \chi$. Hence, by (23.14) and (23.15) with χ replaced by γ, we have

$$(\mathrm{tgt}_\gamma \,\square\, \mathrm{tgt}_\chi^{\leftarrow})(z, \mathbf{u}) = \big(\ z\ ,\ \nabla_{\chi(z)}(\gamma \,\square\, \chi^{\leftarrow})\,\mathbf{u}\ \big) \tag{23.17}$$

for all $z \in \mathrm{Dom}\,\gamma \cap \mathrm{Dom}\,\chi$ and all $\mathbf{u} \in \mathcal{V}_\chi$. It is clear that $\mathrm{tgt}_\gamma \,\square\, \mathrm{tgt}_\chi^{\leftarrow}$ is of class C^{r-1}. Since $\chi, \gamma \in \mathrm{Ch}\mathcal{M}$ were arbitrary, it follows from (23.17) that

$$\{\ \mathrm{tgt}_\alpha \ \big|\ \alpha \in \mathrm{Ch}\mathcal{M}\ \}$$

is a C^{r-1} bundle-atlas of $\mathrm{T}\mathcal{M}$. We consider $\mathrm{T}\mathcal{M}$ has being endowed with the C^{r-1} linear space bundle structure determined by this atlas.

It is also easily seen that $\{\, (\alpha \times \mathbf{1}_{\mathcal{V}_\alpha}) \circ \mathrm{tgt}_\alpha \,|\, \alpha \in \mathrm{Ch}\mathcal{M}\,\}$ is a C^{r-1} manifold-atlas of $\mathrm{T}\mathcal{M}$. If $\chi \in \mathrm{Ch}\mathcal{M}$ then the page of the manifold chart $(\chi \times \mathbf{1}_{\mathcal{V}_\chi}) \circ \mathrm{tgt}_\chi$ is

$$\mathrm{Pag}\ ((\chi \times \mathbf{1}_{\mathcal{V}_\chi}) \circ \mathrm{tgt}_\chi) = \mathrm{Pag}\ \chi \times \mathcal{V}_\chi \tag{23.18}$$

and we have

$$\mathcal{V}_{(\chi \times \mathbf{1}_{\mathcal{V}_\chi}) \circ \mathrm{tgt}_\chi} = (\mathcal{V}_\chi)^2 \tag{23.19}$$

and hence

$$\dim_{\mathbf{t}} \mathrm{T}\mathcal{M} = 2 \dim_{\mathrm{pt}(\mathbf{t})} \mathcal{M} \quad \text{for all} \quad \mathbf{t} \in \mathrm{T}\mathcal{M}. \tag{23.20}$$

It is easily seen that the bundle projection $\mathrm{pt} : \mathrm{T}\mathcal{M} \to \mathcal{M}$ defined by (23.12) is of class C^{r-1}.

Let $r \in$ and C^r-manifolds $\mathcal{M}$ and $\mathcal{M}'$ be given. Let g be a mapping whose domain and codomain are open subsets of $\mathcal{M}$ and $\mathcal{M}'$, respectively. We say that g is **of class** C^s with $s \in 0..r$ if $\chi' \,\square\, g \,\square\, \chi^{\leftarrow}$ is of class C^s in the ordinary sense for all $\chi \in \mathrm{Ch}\mathcal{M}$ and all $\chi' \in \mathrm{Ch}\mathcal{M}'$. This is the case if and only if $\chi' \,\square\, g$ is of class C^s in the sense of Sect.21 for all $\chi' \in \mathrm{Ch}\mathcal{M}'$. Also, g is of class C^s if $\chi' \,\square\, g \,\square\, \chi^{\leftarrow}$ is of class C^s for all χ in some atlas included in $\mathrm{Ch}\mathcal{M}$ and for all

χ' in some atlas included in $\mathrm{Ch}\mathcal{M}'$. The notion of differentiablity of g is defined in a similar way.

Assume that g is differentiable at $x \in \mathcal{M}$. It follows from (23.16) that

$$\nabla_x g := (\nabla_{g(x)}\chi')^{-1}\nabla_{\chi(x)}(\chi' \circ g \circ \chi^{\leftarrow})\nabla_x\chi \tag{23.21}$$

does not depend on the choice of $\chi \in \mathrm{Ch}_x\mathcal{M}$ and $\chi' \in \mathrm{Ch}_{g(x)}\mathcal{M}'$. We call

$$\nabla_x g \ \in \ \mathrm{Lin}\,(\,\mathrm{T}_x\mathcal{M}\,,\,\mathrm{T}_{g(x)}\mathcal{M}'\,) \tag{23.22}$$

the **gradient of** g **at** x. Appropriate versions of the chain rule apply to gradients in this sense. If $\mathcal{M}'$ is an open subset of a flat space $\mathcal{E}'$ with translation space $\mathcal{V}'$, then the gradient $\nabla_x g$ in the sense of (23.22) is related to the gradient $\nabla_x g$ in the sense of (23.4) by

$$(\nabla_x g)\mathbf{t} = (\,g(x)\,,\,(\nabla_x g)\mathbf{t}\,) \qquad \text{for all} \qquad \mathbf{t} \in \mathrm{T}_x\mathcal{M} \tag{23.23}$$

when the identification $\mathrm{T}_{g(x)}\mathcal{M}' \cong \{g(x)\} \times \mathcal{V}'$ is used.

Definition: *A mapping* $\mathbf{h} : \mathcal{M} \to \mathrm{T}\mathcal{M}$ *is called a* **vector-field** *on* $\mathcal{M}$ *if it is a right-inverse of* pt*, i.e. if*

$$\mathbf{h}(x) \in \mathrm{T}_x\mathcal{M} \qquad \text{for all} \qquad x \in \mathcal{M}. \tag{23.24}$$

If $\mathbf{h}$ and $\mathbf{k}$ are vector-fields, then $\mathbf{h}+\mathbf{k}$ is the vector-field defined by value-wise addition, i.e. by $(\mathbf{h}+\mathbf{k})(x) := \mathbf{h}(x)+\mathbf{k}(x)$ for all $x \in \mathcal{M}$. If $\mathbf{h}$ is a vector-field and f a real-valued function on $\mathcal{M}$ (often called a "scalar-field"), then $f\,\mathbf{h}$ is defined by value-wise sacalar multiplication, *i.e.* by $(f\,\mathbf{h})(x) := f(x)\mathbf{h}(x)$ for all $x \in \mathcal{M}$.

The set of all real-valued functions of class C^s, $s \in 0\,..(r-1)$, on $\mathcal{M}$ will be denoted by $\mathrm{C}^s(\mathcal{M})$. The set of all vector-fields of class C^s, $s \in 0\,..(r-1)$, on $\mathcal{M}$ will be denoted by $\mathfrak{X}^s(\mathrm{T}\mathcal{M})$. Using value-wise addition and mutiplication, $\mathrm{C}^s(\mathcal{M})$ acquires the natural structure of a commutative algebra over . The constants form a subalgebra of $\mathrm{C}^s(\mathcal{M})$ that is isomorphic to . Using value-wise addition and mutiplication, $\mathfrak{X}^s(\mathrm{T}\mathcal{M})$ acquires the natural structure of a $\mathrm{C}^s(\mathcal{M})$-module.

Let $\mathbf{h} : \mathcal{M} \to \mathrm{T}\mathcal{M}$ be a vector-field and $\chi \in \mathrm{Ch}\mathcal{M}$. Define $\mathbf{h}^\chi : \mathrm{Dom}\chi \to \mathcal{V}_\chi$ by

$$\mathbf{h}^\chi(y) := (\nabla_y\chi)\mathbf{h}(y) \quad \text{foa all} \quad y \in \mathrm{Dom}\chi. \tag{23.25}$$

Given $x \in \mathrm{Dom}\,\chi$, we define

$$\nabla_x^\chi\mathbf{h} := (\nabla_x\chi)^{-1}\nabla_x\mathbf{h}^\chi \in \mathrm{Lin}\,\mathrm{T}_x\mathcal{M}. \tag{23.26}$$

It is easily seen from $(\nabla_x \chi)^{-1} \nabla_x \mathbf{h}^\chi = (\nabla_x \chi)^{-1} \big(\nabla_\chi \mathbf{h}^\chi(x)\big) \nabla_x \chi$ that $\nabla_x^\chi \mathbf{h}$ is simply the ordinary gradient of $\mathbf{h}^\chi$ in the chart χ, transported from $\operatorname{Lin} \mathcal{V}_\chi$ to $\operatorname{Lin} \mathrm{T}_x\mathcal{M}$ by $\nabla_x \chi$.

A continuous mapping $p : J \to \mathcal{M}$ from some genuine interval $J \in$ into the manifold $\mathcal{M}$ will be called a **process**. If p is differentiable at a given $t \in J$, then

$$\partial_t p := (\nabla_{p(x)} \chi)^{-1} \partial_t (\chi \circ p) \tag{23.27}$$

does not depend on the choice of $\chi \in \mathrm{Ch}_{p(t)}\mathcal{M}$. We call $\partial_t p \in \mathrm{T}_{p(t)}\mathcal{M}$ the **derivative of** p **at** t. If p is differentiable, we define the **derivative** (-process) $p^{\cdot} : J \to \mathrm{T}\mathcal{M}$ by

$$p^{\cdot}(t) := \partial_t\, p \qquad \text{for all} \qquad t \in J. \tag{23.28}$$

24. Tensor Bundles

We now assume that a number $s \in^{\sim}$and a C^s linear-space bundle $(\mathcal{B}, \tau, \mathcal{M})$ are given.

With each analytic tensor functor $\mathbf{\Phi}$ one can construct what is called the associated **$\mathbf{\Phi}$-bundle** of $\mathcal{B}$

$$\mathbf{\Phi}(\mathcal{B}) := \bigcup_{y \in \mathcal{M}} \mathbf{\Phi}(\mathcal{B}_y). \tag{24.1}$$

It has the natural structure of a C^s linear-space bundle over $\mathcal{M}$. For every open subset $\mathcal{P}$ of $\mathcal{M}$, we also use the following notation

$$\mathbf{\Phi}(\tau^{<}(\mathcal{P})) := \bigcup_{y \in \mathcal{P}} \mathbf{\Phi}(\mathcal{B}_y). \tag{24.2}$$

We define the **bundle projection** $\tau^{\mathbf{\Phi}} : \mathbf{\Phi}(\mathcal{B}) \to \mathcal{M}$ of the bundle $\mathbf{\Phi}(\mathcal{B})$ by

$$\tau^{\mathbf{\Phi}}(\mathbf{v}) :\in \big\{\ y \in \mathcal{M}\ \big|\ \mathbf{v} \in \mathbf{\Phi}(\mathcal{B}_y)\ \big\}. \tag{24.3}$$

For every bundle chart $\phi : \tau^{<}(\mathcal{O}_\phi) \to \mathcal{O}_\phi \times \mathcal{V}_\phi$, we have

$$\phi(\mathbf{v}) = \big(\ y\ ,\ \phi\rfloor_y(\mathbf{t})\ \big) \qquad \text{where} \quad y := \tau(\mathbf{t})$$

We define the mapping

$$\mathbf{\Phi}(\phi) : \mathbf{\Phi}(\pi^{<}(\mathcal{O}_\phi)) \to \mathcal{O}_\phi \times \mathbf{\Phi}(\mathcal{V}_\phi) \tag{24.4}$$

by

$$(\mathbf{\Phi}(\phi))(\mathbf{v}) := (\ y\ ,\ \mathbf{\Phi}(\phi\rfloor_y)\mathbf{v}\ ) \qquad \text{when} \quad y := \tau^{\mathbf{\Phi}}(\mathbf{v}). \tag{24.5}$$

It follows from the analyticity of the mapping $(\mathbf{L} \mapsto \mathbf{\Phi}(\mathbf{L}))$ that

$$\{\ \mathbf{\Phi}(\phi)\ \big|\ \phi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})\ \}$$

is a C^s-bundle-atlas of $\mathbf{\Phi}(\mathcal{B})$. It determines the C^s linear-space bundle structure of $(\mathbf{\Phi}(\mathcal{B}), \tau^{\mathbf{\Phi}}, \mathcal{M})$.

The bundle projection $\tau^{\mathbf{\Phi}} : \mathbf{\Phi}(\mathcal{B}) \to \mathcal{M}$ defined by (24.3) is easily seen to be of class C^s.

Notation: *For every $p \in 0..s$, we denote the collection of all C^p cross sections of $\mathbf{\Phi}(\mathcal{B})$ by $\mathfrak{X}^p(\mathbf{\Phi}(\mathcal{B}))$. The collection of all differentiable cross sections of $\mathbf{\Phi}(\mathcal{B})$ is denoted by $\mathfrak{X}(\mathbf{\Phi}(\mathcal{B}))$.*

In the special case $\mathcal{B} = \mathrm{T}\mathcal{M}$, we call $\mathbf{\Phi}(\mathrm{T}\mathcal{M})$ the **tansor bundle** of $\mathcal{M}$ of type $\mathbf{\Phi}$. A cross section of the tensor bundle $\mathbf{\Phi}(\mathrm{T}\mathcal{M})$ is called a **tensor-field** of type $\mathbf{\Phi}$. When $\mathbf{\Phi} := \mathrm{Dl}$ is the duality functor (see Sect.13), we call $\mathrm{Dl}\,(\mathrm{T}\mathcal{M})$ the **cotangent bundle** of $\mathcal{M}$ which will be denoted by $\mathrm{T}^*\mathcal{M}$.

Remark: Let $\mathcal{M}$ be a C^∞-manifold. With every $\mathbf{h} \in \mathfrak{X}^\infty(\mathrm{T}\mathcal{M})$ we can then associate a mapping $\overset{\triangledown}{\mathbf{h}} : \mathrm{C}^\infty(\mathcal{M}) \to \mathrm{C}^\infty(\mathcal{M})$ defined by

$$\overset{\triangledown}{\mathbf{h}}(f) := (\nabla f)\mathbf{h} \qquad \text{for all} \qquad f \in \mathrm{C}^\infty(\mathcal{M}) \tag{24.6}$$

where the gradient ∇f of f is the covector field of class C^∞ given by $\nabla f(x) := \nabla_x f$ for all $x \in \mathrm{Dom}\, f$. It is clear that $\overset{\triangledown}{\mathbf{h}}$ is -linear. By using the product rule $\nabla fg = f\nabla g + g\nabla f$, we have

$$\overset{\triangledown}{\mathbf{h}}(fg) = f\overset{\triangledown}{\mathbf{h}}(g) + g\overset{\triangledown}{\mathbf{h}}(f) \quad \text{for all} \quad f, g \in \mathbf{C}^\infty(\mathcal{M}). \tag{24.7}$$

This shows that $\overset{\triangledown}{\mathbf{h}}$ is a derivation of the module $\mathrm{C}^\infty(\mathcal{M})$. One can prove that every derivation of $\mathrm{C}^\infty(\mathcal{M})$ can be obtained in this manner. (The proof is fairly difficult.) ∎

Let a cross section section $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathcal{B})$ be given. For every bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ we define the mapping

$$\mathbf{H}^\phi : \mathcal{O}_\phi \to \mathbf{\Phi}(\mathcal{V}_\phi)$$

by

$$\mathbf{H}^\phi(y) := \mathbf{\Phi}(\phi\rfloor_y)\mathbf{H}(y), \quad \text{for all} \quad y \in \mathcal{O}_\phi. \tag{24.8}$$

Given $x \in \mathcal{O}_\phi$, we define

$$\nabla_x^\phi \mathbf{H} := \mathbf{\Phi}(\phi\rfloor_x^{-1})\nabla_x \mathbf{H}^\phi \in \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathbf{\Phi}(\mathcal{B}_x)). \tag{24.9}$$

When $\mathbf{\Phi} = \mathrm{Id}$ and $\mathcal{B} = \mathrm{T}\mathcal{M}$, we have $\nabla_x^{\mathrm{tgt}_\chi}\mathbf{h} = \nabla_x^\chi \mathbf{h}$ for all $\chi \in \mathrm{Ch}\mathcal{M}$ and all $x \in \mathrm{Dom}\,\chi$.

One defines value-wise addition of cross sections of $\mathbf{\Phi}(\mathcal{B})$ and value-wise scalar multiplication of a real function on $\mathcal{M}$ and a cross section of $\mathbf{\Phi}(\mathcal{B})$ in the obvious manner. $\mathfrak{X}^p\mathbf{\Phi}(\mathcal{B})$ has the natural structure of a $\mathrm{C}^p(\mathcal{M})$-module, where $\mathrm{C}^p(\mathcal{M})$ is the ring of all real-valued functions of class C^p on $\mathcal{M}$.

Let $(\mathcal{L}_1, \tau_1, \mathcal{M})$ and $(\mathcal{L}_2, \tau_2, \mathcal{M})$ be linear-space bundles over $\mathcal{M}$ and let $\mathcal{L}_1 \times_{\mathcal{M}} \mathcal{L}_2$ be the fiber product bundle of $\mathcal{L}_1$ and $\mathcal{L}_2$. For every tensor bifunctor $\mathbf{\Upsilon}$, it follows form (24.5) that for each bundle chart $\phi_1 \in \mathrm{Ch}(\mathcal{L}_1, \mathcal{M})$ and each buhdle chart $\phi_2 \in \mathrm{Ch}(\mathcal{L}_2, \mathcal{M})$

$$\mathbf{\Upsilon}(\phi_1 \times_{\mathcal{M}} \phi_2)(\mathbf{v}) = \big(\, y\ ,\ \mathbf{\Upsilon}(\varphi\rfloor_y \times \phi\rfloor_y)\mathbf{v}\,\big) \tag{24.10}$$

where $y := (\tau_1 \times_{\mathcal{M}} \tau_2)^{\mathbf{\Upsilon}}(\mathbf{v})$ (see 24.3).

Let a cross section $\mathbf{H} : \mathcal{M} \to \mathbf{\Upsilon}(\mathcal{L}_1 \times_{\mathcal{M}} \mathcal{L}_2)$ be given. For each bundle chart $\phi_1 \in \mathrm{Ch}(\mathcal{L}_1, \mathcal{M})$ and each buhdle chart $\phi_2 \in \mathrm{Ch}(\mathcal{L}_2, \mathcal{M})$, we define the mapping

$$\mathbf{H}^{\phi_1,\phi_2} : \mathcal{O}_\phi \to \mathbf{\Upsilon}(\mathcal{V}_{\phi_1} \times \mathcal{V}_{\phi_2})$$

by

$$\mathbf{H}^{\phi_1,\phi_2}(y) := \mathbf{\Phi}(\phi\big\rfloor_y)\mathbf{H}(y), \quad \text{for all} \quad y \in \mathcal{O}_{\phi_1} \cap \mathcal{O}_{\phi_2}. \tag{24.11}$$

Given $x \in \mathcal{O}_{\phi_1} \cap \mathcal{O}_{\phi_2}$, we define

$$\nabla_x^{\phi_1,\phi_2}\mathbf{H} := \mathbf{\Upsilon}(\phi_1\big\rfloor_x^{-1} \times \phi_2\big\rfloor_x^{-1})\nabla_x \mathbf{H}^{\phi_1,\phi_2} \tag{24.12}$$

which is in $\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathbf{\Upsilon}(\mathcal{L}_{1x} \times \mathcal{L}_{2x}))$.

Chapter 3

Connections

31. Tangent Connectors

We assume that $r \in \widetilde{\ }$ with $r \geq 2$ and a C^r-manifold $\mathcal{M}$ are given. Let a number $s \in 1..r$ and a C^s bundle $(\mathcal{B}, \tau, \mathcal{M})$ be given. We assume that both $\mathcal{M}$ and $\mathcal{B}$ have constant dimensions, and we put

$$n := \dim \mathcal{M} \quad \text{and} \quad m := \dim \mathcal{B} - \dim \mathcal{M}. \tag{31.1}$$

Then $m = \dim \mathcal{B}_x$ for all $x \in \mathcal{M}$.

Recall that for every bundle chart $\phi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})$, we have $\mathrm{ev}_1 \circ \phi(\mathbf{v}) = \tau(\mathbf{v})$ and

$$\phi(\mathbf{v}) = \big(\, z \,,\, \mathrm{ev}_2(\phi(\mathbf{v}))\,\big) \qquad \text{where} \quad z := \tau(\mathbf{v}) \tag{31.2}$$

for all $\mathbf{v} \in \mathrm{Dom}\,\phi$. Moreover, if $\phi, \psi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})$, it follows easily from (31.2) with ϕ replaced by ψ that

$$(\psi \,\square\, \phi^{\leftarrow})(z, \mathbf{u}) = \big(\, z \,,\, \mathrm{ev}_2((\psi \,\square\, \phi^{\leftarrow})(z, \mathbf{u}))\,\big) \tag{31.3}$$

for all $z \in \mathcal{O}_\phi \cap \mathcal{O}_\psi$ and all $\mathbf{u} \in \mathcal{V}_\phi$. $\blacksquare$

Now let $\mathbf{b} \in \mathcal{B}$ be fixed and put $x := \tau(\mathbf{b})$. Let $\mathrm{in}_x : \mathcal{B}_x \to \mathcal{B}$ be the inclusion mapping

$$\mathrm{in}_x := \mathbf{1}_{\mathcal{B}_x \subset \mathcal{B}}. \tag{31.4}$$

Consider the following diagram

$$\mathcal{B}_x \xrightarrow{\mathrm{in}_x} \mathcal{B} \xrightarrow{\tau} \mathcal{M},$$

the composite $\tau \circ \mathrm{in}_x$ is the constant mapping with value x. Taking the gradient of $(\tau \circ \mathrm{in}_x)$ at $\mathbf{b}$, we get $(\nabla_{\mathbf{b}}\tau)(\nabla_{\mathbf{b}}\mathrm{in}_x) = \mathbf{0}$ and hence $\mathrm{Rng}\,\nabla_{\mathbf{b}}\mathrm{in}_x \subset \mathrm{Null}\,\nabla_{\mathbf{b}}\tau$. Indeed, we have $\mathrm{Rng}\,\nabla_{\mathbf{b}}\mathrm{in}_x = \mathrm{Null}\,\nabla_{\mathbf{b}}\tau$ as to be shown in Prop.1.

<u>**Notation:**</u> *We define the* **projection mapping $\mathbf{P}_{\mathbf{b}}$** *at* $\mathbf{b}$ *by*

$$\mathbf{P}_{\mathbf{b}} := \nabla_{\mathbf{b}}\tau \in \mathrm{Lin}\,(\mathrm{T}_{\mathbf{b}}\mathcal{B}, \mathrm{T}_x\mathcal{M}) \tag{31.5}$$

and the **injection mapping $\mathbf{I}_{\mathbf{b}}$** *at* $\mathbf{b}$ *by*

$$\mathbf{I}_{\mathbf{b}} := \nabla_{\mathbf{b}}\mathrm{in}_x \in \mathrm{Lin}\,(\mathrm{T}_{\mathbf{b}}\mathcal{B}_x, \mathrm{T}_{\mathbf{b}}\mathcal{B}). \tag{31.6}$$

Proposition 1: *The projection mapping* $\mathbf{P_b}$ *is surjective, the injection mapping* $\mathbf{I_b}$ *is injective, and we have*

$$\operatorname{Null}\mathbf{P_b} = \operatorname{Rng}\mathbf{I_b} \tag{31.7}$$

i.e.

$$\mathrm{T}_{\mathbf{b}}\mathcal{B}_x \xrightarrow{\mathbf{I_b}} \mathrm{T}_{\mathbf{b}}\mathcal{B} \xrightarrow{\mathbf{P_b}} \mathrm{T}_x\mathcal{M} \tag{31.8}$$

is a short exact sequence.

Proof: Choose a bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$. It follows from (31.2) that

$$(\phi \circ \mathrm{in}_x)(\mathbf{d}) = \left(x \, , \, \phi\rfloor_x(\mathbf{d}) \right) \qquad \text{for all} \quad \mathbf{d} \in \mathcal{B}_x.$$

Using the chain rule and (31.6), we obtain

$$((\nabla_{\mathbf{b}}\phi)\,\mathbf{I_b})\,\mathbf{m} = \Big(\mathbf{0}, \nabla_{\mathbf{b}}\phi\rfloor_x\mathbf{m}\Big) \quad \text{for all} \quad \mathbf{m} \in \mathrm{T}_{\mathbf{b}}\mathcal{B}_x\,. \tag{31.9}$$

Since both $\nabla_{\mathbf{b}}\,\phi$ and $\nabla_{\mathbf{b}}\phi\rfloor_x$ are invertible, it follows that $\operatorname{Null}\mathbf{I_b} = \{\mathbf{0}\}$ and

$$\operatorname{Rng}\mathbf{I_b} = (\nabla_{\mathbf{b}}\phi)^{<}(\{\mathbf{0}\} \times \mathrm{T}_{\mathbf{v}}\mathcal{V}_\phi) \quad \text{where} \quad \mathbf{v} := \mathrm{ev}_2(\phi(\mathbf{b})). \tag{31.10}$$

On the other hand, it follows from (31.2) that

$$(\tau \circ \phi^{\leftarrow})(z,\mathbf{u}) = z \qquad \text{for all} \quad z \in \mathcal{O}_\phi$$

and all $\mathbf{u} \in \mathcal{V}_\phi$. Using the chain rule and (31.5) we conclude that

$$\mathbf{P_b}(\nabla_{\mathbf{b}}\phi)^{-1}(\mathbf{t},\mathbf{w}) = \mathbf{t} \qquad \text{for all} \quad \mathbf{t} \in \mathrm{T}_x\mathcal{M} \tag{31.11}$$

and all $\mathbf{w} \in \mathrm{T}_{\mathbf{v}}\mathcal{V}_\phi$. Since $\nabla_{\mathbf{b}}\phi$ is invertible, it follows that $\operatorname{Rng}\mathbf{P_b} = \mathrm{T}_x\mathcal{M}$ and

$$\operatorname{Null}\mathbf{P_b} = ((\nabla_{\mathbf{b}}\phi)^{-1})_{>}(\{\mathbf{0}\} \times \mathrm{T}_{\mathbf{v}}\mathcal{V}_\phi) \quad \text{where} \quad \mathbf{v} := \mathrm{ev}_2(\phi(\mathbf{b})). \tag{31.12}$$

Since $((\nabla_{\mathbf{b}}\phi)^{-1})_{>} = (\nabla_{\mathbf{b}}\phi)^{<}$, comparison of (31.10) with (31.12) shows that (31.7) holds. ∎

Definition: *A linear right-inverse of the projection-mapping* $\mathbf{P_b}$ *will be called a* **right tangent-connector** *at* $\mathbf{b}$, *a linear left-inverse of the injection-mapping* $\mathbf{I_b}$ *will be called a* **left tangent-connector** *at* $\mathbf{b}$. *The sets*

$$\begin{aligned} \mathrm{Rcon}_{\mathbf{b}}\mathcal{B} &:= \mathrm{Riv}(\mathbf{P_b}) \\ \mathrm{Lcon}_{\mathbf{b}}\mathcal{B} &:= \mathrm{Liv}(\mathbf{I_b}) \end{aligned} \tag{31.13}$$

of allright tangent-connectors at $\mathbf{b}$ *and all left tangent-connectors at* $\mathbf{b}$ *will be called the* **right tangent-connector space** *at* $\mathbf{b}$ *and the* **left tangent-connector space** *at* $\mathbf{b}$*, respectively.*

The right tangent connector space $\mathrm{Rcon}_{\mathbf{b}}\mathcal{B}$ is a flat in $\mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B})$ with direction space

$$\big\{\, \mathbf{I}_{\mathbf{b}}\mathbf{L} \;\big|\; \mathbf{L} \in \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x)\,\big\}, \tag{31.14}$$

and the left tangent connector space $\mathrm{Lcon}_{\mathbf{b}}\mathcal{B}$ is a flat in $\mathrm{Lin}\,(\mathrm{T}_{\mathbf{b}}\mathcal{B}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x)$ with direction space

$$\big\{ -\mathbf{L}\mathbf{P}_{\mathbf{b}} \;\big|\; \mathbf{L} \in \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x)\,\big\}. \tag{31.15}$$

Using the identifications

$$\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x)\{\mathbf{P}_{\mathbf{b}}\} \cong \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x) \cong \{\mathbf{I}_{\mathbf{b}}\}\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}),$$

we consider $\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x)$ as the external translation space of both $\mathrm{Rcon}_{\mathbf{b}}\mathcal{B}$ and $\mathrm{Lcon}_{\mathbf{b}}\mathcal{B}$. Since $\dim \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x) = n\,m$, we have

$$\dim \mathrm{Rcon}_{\mathbf{b}}\mathcal{B} = n\,m = \dim \mathrm{Lcon}_{\mathbf{b}}\mathcal{B}. \tag{31.16}$$

By Prop. 1 of Sect. 14, there is a flat isomorphism

$$\mathbf{\Lambda} : \mathrm{Rcon}_{\mathbf{b}}\mathcal{B} \to \mathrm{Lcon}_{\mathbf{b}}\mathcal{B}$$

which assigns to every $\mathbf{K} \in \mathrm{Rcon}_{\mathbf{b}}\mathcal{B}$ an element $\mathbf{\Lambda}(\mathbf{K}) \in \mathrm{Lcon}_{\mathbf{b}}\mathcal{B}$ such that

$$\{\mathbf{0}\} \;\longleftarrow\; \mathrm{T}_{\mathbf{b}}\mathcal{B}_x \;\underset{\mathbf{\Lambda}(\mathbf{K})}{\longleftarrow}\; \mathrm{T}_{\mathbf{b}}\mathcal{B} \;\underset{\mathbf{K}}{\longleftarrow}\; \mathrm{T}_x\mathcal{M} \;\longleftarrow\; \{\mathbf{0}\} \tag{31.17}$$

is again a short exact sequence. We have

$$\mathbf{K}\mathbf{P}_{\mathbf{b}} + \mathbf{I}_{\mathbf{b}}\mathbf{\Lambda}(\mathbf{K}) = \mathbf{1}_{\mathrm{T}_{\mathbf{b}}\mathcal{B}}. \tag{31.18}$$

Proposition 2: *For each bundle chart* $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$, *let* $\mathbf{A}^{\phi}_{\mathbf{b}}$ *in* $\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B})$ *be defined by*

$$\mathbf{A}^{\phi}_{\mathbf{b}}\,\mathbf{t} := (\nabla_{\mathbf{b}}\phi)^{-1}(\mathbf{t}, \mathbf{0}) \qquad \text{for all} \quad \mathbf{t} \in \mathrm{T}_x\mathcal{M}\ . \tag{31.19}$$

Then $\mathbf{A}^{\phi}_{\mathbf{b}}$ *is a linear right-inverse of* $\mathbf{P}_{\mathbf{b}}$*; i.e.* $\mathbf{A}^{\phi}_{\mathbf{b}} \in \mathrm{Rcon}_{\mathbf{b}}\mathcal{B}$.

Proof : If we substitute $\mathbf{w} := \mathbf{0}$ in (31.11) and use (31.19), we obtain

$$\mathbf{P}_{\mathbf{b}}(\mathbf{A}^{\phi}_{\mathbf{b}}\mathbf{t}) = \mathbf{t} \qquad \text{for all} \quad \mathbf{t} \in \mathrm{T}_x\mathcal{M}$$

which shows that $\mathbf{A}^{\phi}_{\mathbf{b}}$ is a linear right-inverse of $\mathbf{P}_{\mathbf{b}}$. ▮

Proposition 3: *If* $\phi, \psi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$, *then* $\mathbf{A}^{\psi}_{\mathbf{b}}$ *and* $\mathbf{A}^{\phi}_{\mathbf{b}}$ *differ by*

$$\begin{aligned} \mathbf{A}^{\phi}_{\mathbf{b}} - \mathbf{A}^{\psi}_{\mathbf{b}} &= \mathbf{I}_{\mathbf{b}}\, \mathbf{\Gamma}^{\phi,\psi}_{\mathbf{b}} \\ \mathbf{\Lambda}(\mathbf{A}^{\phi}_{\mathbf{b}}) - \mathbf{\Lambda}(\mathbf{A}^{\psi}_{\mathbf{b}}) &= -\mathbf{\Gamma}^{\phi,\psi}_{\mathbf{b}} \mathbf{P}_{\mathbf{b}} \end{aligned} \tag{31.20}$$

where

$$\mathbf{\Gamma}^{\phi,\psi}_{\mathbf{b}} := (\nabla_{\mathbf{b}}\psi\rfloor_x)^{-1}\Big(\mathrm{ev}_2 \circ \nabla_x\big((\psi \,\square\, \phi^{\leftarrow})(\cdot, \phi\rfloor_x \mathbf{b})\big)\Big) \tag{31.21}$$

which belongs to $\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x)$.

Proof : It follows from (31.2) that

$$\phi(\mathbf{b}) = (x, \phi\rfloor_x \mathbf{b}). \tag{31.22}$$

Using (31.3) and (31.22), we obtain

$$\nabla_{\phi(\mathbf{b})}(\psi \,\square\, \phi^{\leftarrow})(\mathbf{t}, \mathbf{0}) = \Big(\mathbf{t}\ ,\ \mathrm{ev}_2\big(\nabla_x\big((\psi \,\square\, \phi^{\leftarrow})(\cdot, \phi\rfloor_x \mathbf{b})\big)\mathbf{t}\big)\Big) \tag{31.23}$$

for all $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$.

In view of (23.16), with x replaced by $\mathbf{b}$, γ by ψ, and χ by ϕ, we have

$$\nabla_{\phi(\mathbf{b})}(\psi \,\square\, \phi^{\leftarrow}) = (\nabla_{\mathbf{b}}\psi)(\nabla_{\mathbf{b}}\phi)^{-1}.$$

If we substitute this formula into (31.23) and use (31.19) and (31.21), we obtain

$$(\nabla_{\mathbf{b}}\psi)(\mathbf{A}^{\phi}_{\mathbf{b}}\mathbf{t}) = \Big(\mathbf{t}\ ,\ \nabla_{\mathbf{b}}\psi\rfloor_x \mathbf{\Gamma}^{\phi,\psi}_{\mathbf{b}}\mathbf{t}\Big)$$

for all $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$. Using (31.19) with ψ replaced by ϕ, we conclude that

$$\mathbf{A}^{\phi}_{\mathbf{b}}\mathbf{t} = \mathbf{A}^{\psi}_{\mathbf{b}}\mathbf{t} + (\nabla_{\mathbf{b}}\psi)^{-1}\Big(\mathbf{0}\ ,\ \nabla_{\mathbf{b}}\psi\rfloor_x \mathbf{\Gamma}^{\phi,\psi}_{\mathbf{b}}\mathbf{t}\Big)$$

for all $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$. The desired result $(31.20)_1$ now follows from (31.9), with ϕ replaced by ψ and $\mathbf{m} := \mathbf{\Gamma}^{\phi,\psi}_{\mathbf{b}}\mathbf{t}$. Equation $(31.20)_2$ follows from $(31.20)_1$ and Prop. 3 of Sect.14. ∎

Notation: *Let* $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ *be given. The mapping*

$$\mathbf{\Gamma}^{\phi}_{\mathbf{b}} : \mathrm{Rcon}_{\mathbf{b}}\mathcal{B} \to \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x)$$

is defined by $\mathbf{\Gamma}^{\phi}_{\mathbf{b}} := \mathbf{\Gamma}^{\mathbf{A}^{\phi}_{\mathbf{b}}}$ *in terms of* (14.10)*; i.e. by*

$$\mathbf{\Gamma}^{\phi}_{\mathbf{b}}(\mathbf{K}) := -\mathbf{\Lambda}(\mathbf{A}^{\phi}_{\mathbf{b}})\mathbf{K} \quad \text{for all} \quad \mathbf{K} \in \mathrm{Rcon}_{\mathbf{b}}\mathcal{B}. \tag{31.24}$$

If $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$, we have, by Prop. 6 of Sect. 14,

$$\begin{aligned} \mathbf{A}^{\phi}_{\mathbf{b}} - \mathbf{K} &= \mathbf{I}_{\mathbf{b}}\, \boldsymbol{\Gamma}^{\phi}_{\mathbf{b}}(\mathbf{K}) \\ \boldsymbol{\Lambda}(\mathbf{A}^{\phi}_{\mathbf{b}}) - \boldsymbol{\Lambda}(\mathbf{K}) &= -\boldsymbol{\Gamma}^{\phi}_{\mathbf{b}}(\mathbf{K})\mathbf{P}_{\mathbf{b}} \end{aligned} \tag{31.25}$$

for all $\mathbf{K} \in \mathrm{Rcon}_{\mathbf{b}}\mathcal{B}$. Moreover; if $\phi, \psi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$, then (31.20) and (31.24) give

$$\boldsymbol{\Gamma}^{\phi}_{\mathbf{b}}(\mathbf{K}) - \boldsymbol{\Gamma}^{\psi}_{\mathbf{b}}(\mathbf{K}) = \boldsymbol{\Gamma}^{\phi,\psi}_{\mathbf{b}} \quad \text{for all} \quad \mathbf{K} \in \mathrm{Rcon}_{\mathbf{b}}\mathcal{B}, \tag{31.26}$$

where $\boldsymbol{\Gamma}^{\phi,\psi}_{\mathbf{b}}$ is defined by (31.21). It follows from (31.26) and $\boldsymbol{\Gamma}^{\psi}_{\mathbf{b}}\big(\mathbf{A}^{\psi}_{\mathbf{b}}\big) = \mathbf{0}$ that $\boldsymbol{\Gamma}^{\phi,\psi}_{\mathbf{b}} = \boldsymbol{\Gamma}^{\phi}_{\mathbf{b}}\big(\mathbf{A}^{\psi}_{\mathbf{b}}\big)$ for all $\phi, \psi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$.

<u>Convention</u> : Assume that $\mathcal{B}$ is a flat-space bundle. Let $\mathbf{b} \in \mathcal{B}$ be given and put $x := \tau(\mathbf{b})$. The fiber $\mathcal{B}_x$ has the structure of a flat space; the translation space of $\mathcal{B}_x$ is denoted by $\mathcal{U}_x$. We may and will use the identification as described in (23.9) and (23.10); i.e. we identify $\mathrm{T}_{\mathbf{b}}\mathcal{B}_x$ with $\mathcal{U}_x$. Then (31.8) becomes

$$\mathcal{U}_x \quad \xrightarrow{\mathbf{I}_{\mathbf{b}}} \quad \mathrm{T}_{\mathbf{b}}\mathcal{B} \quad \xrightarrow{\mathbf{P}_{\mathbf{b}}} \quad \mathrm{T}_x\mathcal{M}. \tag{31.27}$$

In particular, if $\mathcal{B}$ is a linear-space bundle, we have $\mathcal{U}_x = \mathcal{B}_x$ and (31.27) becomes

$$\mathcal{B}_x \quad \xrightarrow{\mathbf{I}_{\mathbf{b}}} \quad \mathrm{T}_{\mathbf{b}}\mathcal{B} \quad \xrightarrow{\mathbf{P}_{\mathbf{b}}} \quad \mathrm{T}_x\mathcal{M}. \tag{31.28}$$

Remark 1: For every bundle chart ϕ in $\mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$, we have

$$\begin{aligned} &\mathbf{P}_{\mathbf{b}} = \mathrm{ev}_1 \circ \nabla_{\mathbf{b}}\phi, & &\mathbf{A}^{\phi}_{\mathbf{b}} = (\nabla_{\mathbf{b}}\phi)^{-1} \circ \mathrm{ins}_1, \\ &\mathbf{I}_{\mathbf{b}} = (\nabla_{\mathbf{b}}\phi)^{-1} \circ \mathrm{ins}_2 \circ \nabla_{\mathbf{b}}\phi\big\rfloor_x, & &\boldsymbol{\Lambda}(\mathbf{A}^{\phi}_{\mathbf{b}}) = (\nabla_{\mathbf{b}}\phi\big\rfloor_x)^{-1}(\mathrm{ev}_2 \circ \nabla_{\mathbf{b}}\phi), \end{aligned} \tag{31.29}$$

where ev_i and ins_i, $i \in 2^{\rceil}$, are evaluations and insertions, respectively.

Proof: Let $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ be given. Using (31.9), (31.19) and also observing $\mathbf{A}^{\phi}_{\mathbf{b}}\mathbf{P}_{\mathbf{b}} + \mathbf{I}_{\mathbf{b}}\boldsymbol{\Lambda}(\mathbf{A}^{\phi}_{\mathbf{b}}) = \mathbf{1}_{\mathrm{T}_{\mathbf{b}}\mathcal{B}}$, we have

$$\nabla_{\mathbf{b}}\phi = \nabla_{\mathbf{b}}\phi\big(\mathbf{A}^{\phi}_{\mathbf{b}}\mathbf{P}_{\mathbf{b}} + \mathbf{I}_{\mathbf{b}}\boldsymbol{\Lambda}(\mathbf{A}^{\phi}_{\mathbf{b}})\big) = \big(\,\mathbf{P}_{\mathbf{b}}\,,\,(\nabla_{\mathbf{b}}\phi)\big\rfloor_x\boldsymbol{\Lambda}(\mathbf{A}^{\phi}_{\mathbf{b}})\big). \tag{31.30}$$

The desired result (31.29) follows from (31.9), (31.19) and (31.30). ∎

If in addition $\phi\big\rfloor_x = \mathbf{1}_{\mathcal{B}_x}$, then we have

$$\mathbf{I}_{\mathbf{b}} = (\nabla_{\mathbf{b}}\phi)^{-1} \circ \mathrm{ins}_2 \qquad \text{and} \qquad \boldsymbol{\Lambda}(\mathbf{A}^{\phi}_{\mathbf{b}}) = (\mathrm{ev}_2 \circ \nabla_{\mathbf{b}}\phi).$$

Remark 2: For every cross section $\mathbf{s} : \mathcal{M} \to \mathcal{B}$, we have $\tau \circ \mathbf{s} = \mathbf{1}_{\mathcal{M}}$. If $\mathbf{s}$ is differentiable at $x \in \mathcal{M}$, then the gradient of $\mathbf{1}_{\mathcal{M}} = \tau \circ \mathbf{s}$ at x gives

$$\mathbf{1}_{\mathrm{T}_x\mathcal{M}} = \nabla_x(\tau \circ \mathbf{s}) = (\nabla_{\mathbf{s}(x)}\tau)(\nabla_x \mathbf{s}) = \mathbf{P}_{\mathbf{s}(x)}\nabla_x \mathbf{s}. \tag{31.31}$$

We see that $\nabla_x \mathbf{s}$ is a right tangent connector at $\mathbf{s}(x)$; i.e. $\nabla_x \mathbf{s} \in \mathrm{Rcon}_{\mathbf{s}(x)}(\mathcal{B})$. ∎

Remark 3: Let $\mathcal{B}$ be a linear space bundle and let $x \in \mathcal{M}$ be given. Denote the zero of the linear space $\mathcal{B}_x$ by $\mathbf{0}_x$. It follows from (31.21) that $\mathbf{\Gamma}^{\phi,\psi}_{\mathbf{0}_x} = \mathbf{0}$ and then from (31.20) that $\mathbf{A}^{\phi}_{\mathbf{0}_x} = \mathbf{A}^{\psi}_{\mathbf{0}_x}$ for all $\phi, \psi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$. This shows that $\left\{ \mathbf{A}^{\phi}_{\mathbf{0}_x} \;\middle|\; \phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M}) \right\}$ is a singleton and hence

$$\left\{ \mathbf{A}^{\phi}_{\mathbf{0}_x} \;\middle|\; \phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M}) \right\} \;\; \mathrm{Rcon}_{\mathbf{0}_x}\mathcal{B}.$$ ∎

Remark 4: For every $\mathbf{b} \in \mathcal{B}$, we define the **vertical space** $\mathrm{V}_{\mathbf{b}}\mathcal{B}$ **of** $\mathcal{B}$ **at** $\mathbf{b}$ by

$$\mathrm{V}_{\mathbf{b}}\mathcal{B} := \mathrm{Null}\,\mathbf{P}_{\mathbf{b}} = \mathrm{Rng}\,\mathbf{I}_{\mathbf{b}} \subset \mathrm{T}_{\mathbf{b}}\mathcal{B}\,. \tag{31.32}$$

Since $\mathbf{I}_{\mathbf{b}}$ is injective, $\mathrm{V}_{\mathbf{b}}\mathcal{B}$ is isomorphic with $\mathrm{T}_{\mathbf{b}}\mathcal{B}_{\tau(\mathbf{b})}$. The sequence

$$\mathrm{V}_{\mathbf{b}}\mathcal{B} \;\hookrightarrow\; \mathrm{T}_{\mathbf{b}}\mathcal{B} \;\xrightarrow{\mathbf{P}_{\mathbf{b}}}\; \mathrm{T}_{\tau(\mathbf{b})}\mathcal{M} \tag{31.33}$$

is a short exact sequence. For every right tangent connector $\mathbf{K} \in \mathrm{Rcon}_{\mathbf{b}}\mathcal{B}$, the range of $\mathbf{K}$

$$\mathrm{H}^{\mathbf{K}}_{\mathbf{b}}\mathcal{B} := \mathrm{Rng}\mathbf{K} \subset \mathrm{T}_{\mathbf{b}}\mathcal{B} \tag{31.34}$$

is called the **horizontal space** of $\mathcal{B}$ at $\mathbf{b}$ relative to $\mathbf{K}$. It is easily seen that $\mathrm{V}_{\mathbf{b}}\mathcal{B}$ and $\mathrm{H}^{\mathbf{K}}_{\mathbf{b}}\mathcal{B}$ are supplementary in $\mathrm{T}_{\mathbf{b}}\mathcal{B}$. ∎

Notes 31

(1) The convention that we made in this section was first introduced by Noll, in 1974, on the tangent bundle $\mathrm{T}\mathcal{M}$ (see [N3]). This convention plays a central role in our development.

(2) The short exact sequence (31.33) can be found in [Sa].

32. Transfer Isomorphisms, Shift Spaces

We assume that $r \in \tilde{}$ with $r \geq 2$ and a C^r-manifold $\mathcal{M}$ are given. Let a number $s \in 1..r$ be given and let $\mathcal{B}$ be a C^s linear-space bundle over $\mathcal{M}$. We assume that both $\mathcal{M}$ and $\mathcal{B}$ have constant dimensions, and put $n := \dim \mathcal{M}$ and $m := \dim \mathcal{B} - \dim \mathcal{M}$. Then

$$m = \dim \mathcal{B}_x \quad \text{for all} \quad x \in \mathcal{M}. \tag{32.1}$$

Now let $x \in \mathcal{M}$ be fixed. We define the **bundle of transfer isomorphisms** of $\mathcal{B}$ from x by

$$\mathrm{Tlis}_x\mathcal{B} := \bigcup_{y \in \mathcal{M}} \mathrm{Lis}(\mathcal{B}_x, \mathcal{B}_y). \tag{32.2}$$

It is endowed with the natural structure of a C^s-fiber bundle as shown below. The corresponding bundle projection $\pi_x : \mathrm{Tlis}_x\mathcal{B} \to \mathcal{M}$ is given by

$$\pi_x(\mathbf{T}) :\in \left\{ \, y \in \mathcal{M} \;\middle|\; \mathbf{T} \in \mathrm{Lis}(\mathcal{B}_x, \mathcal{B}_y) \, \right\} \tag{32.3}$$

and the bundle inclusion $\iota_x : \mathrm{Lis}\,\mathcal{B}_x \to \mathrm{Tlis}_x\,\mathcal{B}$ at x is

$$\iota_x := \mathbf{1}_{\mathrm{Lis}\mathcal{B}_x \subset \mathrm{Tlis}_x\mathcal{B}}. \tag{32.4}$$

For every bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$, we define

$$\mathrm{tlis}_x^{\phi} : \mathrm{Tlis}_x(\mathcal{O}_\phi) \to \mathcal{O}_\phi \times \mathrm{Lis}(\mathcal{B}_x, \mathcal{V}_\phi) \tag{32.5}$$

by

$$\mathrm{tlis}_x^{\phi}(\mathbf{T}) := \left(\, z \, , \, \phi\rfloor_z \mathbf{T} \, \right), \quad \text{where} \quad z := \pi_x(\mathbf{T}). \tag{32.6}$$

It is easily seen that tlis_x^{ϕ} is invertible and that

$$\mathrm{tlis}_x^{\phi\leftarrow}(z, \mathbf{L}) = (\phi\rfloor_z)^{-1}\mathbf{L} \tag{32.7}$$

for all $z \in \mathcal{O}_\phi$ and all $\mathbf{L} \in \mathrm{Lis}(\mathcal{B}_x, \mathcal{V}_\phi)$. Moreover, if $\psi, \phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$, it follows easily from (32.7) and (32.6) with ϕ replaced by ψ that

$$\left(\mathrm{tlis}_x^{\psi} \circ \mathrm{tlis}_x^{\phi\leftarrow}\right)(z, \mathbf{L}) = \left(\, z \, , \, (\psi \diamond \phi)(z)\mathbf{L} \, \right) \tag{32.8}$$

for all $z \in \mathcal{O}_\psi \cap \mathcal{O}_\phi$ and all $\mathbf{L} \in \mathrm{Lis}(\mathcal{B}_x, \mathcal{V}_\phi)$ (See (22.7) for the definition of $\psi \diamond \phi$). It is clear that $\mathrm{tlis}_x^{\psi} \circ \mathrm{tlis}_x^{\phi\leftarrow}$ is of class C^s. Since $\psi, \phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ were arbitrary, it follows that $\left\{ \, \mathrm{tlis}_x^{\alpha} \;\middle|\; \alpha \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M}) \, \right\}$ is a C^s-bundle atlas of $\mathrm{Tlis}_x\mathcal{B}$. We consider $\left(\mathrm{Tlis}_x\mathcal{B} \, , \, \pi_x, \mathcal{M} \right)$ as being endowed with the C^s fiber bundle structure over $\mathcal{M}$ determined by this atlas.

Remark : We may view $\mathrm{Tlis}_x\mathcal{B}$ as a Tran_x-bundle, where Tran_x is the iso-category whose objects are of the form $\mathrm{Lis}(\mathcal{B}_x, \mathcal{V})$ with $\mathcal{V} \in LS$ and whose isomorphisms are of the form

$$(\mathbf{T} \mapsto \mathbf{LT}) : \mathrm{Lis}(\mathcal{B}_x, \mathrm{Dom}\mathbf{L}) \to \mathrm{Lis}(\mathcal{B}_x, \mathrm{Cod}\mathbf{L})$$

with $\mathbf{L} \in \mathrm{LIS}$. ▮

It is easily seen that the mappings π_x and ι_x defined by (32.3) and (32.4) are of class C^s.

We now apply the results of Sect.31 by replacing the ISO-bundle $\mathcal{B}$ there by the bundle $\mathrm{Tlis}_x\mathcal{B}$ and $\mathbf{b} \in \mathcal{B}$ there by $\mathbf{1}_{\mathcal{B}_x} \in \mathrm{Tlis}_x\mathcal{B}$.

Definition: *The* **shift-space** $\mathrm{S}_x\mathcal{B}$ *of* $\mathcal{B}$ *at* $x \in \mathcal{M}$ *is defined to be*

$$\mathrm{S}_x\mathcal{B} := \mathrm{T}_{\mathbf{1}_{\mathcal{B}_x}}\mathrm{Tlis}_x\mathcal{B}. \tag{32.9}$$

We define the **projection mapping** *of* $\mathrm{S}_x\mathcal{B}$ *by*

$$\mathbf{P}_x := \mathbf{P}_{\mathbf{1}_{\mathcal{B}_x}} = \nabla_{\mathbf{1}_{\mathcal{B}_x}}\pi_x \in \mathrm{Lin}\,(\mathrm{S}_x\mathcal{B}, \mathrm{T}_x\mathcal{M}) \tag{32.10}$$

and the **injection mapping** *of* $\mathrm{S}_x\mathcal{B}$ *by*

$$\mathbf{I}_x := \mathbf{I}_{\mathbf{1}_{\mathcal{B}_x}} = \nabla_{\mathbf{1}_{\mathcal{B}_x}}\iota_x \in \mathrm{Lin}\,(\mathrm{Lin}\mathcal{B}_x, \mathrm{S}_x\mathcal{B}) \tag{32.11}$$

in terms of (31.5) *and* (31.6)*; respectively, where* π_x *and* ι_x *are defined by* (32.3) *and* (32.4).

It is clear from (32.5) that

$$\dim(\mathrm{Tlis}_x\mathcal{B}) = \dim(\mathrm{S}_x\mathcal{B}) = n + m^2. \tag{32.12}$$

Proposition 1: *The projection mapping* $\mathbf{P}_x$ *is surjective, the injection mapping* $\mathbf{I}_x$ *is injective, and we have*

$$\mathrm{Null}\,\mathbf{P}_x = \mathrm{Rng}\,\mathbf{I}_x \tag{32.13}$$

i.e.

$$\mathrm{Lin}\,\mathcal{B}_x \xrightarrow{\mathbf{I}_x} \mathcal{S}_x\mathcal{B} \xrightarrow{\mathbf{P}_x} \mathrm{T}_x\mathcal{M} \tag{32.14}$$

is a short exact sequence.

Definition: *A linear right-inverse of the projection-mapping* $\mathbf{P}_x$ *will be called a* **right shift-connector** *(or simply* **right connector***) at* x*, a linear left-inverse* *of the injection-mapping* $\mathbf{I}_x$ *will be called a* **left shift-connector** *(or simply* **left connector***) at* x*. The sets*

$$\begin{aligned}\mathrm{Rcon}_x\mathcal{B} &:= \mathrm{Rcon}_{\mathbf{1}_{\mathcal{B}_x}}\mathrm{Tlis}_x\mathcal{B}\\ \mathrm{Lcon}_x\mathcal{B} &:= \mathrm{Lcon}_{\mathbf{1}_{\mathcal{B}_x}}\mathrm{Tlis}_x\mathcal{B}\end{aligned} \tag{32.15}$$

of all right connectors at x *and all left connector at* x *will be called the* **right connector space** *at* x *and the* **left connector space** *at* x*, respectively.*

The right connector space $\mathrm{Rcon}_x\mathcal{B}$ is a flat in $\mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathcal{S}_x\mathcal{B})$ with direction space

$$\left\{\, \mathbf{I}_x\mathbf{L} \;\middle|\; \mathbf{L} \in \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x)\,\right\}, \tag{32.16}$$

and the left connector space $\mathrm{Lcon}_x\mathcal{B}$ is a flat in $\mathrm{Lin}\,(\mathcal{S}_x\mathcal{B}, \mathrm{Lin}\,\mathcal{B}_x)$ with direction space

$$\left\{\, -\mathbf{L}\mathbf{P}_x \;\middle|\; \mathbf{L} \in \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x)\,\right\}. \tag{32.17}$$

Using the identifications

$$\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x)\{\mathbf{P}_x\} \cong \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x) \cong \{\mathbf{I}_x\}\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x),$$

we consider $\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x)$ as the external translation space of both $\mathrm{Rcon}_x\mathcal{B}$ and $\mathrm{Lcon}_x\mathcal{B}$. Since $\dim \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x) = nm^2$, we have

$$\dim \mathrm{Rcon}_x\mathcal{B} = nm^2 = \dim \mathrm{Lcon}_x\mathcal{B}. \tag{32.18}$$

The flat isomorphism

$$\boldsymbol{\Lambda} : \mathrm{Rcon}_x\mathcal{B} \to \mathrm{Lcon}_x\mathcal{B}$$

assigns to every $\mathbf{K} \in \mathrm{Rcon}_x\mathcal{B}$ an element $\boldsymbol{\Lambda}(\mathbf{K}) \in \mathrm{Lcon}_x\mathcal{B}$ such that

$$\mathrm{Lin}\,\mathcal{B}_x \;\underset{\boldsymbol{\Lambda}(\mathbf{K})}{\longleftarrow}\; \mathcal{S}_x\mathcal{B} \;\underset{\mathbf{K}}{\longleftarrow}\; \mathrm{T}_x\mathcal{M} \tag{32.19}$$

is again a short exact sequence. We have

$$\mathbf{K}\mathbf{P}_x + \mathbf{I}_x\boldsymbol{\Lambda}(\mathbf{K}) = \mathbf{1}_{\mathcal{S}_x\mathcal{B}} \quad \text{for all} \quad \mathbf{K} \in \mathrm{Rcon}_x\mathcal{B}. \tag{32.20}$$

Convention : Since there is one-to-one correspondence between right connectors and left connectors, we shall only deal with one kind of connectors, say right connectors. If we say "connector", we mean a right connector. The notation

$$\mathrm{Con}_x\mathcal{B} := \mathrm{Rcon}_x\mathcal{B}$$

is also used.

Proposition 2: *For each* $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$*, let* $\mathbf{A}_x^\phi \in \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathcal{S}_x\mathcal{B})$ *be defined by* $\mathbf{A}_x^\phi := \mathbf{C}_{\mathbf{1}_{\mathcal{B}_x}}^{\mathrm{tlis}_x^\phi}$ *in terms of* (31.19)*; i.e.*

$$\mathbf{A}_x^\phi\,\mathbf{t} := (\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathrm{tlis}_x^\phi)^{-1}(\mathbf{t},\mathbf{0}) \qquad \text{for all} \quad \mathbf{t} \in \mathrm{T}_x\mathcal{M}\ . \tag{32.21}$$

Then $\mathbf{A}_x^\phi$ *is a linear right-inverse of* $\mathbf{P}_x$*, i.e.* $\mathbf{A}_x^\phi \in \mathrm{Con}_x\mathcal{B}$*.*

Let $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ be given. We have the following short exact sequence

$$\mathrm{Lin}\,\mathcal{B}_x \quad \underset{\mathbf{\Lambda}(\mathbf{A}_x^\phi)}{\longleftarrow} \quad \mathcal{S}_x\mathcal{B} \quad \underset{\mathbf{A}_x^\phi}{\longleftarrow} \quad \mathrm{T}_x\mathcal{M} \tag{32.22}$$

and

$$\mathbf{A}_x^\phi\mathbf{P}_x + \mathbf{I}_x\mathbf{\Lambda}(\mathbf{A}_x^\phi) = \mathbf{1}_{\mathcal{S}_x\mathcal{B}}. \tag{32.23}$$

Proposition 3: *If* $\psi, \phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ *are given, then*

$$\begin{aligned} \mathbf{A}_x^\phi - \mathbf{A}_x^\psi &= \mathbf{I}_x\,\mathbf{\Gamma}_x^{\phi,\psi} \\ \mathbf{\Lambda}(\mathbf{A}_x^\phi) - \mathbf{\Lambda}(\mathbf{A}_x^\psi) &= -\mathbf{\Gamma}_x^{\phi,\psi}\mathbf{P}_x \end{aligned} \tag{32.24}$$

where $\mathbf{\Gamma}_x^{\phi,\psi} := \mathbf{\Gamma}_{\mathbf{1}_{\mathcal{B}_x}}^{\mathrm{tlis}_x^\phi,\mathrm{tlis}_x^\psi}$ *in terms of* (31.21) *is of the form*

$$\mathbf{\Gamma}_x^{\phi,\psi} := (\psi\rfloor_x)^{-1}\big(\nabla_x(\psi \diamond \phi)\big) \circ (\mathbf{1}_{\mathrm{T}_x\mathcal{B}} \times \phi\rfloor_x) \tag{32.25}$$

which belongs to $\mathrm{Lin}\,(\mathrm{T}_x, \mathrm{Lin}\,\mathcal{B}_x)$*. Here, the notation* (22.7) *is used.*

Proof : Applying Prop. 3 in Sect. 32 with ϕ replaced by tlis_x^ϕ and ψ replaced by tlis_x^ψ together with (32.6) and (32.8), we obtain the desired result (32.25). ∎

Notation: *Let* $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ *be given. We define the mapping*

$$\mathbf{\Gamma}_x^\phi : \mathrm{Con}_x\mathcal{B} \to \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x)$$

by $\mathbf{\Gamma}_x^\phi := \mathbf{\Gamma}^{\mathbf{A}_x^\phi} = \mathbf{\Gamma}_{\mathbf{1}_{\mathcal{B}_x}}^{\mathrm{tlis}_x^\phi}$ *in terms of* (14.10) *and* (31.24)*; i.e.*

$$\mathbf{\Gamma}_x^\phi(\mathbf{K}) = -\mathbf{\Lambda}(\mathbf{A}_x^\phi)\mathbf{K} \quad \text{for all} \quad \mathbf{K} \in \mathrm{Con}_x\mathcal{B}. \tag{32.26}$$

If $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$, then (31.25) reduces to

$$\begin{aligned} \mathbf{A}_x^\phi - \mathbf{K} &= \mathbf{I}_x\,\mathbf{\Gamma}_x^\phi(\mathbf{K}) \\ \mathbf{\Lambda}(\mathbf{A}_x^\phi) - \mathbf{\Lambda}(\mathbf{K}) &= -\mathbf{\Gamma}_x^\phi(\mathbf{K})\mathbf{P}_x \end{aligned} \tag{32.27}$$

for all $\mathbf{K} \in \mathrm{Con}_x\mathcal{B}$. Moreover; if $\psi, \phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$, then

$$\mathbf{\Gamma}_x^{\phi}(\mathbf{K}) - \mathbf{\Gamma}_x^{\psi}(\mathbf{K}) = \mathbf{\Gamma}_x^{\phi,\psi} \quad \text{for all} \quad \mathbf{K} \in \mathrm{Con}_x\mathcal{B}, \tag{32.28}$$

where $\mathbf{\Gamma}_x^{\phi,\psi}$ is defined by (32.25). It follows from (32.28) that $\mathbf{\Gamma}_x^{\psi,\phi} = -\mathbf{\Gamma}_x^{\phi,\psi}$ and from $\mathbf{\Gamma}_x^{\psi}\left(\mathbf{A}_x^{\psi}\right) = \mathbf{0}$ that $\mathbf{\Gamma}_x^{\phi}\left(\mathbf{A}_x^{\psi}\right) = \mathbf{\Gamma}_x^{\phi,\psi}$ for all bundle charts $\psi, \phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$.

For every cross section $\mathbf{H} : \mathcal{O} \to \mathrm{Tlis}_x\mathcal{B}$ of the bundle $\mathrm{Tlis}_x\mathcal{B}$, the mapping $\mathbf{T} : \mathcal{M} \to \mathrm{Tlis}_x\mathcal{B}$ defined by

$$\mathbf{T}(y) := \mathbf{H}(y)\mathbf{H}^{-1}(x) \qquad \text{for all} \quad y \in \mathcal{M} \tag{32.29}$$

is a cross section of the bundle $\mathrm{Tlis}_x\mathcal{B}$ with $\mathbf{T}(x) = \mathbf{1}_{\mathcal{B}_x}$.

<u>Definition</u>: *A cross section* $\mathbf{T} : \mathcal{O} \to \mathrm{Tlis}_x\mathcal{B}$ *of the bundle* $\mathrm{Tlis}_x\mathcal{B}$ *such that* $\mathbf{T}(x) = \mathbf{1}_{\mathcal{B}_x}$ *is called a* **transport from** x.

For every bundle chart $\phi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})$, we see that

$$\big(y \mapsto (\phi\rfloor_y)^{-1}\phi\rfloor_x\big) : \mathcal{O}_\phi \to \mathrm{Tlis}_x\mathcal{B}$$

is a transport from x which is of class C^s.

Remark 1: For every $\mathbf{K} \in \mathrm{Con}_x\mathcal{B}$, there is a bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ with $\phi\rfloor_x = \mathbf{1}_{\mathcal{B}_x}$ such that

$$\mathbf{K} = \nabla_x(\phi\rfloor)^{-1} = \mathbf{A}_x^{\phi}. \tag{32.30}$$

Proof: Let $\mathbf{K} \in \mathrm{Con}_x\mathcal{B}$ be given. It is not hard to construct a transport $\mathbf{T} : \mathcal{O} \to \mathrm{Tlis}_x\mathcal{B}$ from x such that (Ask Prof. Noll!!!!!!!!!!!!!!!!!!!!!!)

$$\mathbf{K} = \nabla_x\mathbf{T}. \tag{32.31}$$

There is a bundle chart $\phi : \tau^{<}(\mathcal{O}) \to \mathcal{O} \times \mathcal{B}_x$ induced from $\mathbf{T}$ by

$$\phi(\mathbf{v}) := (\ y\ ,\ \mathbf{T}^{-1}(y)\mathbf{v}) \qquad \text{where} \quad y := \tau(\mathbf{v}) \tag{32.32}$$

for all $\mathbf{v} \in \tau^{<}(\mathcal{O})$. It is easily seen that $(\phi\rfloor)^{-1} = \mathbf{T}$. The first part of (32.30) follows from (32.31). In view of (31.29) we have

$$\begin{aligned}\mathbf{\Lambda}(\mathbf{A}_x^{\phi})\big(\nabla_x(\phi\rfloor)^{-1}\big) &= \big(\mathrm{ev}_2 \circ \nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathrm{tlis}_x^{\phi}\big)\nabla_x(\phi\rfloor)^{-1} \\ &= \mathrm{ev}_2 \circ \nabla_x\big(y \mapsto \mathrm{tlis}_x^{\phi}((\phi\rfloor_y)^{-1})\big).\end{aligned} \tag{32.33}$$

Using (32.6) and ovbserving $\phi\rfloor_y \in \mathrm{Lin}\,(\mathcal{B}_y, \mathcal{B}_x)$, we have

$$\mathrm{tlis}_x^{\phi}((\phi\rfloor_y)^{-1}) = (\ y\ ,\ \phi\rfloor_y(\phi\rfloor_y)^{-1}) = (\ y\ ,\ \mathbf{1}_{\mathcal{B}_x}). \tag{32.34}$$

Taking the gradient of (32.34) at x, we observe that

$$\nabla_x\big(y \mapsto \mathrm{tlis}_x^{\phi}((\phi\rfloor_y)^{-1})\big) = (\mathbf{1}_{\mathrm{T}_x\mathcal{M}}, \mathbf{0}). \tag{32.35}$$

It follows from (32.33) and (32.35) that

$$\mathbf{\Lambda}(\mathbf{A}_x^{\phi})\big(\nabla_x(\phi\rfloor)^{-1}\big) = \mathbf{0}.$$

This can happen only when $\nabla_x(\phi\rfloor)^{-1} = \mathbf{A}_x^{\phi}$. ∎

33. Torsion

Let $r \in \tilde{}$, with $r \geq 2$, and a C^r-manifold $\mathcal{M}$ be given. For every $x \in \mathcal{M}$, we have; as described in Sect. 32 with $\mathcal{B} := \mathrm{T}\mathcal{M}$,

$$\mathrm{Tlis}_x \mathrm{T}\mathcal{M} := \bigcup_{y \in \mathcal{M}} \mathrm{Lis}(\mathrm{T}_x\mathcal{M}, \mathrm{T}_y\mathcal{M}). \tag{33.1}$$

We also have the following short exact sequence

$$\mathrm{Lin}\,\mathrm{T}_x\mathcal{M} \quad \xrightarrow{\mathbf{I}_x} \quad \mathrm{S}_x\mathrm{T}\mathcal{M} \quad \xrightarrow{\mathbf{P}_x} \quad \mathrm{T}_x\mathcal{M}. \tag{33.2}$$

The short exact sequence (33.2) is of the form (15.1) and hence all of the results in Sect.15 can be used here.

For every manifold chart $\chi \in \mathrm{Ch}\mathcal{M}$, the tangent mapping tgt_χ; as defined in (22.13), is a bundle chart of the tangent bundle $\mathrm{T}\mathcal{M}$ such that $\mathrm{ev}_2 \circ \mathrm{tgt}_\chi = \nabla\chi$. Note that not every tangent bundle chart $\phi \in \mathrm{Ch}(\mathrm{T}\mathcal{M}, \mathcal{M})$ can be obtained from the gradient of a manifold chart. To avoid complicated notations, we replace all the superscript of $\phi = \mathrm{tgt}_\chi$ by superscript of χ; i.e. we use the following notation

$$\mathbf{A}_x^\chi := \mathbf{A}_x^{\mathrm{tgt}_\chi}, \quad \mathbf{\Gamma}_x^\chi := \mathbf{\Gamma}_x^{\mathrm{tgt}_\chi} \quad \text{and} \quad \mathbf{\Gamma}_x^{\chi,\gamma} := \mathbf{\Gamma}_x^{\mathrm{tgt}_\chi,\mathrm{tgt}_\gamma} \tag{33.3}$$

for all manifold charts $\chi, \gamma \in \mathrm{Ch}\mathcal{M}$. Given $\chi, \gamma \in \mathrm{Ch}\mathcal{M}$. It is easily seen from (32.25) and (23.16) that

$$\mathbf{\Gamma}_x^{\chi,\gamma} := \left((\nabla_x\gamma)^{-1}\nabla_\chi^{(2)}\gamma(x)\right) \circ (\nabla_x\chi \times \nabla_x\chi). \tag{33.4}$$

It follows from the Theorem on Symmetry of Second Gradients (see Sect.612, [FDS]) that $\mathbf{\Gamma}_x^{\chi,\gamma}$ belongs to the subspace $\mathrm{Sym}_2(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M})$ of $\mathrm{Lin}_2(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M}) \cong \mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathrm{T}_x\mathcal{M})$.

Proposition 1: *There is exactly one flat $\mathcal{F}$ in $\mathrm{Con}_x\mathrm{T}\mathcal{M}$ with direction space $\{\mathbf{I}_x\}\mathrm{Sym}_2(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M})$ which contains $\mathbf{A}_x^\chi$ for every manifold chart $\chi \in \mathrm{Ch}_x\mathcal{M}$, so that*

$$\mathcal{F} = \mathbf{A}_x^\chi + \{\mathbf{I}_x\}\mathrm{Sym}_2(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M}) \quad \text{for all} \quad \chi \in \mathrm{Ch}_x\mathcal{M}. \tag{33.5}$$

Definition: *The* **shift-bracket** $\mathbf{B}_x \in \mathrm{Skw}_2\left(\mathrm{S}_x\mathrm{T}\mathcal{M}^2, \mathrm{T}_x\mathcal{M}\right)$ *of* $\mathrm{S}_x\mathrm{T}\mathcal{M}$ *is defined by*

$$\mathbf{B}_x := \mathbf{B}_\mathcal{F} \tag{33.6}$$

where $\mathbf{B}_\mathcal{F}$ *is defined as in* (15.5).

Definition: *The* **torsion-mapping** $\mathbf{T}_x : \mathrm{Con}_x\mathrm{T}\mathcal{M} \to \mathrm{Skw}_2\left(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M}\right)$ *of* $\mathrm{Con}_x\mathrm{T}\mathcal{M}$ *is defined by*

$$\mathbf{T}_x := \mathbf{T}_\mathcal{F} \tag{33.7}$$

where $\mathbf{T}_{\mathcal{F}}$ *is defined as in* (15.8).

It follows from Prop.3 of Sect.15 that, for every manifold chart $\chi \in \mathrm{Ch}_x\mathcal{M}$, we have

$$\mathbf{T}_x = \mathbf{\Gamma}_x^{\chi} - \mathbf{\Gamma}_x^{\chi\sim} \tag{33.8}$$

where $^{\sim}$ denotes the value-wise switch, so that $\mathbf{\Gamma}_x^{\chi\sim}(\mathbf{K})(\mathbf{s},\mathbf{t}) = \mathbf{\Gamma}_x^{\chi}(\mathbf{K})(\mathbf{t},\mathbf{s})$ for all $\mathbf{K} \in \mathrm{Con}_x\mathcal{M}$ and all $\mathbf{s},\mathbf{t} \in \mathrm{T}_x\mathcal{M}$.

The torsion-mapping $\mathbf{T}_x$ is a surjective flat mapping with $\mathbf{T}_x^{<}(\{\mathbf{0}\}) = \mathcal{F}$ whose gradient

$$\nabla\mathbf{T}_x \in \mathrm{Lin}\big(\ \mathrm{Lin}_2\,(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M})\ ,\ \mathrm{Skw}_2\,(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M})\big) \tag{33.9}$$

is given by

$$(\nabla\mathbf{T}_x)\mathbf{L} = \mathbf{L}^{\sim} - \mathbf{L} \tag{33.10}$$

for all $\mathbf{L} \in \mathrm{Lin}_2\,(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M})$.

Definition: *We say that a connector* $\mathbf{K} \in \mathrm{Con}_x\mathrm{T}\mathcal{M}$ *is* **torsion-free** *(or* **symmetric***) if* $\mathbf{T}_x(\mathbf{K}) = \mathbf{0}$*, i.e.* $\mathbf{K} \in \mathcal{F}$*. The flat of all symmetric connectors will be denoted by* $\mathrm{Scon}_x\mathcal{M} := \mathbf{T}_x^{<}(\{\mathbf{0}\})$.

The mapping

$$\mathbf{S}_x := (\mathbf{1}_{\mathrm{Con}_x\mathrm{T}\mathcal{M}} + \frac{1}{2}\mathbf{I}_x\mathbf{T}_x)\big|^{\mathrm{Scon}_x\mathcal{M}} \tag{33.11}$$

is the projection of $\mathrm{Con}_x\mathrm{T}\mathcal{M}$ onto $\mathrm{Scon}_x\mathcal{M}$ with

$$\mathrm{Null}\,\nabla\mathbf{S}_x = \mathrm{Skw}_2\,(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M}).$$

If $\mathbf{K} \in \mathrm{Con}_x\mathrm{T}\mathcal{M}$, we call $\mathbf{S}_x(\mathbf{K}) = \mathbf{K} + \frac{1}{2}\mathbf{I}_x\big(\mathbf{T}_x(\mathbf{K})\big)$ the **symmetric part** of $\mathbf{K}$.

Theorem : *A connector* $\mathbf{K} \in \mathrm{Con}_x\mathrm{T}\mathcal{M}$ *is symmetric if and only if* $\mathbf{K} = \mathbf{A}_x^{\chi}$ *for some* $\chi \in \mathrm{Ch}_x\mathcal{M}$*. Thus* $\mathrm{Scon}_x\mathcal{M} = \{\,\mathbf{A}_x^{\chi} \mid \chi \in \mathrm{Ch}_x\mathcal{M}\}$.

Proof: Let $\mathbf{K} \in \mathrm{Con}_x\mathcal{M}$ be given. If $\mathbf{K} = \mathbf{A}_x^{\chi}$ for some $\chi \in \mathrm{Ch}_x\mathcal{M}$, then $\mathbf{\Gamma}_x^{\chi}(\mathbf{K}) = \mathbf{0}$ and hence $\mathbf{T}_x(\mathbf{K}) = \mathbf{0}$ by (33.8).

Assume now that $\mathbf{T}_x(\mathbf{K}) = \mathbf{0}$. We choose $\gamma \in \mathrm{Ch}_x\mathcal{M}$ and put

$$\mathbf{L} := \nabla_x\gamma\,\mathbf{\Gamma}_x^{\gamma}(\mathbf{K}) \circ \big((\nabla_x\gamma)^{-1} \times (\nabla_x\gamma)^{-1}\big)\ . \tag{33.12}$$

It follows from (33.8) that $\mathbf{L}$ is symmetric, i.e. that $\mathbf{L} \in \mathrm{Sym}_2(\mathcal{V}_\gamma^2, \mathcal{V}_\gamma)$. We now define the mapping $\alpha : \mathrm{Dom}\,\gamma \to \mathcal{V}_\gamma$ by

$$\alpha(z) := \gamma(z) + \frac{1}{2}\mathbf{L}\big(\,\gamma(z) - \gamma(x)\,,\,\gamma(z) - \gamma(x)\,\big) \quad \text{for all} \quad z \in \mathrm{Dom}\,\gamma\ .$$

Take the gradient at x, we have $\nabla_x\alpha = \nabla_x\gamma$ i.e. that is $(\nabla_x\alpha)(\nabla_x\gamma)^{-1} = \mathbf{1}_{\mathcal{V}_\gamma}$. It follows from the Local Inversion Theorem that there exist an open subset $\mathcal{N}$ of $\operatorname{Dom}\alpha$ such that $\chi := \alpha|_{\mathcal{N}}^{\alpha_>(\mathcal{N})}$ is a bijection of class C^r. It is easily seen that $\chi \in \mathrm{Ch}_x\mathcal{M}$ and that

$$\nabla_\gamma^{(2)}\chi(x) = \mathbf{L}$$

Using (33.12), (32.25) and $\nabla_x\chi = \nabla_x\gamma$, we conclude that

$$\mathbf{\Gamma}_x^\gamma(\mathbf{K}) = (\nabla_x\chi)^{-1}\nabla_\gamma^{(2)}\chi \circ \left(\nabla_x\gamma \times \nabla_x\gamma\right) = \mathbf{\Gamma}_x^{\gamma,\chi} .$$

Hence, by (32.24) and (32.27), we have

$$\mathbf{A}_x^\gamma - \mathbf{A}_x^\chi = \mathbf{I}_x\mathbf{\Gamma}_x^{\gamma,\chi} = \mathbf{I}_x\mathbf{\Gamma}_x^\gamma(\mathbf{K}) = \mathbf{A}_x^\gamma - \mathbf{K} ,$$

which gives $\mathbf{K} = \mathbf{A}_x^\chi$. ∎

34. Connections, Curvature

From now on, in this chapter, we assume a linear-space bundle $(\mathcal{B}, \tau, \mathcal{M})$ of class C^s, $s \geq 2$, is given. We also assume that both $\mathcal{M}$ and $\mathcal{B}$ have constant dimensions, and put $n := \dim\mathcal{M}$ and $m := \dim\mathcal{B} - \dim\mathcal{M}$. Then we have, as in (32.1),

$$m = \dim\mathcal{B}_x \quad \text{for all} \quad x \in \mathcal{M}. \tag{34.1}$$

<u>Definition</u>: *The* **connector bundle** $\operatorname{Con}\mathcal{B}$ *of* $\mathcal{B}$ *is defined to be the union of all the right-connector spaces*

$$\operatorname{Con}\mathcal{B} := \bigcup_{x\in\mathcal{M}} \operatorname{Con}_x\mathcal{B} . \tag{34.2}$$

It is endowed with the structure of a C^{s-1}-flat space bundle over $\mathcal{M}$ as shown below.

If $\mathcal{P}$ is an open subset of $\mathcal{M}$ and $x \in \mathcal{P}$, we can identify $\operatorname{Con}_x\mathcal{A} \cong \operatorname{Con}_x\mathcal{B}$, where $\mathcal{A} := \tau^<(\mathcal{P})$, in the same way as was done for the tangent space. Hence we may regard $\operatorname{Con}\mathcal{A}$ as a subset of $\operatorname{Con}\mathcal{B}$.

Note that the family $(\operatorname{Con}_x\mathcal{B} \,|\, x \in \mathcal{M})$ is disjoint. The bundle projection $\rho : \operatorname{Con}\mathcal{B} \to \mathcal{M}$ is given by

$$\rho(\mathbf{K}) :\in \left\{ y \in \mathcal{M} \;\middle|\; \mathbf{K} \in \operatorname{Con}_x\mathcal{B} \right\}, \tag{34.3}$$

and, for every $x \in \mathcal{M}$, the bundle inclusion $\mathrm{in}_x : \operatorname{Con}_x\mathcal{B} \to \operatorname{Con}\mathcal{B}$ at x is

$$\mathrm{in}_x := \mathbf{1}_{\operatorname{Con}_x\mathcal{B}\subset\operatorname{Con}\mathcal{B}} . \tag{34.4}$$

For every $(\chi, \phi) \in \mathrm{Ch}\mathcal{M} \times \mathrm{Ch}(\mathcal{B}, \mathcal{M})$ we define

$$\mathrm{con}^{(\chi,\phi)} : \mathrm{Con}(\mathrm{Dom}\phi) \to (\mathrm{Dom}\,\chi \cap \mathcal{O}_\phi) \times \mathrm{Lin}(\mathcal{V}_\chi, \mathrm{Lin}\mathcal{V}_\phi) \tag{34.5}$$

by

$$\begin{gathered}\mathrm{con}^{(\chi,\phi)}(\mathbf{H}) := \Big(\ z\ , \phi\big\rfloor_z\ \mathbf{\Lambda}(\mathbf{A}_z^\phi)(\mathbf{H})\ (\nabla_z\chi^{-1} \times \phi\big\rfloor_z^{-1})\ \Big) \\ \text{where} \quad z := \rho(\mathbf{H})\end{gathered} \tag{34.6}$$

for all $\mathbf{H} \in \mathrm{Con}(\mathrm{Dom}\phi)$. It is easily seen that $\mathrm{con}^{(\chi,\phi)}$ is invertible and

$$\mathrm{con}^{(\chi,\phi)^{\leftarrow}}(z, \mathbf{L}) = \mathbf{A}_z^\phi + \mathbf{I}_z\phi\big\rfloor_z^{-1}\ \mathbf{L}\ (\nabla_z\chi \times \phi\big\rfloor_z) \tag{34.7}$$

for all $z \in (\mathrm{Dom}\chi \cap \mathcal{O}_\phi)$ and all $\mathbf{L} \in \mathrm{Lin}(\mathcal{V}_\chi, \mathrm{Lin}\mathcal{V}_\phi)$. Let $(\chi, \phi), (\gamma, \psi) \in \mathrm{Ch}\mathcal{M} \times \mathrm{Ch}(\mathcal{B}, \mathcal{M})$ be given. We easily deduce from (34.7) and (34.6), with (χ, ϕ) replaced by (γ, ψ) and $\mathbf{\Lambda}(\mathbf{A}_z^\psi)(\mathbf{A}_z^\phi) = -\mathbf{\Gamma}_z^{\psi,\phi} = \mathbf{\Gamma}_z^{\phi,\psi}$, that

$$\begin{aligned}&\big(\mathrm{con}^{(\gamma,\psi)} \circ \mathrm{con}^{(\chi,\phi)^{\leftarrow}}\big)(z, \mathbf{L}) \\ &\quad = \Big(\ z\ ,\ \psi\big\rfloor_z\ \mathbf{\Gamma}_z^{\phi,\psi}(\nabla_z\gamma^{-1} \times \psi\big\rfloor_z^{-1}) + \kappa(z)\,\mathbf{L}\,\big(\nabla_z\lambda \times \kappa(z)^{-1}\big)\Big) \\ &\qquad\qquad \text{where} \ \ \lambda := \gamma \circ \chi^{\leftarrow} \ \ \text{and} \quad \kappa := \psi \diamond \phi \ \ (\textit{see } (22.7))\end{aligned} \tag{34.8}$$

for all $z \in (\mathrm{Dom}\chi \cap \mathcal{O}_\phi) \cap (\mathrm{Dom}\gamma \cap \mathcal{O}_\psi)$ and $\mathbf{L} \in \mathrm{Lin}(\mathcal{V}_\chi, \mathrm{Lin}\mathcal{V}_\phi)$. It is clear that $\mathrm{con}^{(\gamma,\psi)} \circ \mathrm{con}^{(\chi,\phi)^{\leftarrow}}$ is of class C^{s-1}. Since $(\gamma, \psi), (\chi, \phi) \in \mathrm{Ch}\mathcal{M} \times \mathrm{Ch}(\mathcal{B}, \mathcal{M})$ were arbitrary, it follows that $\big\{\, \mathrm{con}^{(\alpha,\phi)} \ \big|\ (\alpha, \phi) \in \mathrm{Ch}\mathcal{M} \times \mathrm{Ch}(\mathcal{B}, \mathcal{M}) \,\big\}$ is a C^{s-1}-bundle atlas of $\mathrm{Con}\,\mathcal{B}$; it determines the natural structure of a C^{s-1} flat-space bundle over $\mathcal{M}$.

The mappings ρ and in_x defined by (34.3) and (34.4) are easily seen to be of class C^{s-1}.

<u>**Definition:**</u> *Let $\mathcal{O}$ be an open subset of $\mathcal{M}$. A cross section on $\mathcal{O}$ of the connector bundle* $\mathrm{Con}\,\mathcal{B}$

$$\mathbf{A} : \mathcal{O} \to \mathrm{Con}\,\mathcal{B} \tag{34.9}$$

is called a **connection on** $\mathcal{O}$ **for** *the bundle $\mathcal{B}$. A connection on $\mathcal{M}$ for the bundle $\mathcal{B}$ is simply called a connection for the bundle $\mathcal{B}$. For every bundle chart ϕ in* $\mathrm{Ch}(\mathcal{B}, \mathcal{M})$*, the connection $\mathbf{A}^\phi$ on $\mathcal{O}_\phi$ is defined by*

$$\mathbf{A}^\phi(x) := \mathbf{A}_x^\phi \qquad \text{for all} \quad x \in \mathcal{O}_\phi, \tag{34.10}$$

where $\mathbf{A}_x^\phi$ is given by (32.21).

<u>**Definition:**</u> *The tangent-space of* $\mathrm{Con}\,\mathcal{B}$ *at* $\mathbf{K}$ *is denoted by*

$$\mathrm{T}_{\mathbf{K}}\mathrm{Con}\,\mathcal{B}. \tag{34.11}$$

We define the **projection mapping** *of* $\mathrm{T}_{\mathbf{K}}\mathrm{Con}\,\mathcal{B}$ *by*

$$\mathbf{P}_{\mathbf{K}} := \nabla_{\mathbf{K}}\rho \in \mathrm{Lin}\,(\mathrm{T}_{\mathbf{K}}\mathrm{Con}\,\mathcal{B}, \mathrm{T}_x\mathcal{M}) \tag{34.12}$$

and the **injection mapping** *of* $\mathrm{T}_{\mathbf{K}}\mathrm{Con}\,\mathcal{B}$ *by*

$$\mathbf{I}_{\mathbf{K}} := \nabla_{\mathbf{K}}\mathrm{in}_x \in \mathrm{Lin}\left(\mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x), \mathrm{T}_{\mathbf{K}}\mathrm{Con}\,\mathcal{B}\right) \tag{34.13}$$

where ρ *and* in_x *are defined by* (34.3) *and* (34.4).

It is clear from (34.5) that

$$\dim(\mathrm{Con}\,\mathcal{B}) = \dim(\mathrm{T}_{\mathbf{K}}\mathrm{Con}\,\mathcal{B}) = n + nm^2. \tag{34.14}$$

Proposition 1: *The projection mapping* $\mathbf{P}_{\mathbf{K}}$ *is surjective, the injection mapping* $\mathbf{I}_{\mathbf{K}}$ *is injective, and we have*

$$\mathrm{Null}\,\mathbf{P}_{\mathbf{K}} = \mathrm{Rng}\,\mathbf{I}_{\mathbf{K}} \tag{34.15}$$

i.e.

$$\mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x) \xrightarrow{\mathbf{I}_{\mathbf{K}}} \mathrm{T}_{\mathbf{K}}\mathrm{Con}\,\mathcal{B} \xrightarrow{\mathbf{P}_{\mathbf{K}}} \mathrm{T}_x\mathcal{M} \tag{34.16}$$

is a short exact sequence.

The short exact sequence (34.16) is of the form (15.1) and hence all of the results in Sect.15 can be used here.

Proposition 2: *For each* $(\chi, \phi) \in \mathrm{Ch}_x\mathcal{M} \times \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$, *let*

$$\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)} \in \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{K}}\mathrm{Con}\,\mathcal{B})$$

be defined by $\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)} := \mathbf{A}_{\mathbf{K}}^{\mathrm{con}^{(\chi,\phi)}}$ *in terms of the notation* (32.21)*; i.e.*

$$\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)} := \left(\nabla_{\mathbf{K}}\mathrm{con}^{(\chi,\phi)}\right)^{-1} \circ \mathrm{ins}_1. \tag{34.17}$$

Then $\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)}$ *is a linear right-inverse of* $\mathbf{P}_{\mathbf{K}}$*; i.e.* $\mathbf{P}_{\mathbf{K}}\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)} = \mathbf{1}_{\mathrm{T}_x\mathcal{M}}$.

Proposition 3: *If* $(\gamma,\psi),(\chi,\phi)\in \mathrm{Ch}_x\mathcal{M}\times \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$*, with* $\mathbf{A}_x^{\phi}=\mathbf{K}=\mathbf{A}_x^{\psi}$*, then*

$$\begin{aligned}\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)}-\mathbf{A}_{\mathbf{K}}^{(\gamma,\psi)}&=\mathbf{I}_{\mathbf{K}}\,\mathbf{\Gamma}_{\mathbf{K}}^{(\chi,\phi),(\gamma,\psi)}\\ \mathbf{\Lambda}(\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)})-\mathbf{\Lambda}(\mathbf{A}_{\mathbf{K}}^{(\gamma,\psi)})&=-\mathbf{\Gamma}_{\mathbf{K}}^{(\chi,\phi),(\gamma,\psi)}\mathbf{P}_{\mathbf{K}}\end{aligned}\tag{34.18}$$

where $\mathbf{\Gamma}_{\mathbf{K}}^{(\chi,\phi),(\gamma,\psi)}:=\mathbf{\Gamma}_{\mathbf{K}}^{\mathrm{con}^{(\chi,\phi)},\mathrm{con}^{(\gamma,\psi)}}$ *in terms of the notation* (32.25) *is given by*

$$\mathbf{\Gamma}_{\mathbf{K}}^{(\chi,\phi),(\gamma,\psi)}(\mathbf{t},\mathbf{t}')=(\psi\rfloor_x)^{-1}\big(\nabla^{(2)}_{\gamma(x)}(\psi\diamond\phi)(\nabla_x\gamma\,\mathbf{t},\nabla_x\gamma\,\mathbf{t}')\big)\phi\rfloor_x\tag{34.19}$$

for all $\mathbf{t},\mathbf{t}'\in \mathrm{T}_x\mathcal{M}$*. We have* $\mathbf{\Gamma}_{\mathbf{K}}^{(\chi,\phi),(\gamma,\psi)}\in \mathrm{Sym}_2(\mathrm{T}_x\mathcal{M}^2,\mathrm{Lin}\mathcal{B}_x)$*. Here, the notation* (22.7) *is used.*

Proof: Let $(\gamma,\psi),(\chi,\phi)\in \mathrm{Ch}_x\mathcal{M}\times \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$, with $\mathbf{A}_x^{\phi}=\mathbf{K}=\mathbf{A}_x^{\psi}$, be given. Then, we have $\nabla_x(\psi\diamond\phi)=\mathbf{\Lambda}(\mathbf{A}_x^{\phi})(\mathbf{K})=\mathbf{0}$. It follows from (34.6) that

$$\mathrm{con}^{(\chi,\phi)}\rfloor_x(\mathbf{K})=\mathbf{0}.\tag{34.20}$$

Using (34.8), (34.20) and (33.25), we obtain

$$\begin{aligned}&\big(\mathrm{con}^{(\gamma,\psi)}\,\square\,\mathrm{con}^{(\chi,\phi)\leftarrow}\big)(z,\mathrm{con}^{(\chi,\phi)}\rfloor_x(\mathbf{K}))\\ &\qquad=\Big(z\,,\,\nabla_z(\psi\diamond\phi)\big(\nabla_z\gamma^{-1}\times(\phi\rfloor_z\circ\psi\rfloor_z^{-1})\big)\Big).\end{aligned}\tag{34.21}$$

Taking the gradient of (34.21) with respect to z at x and observing $\nabla_x(\psi\diamond\phi)=\mathbf{0}$, we have

$$\begin{aligned}&\mathrm{ev}_2\Big(\nabla_x\big((\mathrm{con}^{(\gamma,\psi)}\,\square\,\mathrm{con}^{(\chi,\phi)\leftarrow})\big(\cdot,\mathrm{con}^{(\chi,\phi)}\rfloor_x(\mathbf{K})\big)\big)\mathbf{t}\Big)\\ &\qquad=\big((\nabla^{(2)}_{\gamma(x)}(\psi\diamond\phi)\big)\nabla_x\gamma\,\mathbf{t}\big)(\mathbf{1}_{\mathcal{V}_\gamma}\times(\phi\rfloor_x\circ\psi\rfloor_x^{-1}))\end{aligned}\tag{34.22}$$

for all $\mathbf{t}\in \mathrm{T}_x\mathcal{M}$. Using (34.22), (34.6) with (χ,ϕ) replaced by (γ,ψ) and applying Prop. 3 in Sect. 32 with ϕ replaced by $\mathrm{con}^{(\chi,\phi)}$ and ψ replaced by $\mathrm{con}^{(\gamma,\psi)}$, we obtain the desired result (34.19). ∎

If $\phi,\psi\in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$, with $\mathbf{A}_x^{\phi}=\mathbf{K}=\mathbf{A}_x^{\psi}$, we have $\mathbf{\Gamma}_x^{\phi,\psi}=\mathbf{0}$ by (33.25). It follows from (21.9) that the right hand side of (34.19) does not depend on the manifold charts $\chi,\gamma\in \mathrm{Ch}_x\mathcal{M}$. In particular, when $\psi=\phi$ we have $\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)}=\mathbf{A}_{\mathbf{K}}^{(\gamma,\phi)}$ for all manifold charts $\chi,\gamma\in \mathrm{Ch}_x\mathcal{M}$.

By using the definition of the gradient

$$\nabla_x\mathbf{A}^{\phi}=(\nabla_{\mathbf{K}}\mathrm{con}^{\chi,\phi})^{-1}\nabla_{\chi(x)}\big(\mathrm{con}^{\chi,\phi}\,\square\,\mathbf{A}^{\phi}\,\square\,\chi^{\leftarrow}\big)\nabla_x\chi$$

and (34.6), we can easily seen that for every bundle chart $\phi\in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ with $\mathbf{A}_x^{\phi}=\mathbf{K}$

$$\nabla_x\mathbf{A}^{\phi}=\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)}\qquad\text{for all}\quad \chi\in \mathrm{Ch}_x\mathcal{M}.\tag{34.23}$$

for all bundle charts $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ with $\mathbf{A}_x^\phi = \mathbf{K}$.

Proof: The assertion follows from (34.23) together with (34.18) and (34.19). ▮

Definition: *The* **bracket** $\mathbf{B}_\mathbf{K} \in \mathrm{Skw}_2\,(\mathrm{T}_\mathbf{K}\mathrm{Con}\,\mathcal{B}^2, \mathrm{T}_x\mathcal{M})$ *of* $\mathrm{T}_\mathbf{K}\mathrm{Con}\,\mathcal{B}$ *is defined by*

$$\mathbf{B}_\mathbf{K} := \mathbf{B}_{\mathcal{F}_\mathbf{K}} \tag{34.25}$$

where $\mathbf{B}_{\mathcal{F}_\mathbf{K}}$ *is defined as in* (15.5).

Definition: *Let* $\mathbf{A} : \mathcal{M} \to \mathrm{Con}\,\mathcal{B}$ *be a connection which is differentiable at* x*. The* **curvature of A at** x*, denoted by*

$$\mathbf{R}_x(\mathbf{A}) \in \mathrm{Skw}_2\big(\mathrm{T}_x\mathcal{M}^2, \mathrm{Lin}\mathcal{B}_x\big), \tag{34.26}$$

is defined by

$$\mathbf{R}_x(\mathbf{A}) := \mathbf{T}_{\mathcal{F}_{\mathbf{A}(x)}}(\nabla_x\mathbf{A}) \tag{34.27}$$

where $\mathbf{T}_{\mathcal{F}_{\mathbf{A}(x)}}$ *is defined as in* (15.8).

If $\mathbf{A}$ *is differentiable, then the mapping* $\mathbf{R}(\mathbf{A}) : \mathcal{M} \to \mathrm{Skw}_2(\,\mathrm{Tan}\mathcal{M}^2\,, \mathrm{Lin}\,\mathcal{B}\,)$ *defined by*

$$\mathbf{R}(\mathbf{A})(x) := \mathbf{R}_x(\mathbf{A}) \qquad \text{for all} \qquad x \in \mathcal{M}$$

is called the **curvature field** *of the connection* $\mathbf{A}$.

A fomula for the curvature field $\mathbf{R}(\mathbf{A})$ in terms of covariant gradients will be given in Prop. 5. If the connection $\mathbf{A}$ is of class C^p, with $p \in 1..s-1$, then $\nabla\mathbf{A}$ is of class C^{p-1}, and so is the curvature field $\mathbf{R}(\mathbf{A})$.

More generally, if $\phi, \psi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$, without assuming that $\mathbf{A}_x^\phi = \mathbf{K} = \mathbf{A}_x^\psi$, then Eq. (34.19) must be replaced by

$$\begin{aligned}
&\mathbf{\Gamma}_\mathbf{K}^{(\chi,\phi),(\gamma,\psi)}(\mathbf{t},\mathbf{t}')\\
&\quad = -\mathbf{\Gamma}_x^{\phi,\psi}(\mathbf{t})\mathbf{\Gamma}_x^{\phi}(\mathbf{K})(\mathbf{t}') + \mathbf{\Gamma}_x^{\phi}(\mathbf{K})(\mathbf{t}')\mathbf{\Gamma}_x^{\phi,\psi}(\mathbf{t}) + \mathbf{\Gamma}_x^{\phi}(\mathbf{K})\mathbf{\Gamma}_x^{\chi,\gamma}(\mathbf{t},\mathbf{t}')\\
&\qquad - \mathbf{\Gamma}_x^{\phi,\psi}(\mathbf{t}')\mathbf{\Gamma}_x^{\phi,\psi}(\mathbf{t}) + (\psi\rfloor_x)^{-1}\big(\nabla_\gamma^{(2)}(\psi\diamond\phi)\big)(x)(\nabla_x\gamma\,\mathbf{t}, \nabla_x\gamma\,\mathbf{t}')\phi\rfloor_x
\end{aligned} \tag{34.28}$$

for all $\mathbf{t}, \mathbf{t}' \in \mathrm{T}_x\mathcal{M}$. If one of those two bundle charts, say ϕ, satisfies $\mathbf{A}_x^\phi = \mathbf{K}$, then it follows from (34.28), $\mathbf{\Gamma}_x^\phi(\mathbf{K}) = \mathbf{0}$ and $-\mathbf{\Gamma}_x^{\phi,\psi} = \mathbf{\Gamma}_x^\psi(\mathbf{K})$ that

$$\begin{aligned}
&\mathbf{\Gamma}_\mathbf{K}^{(\chi,\phi),(\gamma,\psi)}(\mathbf{t},\mathbf{t}')\\
&\quad = -\mathbf{\Gamma}_x^{\psi}(\mathbf{K})\mathbf{t}'\mathbf{\Gamma}_x^{\psi}(\mathbf{K})\mathbf{t} + (\psi\rfloor_x)^{-1}\big(\nabla_\gamma^{(2)}(\psi\diamond\phi)\big)(x)(\nabla_x\gamma\,\mathbf{t}, \nabla_x\gamma\,\mathbf{t}')\phi\rfloor_x
\end{aligned} \tag{34.29}$$

for all $\mathbf{t}, \mathbf{t}' \in \mathrm{T}_x\mathcal{M}$.

Proposition 5: *Let* $\mathbf{A} : \mathcal{M} \to \mathrm{Con}\,\mathcal{B}$ *be a connection that is differentiable at* $x \in \mathcal{M}$. *The curvature of* $\mathbf{A}$ *at* x *is given by*

$$\begin{aligned}\big(\mathbf{R}_x(\mathbf{A})\big)(\mathbf{s},\mathbf{t}) &= (\nabla_x^{\gamma,\psi}\mathbf{\Gamma}^{\psi}(\mathbf{A}))(\mathbf{s},\mathbf{t}) - (\nabla_x^{\gamma,\psi}\mathbf{\Gamma}^{\psi}(\mathbf{A}))(\mathbf{t},\mathbf{s}) \\ &\quad + \Big(\mathbf{\Gamma}_x^{\psi}(\mathbf{A}(x))\mathbf{s}\mathbf{\Gamma}_x^{\psi}(\mathbf{A}(x))\mathbf{t} - \mathbf{\Gamma}_x^{\psi}(\mathbf{A}(x))\mathbf{t}\mathbf{\Gamma}_x^{\psi}(\mathbf{A}(x))\mathbf{s}\Big)\end{aligned} \tag{34.30}$$

for all $(\gamma,\psi) \in \mathrm{Ch}_x\mathcal{M} \times \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ *and all* $\mathbf{s},\mathbf{t} \in \mathrm{T}_x\mathcal{M}$.

Proof: Let a bundle chart $(\gamma,\psi) \in \mathrm{Ch}_x\mathcal{M} \times \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ be given. It follows from (42.6) and $\mathbf{\Lambda}(\mathbf{A}_z^{\psi})(\mathbf{A}(z)) = -\mathbf{\Gamma}_z^{\psi}(\mathbf{A}(z))$ that

$$\mathrm{con}^{(\gamma,\psi)} \circ \mathbf{A}(z) = \Big(\ z\ , -\psi\big\rfloor_z \mathbf{\Gamma}_z^{\psi}(\mathbf{A}(z))\ (\nabla_z\gamma^{-1} \times \psi\big\rfloor_z^{-1})\ \Big) \tag{34.31}$$

In view of (32.29), we have

$$\begin{aligned}\mathbf{\Lambda}(\mathbf{A}_{\mathbf{A}(x)}^{(\gamma,\psi)})(\nabla_x\mathbf{A}) &= \mathrm{con}^{(\gamma,\psi)}\big\rfloor_x^{-1}\big(\mathrm{ev}_2 \circ \nabla_{\mathbf{A}(x)}\big(\mathrm{con}^{(\gamma,\psi)}\big)\big)\big(\nabla_x\mathbf{A}\big) \\ &= \mathrm{con}^{(\gamma,\psi)}\big\rfloor_x^{-1}\mathrm{ev}_2 \circ \big(\nabla_x\big(\mathrm{con}^{(\gamma,\psi)} \circ \mathbf{A}\big)\big) \\ &= \nabla_x\Big(z \mapsto -\psi\big\rfloor_x^{-1}\psi\big\rfloor_z\ \mathbf{\Gamma}_z^{\psi}(\mathbf{A}(z))(\nabla_z\gamma^{-1}\nabla_x\gamma \times \psi\big\rfloor_z^{-1}\psi\big\rfloor_x)\Big)\end{aligned} \tag{34.32}$$

By using

$$\mathbf{A}_x^{\gamma} = \nabla_x(z \mapsto \nabla_z\gamma^{-1}\nabla_x\gamma) \quad , \quad \mathbf{A}_x^{\psi} = \nabla_x(z \to \psi\big\rfloor_z^{-1}\psi\big\rfloor_x)$$

and (42.38), we observe that

$$\begin{aligned}\mathbf{\Lambda}(\mathbf{A}_{\mathbf{A}(x)}^{(\gamma,\psi)})(\nabla_x\mathbf{A}) &= \nabla_x\Big(z \mapsto -\psi\big\rfloor_x^{-1}\psi\big\rfloor_z\ \mathbf{\Gamma}_z^{\psi}(\mathbf{A}(z))(\nabla_z\gamma^{-1}\nabla_x\gamma \times \psi\big\rfloor_z^{-1}\psi\big\rfloor_x)\Big) \\ &= -\big(\Box_x\mathbf{\Gamma}^{\psi}(\mathbf{A})\big)(\mathbf{A}_x^{\gamma},\mathbf{A}_x^{\psi}) \\ &= -\nabla_x^{\gamma,\psi}\mathbf{\Gamma}^{\psi}(\mathbf{A}).\end{aligned}$$

Together with (42.27) and (42.29), we prove (34.12). ∎

Remark : When the linear-space bundle $\mathcal{B}$ is the tangent bundle $\mathrm{T}\mathcal{M}$, we have

$$\begin{aligned}\big(\mathbf{R}_x(\mathbf{A})\big)(\mathbf{s},\mathbf{t}) &= (\nabla_x^{\chi}\mathbf{\Gamma}^{\chi}(\mathbf{A}))(\mathbf{s},\mathbf{t}) - (\nabla_x^{\chi}\mathbf{\Gamma}^{\chi}(\mathbf{A}))(\mathbf{t},\mathbf{s}) \\ &\quad + \Big(\mathbf{\Gamma}_x^{\chi}(\mathbf{A}(x))\mathbf{s}\mathbf{\Gamma}_x^{\chi}(\mathbf{A}(x))\mathbf{t} - \mathbf{\Gamma}_x^{\chi}(\mathbf{A}(x))\mathbf{t}\mathbf{\Gamma}_x^{\chi}(\mathbf{A}(x))\mathbf{s}\Big)\end{aligned} \tag{34.33}$$

for all manifold chart $\chi \in \mathrm{Ch}_x\mathcal{M}$ and all $\mathbf{s},\mathbf{t} \in \mathrm{T}_x\mathcal{M}$.

If a transport $\mathbf{T} : \mathcal{M} \to \mathrm{Tlis}_x\mathcal{M}$ from x is differentiable at y, we define the **connector-gradient**, $\nabla\!\!\!\!\nabla_y\ \mathbf{T} \in \mathrm{Lin}\,(\mathcal{T}_y,\mathcal{S}_y)$, of $\mathbf{T}$ at y by

$$\nabla\!\!\!\!\nabla_y\ \mathbf{T} := \nabla_y\big(z \mapsto \mathbf{T}(z)\mathbf{T}(y)^{-1}\big). \tag{34.34}$$

Theorem : *A connection* $\mathbf{A} : \mathcal{M} \to \mathrm{Con}\mathcal{B}$ *is curvature-free if and only if, locally* $\mathbf{A}$ *agrees with* $\mathbf{A}^{\phi}$ *for some bundle chart* $\phi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})$*. In other word, for every* $x \in \mathcal{M}$*, there is an open neighbourhood* $\mathcal{N}_x$ *of* x *and a transport* $\mathbf{T} : \mathcal{N}_x \to \mathrm{Tlis}_x\mathcal{M}$ *from* x *such that* $\nabla \mathbf{T} = \mathbf{A}\big|_{\mathcal{N}_x}$

Proof: Ask Prof. Noll!!!!!!!!!!!!!!!!!!

35. Parallelisms, Geodesics

Let a connector $\mathbf{K} \in \mathrm{Con}\,\mathcal{B}$ be given and put $x := \rho(\mathbf{K})$.

We now apply the results of Sect. 32 by replacing the ISO-bundle there by the flat-space bundle $\mathrm{Con}\,\mathcal{B}$ and $\mathbf{b} \in \mathcal{B}$ there by $\mathbf{K}$.

<u>**Definition**</u>**:** *The* **shift bundle** $\mathrm{S}\mathcal{B}$ *of* $(\mathcal{B}, \tau, \mathcal{M})$ *is defined to be the union of all the shift spaces of* $\mathcal{B}$:

$$\mathrm{S}\mathcal{B} := \bigcup_{y \in \mathcal{M}} \mathrm{S}_y\mathcal{B}. \tag{35.1}$$

It is endowed with the structure of a C^{r-2}*-manifold.*

We defined the mapping $\sigma : \mathrm{S}\mathcal{B} \to \mathcal{M}$ by

$$\sigma(\mathbf{s}) :\in \left\{ \, y \in \mathcal{M} \;\middle|\; \mathbf{s} \in \mathrm{S}_y\mathcal{B} \, \right\}, \tag{35.2}$$

and every $y \in \mathcal{M}$ the mapping $\mathrm{in}_y : \mathrm{S}_y\mathcal{B} \to \mathrm{S}\mathcal{B}$ by

$$\mathrm{in}_y := \mathbf{1}_{\mathrm{S}_y\mathcal{B} \subset \mathrm{S}\mathcal{B}} \; . \tag{35.3}$$

We define the **projection** $\mathbf{P} : \mathrm{S}\mathcal{B} \to \mathrm{T}\mathcal{M}$ by

$$\mathbf{P}(\mathbf{s}) := \mathbf{P}_{\sigma(\mathbf{s})}\mathbf{s} \qquad \text{for all} \qquad \mathbf{s} \in \mathrm{S}\mathcal{B} \tag{35.4}$$

and the **injection** $\mathbf{I} : \mathrm{Lin}\,\mathcal{B} \to \mathrm{S}\mathcal{B}$ by

$$\mathbf{I}(\mathbf{L}) := \mathbf{I}_{\tau^{\mathrm{Ln}}(\mathbf{L})}\,\mathbf{L} \qquad \text{for all} \qquad \mathbf{L} \in \mathrm{Lin}\,\mathcal{B} \tag{35.5}$$

where Ln is the lineon functor (see Sect.13) and

$$\mathrm{Lin}\,\mathcal{B} := \mathrm{Ln}(\mathcal{B}) = \bigcup_{y \in \mathcal{M}} \mathrm{Lin}\,\mathcal{B}_y. \tag{35.6}$$

We have

$$\mathrm{pt}(\mathbf{P}(\mathbf{s})) = \sigma(\mathbf{s}) \qquad \text{for all} \qquad \mathbf{s} \in \mathrm{S}\mathcal{B} \tag{35.7}$$

and

$$\sigma(\mathbf{I}\,\mathbf{L}) = \tau^{\mathrm{Ln}}(\mathbf{L}) \qquad \text{for all} \qquad \mathbf{L} \in \mathrm{Lin}\,\mathcal{B}. \tag{35.8}$$

It is easily seen that $\mathbf{P}$ and $\mathbf{I}$ are of class C^{r-2}.

We now fix $x \in \mathcal{M}$ and consider the bundle $\mathrm{Tlis}_x\mathcal{B}$ of transfer-isomorphism from x as defined by (32.2). A mapping of the type

$$\mathbf{T} : [0,d] \to \mathrm{Tlis}_x\mathcal{B} \qquad \text{with} \qquad \mathbf{T}(0) = \mathbf{1}_{\mathcal{B}_x}\ , \tag{35.9}$$

where $d \in {}^{\times}$, will be called a **transfer-process** of $\mathcal{B}$ from x. If $\mathbf{T}$ is differentiable at a given $t \in [0,d]$, we defined the **shift-derivative** $\mathrm{sd}_t\mathbf{T} \in \mathrm{S}_{\pi_x(\mathbf{T}(t))}\mathcal{B}$ at t of $\mathbf{T}$ by

$$\mathrm{sd}_t\mathbf{T} := \partial_t \left(s \mapsto \mathbf{T}(s)\mathbf{T}(t)^{-1} \right)\ . \tag{35.10}$$

We have

$$\sigma\left(\mathrm{sd}_t\mathbf{T}\right) = \pi_x\left(\mathbf{T}(t)\right)\ , \tag{35.11}$$

when π_x is defined by (32.3). If $\mathbf{T}$ is differentiable, we define the **shift-derivative** (-process) $\mathrm{sd}\mathbf{T} : [0,d] \to \mathrm{S}\mathcal{B}$ by

$$\left(\mathrm{sd}\mathbf{T}\right)(t) := \mathrm{sd}_t\mathbf{T} \qquad \text{for all} \qquad t \in [0,d]\ . \tag{35.12}$$

If $\mathbf{T}$ is of class C^s, $s \in 1..(r-2)$, then $\mathrm{sd}\mathbf{T}$ is of class C^{s-1}.

Proposition 1: *Let $\mathbf{T} : [0,d] \to \mathrm{Tlis}_x\mathcal{B}$ be a transfer-process of $\mathcal{B}$ from x and put*

$$p := \pi_x \circ \mathbf{T} = \sigma \circ (\mathrm{sd}\mathbf{T}) : [0,d] \to \mathcal{M}. \tag{35.13}$$

Then p is differentiable and

$$\mathbf{P} \circ (\mathrm{sd}\mathbf{T}) = p^{\cdot}\ . \tag{35.14}$$

Proof: Let $t \in [0,d]$ be given and put $y := p(t)$. Then $\mathbf{T}(s)\mathbf{T}(t)^{-1} \in \mathrm{Tlis}_y\mathcal{B}$ and

$$\pi_y\left(\mathbf{T}(s)\mathbf{T}(t)^{-1}\right) = \pi_x\left(\mathbf{T}(s)\right) = p(s)$$

for all $s \in [0,d]$. Differentiation with respect to s at t, using (35.10), (32.10), and the chain rule, gives $\mathbf{P}_y(\mathrm{sd}_t\mathbf{T}) = p^{\cdot}(t)$. Since $t \in [0,d]$ was arbitrary, (35.14) follows. ∎

Proposition 2: *Let* $\mathbf{T}$ *be a differentiable transfer-process from* x *and let* p *be defined as in Prop. 1. Assume, moreover, that* $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ *is a chart such that* $\mathrm{Rng}\, p \subset \mathcal{O}_\phi$. *If we define* $\mathbf{H} : [0, d] \to \mathrm{Lis}\mathcal{B}_x$ *and* $\mathbf{V} : [0, d] \to \mathrm{Lin}\mathcal{B}_x$ *by*

$$\mathbf{H}(t) := (\phi\rfloor_y)\, \mathbf{T}(t) \tag{35.15}$$

and

$$\mathbf{V}(t) := \phi\rfloor_y \left(\mathbf{\Lambda}(\mathbf{A}_y^\phi)(\mathrm{sd}_t\mathbf{T})\right) (\phi\rfloor_y)^{-1} \tag{35.16}$$

when $y := p(t)$ *and* $t \in [0, d\,]$, *then*

$$\mathbf{H}^{\boldsymbol{\cdot}} = \mathbf{V}\mathbf{H} \quad , \quad \mathbf{H}(0) = \mathbf{1}_{\mathcal{B}_x}\ . \tag{35.17}$$

Proof: Let $t \in [0, d]$ be given and put $y := p(t)$. Using (32.6) with x replaced by y and $\mathbf{T}$ by $\mathbf{T}(s)\mathbf{T}(t)^{-1}$, we obtain from (35.15) that

$$\mathrm{tlis}_y^\phi(\mathbf{T}(s)\mathbf{T}(t)^{-1}) = \Big(p(s)\ ,\ \phi\rfloor_y\mathbf{H}(s)\mathbf{H}(t)^{-1}(\phi\rfloor_y)^{-1}\Big) \quad \text{for all} \quad s \in [0, d].$$

In view of (31.30) with ϕ replaced by tlis_y^ϕ and (35.10) we conclude that

$$\left(\nabla_{\mathbf{1}_{\mathrm{T}_y}}\mathrm{tlis}_y^\phi\right)(\mathrm{sd}_t\mathbf{T}) = \left(p^{\boldsymbol{\cdot}}(t)\ ,\ \phi\rfloor_y(\mathbf{H}^{\boldsymbol{\cdot}}\mathbf{H}^{-1})(t)(\phi\rfloor_y)^{-1}\right).$$

Comparing this result with (31.29) and (35.16), and using the injectivity of $\nabla_{\mathbf{1}_{\mathrm{T}_x}}\mathrm{tlis}_y^\phi$, we obtain $(\mathbf{H}^{\boldsymbol{\cdot}}\mathbf{H}^{-1})(t) = \mathbf{V}(t)$. Since $t \in [0, d]$ was arbitrtary, $(35.17)_1$ follows. Since both $\phi\rfloor_x = \mathbf{1}_{\mathcal{B}_x}$ and $\mathbf{T}(0) = \mathbf{1}_{\mathcal{B}_x}$, $(35.17)_2$ is a direct consequence of (35.15). ∎

Theorem on Shift-Processes: *Let* $\mathbf{U} : [0, d] \to \mathrm{S}\mathcal{B}$, *with* $d \in {}^\times$, *be a continuous shift-process of* $\mathcal{B}$ *such that* $p := \sigma \circ \mathbf{U}$ *is differentiable and*

$$\mathbf{P} \circ \mathbf{U} = p^{\boldsymbol{\cdot}} : [0, d] \to \mathrm{Tan}\mathcal{M}\ . \tag{35.18}$$

Then there exists exactly one transfer-process $\mathbf{T} : [0, d] \to \mathrm{Tlis}_x\mathcal{B}$ *of* $\mathcal{B}$ *from* $x := p(0)$, *of class* C^1, *such that* $\mathrm{sd}\mathbf{T} = \mathbf{U}$.

Proof: Assume first that $\phi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})$ can be chosen such that $\mathrm{Rng}\, p \subset \mathrm{Dom}\, \chi$. Define $\overline{\mathbf{V}} : [0, d] \to \mathrm{Lin}\mathcal{V}_\phi$ by

$$\overline{\mathbf{V}}(t) := (\phi\rfloor_y) \left(\mathbf{\Lambda}(\mathbf{A}_y^\phi)\mathbf{U}(t)\right) (\phi\rfloor_y)^{-1} \quad \text{when} \quad y := p(t). \tag{35.19}$$

Since $\mathbf{U}$ is continuous, so is $\overline{\mathbf{V}}$. Let $\overline{\mathbf{H}} : [0, d] \to \mathrm{Lin}\mathcal{V}_\phi$ be the unique solution of the initial value problem

$$?\, \overline{\mathbf{H}} \quad , \quad \overline{\mathbf{H}}^{\boldsymbol{\cdot}} = \overline{\mathbf{V}}\, \overline{\mathbf{H}} \quad , \quad \overline{\mathbf{H}}(0) = \mathbf{1}_{\mathcal{V}_\phi}\ . \tag{35.20}$$

This solution is of class C^1.

Now, if $\mathbf{T}$ is a process that satisfies the conditions, then $\overline{\mathbf{V}}$, as defined by (35.19), coincides with $\mathbf{V}$, as defined by (35.16). Therefore, by Prop. 2, we have $\mathbf{H} = \overline{\mathbf{H}}$ and hence $\mathbf{T}$ must be given by

$$\mathbf{T}(t) = (\phi\rfloor_{p(t)})^{-1}\overline{\mathbf{H}}(t)\phi\rfloor_{x} \qquad \text{for all} \qquad t \in [0,d]. \tag{35.21}$$

On the other hand, if we *define* $\mathbf{T}$ by (35.21) and then $\mathbf{H}$ and $\mathbf{V}$ by (35.15) and (35.16), we have $\pi_x \circ \mathbf{T} = p$, $\overline{\mathbf{H}} = \mathbf{H}$, and $\overline{\mathbf{V}} = \mathbf{V}$. Thus, using (31.30) with ϕ replaced by tlis_y^{ϕ} and (35.19), we conclude that

$$\big(\nabla_{\mathbf{1}_{\mathcal{B}_y}} \mathrm{tlis}_y^{\phi}\big)(\mathrm{sd}_t\mathbf{T}) = \big(\nabla_{\mathbf{1}_{\mathcal{B}_y}} \mathrm{tlis}_y^{\phi}\big)(\mathbf{U}(t)) \quad \text{when} \quad y := p(t)$$

for all $t \in [0,d]$. Since $\nabla_{\mathbf{1}_{\mathcal{B}_y}} \mathrm{tlis}_y^{\phi}$ is injective for all $y \in \mathcal{M}$, we conclude that $\mathbf{U} = \mathrm{sd}\mathbf{T}$.

There need not be a single bundle chart $\phi \in \mathrm{Ch}(\mathcal{B},\mathcal{M})$ such that $\mathrm{Rng}\, p \subset \mathrm{Dom}\,\chi$. However, since $\mathrm{Rng}\, p$ is a compact subset of $\mathcal{M}$, we can find a finite set $\mathfrak{F} \subset \mathrm{Ch}\mathcal{M}$ such that

$$\mathrm{Rng}\, p \subset \bigcup_{\chi\in\mathfrak{F}} \mathrm{Dom}\,\chi.$$

We can then determine a strictly isotone list $\big(a_i \mid i \in (m+1)^{[}\big)$ in such that $a_0 = 0$, $a_m = d$ and such that, for each $i \in m^{[}$, $p_{>}([a_i, a_{i+1}])$ is included in a single chart belonging to $\mathfrak{F}$. By applying the result already proved, for each $i \in m^{[}$, to the case when $\mathbf{U}$ is replaced by

$$\big(\, t \mapsto \mathbf{U}(a_i + t)\,\big) : [\,0\,,\, a_{i+1} - a_i] \to \mathrm{S}\mathcal{B},$$

one easily sees that the assertion of the theorem is valid in general. ∎

We assume now that a continuous connection $\mathbf{C}$ is prescribed.

Let $d \in {}^{\times}$ and a process $p : [0,d] \to \mathcal{M}$ of class C^1 be given and put $x := p(0)$. We define the shift process $\mathbf{U} : [0,d] \to \mathrm{S}\mathcal{B}$ by

$$\mathbf{U}(t) := \mathbf{C}\big(p(t)\big)p^{\cdot}(t) \quad \text{for all} \quad t \in [0,d]. \tag{35.22}$$

Clearly, $\mathbf{U}$ is continuous and, since $\mathbf{P}_y\mathbf{C}(y) = \mathbf{1}_{\mathrm{T}_y}$ for all $y \in \mathcal{M}$, (35.18) is valid. Hence, by the Theorem on Shift Processes there is a unique transfer process $\mathbf{T} : [0,d] \to \mathrm{Tlis}_x\mathcal{B}$ of class C^1 such that

$$\mathrm{sd}\mathbf{T} = (\mathbf{C} \circ p)p^{\cdot}\ . \tag{35.23}$$

This process is called the **parallelism along** p for the connection $\mathbf{C}$.

Let $\mathbf{H} : [0,d\,] \to \mathbf{\Phi}(\mathcal{B})$ be a process on $\mathbf{\Phi}(\mathcal{B})$ and put $p := \tau \circ \mathbf{H}$. We say that $\mathbf{H}$ is a **parallel process** for $\mathbf{C}$ if $\mathbf{H}(0) \neq \mathbf{0}$ and if

$$\mathbf{H}(t) = \mathbf{\Phi}(\mathbf{T}(t))\mathbf{H}(0) \qquad \text{for all} \quad t \in [0,d\,] \tag{35.24}$$

where $\mathbf{T}$ is the parallelism along p for $\mathbf{C}$.

Let $\mathbf{H} : [0, d\,] \to \mathbf{\Phi}(\mathcal{B})$ be a process on $\mathbf{\Phi}(\mathcal{B})$ and let $\mathbf{T}$ be the parallelism along $p := \tau^{\mathbf{\Phi}} \circ \mathbf{H}$ for the connection $\mathbf{C}$. Given $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ that satisfies $\mathrm{Rng}\, p \subset \mathcal{O}_\phi$. Define $\left(\mathbf{H}^{\phi\rfloor}\right)^\bullet : [0, d\,] \to \tau^<(\mathrm{Rng}p)$ and $\left(\mathbf{H}^{T}\right)^\bullet : [0, d\,] \to \tau^<(\mathrm{Rng}p)$ by

$$\begin{aligned} \left(\mathbf{H}^{\phi\rfloor}\right)^\bullet(t) &:= \partial_t\big(\; s \mapsto \mathbf{\Phi}(\phi\rfloor_{p(t)}^{-1}\phi\rfloor_{p(s)})\mathbf{H}(s)\;\big) \\ \left(\mathbf{H}^{T}\right)^\bullet(t) &:= \partial_t\big(\; s \mapsto \mathbf{\Phi}(\mathbf{T}(t)\mathbf{T}^{-1}(s))\mathbf{H}(s)\;\big) \end{aligned} \tag{35.25}$$

for all $t \in [0, d\,]$.

Proposition 3: *A process $\mathbf{H} : [0, d] \to \mathbf{\Phi}(\mathcal{B})$ is parallel with respect to $\mathbf{C}$ if and only if $\mathbf{H}$ is of class C^1 and satisfies the differential equation*

$$\mathbf{0} = \left(\mathbf{H}^{T}\right)^\bullet = \left(\mathbf{H}^{\phi\rfloor}\right)^\bullet + \mathbf{\Phi}^\bullet\big(\left(\mathbf{\Gamma}^\phi(\mathbf{C}) \circ p\right) p^\bullet\big)\mathbf{H}. \tag{35.26}$$

We assume now that the linear space bundle $\mathcal{B}$ is the tangent bundle $\mathrm{T}\mathcal{M}$ and that a continuous connection $\mathbf{C} : \mathcal{M} \to \mathrm{ConT}\mathcal{M}$ for $\mathrm{T}\mathcal{M}$ is prescribed.

We say that $p : [0, d] \to \mathcal{M}$ is a **geodesic process** for $\mathbf{C}$ if $p^\bullet(0) \neq \mathbf{0}$ and if

$$\mathbf{T}(t)p^\bullet(0) = p^\bullet(t) \quad \text{for all} \quad t \in [0, d], \tag{35.28}$$

where $\mathbf{T}$ is the parallelism along p for $\mathbf{C}$, i.e. $p^\bullet$ is parallel with respect to the parallelism $\mathbf{T}$.

Let $p : [0, d\,] \to \mathcal{M}$ be a process of class C^1 such that $p^\bullet(0) \neq \mathbf{0}$ and given $\chi \in \mathrm{Ch}\mathcal{M}$ that satisfies $\mathrm{Rng}\, p \subset \mathrm{Dom}\,\chi$. Define $\overline{p} : [0, d\,] \to \mathrm{Cod}\,\chi$ by $\overline{p} := \chi \circ p$ and $\overline{\mathbf{\Gamma}} : \mathrm{Cod}\,\chi \to \mathrm{Lin}_2\,(\mathcal{V}_\chi^2, \mathcal{V}_\chi)$ by

$$\overline{\mathbf{\Gamma}}(z) := \nabla_y\chi\,\mathbf{\Gamma}_y^\chi(\mathbf{C}(y)) \circ (\nabla_y\chi^{-1} \times \nabla_y\chi^{-1}) \quad \text{when} \quad y := \chi^{\leftarrow}(z), \tag{35.29}$$

where $\mathbf{\Gamma}_y^\chi$ is defined by (33.3).

Proposition 4: *The process p is a geodedic process if and only if $\overline{p}$ is of class C^2 and satisfies the differential equation*

$$\overline{p}^{\bullet\bullet} + \left(\overline{\mathbf{\Gamma}} \circ \overline{p}\right)(\overline{p}^{\,\bullet}, \overline{p}^{\,\bullet}) = \mathbf{0}\ . \tag{35.30}$$

Geodesic Deviations: **Study the derivative of (35.26)???**

36. Holonomy

Let a continuous connection $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\mathcal{B}$ be given. For every C^1 process $p : [0, d_p] \to \mathcal{M}$ there is exactly one parallelism $\mathbf{T}_p : [0, d_p] \to \mathrm{Tlis}_x\mathcal{B}$ from $x := p(0)$ along p for the connection $\mathbf{C}$. The **reverse process** $p^- : [0, d_p] \to \mathcal{M}$ of $p : [0, d_p] \to \mathcal{M}$ is given by

$$p^-(t) := p(d_p - t) \qquad \text{for all} \quad t \in [0, d_p].$$

Proposition 1: *Let $p^- : [0, d_p] \to \mathcal{M}$ be the reverse process of a C^1 process $p : [0, d_p] \to \mathcal{M}$. We have*

$$\mathbf{T}_{p^-}(t) = \mathbf{T}_p(d_p - t)\mathbf{T}_p^{-1}(d_p) \quad \text{for all} \quad t \in [0, d_p]. \tag{36.1}$$

Let C^1 processes $p : [0, d_p] \to \mathcal{M}$ and , $q : [0, d_q] \to \mathcal{M}$ with $q(0) = p(d_p)$ be given. We define the **continuation process** $q * p : [0, d_p + d_q] \to \mathcal{M}$ of p with q by

$$(q * p)(t) := \begin{cases} p(t) & t \in [0, d_p], \\ q(t - d_p) & t \in [d_p, d_p + d_q]. \end{cases} \tag{36.2}$$

If in addition that $\overset{\bullet}{q}(0) = \overset{\bullet}{p}(d_p)$, then the continuation process $q * p$ is of class C^1 and

$$\mathbf{T}_{q*p}(t) = \begin{cases} \mathbf{T}_p(t) & t \in [0, d_p], \\ \mathbf{T}_q(t - d_p)\mathbf{T}_p(d_p) & t \in [d_p, d_p + d_q]. \end{cases} \tag{36.3}$$

<u>**Definition**</u>: *For every pair of C^1 processes $p : [0, d_p] \to \mathcal{M}$ and , $q : [0, d_q] \to \mathcal{M}$ with $q(0) = p(d_p)$ be given. We define the piecewise parallelism (along $q * p$)*

$$\mathbf{T}_{q*p} : [0, d_p + d_q] \to \mathrm{Tlis}_x\mathcal{B} \quad \text{where} \quad x := p(0)$$

by

$$\mathbf{T}_{q*p}(t) := \begin{cases} \mathbf{T}_p(t) & t \in [0, d_p], \\ \mathbf{T}_q(t - d_p)\mathbf{T}_p(d_p) & t \in [d_p, d_p + d_q]. \end{cases} \tag{36.4}$$

In view of (36.1), if $q := p^-$ we have $\mathbf{T}_{p^-}(t - d_p)\mathbf{T}_p(d_p) = \mathbf{T}_p(2d_p - t)$ and hence

$$\mathbf{T}_{-p*p}(t) := \begin{cases} \mathbf{T}_p(t) & t \in [0, d_p], \\ \mathbf{T}_p(2d_p - t) & t \in [d_p, 2d_p]. \end{cases} \tag{36.5}$$

In particular, $\mathbf{T}_{p^- * p}(2d_p) = \mathbf{T}_{-p*p}(0) = \mathbf{1}_{\mathcal{B}_x}$.

Let $\mathcal{O}$ be an open neighboorhood of $x \in \mathcal{M}$ and let $\mathcal{L}(\mathcal{O}, x)$ be the set of all piecewise C^1 loops $p : [0, d_p] \to \mathcal{M}$ at x with $\mathrm{Rng}p \subset \mathcal{O}$. It is easily seen that $(\mathcal{L}(\mathcal{O}, x), *)$ is a group. We also use the following notation

$$\mathcal{H}(\mathcal{O}, x) := \{\mathbf{T}_p(d_p) \,|\, p \in \mathcal{L}(\mathcal{O}, x)\}. \tag{36.6}$$

Proposition 3: *For every $q, p \in \mathcal{L}(\mathcal{O}, x)$, we have*

$$\mathbf{T}_{q*p}(d_p + d_q) = \mathbf{T}_q(d_q)\mathbf{T}_p(d_p). \tag{36.7}$$

Hence $\mathcal{H}(\mathcal{O}, x)$ is a subgroup of $\mathrm{Lis}\mathcal{B}_x$, *which is called the* **holonomy group** *on $\mathcal{O}$ of the connection* $\mathbf{C}$ **at** x.

Let $\mathbf{T} : \mathcal{M} \to \mathrm{Tlis}_x\mathcal{M}$ be a transport from $x \in \mathcal{M}$ of class C^1. For every differentiable process $\lambda : [0, 1] \to \mathcal{M}$, we see that $\mathbf{T} \circ \lambda : [0, 1] \to \mathrm{Tlis}_x\mathcal{M}$ is a transfer process from x and

$$\mathrm{sd}\mathbf{T} = ((\nabla\mathbf{T}) \circ \lambda)\lambda^{\bullet}.$$

Hence $\mathbf{T} \circ \lambda$ is the parallelism along λ for the connection $\nabla\mathbf{T}$. For every $t \in [0, 1]$, $(\mathbf{T} \circ \lambda)(t) = \mathbf{T}(\lambda(t))$ depends on, of course, only on the point $y := \lambda(t)$, not on the process λ. When λ is closed, beginning and ending at $\lambda(0) = x = \lambda(1)$, then

$$(\mathbf{T} \circ \lambda)(1) = \mathbf{T}(x) = \mathbf{1}_{\mathcal{B}_x}.$$

The following theorem is a immediated consequence of the above discussion and the Theorem of Sect.34.

Theorem : *A continuous connection $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\mathcal{B}$ is curvature-free; i.e. $\mathbf{R}(\mathbf{C}) = \mathbf{0}$ if and only if locally the holonomy groups are $\mathcal{H}(\mathcal{O}, x) = \{\mathbf{1}_{\mathcal{B}_x}\}$ for some open subset set $\mathcal{O}$ of $\mathcal{M}$ and all $x \in \mathcal{M}$.*

Question ?: Does there exist a connection $\mathbf{C}$ such that $\mathcal{H}(\mathcal{O}, x) = \mathrm{Lis}\mathcal{B}_x$ for some x?

Chapter 4

Gradients.

In this chapter, we assume a linear-space bundle $(\mathcal{B}, \tau, \mathcal{M})$ of class C^s, $s \geq 2$, is given. We also assume that both $\mathcal{M}$ and $\mathcal{B}$ have constant dimensions, and put $n := \dim \mathcal{M}$ and $m := \dim \mathcal{B} - \dim \mathcal{M}$. Then we have, as in (32.1), $m = \dim \mathcal{B}_x$ for all $x \in \mathcal{M}$.

41. Shift Gradients

Let $x \in \mathcal{M}$ be fixed.

Let $\mathbf{\Phi}$ be an analytic tensor functor and let $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathcal{B})$ be a cross section of $\mathbf{\Phi}(\mathcal{B})$ that is differentiable at x. We define the mapping

$$\widehat{\mathbf{H}} : \mathrm{Tlis}_x\mathcal{B} \to \mathbf{\Phi}(\mathcal{B}_x) \tag{41.1}$$

by

$$\widehat{\mathbf{H}}(\mathbf{T}) := \mathbf{\Phi}(\mathbf{T})^{-1}\mathbf{H}(\pi_x(\mathbf{T})) \quad \text{for all} \quad \mathbf{T} \in \mathrm{Tlis}_x\mathcal{B}, \tag{41.2}$$

where π_x is defined by (32.3). Since $\mathbf{\Phi}$ is analytic, it is clear that $\widehat{\mathbf{H}}$ is differentiable at $\mathbf{1}_{\mathcal{B}_x}$.

Difinition: *The* **shift-gradient** *of* $\mathbf{H}$ *at* x *is the linear mapping*

$$\Box_x\mathbf{H} \in \mathrm{Lin}\left(\mathrm{S}_x\mathcal{B}, \mathbf{\Phi}(\mathcal{B}_x)\right)$$

defined by

$$\Box_x\mathbf{H} := \nabla_{\mathbf{1}_{\mathcal{B}_x}}\widehat{\mathbf{H}}, \tag{41.3}$$

where $\widehat{\mathbf{H}}$ *is given by* (41.2).

For every bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$, the spaces $\mathrm{Rng}\,\mathbf{I}_x$ and $\mathrm{Rng}\,\mathbf{A}_x^{\phi}$ are supplymentary in $\mathrm{S}_x\mathcal{B}$. Hence, for every $\mathbf{s} \in \mathrm{S}_x\mathcal{B}$ there is exactly one pair $(\mathbf{M}, \mathbf{t}) \in \mathrm{Lin}\,\mathcal{B}_x \times \mathrm{T}_x\mathcal{M}$ such that $\mathbf{s} = \mathbf{I}_x\mathbf{M} + \mathbf{A}_x^{\phi}\mathbf{t}$ and thus

$$(\Box_x\mathbf{H})\mathbf{s} = (\Box_x\mathbf{H})\mathbf{I}_x\mathbf{M} + (\Box_x\mathbf{H})\mathbf{A}_x^{\phi}\mathbf{t}.$$

Proposition 1: *We have*

$$(\Box_x\mathbf{H})\mathbf{I}_x\mathbf{M} = -(\mathbf{\Phi}_x^{\bullet}\mathbf{M})\mathbf{H}(x) \quad \text{for all} \quad \mathbf{M} \in \mathrm{Lin}\,\mathcal{B}_x, \tag{41.4}$$

where $\mathbf{\Phi}_x^{\bullet} \in \mathrm{Lin}\,(\mathrm{Lin}\,\mathcal{B}_x, \mathrm{Lin}\,\mathbf{\Phi}(\mathcal{B}_x))$ *is defined to be the gradient of the mapping* $(\mathbf{L} \mapsto \mathbf{\Phi}(\mathbf{L})) : \mathrm{Lis}\,\mathcal{B}_x \to \mathrm{Lis}\,(\mathbf{\Phi}(\mathcal{B}_x))$ *at* $\mathbf{1}_{\mathcal{B}_x}$.

Proof: In view of (32.4) and (41.2) we have $\widehat{\mathbf{H}} \circ \iota_x : \mathrm{Lis}\,\mathcal{B}_x \to \mathbf{\Phi}(\mathcal{B}_x)$ and

$$(\widehat{\mathbf{H}} \circ \iota_x)(\mathbf{L}) = \mathbf{\Phi}(\mathbf{L})^{-1}\mathbf{H}(x) \quad \text{for all} \quad \mathbf{L} \in \mathrm{Lis}\,\mathcal{B}_x.$$

Taking the gradient of $(\widehat{\mathbf{H}}_x \circ \iota_x)$ at $\mathbf{1}_{\mathcal{B}_x}$ and using (32.11) and (41.3), we obtain the desired result (41.4). ∎

Example 1: Let $\mathcal{B}^* := \mathrm{Dl}\,(\mathcal{B})$, where Dl is the duality functor.

Let $\mathbf{h}$ be a cross section of $\mathcal{B}$, let $\boldsymbol{\omega}$ be a cross section of $\mathcal{B}^*$, let $\mathbf{L}$ be a cross section of $\mathrm{Lin}\,\mathcal{B}$, let $\mathbf{G}$ be a cross section of $\mathrm{Lin}\,(\mathcal{B}, \mathcal{B}^*) \cong \mathrm{Lin}_2(\mathcal{B}^2,)$ and

let $\mathbf{T}$ be a cross section of $\mathrm{Lin}\,(\mathcal{B}, \mathrm{Lin}\,\mathcal{B}) \cong \mathrm{Lin}_2(\mathcal{B}^2, \mathcal{B})$. Assume that all of these cross sections are differentiable at x. Then

$$(\Box_x \mathbf{h})\mathbf{I}_x\mathbf{M} = -\mathbf{M}\mathbf{h}(x); \tag{41.5}$$

$$(\Box_x \boldsymbol{\omega})\mathbf{I}_x\mathbf{M} = \boldsymbol{\omega}(x)\mathbf{M}; \tag{41.6}$$

$$(\Box_x \mathbf{L})\mathbf{I}_x\mathbf{M} = \mathbf{L}(x)\mathbf{M} - \mathbf{M}\mathbf{L}(x); \tag{41.7}$$

$$(\Box_x \mathbf{G})\mathbf{I}_x\mathbf{M} = \mathbf{G}(x) \circ (\mathbf{M} \times \mathbf{1}_{\mathcal{B}_x}) + \mathbf{G}(x) \circ (\mathbf{1}_{\mathcal{B}_x} \times \mathbf{M}) \tag{41.8}$$

and

$$(\Box_x \mathbf{T})\mathbf{I}_x\mathbf{M} = \mathbf{T}(x) \circ (\mathbf{M} \times \mathbf{1}_{\mathcal{B}_x}) + \mathbf{T}(x) \circ (\mathbf{1}_{\mathcal{B}_x} \times \mathbf{M}) - \mathbf{M}\mathbf{T}(x) \tag{41.9}$$

for all $\mathbf{M} \in \mathrm{Lin}\,\mathcal{B}_x$.

Let a bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ be given. We define the mapping

$$\mathbf{H}^\phi : \mathcal{O}_\phi \to \mathbf{\Phi}(\mathcal{V}_\phi)$$

by

$$\mathbf{H}^\phi(y) := \mathbf{\Phi}(\phi\rfloor_y)\mathbf{H}(y), \quad \text{for all} \quad y \in \mathcal{O}_\phi. \tag{41.10}$$

Proposition 2: *We have*

$$(\Box_x \mathbf{H})\mathbf{A}_x^\phi = \nabla_x^\phi \mathbf{H} = \mathbf{\Lambda}\big(\mathbf{A}_{\mathbf{H}(x)}^{\mathbf{\Phi}(\phi)}\big)\nabla_x \mathbf{H} \tag{41.11}$$

where $\mathbf{\Phi}(\phi)$ *is defined by* (24.5), $\nabla_x^\phi \mathbf{H}$ *is described in* (24.9) *and* $\mathbf{A}_{\mathbf{H}(x)}^{\mathbf{\Phi}(\phi)}$ *is defined in terms of* (31.19).

Proof: Let $y \in \mathcal{O}_\phi$ be given. Substituting $\mathbf{T} := (\phi\rfloor_y)^{-1}\phi\rfloor_x$ in (41.2) gives

$$\begin{aligned}\widehat{\mathbf{H}}((\phi\rfloor_y)^{-1}\phi\rfloor_x) &= \mathbf{\Phi}((\phi\rfloor_y)^{-1}\phi\rfloor_x)^{-1}\mathbf{H}(y)\\ &= \mathbf{\Phi}(\phi\rfloor_x)^{-1}\mathbf{\Phi}(\phi\rfloor_y)\mathbf{H}(y) = \mathbf{\Phi}(\phi\rfloor_x)^{-1}\mathbf{H}^\phi(y).\end{aligned}$$

Since $\operatorname{tlis}_x^{\phi\leftarrow}(y, \phi\rfloor_x) = (\phi\rfloor_y)^{-1}\phi\rfloor_x$ by (32.7), we obtain

$$(\widehat{\mathbf{H}} \circ \operatorname{tlis}_x^{\phi\leftarrow})(y, \phi\rfloor_x) = \mathbf{\Phi}(\phi\rfloor_x)^{-1}\mathbf{H}^{\phi}(y) \qquad \text{for all} \quad y \in \mathcal{O}_\phi .$$

Taking the gradient with respect to y at x and observing (51.2) gives

$$(\nabla_{\mathbf{1}_{\mathcal{B}_x}} \widehat{\mathbf{H}})(\nabla_{\mathbf{1}_{\mathcal{B}_x}} \operatorname{tlis}_x^{\phi})^{-1}(\mathbf{t}, \mathbf{0}) = \mathbf{\Phi}(\phi\rfloor_x)^{-1}(\nabla_x \mathbf{H}^{\phi})\, \mathbf{t}$$

for all $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$. In view of definition (32.19) and (24.9) we obtain the first equality of the desired result (41.11).

It follows from (41.2), (41.3) and (31.29) with ϕ replaced by $\mathbf{\Phi}(\phi)$ that

$$\begin{aligned}
(\Box_x \mathbf{H})\mathbf{A}_x^{\phi} &= (\nabla_{\mathbf{1}_{\mathcal{B}_x}} \widehat{\mathbf{H}})\nabla_x(\phi\rfloor^{-1}\phi\rfloor_x) \\
&= \nabla_x\big(y \mapsto \mathbf{\Phi}(\phi\rfloor_x^{-1}\phi\rfloor_y)\mathbf{H}(y)\big) \\
&= \big(\mathbf{\Phi}(\phi)\big)\rfloor_x^{-1}\big(\mathrm{ev}_2 \circ \nabla_{\mathbf{H}(x)}\mathbf{\Phi}(\phi)\big)\nabla_x \mathbf{H} \\
&= \mathbf{\Lambda}\big(\mathbf{A}_{\mathbf{H}(x)}^{\mathbf{\Phi}(\phi)}\big)\nabla_x \mathbf{H}.
\end{aligned}$$

Since $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ was arbitrary, the second part of (41.11) follows. ∎

The results of Props. 1 and 2 give the following commutative diagram

$$\begin{tikzcd}
\operatorname{Lin}\mathcal{B}_x \arrow[r, "\mathbf{I}_x"] \arrow[d, "-(\mathbf{\Phi}_x^{\bullet})^{\sim}{}_{\mathbf{H}(x)}"'] \arrow[dr, phantom, "(1)"] & \mathrm{S}_x\mathcal{B} \arrow[dl, "\Box_x \mathbf{H}"] \arrow[dr, phantom, "(2)"] & \mathrm{T}_x\mathcal{M} \arrow[l, "\mathbf{A}_x^{\phi}"'] \arrow[d, equal] \\
\mathbf{\Phi}(\mathcal{B}_x) & \mathrm{T}_{\mathbf{H}(x)}\mathbf{\Phi}(\mathcal{B}) \arrow[l, "\mathbf{\Lambda}\left(\mathbf{A}_{\mathbf{H}(x)}^{\mathbf{\Phi}(\phi)}\right)"] & \mathrm{T}_x\mathcal{M} \arrow[l, "\nabla_x \mathbf{H}"]
\end{tikzcd} \quad . \tag{41.12}$$

Prop. 1 and Prop. 2 are illustrated by (1) and (2) in the diagram, respectively.

Let tensor functors $\mathbf{\Phi}_1$, $\mathbf{\Phi}_2$ and $\mathbf{\Psi}$ and a natural bilinear assignment $B : (\mathbf{\Phi}_1, \mathbf{\Phi}_2) \to \mathbf{\Psi}$ be given. Also, let $\mathbf{H}_1 : \mathcal{M} \to \mathbf{\Phi}_1(\mathcal{B})$ be a cross section of $\mathbf{\Phi}_1(\mathcal{B})$ and let $\mathbf{H}_2 : \mathcal{M} \to \mathbf{\Phi}_2(\mathcal{B})$ be a cross section of $\mathbf{\Phi}_2(\mathcal{B})$. Then the mapping $B(\mathbf{H}_1, \mathbf{H}_2) : \mathcal{M} \to \mathbf{\Psi}$ defined by

$$B(\mathbf{H}_1, \mathbf{H}_2)(x) := B_{\mathcal{B}_x}(\mathbf{H}_1(x), \mathbf{H}_2(x)) \quad \text{for all} \quad x \in \mathcal{M} \tag{41.13}$$

is a cross section of $\mathbf{\Psi}(\mathcal{B})$.

General Product Rule

If $\mathbf{H}_1$ and $\mathbf{H}_2$ are differentiable at x, then $B(\mathbf{H}_1, \mathbf{H}_2)$ is also differentiable at x and we have

$$\big(\Box_x B(\mathbf{H}_1, \mathbf{H}_2)\big)\mathbf{s} = B_{\mathcal{B}_x}\big((\Box_x \mathbf{H}_1)\mathbf{s}, \mathbf{H}_2(x)\big) + B_{\mathcal{B}_x}\big(\mathbf{H}_1(x), (\Box_x \mathbf{H}_2)\mathbf{s}\big) \tag{41.14}$$

for all $\mathbf{s} \in \mathrm{S}_x\mathcal{B}$.

Proof: Put $\mathbf{H} := B(\mathbf{H}_1, \mathbf{H}_2)$ in (41.2), we have

$$\begin{aligned}\widehat{\mathbf{H}}(\mathbf{T}) &= B_{\mathcal{B}_x}\big(\mathbf{\Phi}_1(\mathbf{T}^{-1})\mathbf{H}_1(\pi_x(\mathbf{T})), \mathbf{\Phi}_2(\mathbf{T}^{-1})\mathbf{H}_2(\pi_x(\mathbf{T}))\big) \\ &= B_{\mathcal{B}_x}\big(\widehat{\mathbf{H}}_1(\mathbf{T}), \widehat{\mathbf{H}}_2(\mathbf{T})\big)\end{aligned}$$

for all $\mathbf{T} \in \mathrm{Tlis}_x\mathcal{B}$. Since B is bilinear, the desired result (41.14) follows from (41.3) together with the General Product Rule in flat spaces [FDS]. ∎

Example 2:

Let f be a scalar field, and let $\mathbf{h} : \mathcal{M} \to \mathcal{B}$ be a cross section of $\mathcal{B}$ and $\mathbf{H} : \mathcal{M} \to \mathrm{Lin}\,\mathcal{B}$ be a cross section of $\mathrm{Lin}\,\mathcal{B}$ that are differentiable at x. Then $f\mathbf{H}$ and $\mathbf{Hh}$ defined value-wise are also differentiable at x, and we have

$$(\Box_x f\mathbf{H})\mathbf{s} = ((\Box_x f)\mathbf{s})\mathbf{H}(x) + f(x)\,(\Box_x\mathbf{H})\mathbf{s} \tag{41.15}$$

and

$$\Box_x(\mathbf{Hh})\mathbf{s} = ((\Box_x\mathbf{H})\mathbf{s})\mathbf{h}(x) + \mathbf{H}(x)(\Box_x\mathbf{h})\mathbf{s} \tag{41.16}$$

for all $\mathbf{s} \in \mathrm{S}_x\mathcal{B}$. ∎

Example 3:

Let $\boldsymbol{\omega} : \mathcal{M} \to \mathrm{Skw}_p(\mathcal{B}^{\,p},)$ be a skew-p-form field and $\boldsymbol{\tau} : \mathcal{M} \to \mathrm{Skw}_q(\mathcal{B}^{\,q},)$ a skew-q-form field that are differentiable at x. Then $\boldsymbol{\omega} \wedge \boldsymbol{\tau}$ is a skew-$(p+q)$-form field which is also differentiable at x and we have

$$(\Box_x(\boldsymbol{\omega} \wedge \boldsymbol{\tau}))\mathbf{s} = (\Box_x\boldsymbol{\omega})\mathbf{s} \wedge \boldsymbol{\tau} + \boldsymbol{\omega} \wedge (\Box_x\boldsymbol{\tau})\mathbf{s} \tag{41.17}$$

for all $\mathbf{s} \in \mathrm{S}_x\mathcal{B}$. ∎

Let $\mathcal{L}$, and $\mathcal{L}'$ be linear-space bundles over $\mathcal{M}$. For every $x \in \mathcal{M}$, we denote the fiber product bundle (see Sect.22) of $(\mathrm{Tlis}_x\mathcal{L}, \pi_x, \mathcal{M})$ and $(\mathrm{Tlis}_x\mathcal{L}', \pi'_x, \mathcal{M})$ by

$$\Big(\mathrm{Tlis}_x\mathcal{L} \times_{\mathcal{M}} \mathrm{Tlis}_x\mathcal{L}' \;,\; \pi_x \times_{\mathcal{M}} \pi'_x \;,\; \mathcal{M}\Big). \tag{41.18}$$

Taking the gradient of the mapping

$$\pi_x \times_{\mathcal{M}} \pi'_x \;:\; \mathrm{Tlis}_x\mathcal{L} \times_{\mathcal{M}} \mathrm{Tlis}_x\mathcal{L}' \quad\longrightarrow\quad \mathcal{M} \tag{41.19}$$

at $\mathbf{1}_{\mathcal{L}_x} \times \mathbf{1}_{\mathcal{L}'_x}$, we have

$$\mathbf{P}_x \times_{\mathrm{T}_x\mathcal{M}} \mathbf{P}'_x \;:\; \mathrm{S}_x\mathcal{L} \times_{\mathrm{T}_x\mathcal{M}} \mathrm{S}_x\mathcal{L}' \quad\longrightarrow\quad \mathrm{T}_x\mathcal{M} \tag{41.20}$$

where $\mathbf{P}_x = \nabla_{\mathbf{1}_{\mathcal{L}_x}}\pi_x$ and $\mathbf{P}'_x = \nabla_{\mathbf{1}_{\mathcal{L}'_x}}\pi'_x$. It follows from

$$\pi_x \times_{\mathcal{M}} \pi'_x = \pi_x \circ \mathrm{ev}_1 = \pi'_x \circ \mathrm{ev}_2$$

that

$$(\mathbf{P}_x \times_{\mathrm{T}_x\mathcal{M}} \mathbf{P}'_x)(\mathbf{s}, \mathbf{s}') = \mathbf{P}_x\mathbf{s} = \mathbf{P}'_x(\mathbf{s}') \tag{41.21}$$

for all $(\mathbf{s}, \mathbf{s}') \in \mathrm{S}_x\mathcal{L} \times_{\mathrm{T}_x\mathcal{M}} \mathrm{S}_x\mathcal{L}'$.

Let $\boldsymbol{\Upsilon}$ be a tensor bifunctor and let $\mathbf{H}$ be a cross section of $\boldsymbol{\Upsilon}(\mathcal{L} \times_{\mathcal{M}} \mathcal{L}')$ which is differentiable at x. We define a mapping

$$\widehat{\mathbf{H}} : \mathrm{Tlis}_x\mathcal{L} \times_{\mathcal{M}} \mathrm{Tlis}_x\mathcal{L}' \to \boldsymbol{\Upsilon}(\mathcal{L}_x \times \mathcal{L}'_x) \tag{41.22}$$

by

$$\begin{aligned} \widehat{\mathbf{H}}\Big(\mathbf{T} \times \mathbf{T}'\Big) &:= \boldsymbol{\Upsilon}(\mathbf{T} \times \mathbf{T}')^{-1}\,\mathbf{H}(y) \\ &\quad \text{where} \quad y := \pi_x(\mathbf{T}) = \pi'_x(\mathbf{T}') \end{aligned} \tag{41.23}$$

for all $\mathbf{T} \times \mathbf{T}' \in \mathrm{Tlis}_x\mathcal{L} \times_{\mathcal{M}} \mathrm{Tlis}_x\mathcal{L}'$. The shift-gradient of $\mathbf{H}$ at x is the linear mapping

$$\square_x\mathbf{H} : \mathrm{S}_x\mathcal{L} \times_{\mathrm{T}_x\mathcal{M}} \mathrm{S}_x\mathcal{L}' \to \boldsymbol{\Upsilon}(\mathcal{L}_x \times \mathcal{L}'_x) \tag{41.24}$$

defined in (41.3); i.e.

$$\square_x\mathbf{H} = \nabla_{\mathbf{1}_{\mathcal{P}_x}}\widehat{\mathbf{H}}, \tag{41.25}$$

where $\mathbf{1}_{\mathcal{P}_x} := \mathbf{1}_{\mathcal{L}_x} \times \mathbf{1}_{\mathcal{L}'_x}$. We also use the following notations

$$\mathbf{I}_x := \nabla_{\mathbf{1}_{\mathcal{L}_x}}\mathrm{in}_x \quad \text{and} \quad \mathbf{I}'_x := \nabla_{\mathbf{1}_{\mathcal{L}'_x}}\mathrm{in}'_x$$

where $\mathrm{in}_x := \mathbf{1}_{\mathcal{L}_x \subset \mathcal{L}}$ and $\mathrm{in}'_x := \mathbf{1}_{\mathcal{L}'_x \subset \mathcal{L}'}$ are inclusion mappings.

Proposition 3: *We have*

$$(\square_x\mathbf{H})(\mathbf{I}_x\mathbf{M}, \mathbf{I}'_x\mathbf{M}') = -\overset{\bullet}{\boldsymbol{\Upsilon}}_x(\mathbf{M} \times \mathbf{M}')\mathbf{H}(x) \tag{41.26}$$

for all $\mathbf{M} \in \mathrm{Lin}\,\mathcal{L}_x$ *and all* $\mathbf{M}' \in \mathrm{Lin}\,\mathcal{L}'_x$, *where* $\overset{\bullet}{\boldsymbol{\Upsilon}}_x$ *is the gradient of the mapping* $\big(\mathbf{L} \times \mathbf{L}' \mapsto \boldsymbol{\Upsilon}(\mathbf{L} \times \mathbf{L}')\big)$ *at* $\mathbf{1}_{\mathcal{L}_x} \times \mathbf{1}_{\mathcal{L}'_x}$.

Example 4:

Let $\boldsymbol{\Phi}$ be a analytic tensor functor and let $\mathcal{L} := \mathrm{T}\mathcal{M}$ and $\mathcal{L}' := \mathcal{B}$. If $\mathbf{L} : \mathcal{M} \to \mathrm{Lin}\,(\mathrm{T}\mathcal{M}, \boldsymbol{\Phi}(\mathcal{B}))$ and $\mathbf{T} : \mathcal{M} \to \mathrm{Lin}_2\,(\mathrm{T}\mathcal{M}^2, \boldsymbol{\Phi}(\mathcal{B}))$ are cross sections that are differentiable at x, we have

$$\begin{aligned} \square_x\mathbf{L} &: \mathrm{S}_x\mathrm{T}\mathcal{M} \times_{\mathrm{T}_x\mathcal{M}} \mathrm{S}_x\mathcal{B} \to \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \boldsymbol{\Phi}(\mathcal{B}_x)) \\ \square_x\mathbf{T} &: \mathrm{S}_x\mathrm{T}\mathcal{M} \times_{\mathrm{T}_x\mathcal{M}} \mathrm{S}_x\mathcal{B} \to \mathrm{Lin}_2\,(\mathrm{T}_x\mathcal{M}^2, \boldsymbol{\Phi}(\mathcal{B}_x)) \end{aligned}$$

and

$$\begin{aligned} (\square_x\mathbf{L})(\mathbf{I}_x\mathbf{M}, \mathbf{I}'_x\mathbf{M}') &= \mathbf{L}(x)\mathbf{M} - \overset{\bullet}{\boldsymbol{\Phi}}_x(\mathbf{M}')\mathbf{L}(x) \\ (\square_x\mathbf{T})(\mathbf{I}_x\mathbf{M}, \mathbf{I}'_x\mathbf{M}') &= \mathbf{T}(x)\mathbf{M} + \mathbf{T}(x)^{\sim}\mathbf{M} - \overset{\bullet}{\boldsymbol{\Phi}}_x(\mathbf{M}')\mathbf{T}(x) \end{aligned} \tag{41.27}$$

for all $\mathbf{M} \in \operatorname{Lin} \mathrm{T}_x\mathcal{M}$ and $\mathbf{M}' \in \operatorname{Lin} \mathcal{B}_x$.

Proposition 4: *We have*

$$(\Box_x \mathbf{H})(\mathbf{A}_x^\theta, \mathbf{A}_x^\phi) = \nabla_x^{\phi_1,\phi_2}\mathbf{H}, \tag{41.28}$$

where $\nabla_x^{\phi_1,\phi_2}\mathbf{H}$ *is described in* (24.12)*, for all bundle charts* $\theta \in \mathrm{Ch}_x(\mathcal{L}, \mathcal{M})$ *and* $\phi \in \mathrm{Ch}_x(\mathcal{L}', \mathcal{M})$.

42. Covariant Gradients

Let $x \in \mathcal{M}$ and a connector $\mathbf{K} \in \mathrm{Con}\,_x\mathcal{B}$ be given.

Let $\mathbf{\Phi}$ be a tensor functor and $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathcal{B})$ be a cross section of $\mathbf{\Phi}(\mathcal{B})$ that is differentiable at x.

<u>Definition</u> : *We define the* **covariant gradient** *of* $\mathbf{H}$ *relative to* $\mathbf{K}$ *by*

$$\nabla_{\mathbf{K}}\mathbf{H} := (\Box_x \mathbf{H})\mathbf{K} \in \operatorname{Lin}\big(\mathrm{T}_x\mathcal{M}, \mathbf{\Phi}(\mathcal{B}_x)\big), \tag{42.1}$$

where $\Box_x \mathbf{H}$ *is the shift-gradient of* $\mathbf{H}$ *at* x *as defined by* (41.3)*.*

Given a bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$. It follows from (41.11) and (42.1) that

$$\nabla_{\mathbf{A}_x^\phi}\mathbf{H} = \nabla_x^\phi \mathbf{H}.$$

If $f : \mathcal{M} \to$ is a scalar field differentiable at x, then we have $\Box_x f = \nabla_x f\, \mathbf{P}_x$ and hence

$$\nabla_{\mathbf{K}} f = \nabla_x f \qquad \text{for all} \quad \mathbf{K} \in \mathrm{Con}\,_x\mathcal{B}. \tag{42.2}$$

Proposition 1: For every bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ we have

$$(\nabla_{\mathbf{K}}\mathbf{H})\mathbf{t} = (\nabla_x^\phi \mathbf{H})\mathbf{t} + \overset{\bullet}{\mathbf{\Phi}}_x\big(\mathbf{\Gamma}_x^\phi(\mathbf{K})\mathbf{t}\big)\mathbf{H}(x) \quad \text{for all} \quad \mathbf{t} \in \mathrm{T}_x\mathcal{M}, \tag{42.3}$$

where $\overset{\bullet}{\mathbf{\Phi}}_x \in \operatorname{Lin}\left(\operatorname{Lin}\mathcal{B}_x, \operatorname{Lin}\mathbf{\Phi}(\mathcal{B}_x)\right)$ is defined as in Prop. 1 of Sect.41.

Proof: By (32.27), we have

$$\begin{aligned}(\Box_x \mathbf{H})\mathbf{K}\mathbf{t} &= (\Box_x \mathbf{H})\mathbf{A}_x^\phi \mathbf{t} + \Box_x \mathbf{H}(\mathbf{K} - \mathbf{A}_x^\phi)\mathbf{t} \\ &= (\Box_x \mathbf{H})\mathbf{A}_x^\phi \mathbf{t} - \Box_x \mathbf{H}\big(\mathbf{I}_x \mathbf{\Gamma}_x^\phi(\mathbf{K})\mathbf{t}\big)\end{aligned}$$

for all $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$. Using (32.4), we obtain

$$(\Box_x \mathbf{H})\mathbf{K}\mathbf{t} = (\Box_x \mathbf{H})\mathbf{A}_x^\phi \mathbf{t} + \overset{\bullet}{\mathbf{\Phi}}_x\big(\mathbf{\Gamma}_x^\phi(\mathbf{K})\mathbf{t}\big)\mathbf{H}(x).$$

The result (42.3) follows from the definition (42.1). ∎

Example 1:

Let $\mathbf{h}$ be a cross section of $\mathcal{B}$, let $\boldsymbol{\omega}$ be a cross section of $\mathcal{B}^*$, let $\mathbf{L}$ be a cross section of $\operatorname{Lin}\mathcal{B}$, let $\mathbf{G}$ be a cross section of $\operatorname{Lin}(\mathcal{B},\mathcal{B}^*) \cong \operatorname{Lin}_2(\mathcal{B}^2,)$, and

let $\mathbf{T}$ be a cross section of $\operatorname{Lin}(\mathcal{B},\operatorname{Lin}\mathcal{B}) \cong \operatorname{Lin}_2(\mathcal{B}^2,\mathcal{B})$. If these cross sections are differentiable at x, we have

$$(\nabla_{\mathbf{K}}\mathbf{h})\mathbf{t} = (\nabla_x^{\phi}\mathbf{h})\mathbf{t} + \boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})(\mathbf{t},\mathbf{h}(x)); \tag{42.4}$$

$$(\nabla_{\mathbf{K}}\boldsymbol{\omega})\mathbf{t} = (\nabla_x^{\phi}\boldsymbol{\omega})\mathbf{t} - \boldsymbol{\omega}(x)\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})\mathbf{t}; \tag{42.5}$$

$$(\nabla_{\mathbf{K}}\mathbf{L})\mathbf{t} = (\nabla_x^{\phi}\mathbf{L})\mathbf{t} - \mathbf{L}(x)\big(\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})\mathbf{t}\big) + \big(\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})\mathbf{t}\big)\mathbf{L}(x); \tag{42.6}$$

$$\nabla_{\mathbf{K}}\mathbf{G}(\mathbf{t},\mathbf{b}) = (\nabla_x^{\phi}\mathbf{G})(\mathbf{t},\mathbf{b}) - \big(\mathbf{G}(x)\mathbf{b}\big)\big(\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})\mathbf{t}\big) - \mathbf{G}(x)\big(\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})(\mathbf{t},\mathbf{b})\big) \tag{42.7}$$

and

$$\begin{aligned}\nabla_{\mathbf{K}}\mathbf{T}(\mathbf{t},\mathbf{b}) = {}& (\nabla_x^{\phi}\mathbf{T})(\mathbf{t},\mathbf{b}) - \big(\mathbf{T}(x)\mathbf{b}\big)\big(\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})\mathbf{t}\big) - \mathbf{T}(x)\big(\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})(\mathbf{t},\mathbf{b})\big)\\ &+ \big(\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})\mathbf{t}\big)\big(\mathbf{T}(x)\mathbf{b}\big)\end{aligned} \tag{42.8}$$

for all $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$ and all $\mathbf{b} \in \mathcal{B}_x$.

General Product Rule

Let $\mathbf{H}_1, \mathbf{H}_2$ *be cross sections as given in the General Product Rule of Sect. 41, then we have*

$$\nabla_{\mathbf{K}}B(\mathbf{H}_1,\mathbf{H}_2)\mathbf{t} = B_{\mathcal{B}_x}\big((\nabla_{\mathbf{K}}\mathbf{H}_1)\mathbf{t},\mathbf{H}_2(x)\big) + B_{\mathcal{B}_x}\big(\mathbf{H}_1(x),(\nabla_{\mathbf{K}}\mathbf{H}_2)\mathbf{t}\big) \tag{42.9}$$

for all $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$.

Proof: Substituting $\mathbf{s} := \mathbf{K}\mathbf{t}$ in (41.14) and observing (42.1), we obtain (42.9).

The formulas (41.15), (41.16) and (41.17) remain valid if the shift gradient $\Box_x$ there is replaced by the covariant gradient $\nabla_{\mathbf{K}}$ and $\mathbf{s} \in \mathcal{S}_x\mathcal{B}$ by $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$.

Let $\mathcal{L}$ and $\mathcal{L}'$ be linear-space bundles over $\mathcal{M}$. Let $\boldsymbol{\Upsilon}$ be a tensor bifunctor and let $\mathbf{H} : \mathcal{M} \to \boldsymbol{\Upsilon}(\mathcal{L}\times_{\mathcal{M}}\mathcal{L}')$ be a cross section of $\boldsymbol{\Upsilon}(\mathcal{L}\times_{\mathcal{M}}\mathcal{L}')$ which is differentiable at x. Let a pair of connectors $(\mathbf{K},\mathbf{K}') \in \operatorname{Con}_x\mathcal{L} \times \operatorname{Con}_x\mathcal{L}'$ be given.

<u>**Definition**</u>**:** *The* **covariant-gradient of H at** x **relative to** $(\mathbf{K},\mathbf{K}')$ *is defined by*

$$\nabla_{(\mathbf{K},\mathbf{K}')}\mathbf{H} := (\Box_x\mathbf{H})(\mathbf{K},\mathbf{K}') \tag{42.10}$$

which is in $\operatorname{Lin}\big(\mathrm{T}_x\mathcal{M},\boldsymbol{\Upsilon}(\mathcal{L}_x \times \mathcal{L}'_x)\big)$.

Proposition 2: *For every* $(\mathbf{K},\mathbf{K}') \in \mathrm{Con}_x\mathcal{L} \times \mathrm{Con}_x\mathcal{L}'$ *and all bundle charts* $\phi \in \mathrm{Ch}_x(\mathcal{L},\mathcal{M})$ *and* $\phi' \in \mathrm{Ch}_x(\mathcal{L}',\mathcal{M})$ *we have*

$$(\nabla_{(\mathbf{K},\mathbf{K}')}\mathbf{H})\mathbf{t} = (\nabla_x^{\phi,\phi'}\mathbf{H})\mathbf{t} + \mathbf{\Upsilon}_x^{\bullet}\big(\mathbf{\Gamma}_x^{\phi}(\mathbf{K})\mathbf{t} \times \mathbf{\Gamma}_x^{\phi'}(\mathbf{K}')\mathbf{t}\big)\mathbf{H}(x) \tag{42.11}$$

for all $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$*, where* $\mathbf{\Upsilon}_x^{\bullet}$ *is described in Prop. 3 of Sect. 41.*

Proof: Equation (42.11) follows from $\mathbf{K} = \mathbf{A}_x^{\phi} - \mathbf{I}_x\mathbf{\Gamma}_x^{\phi}(\mathbf{K})$, $\mathbf{K}' = \mathbf{A}_x^{\phi'} - \mathbf{I}_x\mathbf{\Gamma}_x^{\phi'}(\mathbf{K}')$, (42.10) and (41.28). ∎

43. Alternating Covariant Gradients

Let a number $p \in$, with $p \geq 1$, connections $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\,\mathrm{T}\mathcal{M}$ and $\mathbf{D} : \mathcal{M} \to \mathrm{Con}\,\mathcal{B}$ of class C^1 be given.

Let $\mathbf{\Phi}$ be an analytic tensor functor. For every differentiable $\Phi(\mathcal{B})$-valued skew-p-linear field $\mathbf{S} : \mathcal{M} \to \mathrm{Skw}_p(\mathrm{T}\mathcal{M}^p, \Phi(\mathcal{B}))$, the covariant gradient of $\mathbf{S}$ at $x \in \mathcal{M}$ relative to $(\mathbf{C},\mathbf{D})$ is the mapping

$$\nabla_{(\mathbf{C}(x),\mathbf{D}(x))}\mathbf{S} : \mathcal{M} \to \mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathrm{Skw}_p(\mathrm{T}_x\mathcal{M}^p, \Phi(\mathcal{B}_x)).$$

Taking the alternating part of $\nabla_{(\mathbf{C}(x),\mathbf{D}(x))}\mathbf{S}$, we obtain the skew $(p+1)$-linear mapping

$$\mathrm{Alt}\,(\nabla_{(\mathbf{C}(x),\mathbf{D}(x))}\mathbf{S}) \in \mathrm{Skw}_{p+1}(\mathrm{T}_x\mathcal{M}^{p+1}, \Phi(\mathcal{B}_x)). \tag{43.1}$$

Proposition 1: *Let* $x \in \mathcal{M}$ *be given. For every manifold chart* $\chi \in \mathrm{Ch}_x\mathcal{M}$ *and every bundle chart* $\phi \in \mathrm{Ch}_x(\mathcal{M},\mathcal{B})$*, we have*

$$\begin{aligned}
&(p+1)\mathrm{Alt}\,(\nabla_{(\mathbf{C}(x),\mathbf{D}(x))}\mathbf{S})(\mathbf{v}) \\
&= (p+1)\mathrm{Alt}\Big(\nabla_x^{\chi,\phi}\mathbf{S} + \big(\Phi_x^{\bullet}(\mathbf{\Gamma}_x^{\phi}(\mathbf{D}(x)))\tilde{}\,\mathbf{S}(x)\big)\Big)(\mathbf{v}) \\
&\qquad - \sum_{1<i<j<p+1} (-1)^{i+j-1}\mathbf{S}(x)\big(\mathbf{T}_x(\mathbf{C}(x))(\mathbf{v}_i,\mathbf{v}_j), \mathrm{del}_{(i,j)}\mathbf{v}\big)
\end{aligned} \tag{43.2}$$

where $\mathrm{del}_{(i,j)} : \mathcal{V}^{p+1} \to \mathcal{V}^{p-1}$ *is defined by* $\mathrm{del}_{(i,j)} := \mathrm{del}_j \circ \mathrm{del}_i$*,* $i < j$*, for all* $\mathbf{v} \in \mathrm{T}_x\mathcal{M}^{p+1}$.

Proof: Let $\chi \in \mathrm{Ch}_x\mathcal{M}$ and $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ be given. We have

$$\mathbf{C}(x) = \mathbf{A}_x^{\chi} - \mathbf{I}_x\mathbf{\Gamma}_x^{\chi}(\mathbf{C}(x)) \quad \text{and} \quad \mathbf{D}(x) = \mathbf{A}_x^{\phi} - \mathbf{I}_x\mathbf{\Gamma}_x^{\phi}(\mathbf{D}(x)).$$

For every $i \in (p+1)^]$, (42.11) gives

$$\nabla_{(\mathbf{C}(x),\mathbf{D}(x))}\mathbf{S}(\mathbf{v}_i, \mathrm{del}_i\mathbf{v}) = \nabla_x^{\chi,\phi}\mathbf{S}(\mathbf{v}_i, \mathrm{del}_i\mathbf{v}) + \Phi_x^\bullet(\boldsymbol{\Gamma}_x^\phi(\mathbf{D}(x)\mathbf{v}_i)\mathbf{S}(x)(\mathrm{del}_i\mathbf{v}) - \sum_{j\in(p+1)^]\setminus\{i\}} \mathbf{S}(x)(\mathrm{del}_{(i,j)}\mathbf{v}).j)\boldsymbol{\Gamma}_x^\chi(\mathbf{C}(x))(\mathbf{v}_i, \mathbf{v}_j) \tag{43.2}$$

for all $\mathbf{v} \in (\mathrm{T}_x\mathcal{M})^{\times(p+1)}$. Sum up and rearrange all the terms, we obtain the desired formula by observing that $\mathbf{T}_x = \boldsymbol{\Gamma}_x^\chi - \boldsymbol{\Gamma}_x^{\chi\sim}$. ∎

Prop.1 has several applications. The first application is given in the following Prop.2. The second kind of applications are Bianchi identities in Sect.44 and the third application leads to the definition of exterior differential in Sect.45.

For every cross section $\mathbf{H} : \mathcal{M} \to \Phi(\mathcal{B})$ of class C^p, $p \geq 2$, we define the **covariant gradient-mapping of H relative to D**

$$\nabla_{\mathbf{D}}\mathbf{H} : \mathcal{M} \to \mathrm{Lin}(\mathrm{T}\mathcal{M}, \Phi(\mathcal{B}))$$

by

$$\nabla_{\mathbf{D}}\mathbf{H}(y) := \nabla_{\mathbf{D}(y)}\mathbf{H} \quad \text{for all} \quad y \in \mathcal{M}. \tag{43.3}$$

The **second covariant gradient-mapping of H relative to** $(\mathbf{C}, \mathbf{D})$ is defined by

$$\nabla^{(2)}_{(\mathbf{C},\mathbf{D})}\mathbf{H} := \nabla_{(\mathbf{C},\mathbf{D})}(\nabla_{\mathbf{D}}\mathbf{H}) : \mathcal{M} \to \mathrm{Lin}_2\big(\mathrm{T}\mathcal{M}^2\,,\ \Phi(\mathcal{B})\big). \tag{43.4}$$

The second covarient gradient-mapping $\nabla^{(2)}_{(\mathbf{C},\mathbf{D})}\mathbf{H}$ is not necessarily symmetric. Indeed, we have the following:

Proposition 2: *We have*

$$\nabla^{(2)}_{(\mathbf{C},\mathbf{D})}\mathbf{H} - (\nabla^{(2)}_{(\mathbf{C},\mathbf{D})}\mathbf{H})^\sim = \Phi^\bullet(\mathbf{R}(\mathbf{D})(\cdot,\cdot))\mathbf{H} - \big(\nabla_{\mathbf{D}}\mathbf{H}\big)\mathbf{T}(\mathbf{C}) \tag{43.5}$$

where, for each $x \in \mathcal{M}$, $\Phi^\bullet(x) := \Phi_x^\bullet \in \mathrm{Lin}\,(\mathrm{Lin}\,\mathcal{B}_x, \mathrm{Lin}\,\Phi(\mathcal{B}_x))$ *is defined as in Prop. 1 of Sect. 42.*

Proof: Let $x \in \mathcal{M}$ be given. Choose $\chi \in \mathrm{Ch}_x\mathcal{M}$ and $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$. Applying Prop. 1 with $\mathbf{H}$ replaced by $\nabla_{\mathbf{D}(x)}\mathbf{H}$ and Φ replaced by $\mathrm{Lin} \circ (\mathrm{Id}, \Phi)$ (see [N2]), we have

$$\begin{aligned}\nabla^{(2)}_{(\mathbf{C}(x),\mathbf{D}(x))}\mathbf{H}(\mathbf{u},\mathbf{v}) &- \nabla^{(2)}_{(\mathbf{C}(x),\mathbf{D}(x))}\mathbf{H}(\mathbf{v},\mathbf{u}) + \big(\nabla_{\mathbf{D}(x)}\mathbf{H}\big)\mathbf{T}_x(\mathbf{C}(x))(\mathbf{u},\mathbf{v}) \\ &= (\nabla_{(\mathbf{A}_x^\chi,\mathbf{A}_x^\phi)}\nabla_{\mathbf{D}}\mathbf{H})(\mathbf{u},\mathbf{v}) - (\nabla_{(\mathbf{A}_x^\chi,\mathbf{A}_x^\phi)}\nabla_{\mathbf{D}}\mathbf{H})(\mathbf{v},\mathbf{u}) \\ &\quad + \Phi_x^\bullet(\boldsymbol{\Gamma}_x^\phi(\mathbf{D}(x))\mathbf{u})(\nabla_{\mathbf{D}(x)}\mathbf{H})\mathbf{v} - \Phi_x^\bullet(\boldsymbol{\Gamma}_x^\phi(\mathbf{D}(x))\mathbf{v})(\nabla_{\mathbf{D}(x)}\mathbf{H})\mathbf{u}\end{aligned} \tag{43.6}$$

for all $\mathbf{u}, \mathbf{v} \in \mathrm{T}_x\mathcal{M}$. Observing $\nabla_{\mathbf{D}}\mathbf{H} = \nabla_{\mathbf{C}^\phi}\mathbf{H} + \Phi^\bullet_x(\mathbf{\Gamma}^\phi(\mathbf{D}))$, we have

$$\nabla_{(\mathbf{A}^\chi_x, \mathbf{A}^\phi_x)}\nabla_{\mathbf{D}(x)}\mathbf{H}(\mathbf{u}, \mathbf{v}) = \nabla^{(2)}_{(\mathbf{A}^\chi_x, \mathbf{A}^\phi_x)}\mathbf{H}(\mathbf{u}, \mathbf{v}) + \nabla_{(\mathbf{A}^\chi_x, \mathbf{A}^\phi_x)}\Phi^\bullet_x(\mathbf{\Gamma}^\phi(\mathbf{D}))\tilde{\ }\mathbf{H}(\mathbf{u}, \mathbf{v}). \tag{43.7}$$

for all $\mathbf{u}, \mathbf{v} \in \mathrm{T}_x\mathcal{M}$. Since $\Phi^\bullet_x$ is a natural linear assignment, the second term on the right handside of the equality in (43.7) is

$$\begin{aligned}
&(\nabla_{(\mathbf{A}^\chi_x, \mathbf{A}^\phi_x)}\Phi^\bullet_x(\mathbf{\Gamma}^\phi(\mathbf{D}))\tilde{\ }\mathbf{H})(\mathbf{u}, \mathbf{v}) \\
&\qquad = \Phi^\bullet_x(\nabla_{(\mathbf{A}^\chi_x, \mathbf{A}^\phi_x)}\mathbf{\Gamma}^\phi(\mathbf{D})(\mathbf{u}, \mathbf{v}))\mathbf{H}(x) + \Phi^\bullet_x(\mathbf{\Gamma}^\phi_x(\mathbf{D}(x))\mathbf{v})(\nabla_{\mathbf{A}^\phi_x}\mathbf{H})\mathbf{u}.
\end{aligned} \tag{43.8}$$

We also have, the third term on the right hand side of the equality (43.6) satisfies

$$\begin{aligned}
&\Phi^\bullet_x(\mathbf{\Gamma}^\phi_x(\mathbf{D}(x))\mathbf{u})(\nabla_{\mathbf{D}(x)}\mathbf{H})\mathbf{v} \\
&\qquad = \Phi^\bullet_x(\mathbf{\Gamma}^\phi_x(\mathbf{D}(x))\mathbf{u})\big(\nabla_{\mathbf{A}^\phi_x}\mathbf{H} + \Phi^\bullet_x(\mathbf{\Gamma}^\phi_x(\mathbf{D}(x))\big)\mathbf{v} \\
&\qquad = \Phi^\bullet_x(\mathbf{\Gamma}^\phi_x(\mathbf{D}(x))\mathbf{u})\nabla_{\mathbf{C}^\phi}\mathbf{H}\mathbf{v} + \Phi^\bullet_x(\mathbf{\Gamma}^\phi_x(\mathbf{D}(x))\mathbf{u})\Phi^\bullet_x(\mathbf{\Gamma}^\phi_x(\mathbf{D}(x))\mathbf{v}) \\
&\qquad = \Phi^\bullet_x(\mathbf{\Gamma}^\phi_x(\mathbf{D}(x))\mathbf{u})\nabla_{\mathbf{C}^\phi}\mathbf{H}\mathbf{v} + \Phi^\bullet_x(\mathbf{\Gamma}^\phi_x(\mathbf{D}(x))\mathbf{u}\mathbf{\Gamma}^\phi_x(\mathbf{D}(x))\mathbf{v}).
\end{aligned} \tag{43.9}$$

Combining (43.6) to (43.9) with (43.2) and observing that

$$\nabla^{(2)}_{(\mathbf{A}^\chi_x, \mathbf{A}^\phi_x)}\mathbf{H} = \Phi(\phi\rfloor_x)^{-1}\left(\nabla^{(2)}_\chi\mathbf{H}^\phi\right)(\nabla_x\chi \times \nabla_x\chi) \tag{43.10}$$

is symmetric and $x \in \mathcal{M}$ was arbitrary, we obtain (43.5). ∎

Remark: When the given bundle $\mathcal{B}$ is the tangent bundle $\mathrm{T}\mathcal{M}$, then we only need one connection say; the connection $\mathbf{C}$. If this is the case, we have

$$\nabla^{(2)}_{\mathbf{C}}\mathbf{H} - (\nabla^{(2)}_{\mathbf{C}}\mathbf{H})^{\sim} = \Phi^\bullet(\mathbf{R}(\mathbf{C})(\cdot,\cdot))\mathbf{H} - \big(\nabla_{\mathbf{C}}\mathbf{H}\big)\mathbf{T}(\mathbf{C}). \tag{43.11}$$

44. Bianchi Identities

Let connections $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\,\mathrm{T}\mathcal{M}$ and $\mathbf{D} : \mathcal{M} \to \mathrm{Con}\,\mathcal{B}$ of class C^1 be given. Both of the torsion field $\mathbf{T}(\mathbf{C}) : \mathcal{M} \to \mathrm{Skw}_2(\mathrm{T}\mathcal{M}^2, \mathrm{T}\mathcal{M})$ of the connection $\mathbf{C}$ and the curvature field $\mathbf{R}(\mathbf{D}) : \mathcal{M} \to \mathrm{Skw}_2(\mathrm{T}\mathcal{M}^2, \mathrm{Lin}\mathcal{B})$ of the connection $\mathbf{D}$ are skew-2-linear fields. Applying Prop.1 of Sect.43, the alternating part of $\nabla_{\mathbf{C}}\mathbf{T}(\mathbf{C})$ gives the **first Bianchi idetity** and the alternating part of $\nabla_{(\mathbf{C},\mathbf{D})}\mathbf{R}(\mathbf{D})$ gives the **second Bianchi idetity**.

Proposition 1: (**First Bianchi idetity**) *We have*

$$\mathrm{Alt}\,(\nabla_{\mathbf{C}}\mathbf{T}(\mathbf{C}) + \mathbf{T}(\mathbf{C})\mathbf{T}(\mathbf{C})) = \mathrm{Alt}\,(\mathbf{R}(\mathbf{C})) \tag{44.1}$$

where $\mathbf{T}(\mathbf{C})\mathbf{T}(\mathbf{C})$ *is regarded as a cross section of* $\mathrm{Skw}_2(\mathrm{T}\mathcal{M}^2, \mathrm{Lin}\mathrm{T}\mathcal{M})$.

Proof: Applying Prop.1 of Sect.43, we have

$$\mathrm{Alt}\,(\nabla_{\mathbf{C}}\mathbf{T}(\mathbf{C}) + \mathbf{T}(\mathbf{C})\mathbf{T}(\mathbf{C})) = \mathrm{Alt}\,(\nabla_{\mathbf{C}^\chi}\mathbf{T}(\mathbf{C}) + \mathbf{\Gamma}^\chi(\mathbf{C})^\sim\mathbf{T}(\mathbf{C})). \tag{44.2}$$

Using (33.8) and (34.30), we see that

$$\mathrm{Alt}\,(\nabla_{\mathbf{C}^\chi}\mathbf{T}(\mathbf{C}) + \mathbf{\Gamma}^\chi(\mathbf{C})^\sim\mathbf{T}(\mathbf{C})) = \mathrm{Alt}\,(\mathbf{R}(\mathbf{C})). \tag{44.3}$$

The desire result (44.1) follows from (44.2) and (44.3). ∎

Remark 1: When $\mathbf{C}$ is curvature-free (but not necessary torsion free), Eq. (44.1) reduces to

$$\mathrm{Alt}\,(\nabla_{\mathbf{C}}\mathbf{T}(\mathbf{C}) + \mathbf{T}(\mathbf{C})\mathbf{T}(\mathbf{C})) = \mathbf{0}. \tag{44.4}$$

If in addition that $\mathrm{Alt}\,(\nabla_{\mathbf{C}}\mathbf{T}(\mathbf{C})) = \mathbf{0}$, then

$$\mathrm{Alt}\,(\mathbf{T}(\mathbf{C})\mathbf{T}(\mathbf{C})) = \mathbf{0}; \tag{44.5}$$

that is $\mathbf{T}(\mathbf{C})$ satisfies Jacobi identity (cf. Lie Group, Prop.7 of Sect.44). ∎

Proposition 2: (**Second Bianchi idetity**) *We have*

$$\mathrm{Alt}\,(\nabla_{(\mathbf{C},\mathbf{D})}\mathbf{R}(\mathbf{D}) + \mathbf{R}(\mathbf{D})\mathbf{T}(\mathbf{C})) = \mathbf{0}. \tag{44.6}$$

where $\mathbf{R}(\mathbf{D})\mathbf{T}(\mathbf{C})$ *is regarded as a cross section of* $\mathrm{Skw}_2(\mathrm{T}\mathcal{M}^2, \mathrm{Lin}(\mathrm{T}\mathcal{M}, \mathrm{Lin}\mathcal{B}))$.

Proof: Applying Prop.1 of Sect.43, we have

$$\begin{aligned}&\mathrm{Alt}\,(\nabla_{(\mathbf{C},\mathbf{D})}\mathbf{R} + \mathbf{R}_x(\mathbf{C})(\mathbf{T}_x(\mathbf{C})))\\ &\quad = \mathrm{Alt}\,(\nabla_{(\mathbf{A}_x^\chi,\mathbf{A}_x^\phi)}\mathbf{R} + \mathbf{\Gamma}_x^\phi(\mathbf{D})^\sim\mathbf{R}_x(\mathbf{C}) - \mathbf{R}_x(\mathbf{C})(\cdot,\cdot)\mathbf{\Gamma}_x^\phi(\mathbf{D})).\end{aligned} \tag{44.7}$$

Applying Prop.5 of Sect.34, we obtain

$$\begin{aligned}
&\operatorname{Alt}\left(\nabla_{(\mathbf{A}^\chi_x,\mathbf{A}^\phi_x)}\mathbf{R} + \mathbf{\Gamma}^\phi_x(\mathbf{D})^\sim \mathbf{R}_x(\mathbf{C}) - \mathbf{R}_x(\mathbf{C})(\cdot,\cdot)\mathbf{\Gamma}^\phi_x(\mathbf{D})\right) \\
&\qquad = \operatorname{Alt}\Big(\nabla^{(2)}_{(\mathbf{A}^\chi_x,\mathbf{A}^\phi_x)}\mathbf{\Gamma}^\phi(\mathbf{D}) - \big(\nabla^{(2)}_{(\mathbf{A}^\chi_x,\mathbf{A}^\phi_x)}\mathbf{\Gamma}^\phi(\mathbf{D})\big)^\sim\Big).
\end{aligned} \tag{44.8}$$

In view of (44.5), we observe that

$$\nabla^{(2)}_{(\mathbf{A}^\chi_x,\mathbf{A}^\phi_x)}\mathbf{\Gamma}^\phi(\mathbf{D}) - \big(\nabla^{(2)}_{(\mathbf{A}^\chi_x,\mathbf{A}^\phi_x)}\mathbf{\Gamma}^\phi(\mathbf{D})\big)^\sim = \mathbf{0}. \tag{44.9}$$

The desired result follows from (44.7), (44.8) and (44.9). ▮

Remark 2: When the given linear-space bundle is the tangent bundle $\mathcal{B} := \mathrm{T}\mathcal{M}$ of $\mathcal{M}$, the Bianchi identities can be found in literatures (see [P]) as

$$\begin{aligned}
&(\nabla_{\mathbf{C}}\mathbf{T}(\mathbf{C}))(\mathbf{U},\mathbf{V},\mathbf{W}) + (\nabla_{\mathbf{C}}\mathbf{T}(\mathbf{C}))(\mathbf{V},\mathbf{W},\mathbf{U}) + (\nabla_{\mathbf{C}}\mathbf{T}(\mathbf{C}))(\mathbf{W},\mathbf{U},\mathbf{V}) \\
&+\mathbf{T}(\mathbf{C})(\mathbf{T}(\mathbf{C})(\mathbf{U},\mathbf{V}),\mathbf{W}) + \mathbf{T}(\mathbf{C})(\mathbf{T}(\mathbf{C})(\mathbf{V},\mathbf{W}),\mathbf{U}) + \mathbf{T}(\mathbf{C})(\mathbf{T}(\mathbf{C})(\mathbf{W},\mathbf{U}),\mathbf{V}) \\
&= \mathbf{R}(\mathbf{C})(\mathbf{U},\mathbf{V},\mathbf{W}) + \mathbf{R}(\mathbf{C})(\mathbf{V},\mathbf{W},\mathbf{U}) + \mathbf{R}(\mathbf{C})(\mathbf{W},\mathbf{U},\mathbf{V})
\end{aligned} \tag{44.10}$$

and

$$\begin{aligned}
&(\nabla_{\mathbf{C}}\mathbf{R}(\mathbf{C}))(\mathbf{U},\mathbf{V},\mathbf{W}) + (\nabla_{\mathbf{C}}\mathbf{R}(\mathbf{C}))(\mathbf{V},\mathbf{W},\mathbf{U}) + (\nabla_{\mathbf{C}}\mathbf{R}(\mathbf{C}))(\mathbf{W},\mathbf{U},\mathbf{V}) \\
&+\mathbf{R}(\mathbf{C})(\mathbf{T}(\mathbf{C})(\mathbf{U},\mathbf{V}),\mathbf{W}) + \mathbf{R}(\mathbf{C})(\mathbf{T}(\mathbf{C})(\mathbf{V},\mathbf{W}),\mathbf{U}) + \mathbf{R}(\mathbf{C})(\mathbf{T}(\mathbf{C})(\mathbf{W},\mathbf{U}),\mathbf{V}) \\
&= \mathbf{0}
\end{aligned} \tag{44.11}$$

for all vector fields $\mathbf{U},\mathbf{V},\mathbf{W} \in \mathfrak{X}\mathrm{T}\mathcal{M}$. ▮

Remark 3: Most of the literatures, especially in physics, only deal with the special case : in the absence of torsion. Under this assumption, the Bianchi identities becomes

$$\operatorname{Alt}(\mathbf{R}(\mathbf{C})) = \mathbf{0} \tag{44.12}$$

and

$$\operatorname{Alt}(\nabla_{\mathbf{C}}\mathbf{R}(\mathbf{C})) = \mathbf{0}. \tag{44.13}$$

45. Differential Forms

Let $p \in$ and a differentiable $\mathcal{W}$-valued skew p-linear field $\boldsymbol{\omega}$ be given.

In this section, we apply Prop.1 of Sect.43 with the tensor functor $\mathbf{\Phi} := \mathrm{Tr}_{\mathcal{W}}$, the trival functor for a linear space $\mathcal{W}$ (see Sect.13).

Proposition 1: *For every $x \in \mathcal{M}$, we have*

$$\mathrm{Alt}\,(\nabla_x^{\chi}\boldsymbol{\omega}) = \mathrm{Alt}\,(\nabla_x^{\gamma}\boldsymbol{\omega}) \tag{45.1}$$

for all manifold charts $\chi, \gamma \in \mathrm{Ch}_x\mathcal{M}$.

Proof: The desire result (45.1) follows from Prop.1 of Sect.43 with $(\mathrm{Tr}_{\mathcal{W}})_x^{\bullet} = \mathbf{0}$ and $\mathbf{T}_x(\mathbf{A}_x^{\chi}) = \mathbf{0} = \mathbf{T}_x(\mathbf{A}_x^{\gamma})$ (see Theorem in Sect.33) for all manifold charts $\chi, \gamma \in \mathrm{Ch}_x\mathcal{M}$.

<u>**Definition**</u> **:** *The* p^{th}-**exterior differential at** $x \in \mathcal{M}$

$$\boldsymbol{d}_x^p : \mathfrak{X}(\mathrm{Skw}_p(\mathrm{T}\mathcal{M}^p,)) \to \mathrm{Skw}_{p+1}(\mathrm{T}_x\mathcal{M}^{p+1},) \tag{45.2}$$

is defined by

$$\boldsymbol{d}_x^p\boldsymbol{\omega} := \frac{1}{p!}\mathrm{Alt}\,(\nabla_x^{\chi}\boldsymbol{\omega}) \quad \text{for all} \quad \boldsymbol{\omega} \in \mathfrak{X}(\mathrm{Skw}_p(\mathrm{T}\mathcal{M}^p,)) \tag{45.3}$$

which is valid for all manifold chart $\chi \in \mathrm{Ch}_x\mathcal{M}$.

The p^{th}-**exterior differential**

$$\boldsymbol{d}^p : \mathfrak{X}^s(\mathrm{Skw}_p(\mathrm{T}\mathcal{M}^p,)) \to \mathfrak{X}^{s-1}(\mathrm{Skw}_{p+1}(\mathrm{T}\mathcal{M}^{p+1},)) \tag{45.4}$$

is defined by

$$\boldsymbol{d}^p(x) := \boldsymbol{d}_x^p \quad \text{for all} \quad x \in \mathcal{M}. \tag{45.5}$$

Remark : If $\mathcal{M}$ be the underline manifold of a flat space $\mathcal{E}$, then $\nabla\boldsymbol{\omega} = \nabla^{\chi}\boldsymbol{\omega}$ for all manifold chart χ. The definition (45.3) of exterior differential at x becomes

$$\boldsymbol{d}^p\boldsymbol{\omega} = \frac{1}{p!}\mathrm{Alt}\,(\nabla\boldsymbol{\omega}). \tag{45.6}$$

Equation (45.6) can be found in Sect.2.3 of [CH] and in Sect.51 of [B-W]. ∎

Proposition 2: *Let $\mathcal{W}$ be a linear space and let $\boldsymbol{\omega} : \mathcal{M} \to \mathrm{Skw}_p(\mathrm{T}\mathcal{M}^p, \mathcal{W})$ be a differentiable $\mathcal{W}$-valued skew p-linear field. For every $x \in \mathcal{M}$,we have*

$$\begin{aligned} \boldsymbol{d}_x^p\boldsymbol{\omega}(\mathbf{v}) = {} & (\frac{1}{p!}\mathrm{Alt}\,(\nabla_{\mathbf{C}(x)}\boldsymbol{\omega}))\mathbf{v} \\ & + \sum_{1 \le i < j \le p+1} (-1)^{i+j-1}\boldsymbol{\omega}(x)\big(\mathbf{T}_x(\mathbf{C}(x))(\mathbf{v}_i, \mathbf{v}_j), \mathrm{del}_{(i,j)}\mathbf{v}\big) \end{aligned} \tag{45.7}$$

for all connection $\mathbf{C}$ and all $\mathbf{v} \in \mathrm{T}_x\mathcal{M}^{p+1}$.

Proposition 3: *We have*

$$\boldsymbol{d}^{p+1} \circ \boldsymbol{d}^{p} = \mathbf{0}. \tag{45.7}$$

46. Lie gradients, Lie brackets

In this section, we only deal with the tangent bundle of a given C^s-manifold $\mathcal{M}$, where $2 \leq s \in \tilde{\ }$.

We assume that a vector-field $\mathbf{h}$ is given and that $\mathbf{h}$ is differentiable at x.

Proposition 1: *There is exactly one shift, which is called the* **shift** *of* $\mathbf{h}$ *at* x *and is denoted by* $\triangleright_x \mathbf{h} \in \mathrm{S}_x\mathrm{T}\mathcal{M}$, *such that*

$$\mathbf{B}_x (\triangleright_x \mathbf{h}) = \square_x \mathbf{h}, \tag{46.1}$$

where $\mathbf{B}_x$ *is given in* (33.6) *and* $\square_x \mathbf{h} \in \mathrm{Lin}\,(\mathrm{S}_x\mathrm{T}\mathcal{M}, \mathrm{T}_x\mathcal{M})$ *is the shift-gradient of* $\mathbf{h}$ *as defined by* (41.3). *We have*

$$\mathbf{P}_x (\triangleright_x \mathbf{h}) = \mathbf{h}(x) \tag{46.2}$$

Proof: The injectivity of $\mathbf{B}_x$ (see Prop. 2 of Sect.15) shows that there is at most one $\triangleright_x \mathbf{h} \in \mathrm{S}_x\mathrm{T}\mathcal{M}$ with the property (46.1).

We now choose $\chi \in \mathrm{Ch}_x\mathcal{M}$ and define

$$\triangleright_x \mathbf{h} := \mathbf{I}_x \left((\square_x \mathbf{h})\, \mathbf{A}_x^\chi \right) + \mathbf{A}_x^\chi\, \mathbf{h}(x). \tag{46.3}$$

By $(15.6)_1$ and (32.23) we have

$$\begin{aligned} \mathbf{B}_x (\triangleright_x \mathbf{h}) &= (\square_x \mathbf{h})(\mathbf{A}_x^\chi\, \mathbf{P}_x) + \mathbf{B}_x \left(\mathbf{A}_x^\chi\, \mathbf{h}(x) \right) \\ &= \square_x \mathbf{h}\, (\mathbf{1}_{\mathrm{S}_x\mathrm{T}\mathcal{M}} - \mathbf{I}_x\, \mathbf{\Lambda}(\mathbf{A}_x^\chi)) + \mathbf{B}_x \left(\mathbf{A}_x^\chi\, \mathbf{h}(x) \right). \end{aligned} \tag{46.4}$$

It follows from (41.4) and $(15.6)_2$ that

$$\begin{aligned} \square_x \mathbf{h} \Big(\mathbf{I}_x \big(\mathbf{\Lambda}(\mathbf{A}_x^\chi)(\mathbf{s}) \big) \Big) &= -\mathbf{\Lambda}(\mathbf{A}_x^\chi)(\mathbf{s})\, \mathbf{h}(x) \\ &= -\mathbf{B}_x\, (\mathbf{s}) \big(\mathbf{A}_x^\chi\, \mathbf{h}(x) \big) = \Big(\mathbf{B}_x \left(\mathbf{A}_x^\chi\, \mathbf{h}(x) \right) \Big)(\mathbf{s}) \end{aligned}$$

holds for all $\mathbf{s} \in \mathrm{S}_x\mathrm{T}\mathcal{M}$. Hence (46.4) reduces to (46.1). Applying $\mathbf{P}_x$ to (46.3) and observing $\mathbf{P}_x\, \mathbf{I}_x = \mathbf{0}$ and $\mathbf{P}_x\, \mathbf{A}_x^\chi = \mathbf{1}_{\mathrm{T}_x\mathcal{M}}$ yields (46.2). ∎

Proposition 2: *Let* $\chi \in \mathrm{Ch}_x\mathcal{M}$ *be given. The shift* $\triangleright_x \mathbf{h}$ *of* $\mathbf{h}$ *at* x *satisfies*

$$\mathbf{\Lambda}(\mathbf{A}_x^\chi)(\triangleright_x \mathbf{h}) = \nabla_x^\chi \mathbf{h} \tag{46.5}$$

Proof: The equality follows by operating on (44.3) with $\mathbf{\Lambda}(\mathbf{A}_x^\chi)$ and observing $\mathbf{\Lambda}(\mathbf{A}_x^\chi)\mathbf{I}_x = \mathbf{1}_{\mathrm{Lin}\mathrm{T}_x\mathcal{M}}$ and $\mathbf{\Lambda}(\mathbf{A}_x^\chi)\mathbf{A}_x^\chi = \mathbf{0}$. ∎

For every manifold chart $\chi \in \mathrm{Ch}_x\mathcal{M}$, we have

$$\mathbf{A}_x^\chi \mathbf{h}(x) + \mathbf{I}_x \Box_x \mathbf{h} \mathbf{A}_x^\chi = \left(\nabla_{\mathbf{1}_{\mathrm{T}_x\mathcal{M}}} \mathrm{tlis}_x^\chi\right)^{-1}\left(\, \mathbf{h}^\chi(x)\,,\ \nabla_x \mathbf{h}^\chi\,\right). \tag{46.6}$$

In view of (46.3), we have

$$\rhd_x \mathbf{h} = \left(\nabla_{\mathbf{1}_{\mathrm{T}_x\mathcal{M}}} \mathrm{tlis}_x^\chi\right)^{-1}\left(\, \mathbf{h}^\chi(x)\,,\ \nabla_x \mathbf{h}^\chi\,\right)$$

for every manifold chart $\chi \in \mathrm{Ch}_x\mathcal{M}$.

Remark: By (46.1) and the injectivity of $\mathbf{B}_x$, we have

$$\rhd_x \mathbf{k} = \mathbf{0} \quad \text{if and only if} \quad \Box_x \mathbf{k} = \mathbf{0} \tag{46.7}$$

Proposition 3: *If $f : \mathcal{M} \to$ is differentiable at x, so is the vector-field $f\,\mathbf{h}$ and we have*

$$\rhd_x (f\,\mathbf{h}) = f(x) \rhd_x \mathbf{h} + \mathbf{I}_x\,(\mathbf{h}(x) \otimes \nabla_x f). \tag{46.8}$$

Proof: It follows from $(15.6)_1$ with $\mathbf{M} := \mathbf{h}(x) \otimes \nabla_x f$ that

$$\mathbf{B}_x\left(\mathbf{I}_x\,(\mathbf{h}(x) \otimes \nabla_x f)\right) = (\mathbf{h}(x) \otimes \nabla_x f)\,\mathbf{P}_x = \mathbf{h}(x) \otimes \mathbf{P}_x^\top \nabla_x f.$$

In view of (46.4) and (41.15), it follows that

$$\begin{aligned}\mathbf{B}_x\left(\rhd_x (f\,\mathbf{h})\right) = \Box_x (f\,\mathbf{h}) &= f(x) \Box_x \mathbf{h} + \mathbf{h}(x) \otimes \mathbf{P}_x^\top \nabla_x f \\ &= \mathbf{B}_x\left(f(x) \rhd_x \mathbf{h} + \mathbf{I}_x\,(\mathbf{h}(x) \otimes \nabla_x f)\right)\end{aligned}$$

Since $\mathbf{B}_x$ is injective, (46.8) follows. ▮

Let $\mathbf{\Phi}$ be a functor as described in Sect.13 and let $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathrm{T}\mathcal{M})$ be a tensor-field that is differentiable at x. Also, let $\mathbf{k}$ be a vector-field that is differentiable at x.

<u>**Definition**</u>**:** *The* **Lie-gradient** *of* $\mathbf{H}$ **with respect to** $\mathbf{k}$ **at** x *is defined by*

$$(\mathrm{Lie}_{\mathbf{k}}\mathbf{H})_x := \Box_x \mathbf{H}(\rhd_x \mathbf{k}), \tag{46.9}$$

where $\Box_x \mathbf{H}$ is the shift-gradient of $\mathbf{H}$ at x as defined by (41.3) and where $\rhd_x \mathbf{k}$ is the shift of $\mathbf{k}$ at x as determined by (46.1).

Proposition 4: *Let $f : \mathcal{M} \to$ and $\mathbf{H}$ be differentiable at x. We have*

$$\begin{aligned}\left(\mathrm{Lie}_{\mathbf{k}} f\,\mathbf{H}\right)_x &= f(x)\left(\mathrm{Lie}_{\mathbf{k}}\mathbf{H}\right)_x + \left((\nabla_x f)\,\mathbf{k}(x)\right)\mathbf{H}(x); \\ \left(\mathrm{Lie}_{f\mathbf{k}}\mathbf{H}\right)_x &= f(x)\left(\mathrm{Lie}_{\mathbf{k}}\mathbf{H}\right)_x + \left(\mathbf{\Phi}_x^\bullet\left(\mathbf{k}(x) \otimes \nabla_x f\right)\right)\mathbf{H}(x),\end{aligned} \tag{46.9}$$

where $\mathbf{\Phi}_x^\bullet \in \mathrm{Lin}\left(\mathrm{Lin}\mathrm{T}_x, \mathrm{Lin}\mathbf{\Phi}(\mathrm{T}_x)\right)$ is defined as in Prop.1 of Sect.41.

General Product Rule

Let $\mathbf{H}_1, \mathbf{H}_2$ be cross sections as given in the General Product Rule of Sect.41, then we have

$$(\mathrm{Lie}_{\mathbf{k}} B(\mathbf{H}_1, \mathbf{H}_2))_x = B_{\mathcal{B}_x}\big((\mathrm{Lie}_{\mathbf{k}}\mathbf{H}_1)_x, \mathbf{H}_2(x)\big) + B_{\mathcal{B}_x}\big(\mathbf{H}_1(x), (\mathrm{Lie}_{\mathbf{k}}\mathbf{H}_2)_x\big). \tag{46.10}$$

Remark: We have

$$(\mathrm{Lie}_{\mathbf{k}}\mathbf{H})_x = (\nabla_{\mathbf{K}}\mathbf{H})\mathbf{k}(x) + \mathbf{\Phi}^{\bullet}\big(\mathbf{T}_x(\mathbf{K})\mathbf{k}(x) + \nabla_{\mathbf{K}}\mathbf{k}\big)\mathbf{H}(x)$$

for all $\mathbf{K} \in \mathrm{Com}_x(\mathrm{T}\mathcal{M})$. ∎

We now assume that two vector-fields $\mathbf{h}$ and $\mathbf{k}$, both are differentiable at x, are given.

<u>Definition</u>: *The* **Lie-bracket** *of* $\mathbf{h}$ *with* $\mathbf{k}$ *at* x *is defined by*

$$[\![\, \mathbf{k}\ ,\ \mathbf{h}\,]\!]_x := \mathbf{B}_x(\rhd_x \mathbf{h}, \rhd_x \mathbf{k}). \tag{46.11}$$

It follows from (46.1), (46.9) and (46.11) that

$$[\![\, \mathbf{k}\ ,\ \mathbf{h}\,]\!]_x = (\mathrm{Lie}_{\mathbf{k}}\mathbf{h})_x \tag{46.12}$$

Proposition 5: *We have*

$$[\![\, \mathbf{k}\ ,\ \mathbf{h}\,]\!]_x = -[\![\, \mathbf{h}\ ,\ \mathbf{k}\,]\!]_x. \tag{46.13}$$

If $f : \mathcal{M} \to$ is differentiable at x, then

$$[\![\, f\,\mathbf{h}\ ,\ \mathbf{k}\,]\!]_x = f(x)[\![\, \mathbf{h}\ ,\ \mathbf{k}\,]\!]_x - \big((\nabla_x f)\,\mathbf{k}(x)\big)\,\mathbf{h}(x). \tag{46.14}$$

Proof: (46.13) follows from the skewness of $\mathbf{B}_x$. Substitution of $f\mathbf{h}$ for $\mathbf{h}$ in (46.11) and use of (46.8) gives

$$[\![\, f\,\mathbf{h}\ ,\ \mathbf{k}\,]\!]_x = f(x)[\![\, \mathbf{h}\ ,\ \mathbf{k}\,]\!]_x - \mathbf{B}_x\left(\mathbf{I}_x\,(\mathbf{h}(x) \otimes \nabla_x f), \rhd_x \mathbf{k}\right)$$

and hence, by $(15.6)_1$,

$$[\![\, f\,\mathbf{h}\ ,\ \mathbf{k}\,]\!]_x = f(x)[\![\, \mathbf{h}\ ,\ \mathbf{k}\,]\!]_x - (\mathbf{h}(x) \otimes \nabla_x f)(\mathbf{P}_x \rhd_x \mathbf{k})$$

The desired result (46.14) now follows from (46.2). ∎

Remark: Let $r=\infty$, let $\mathbf{h},\mathbf{k}\in\mathfrak{X}^\infty\mathcal{M}$ and let $\overset{\triangledown}{\mathbf{h}}$ and $\overset{\triangledown}{\mathbf{k}}$ be the mappings from $C^\infty(\mathcal{M})$ to $C^\infty(\mathcal{M})$ defined by (24.6). One can easily show that the mapping $[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]^{\triangledown}: C^\infty(\mathcal{M})\to C^\infty(\mathcal{M})$ corresponding to $[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]^{\triangledown}$ is given by

$$[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]^{\triangledown}=\overset{\triangledown}{\mathbf{h}}\circ\overset{\triangledown}{\mathbf{k}}-\overset{\triangledown}{\mathbf{k}}\circ\overset{\triangledown}{\mathbf{h}} \tag{46.15}$$

If $f\in C^\infty(\mathcal{M})$, we then have

$$[\![\,f\mathbf{h}\,,\,\mathbf{k}\,]\!]^{\triangledown}=f[\![\,\overset{\triangledown}{\mathbf{h}}\,,\,\overset{\triangledown}{\mathbf{k}}\,]\!]-\overset{\triangledown}{\mathbf{k}}(f)\overset{\triangledown}{\mathbf{h}}, \tag{46.16}$$

which can be derived from (46.14) or directly from (46.15). ∎

Proposition 6: *If both* $\mathbf{h}$ *and* $\mathbf{k}$ *are vector-fields that are differentiable at* x*, then have*

$$[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]_x=(\nabla_x^\chi\mathbf{k})\,\mathbf{h}(x)-(\nabla_x^\chi\mathbf{h})\,\mathbf{k}(x). \tag{46.17}$$

for every manifold chart $\chi\in\mathbf{Ch}_x\mathcal{M}$ *where* $\nabla_x^\chi\mathbf{k}$ *and* $\nabla_x^\chi\mathbf{h}$ *be defined according to* (23.26)*. Moreover, we have*

$$(\nabla_{\mathbf{K}}\mathbf{k})\mathbf{h}(x)-(\nabla_{\mathbf{K}}\mathbf{h})\mathbf{k}(x)=[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]_x+\mathbf{T}_x(\mathbf{K})(\mathbf{h},\mathbf{k}) \tag{46.18}$$

for all $\mathbf{K}\in\mathrm{Con}_x\mathrm{T}\mathcal{M}$.

Proof: If we substitute $\mathbf{s}:=\triangleright_x\mathbf{h}$ and $\mathbf{s}':=\triangleright_x\mathbf{k}$ in (33.6) and (12.5) we obtain from (46.11) that

$$[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]_x=-\mathbf{D}_x^\chi(\triangleright_x\mathbf{h})\mathbf{P}_x(\triangleright_x\mathbf{k})+\mathbf{D}_x^\chi(\triangleright_x\mathbf{k})\mathbf{P}_x(\triangleright_x\mathbf{h})$$

The desired result (46.17) follows now from (46.5) and (46.2).

By (42.3) we have

$$(\nabla_{\mathbf{K}}\mathbf{h})\mathbf{k}(x)=(\nabla_x^\chi\mathbf{h})\mathbf{k}(x)+\mathbf{\Gamma}_x^\chi(\mathbf{K})\big(\mathbf{k}(x),\mathbf{h}(x)\big).$$

Interchanging $\mathbf{h}$ and $\mathbf{k}$ and taking the difference, we obtain (46.18) from (46.17) and (33.8). ∎

Let $s\in 1..(r-1)$ and $\mathbf{h},\mathbf{k}\in\mathfrak{X}^s\mathrm{T}\mathcal{M}$ be given. Then the vector-field $[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]$ is defined by

$$[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!](x):=[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]_x\qquad\text{for all}\quad x\in\mathcal{M} \tag{46.19}$$

It is clear from **Proposition 5** that $[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]\in\mathfrak{X}^{s-1}\mathrm{T}\mathcal{M}$. Using (23.6), it follows from (46.17) and the definition (23.35) that

$$[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]^\chi=(\nabla_\chi\mathbf{k}^\chi)\,\mathbf{h}^\chi-(\nabla_\chi\mathbf{h}^\chi)\,\mathbf{k}^\chi. \tag{46.20}$$

Proposition 7: (**Jacobi identity**): *Let* $s \in 2..(r-1)$ *and* $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3 \in \mathfrak{X}^s\mathrm{T}\mathcal{M}$ *be given, then*

$$[\![\, [\![\, \mathbf{h}_1 \,,\, \mathbf{h}_2 \,]\!] \,,\, \mathbf{h}_3 \,]\!] + [\![\, [\![\, \mathbf{h}_2 \,,\, \mathbf{h}_3 \,]\!] \,,\, \mathbf{h}_1 \,]\!] + [\![\, [\![\, \mathbf{h}_3 \,,\, \mathbf{h}_1 \,]\!] \,,\, \mathbf{h}_2 \,]\!] = \mathbf{0} \qquad (46.21)$$

Proof: A straightforward but somewhat tedious calculation, using (46.20) and the Symmetry Theorem for Second Gradients, yields the desired result (46.21). ▮

If $\mathcal{M}$ is a C^∞ manifold, then $\mathfrak{X}^\infty\mathrm{T}\mathcal{M}$ together with the bilinear mapping

$$[\![\ ,\ ]\!] : \mathfrak{X}^\infty\mathrm{T}\mathcal{M} \times \mathfrak{X}^\infty\mathrm{T}\mathcal{M} \longrightarrow \mathfrak{X}^\infty\mathrm{T}\mathcal{M}$$

given in (46.21) is a Lie algebra, as defined in Sect.11.

47. Transport Systems

We assume that $r \in {}^{\sim}$ with $r \geq 2$ and a C^r-manifold $\mathcal{M}$ are given. Let $(\mathcal{B}, \tau, \mathcal{M})$ be a C^s linear-space bundle, $s \in 0..r$.

We define the **bundle of transfer isomorphisms** of $\mathcal{B}$ by

$$\mathrm{Tlis}\,\mathcal{B} := \bigcup_{x\in\mathcal{M}} \mathrm{Tlis}_x\mathcal{B} = \bigcup_{x,y\in\mathcal{M}} \mathrm{Lis}(\mathcal{B}_x, \mathcal{B}_y). \qquad (47.1)$$

It is endowed with the natural structure of a C^s-fiber bundle over $\mathcal{M}\times\mathcal{M}$ whose bundle projection $\pi : \mathrm{Tlis}\,\mathcal{B} \to \mathcal{M}\times\mathcal{M}$ is

$$\pi(\mathbf{T}) :\in \{\ (x,y) \in \mathcal{M}\times\mathcal{M} \ \big|\ \mathbf{T} \in \mathrm{Lis}(\mathcal{B}_x, \mathcal{B}_y)\ \}. \qquad (47.2)$$

<u>Definition</u>: *A subset* $\mathfrak{T}$ *of* $\mathrm{Tlis}\,\mathcal{B}$ *is called a* C^s **transport structure for** $\mathcal{B}$ *if* $\mathfrak{T}$ *is a* C^s*-submanifold of* $\mathrm{Tlis}\,\mathcal{B}$ *such that*

(T1) *for all* $\mathbf{A} \in \mathfrak{T}$, $\mathbf{A}^{-1} \in \mathfrak{T}$,

(T2) *for all* $\mathbf{A}, \mathbf{B} \in \mathfrak{T}$ *such that* $\mathrm{Cod}\,\mathbf{A} = \mathrm{Dom}\,\mathbf{B}$, $\mathbf{BA} \in \mathfrak{T}$,

(T3) *for all* $x, y \in \mathcal{M}$, $\mathfrak{T} \cap \mathrm{Lis}(\mathcal{B}_x, \mathcal{B}_y) \neq \{\ \}$.

It can be shown that $\mathfrak{T}_x := \mathfrak{T} \cap \mathrm{Tlis}_x\,\mathcal{B}$ is a C^s-submanifold of $\mathrm{Tlis}_x\,\mathcal{B}$.

Theorem on Transport Structure and Parallelisms

Let $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\mathcal{B}$ *be a connection of class* C^s*. Define*

$$\mathfrak{F} := \{\mathbf{A} \in \mathrm{Tlis}\,\mathcal{B} \mid \cdots\cdots\cdots\cdots\}.$$

Then $\mathfrak{F}$ *is a transport structure for* $\mathcal{B}$*.*

Proof:

A cross section $\mathbf{F} : \mathcal{M} \times \mathcal{M} \to \mathfrak{T}$ is called a (global) **transport system for** $\mathcal{B}$ if

$$\mathbf{F}(x,z) = \mathbf{F}(y,z)\mathbf{F}(x,y) \qquad \text{for all} \quad x,y,z \in \mathcal{M} \tag{47.3}$$

and

$$\mathbf{F}(x,x) = \mathbf{1}_{\mathcal{B}_x} \qquad \text{for all} \quad x \in \mathcal{M}. \tag{47.4}$$

Recall that a cross section $\mathbf{T} : \mathcal{M} \to \mathrm{Tlis}_x\mathcal{B}$ of the bundle $\mathrm{Tlis}_x\mathcal{B}$, $x \in \mathcal{M}$, with

$$\mathbf{T}(x) = \mathbf{1}_{\mathcal{B}_x} \tag{47.5}$$

is called a transport from x. It follows from (47.3), (47.4) and (47.5) that, for each $x \in \mathcal{M}$, the mapping $\mathbf{F}(x,\cdot) : \mathcal{M} \to \mathrm{Tlis}_x\mathcal{B}$ is a transport from x. Moreover, we have

$$\mathbf{F}(y,\cdot) = \mathbf{F}(x,\cdot)\mathbf{F}(y,x) \quad \text{for all} \quad x,y \in \mathcal{M}. \tag{47.6}$$

Conversely, let $x \in \mathcal{M}$ and a transport $\mathbf{F}_x : \mathcal{M} \to \mathrm{Tlis}_x\mathcal{B}$ from x be given. For each $y \in \mathcal{M}$, we obtain a transport $\mathbf{F}_y : \mathcal{M} \to \mathrm{Tlis}_y\mathcal{B}$ from y by

$$\mathbf{F}_y(z) := \mathbf{F}_x(z)\mathbf{F}_x(y)^{-1} \quad \text{for all} \quad z \in \mathcal{M}. \tag{47.7}$$

and, a transport system $\mathbf{F} : \mathcal{M} \times \mathcal{M} \to \mathrm{Tlis}\,\mathcal{B}$ by

$$\mathbf{F}(y,z) := \mathbf{F}_x(z)\mathbf{F}_x(y)^{-1} \quad \text{for all} \quad y,z \in \mathcal{M}. \tag{47.8}$$

We conclude that, for each $x \in \mathcal{M}$, there is one to one correspondent between the set of all transports from x for $\mathcal{B}$ and the set of all transport systems for $\mathcal{B}$.

Every transport system $\mathbf{F} : \mathcal{M} \times \mathcal{M} \to \mathrm{Tlis}\,\mathcal{B}$ induces a connection $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\mathcal{B}$ by

$$\mathbf{C}(y) := \nabla_{\mathbf{1}_{\mathcal{B}_y}}\mathbf{F}(y,\cdot) \quad \text{for all} \quad y \in \mathcal{M}. \tag{47.9}$$

Let a transport system $\mathbf{F} : \mathcal{M} \times \mathcal{M} \to \mathrm{Tlis}\,\mathcal{B}$ for $\mathcal{B}$, a tensor functor $\mathbf{\Phi}$ and a cross section $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathcal{B})$ be given. We say that $\mathbf{H}$ is **parallel with respect to F** if

$$\mathbf{H}(y) = \mathbf{\Phi}(\mathbf{F}(x,y))\mathbf{H}(x) \quad \text{for all} \quad x,y \in \mathcal{M}. \tag{47.10}$$

Proposition 1: *Let* $\mathbf{C}$ *be the connection induced by a transport system* $\mathbf{F}$*, as given in* (47.9)*. Let* $\mathbf{H} : \mathcal{O} \to \mathbf{\Phi}(\mathcal{B})$ *be a cross section of class* C^1*. If* $\mathbf{H}$ *is parallel with respect to* $\mathbf{F}$*, then* $\nabla_{\mathbf{C}}\mathbf{H} = \mathbf{0}$*. Conversely, if* $\nabla_{\mathbf{C}}\mathbf{H} = \mathbf{0}$ *and if* $\mathcal{M}$ *is connected then* $\mathbf{H}$ *is parallel with respect to* $\mathbf{F}$*.*

Proof: Fix $x \in \mathcal{M}$ and let $\mathbf{T} := \mathbf{F}(x, \cdot)$. Let $y \in \mathcal{M}$ be given and define $\widehat{\mathbf{H}}_y : \mathrm{Tlis}_y\mathcal{B} \to \mathcal{B}_y$ in accord with (41.2). Then

$$\widehat{\mathbf{H}}_y(\mathbf{T}(z)\mathbf{T}(y)^{-1}) = \mathbf{\Phi}(\mathbf{T}(y)\mathbf{T}(z)^{-1})\mathbf{H}(z) \quad \text{for all} \quad z \in \mathcal{M}.$$

Differentiation with respect to z at y gives, using (42.1), (41.3), (47.9), and the chain rule,

$$(\nabla_{\mathbf{C}}\mathbf{H})(y) = (\Box_y\mathbf{H})\mathbf{C}(y) = \mathbf{\Phi}(\mathbf{T}(y))\nabla_y\widetilde{\mathbf{H}}, \tag{47.11}$$

where $\widetilde{\mathbf{H}} : \mathcal{M} \to \mathbf{\Phi}(\mathcal{B}_x)$ is defined by $\widetilde{\mathbf{H}}(z) := \mathbf{\Phi}(\mathbf{T}(z)^{-1})\mathbf{H}(z)$ for all $z \in \mathcal{M}$. Since $y \in \mathcal{M}$ was arbitrary and since $\mathbf{\Phi}(\mathbf{T}(y))$ is invertible, we conclude from (47.11) that $\nabla_{\mathbf{C}}\mathbf{H} = \mathbf{0}$, if and only if $\nabla\widetilde{\mathbf{H}} = \mathbf{0}$. Now if $\mathbf{H} = \mathbf{\Phi}(\mathbf{T})\mathbf{v}$ for some $\mathbf{v} \in \mathbf{\Phi}(\mathcal{B}_x)$, then $\widetilde{\mathbf{H}}$ is a constant and hence $\nabla\widetilde{\mathbf{H}} = 0$. Conversely if $\mathcal{M}$ is connected and $\nabla\widetilde{\mathbf{H}} = 0$, then $\widetilde{\mathbf{H}}$ is a constant and hence $\mathbf{H} = \mathbf{\Phi}(\mathbf{T})\mathbf{v}$ for some $\mathbf{v} \in \mathbf{\Phi}(\mathcal{B}_x)$. ∎

Remark : Let a connection $\mathbf{C}$, not necessarily induced by a transport system, be given. Then the condition $\nabla_{\mathbf{C}}\mathbf{H} = \mathbf{0}$ does not equivalent to to the condition that $\mathbf{H}$ is parallel with respective to a transport system. ∎

Proposition 2: *Let* $\mathbf{T} : [0, d] \to \mathrm{Tlis}_x\mathcal{B}$ *be a differentiable transfer process from* x*, and put* $p := \pi_x \circ \mathbf{T} : [0, d] \to \mathcal{M}$*. For every differentiable cross section* $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathcal{B})$*, we have*

$$(\Box_{p(t)}\mathbf{H})(\mathrm{sd}_t\mathbf{T}) = \partial_t\big(\, s \mapsto \mathbf{\Phi}(\mathbf{T}(t)\mathbf{T}^{-1}(s))\mathbf{H}(p(s))\,\big) \tag{47.12}$$

for all $t \in [0, d]$*, the derivative* (47.12) *may be interpreted, roughly, as the* **rate of change of H at** $p(t)$ **relative to** *the transfer process* $\mathbf{T}$*.*

Let $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\mathcal{B}$ be a continuous connection and $p : [0, d] \to \mathcal{M}$ be a process of class C^1, with $x = p(0)$. Let $\mathbf{T}$ be the parallelism along p for the connection $\mathbf{C}$. It follows from (35.23), $\mathrm{sd}\mathbf{T} = (\mathbf{C} \circ p)p^{\bullet}$, that

$$(\nabla_{\mathbf{C}(p(t))}\mathbf{H})p^{\bullet}(t) = (\Box_{p(t)}\mathbf{H})(\mathrm{sd}_t\mathbf{T}). \tag{47.13}$$

This result does not depend on the choice of the process p, and hence does not depend on the parallelism $\mathbf{T}$ along p.

Proposition 3: *Let* $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\mathcal{B}$ *be a continuous connection and let the cross section* $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathcal{B})$ *be differentiable. Then* $\nabla_{\mathbf{C}}\mathbf{H} = \mathbf{0}$ *if and only if, for every differentiable process* $p : [0, d] \to \mathcal{M}$,

$$((\square\,\mathbf{H}) \circ p)(\mathrm{sd}\mathbf{T}) = \mathbf{0} \tag{47.14}$$

where $\mathbf{T}$ *is the parallelism along* p *for* $\mathbf{C}$*.*

Let $x \in \mathcal{M}$ and a continuous vector field $\mathbf{k} : \mathcal{M} \to \mathrm{T}\mathcal{M}$ be given. By the maximum **local flow** for $\mathbf{k}$ at x we mean a mapping

$$\alpha : I \times \mathcal{D} \to \mathcal{M}$$

where I is an open interval containing 0,and $\mathcal{D}$ containing x, and $\mathcal{D}$ is an open subset of $\mathcal{M}$ containing x, such that for every $y \in \mathcal{D}$ the mapping $\alpha(\cdot, y) : I \to \mathcal{M}$ is the maximum integral process (integral curve) of $\mathbf{k}$ with the initial condition y; i.e. $\alpha(0, y) = y$ and $\mathbf{k}\big(\alpha(t, y)\big) = (\alpha^{\bullet}(\cdot, y))(t)$.

Let $x \in \mathcal{M}$ and a continuous vector field $\mathbf{k} : \mathcal{M} \to \mathrm{T}\mathcal{M}$ be given. It is a well known theorem in O.D.E. (see Sect.1 of Ch.4, [L]) that there is a maximum local flow

$$\alpha : I \times \mathcal{D} \to \mathcal{M}$$

for $\mathbf{k}$ at x. We may define a mapping $\mathbf{L}_{\mathbf{k}} : I \to \mathrm{Tlis}_x\mathcal{M}$ by

$$\mathbf{L}_{\mathbf{k}}(t) := \nabla_x \alpha(t, \cdot) \quad \text{for all} \quad t \in I.$$

It is clear that

$$\mathbf{L}_{\mathbf{k}>}(I) = \bigcup_{y \in \alpha(\cdot, x)_>(I)} \mathrm{Lis}(\mathrm{T}_x, \mathrm{T}_y).$$

Since $\mathbf{L}_{\mathbf{k}}(0) = \mathbf{1}_{\mathrm{T}_x}$, $\mathbf{L}_{\mathbf{k}}$ is a transfer process from x. We shall call $\mathbf{L}_{\mathbf{k}}$ the **Lie transfer process** from x of the vector-field $\mathbf{k}$.

Proposition 4: *Let* $x \in \mathcal{M}$ *and a vector field* $\mathbf{k} : \mathcal{M} \to \mathrm{T}\mathcal{M}$ *be given. Let* $\mathbf{L}_{\mathbf{k}}$ *be the Lie transfer process from* x *of* $\mathbf{k}$*. We have* $\mathrm{sd}_0\mathbf{L}_{\mathbf{k}} = \rhd_x \mathbf{k}$ *and*

$$(\mathrm{Lie}_{\mathbf{k}}\mathbf{H})(x) = \partial_0\big(t \mapsto \mathbf{\Phi}(\mathbf{L}_{\mathbf{k}}(t)^{-1})\mathbf{H}(p(t))\big). \tag{47.15}$$

Proof: Define the processes $\mathbf{H} : I \to \mathrm{Lis}\mathcal{V}_\chi$ and $\mathbf{V} : I \to \mathrm{Lis}\mathcal{V}_\chi$ by

$$\begin{aligned}
\mathbf{H}(t) &: = \nabla_{\alpha_x(t)}\chi \nabla_x \alpha_t (\nabla_x \chi)^{-1} = \nabla_{\alpha_x(t)}\chi \mathbf{L}_{\mathbf{k}}(t)(\nabla_x\chi)^{-1} \\
\mathbf{V}(t) &: = \nabla_{\alpha_x(t)}\chi (\mathbf{D}^{\chi}_{\alpha_x(t)} \underset{\alpha_x(t)}{\rhd} \mathbf{k})(\nabla_{\alpha_x(t)}\chi)^{-1}
\end{aligned}$$

Taking the gradient of $\mathbf{H}$ at 0 and observing $\mathbf{D}^{\chi}_{\alpha_x(t)} \rhd_{\alpha_x(t)} \mathbf{k} = (\nabla_{\alpha_x(t)}\chi)^{-1}\nabla_{\alpha_x(t)}\mathbf{k}^{\chi}$, we have

$$\begin{aligned}
\mathbf{H}^{\cdot}(t) &= \partial_t\big(s \mapsto \nabla_{\alpha_x(s)}\chi\nabla_x\alpha_s(\nabla_x\chi)^{-1}\big) \\
&= \partial_t\big(s \mapsto (\nabla_x\alpha_s)\big)^{\chi}(\nabla_x\chi)^{-1} \\
&= \nabla_x\big(\partial_t(s \mapsto \alpha_s)\big)^{\chi}(\nabla_x\chi)^{-1} \\
&= \nabla_x(\mathbf{k}^{\chi} \circ \alpha_t)(\nabla_x\chi)^{-1} \\
&= \nabla_{\alpha_x(t)}\mathbf{k}^{\chi}\nabla_x\alpha_t(\nabla_x\chi)^{-1} \\
&= \big(\nabla_{\alpha_x(t)}\chi\big((\nabla_{\alpha_x(t)}\chi)^{-1}\nabla_{\alpha_x(t)}\mathbf{k}^{\chi}\big)(\nabla_{\alpha_x(t)}\chi)^{-1}\big)\big(\nabla_{\alpha_x(t)}\chi\nabla_x\alpha_t(\nabla_x\chi)^{-1}\big) \\
&= \big(\nabla_{\alpha_x(t)}\chi(\mathbf{D}^{\chi}_{\alpha_x(t)} \underset{\alpha_x(t)}{\rhd} \mathbf{k})(\nabla_{\alpha_x(t)}\chi)^{-1}\big)\big(\nabla_{\alpha_x(t)}\chi\nabla_x\alpha_t(\nabla_x\chi)^{-1}\big) \\
&= (\mathbf{V}\mathbf{H})(t).
\end{aligned}$$

This shows that $\mathbf{L}_{\mathbf{k}}$ is the only transfer process from x such that $\mathrm{sd}\mathbf{L}_{\mathbf{k}} = (\rhd\,\mathbf{k}) \circ \alpha_x$. Since $\alpha_x(0) = x$, we have $\mathrm{sd}_0\mathbf{L}_{\mathbf{k}} = \rhd_x \mathbf{k}$. The assertion follows by applying Prop.2. ∎

48. Lie Group

Definition: *A* **Lie group** *is a set $\mathcal{G}$ endowed both with the structure of a group and with the structure of a C^{ω}-manifold in such a way that the group-operation and the group-inversion are analytic mappings.*

We use multiplicative notation and terminology for the group $\mathcal{G}$ and denote its unity by u.

For every $x \in \mathcal{G}$, we define the **left-multiplication** $\mathrm{le}_x : \mathcal{G} \to \mathcal{G}$ by

$$\mathrm{le}_x(y) := xy \qquad \text{for all} \qquad y \in \mathcal{G}. \tag{48.1}$$

$\mathrm{le}_x : \mathcal{G} \to \mathcal{G}$, is invertible for all $x \in \mathcal{G}$; in fact,

$$(x \mapsto \mathrm{le}_x) : \mathcal{G} \to \mathrm{Perm}\,\mathcal{G} \tag{48.2}$$

is an injective group-homomorphism, *i.e.* we have

$$\mathrm{le}_u = \mathbf{1}_{\mathcal{G}} \quad , \quad \mathrm{le}_{xy} = \mathrm{le}_x \circ \mathrm{le}_y \quad , \quad \mathrm{le}_{x^{-1}} = \mathrm{le}_x^{\leftarrow} \tag{48.3}$$

for all $x, y \in \mathcal{G}$. Also, when $x \in \mathcal{G}$ is given, le_x is analytic and we have

$$\nabla_y\mathrm{le}_x \in \mathrm{Lis}(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{xy}\mathcal{M}) \subset \mathrm{Tlis}_y\mathcal{G} \tag{48.4}$$

for all $y \in \mathcal{G}$. We define the analytic mapping

$$\mathbf{G} : \mathcal{G} \to \mathrm{Tlis}_u\mathcal{G} \tag{48.5}$$

by

$$\mathbf{G}(x) := \nabla_u \mathrm{le}_x \qquad \text{for all} \qquad x \in \mathcal{G}. \tag{48.6}$$

Taking the gradient of $(48.18)_2$ at u gives

$$\mathbf{G}(xy) := (\nabla_y \mathrm{le}_x)\mathbf{G}(y) \qquad \text{for all} \qquad x, y \in \mathcal{G}. \tag{48.7}$$

For every $\mathbf{t} \in \mathrm{T}_u\mathcal{M}$, we define the analytic vector field $\mathbf{Gt} : \mathcal{G} \to \mathrm{T}\,\mathcal{G}$ by

$$(\mathbf{Gt})(y) = \mathbf{G}(y)\mathbf{t} \qquad \text{for all} \qquad y \in \mathcal{G}. \tag{48.8}$$

We have

$$\mathbf{G}(u) = \mathbf{1}_{\mathrm{T}_u\mathcal{M}} \quad \text{and} \quad (\mathbf{Gt})(u) = \mathbf{t} \quad \text{for all} \quad \mathbf{t} \in \mathrm{T}_u\mathcal{M}. \tag{48.9}$$

Proposition 5: *For all* $\mathbf{t}, \mathbf{s} \in \mathrm{T}_u\mathcal{M}$ *we have*

$$[\![\, \mathbf{Gt}\, ,\ \mathbf{Gs} \,]\!] = \mathbf{G}\, [\![\, \mathbf{Gt}\, ,\ \mathbf{Gs} \,]\!]_u \tag{48.10}$$

Proof: Let $\mathbf{t} \in \mathrm{T}_u\mathcal{M}$ and $x \in \mathcal{G}$ be given and choose $\chi \in \mathrm{Ch}_x\mathcal{G}$. Since le_x is analytic and invertible and $\mathrm{le}_x(u) = x$, we have $\chi \circ \mathrm{le}_x \in \mathrm{Ch}_u\mathcal{G}$. Using the chain rule and (48.22), we obtain

$$\nabla_y(\chi \circ \mathrm{le}_x) = (\nabla_{xy}\chi)\nabla_y \mathrm{le}_x = (\nabla_{xy}\chi)\mathbf{G}(xy)\mathbf{G}_(y)^{-1} \quad \text{for all} \quad y \in \mathcal{G}. \tag{48.11}$$

Using the definitions (48.23) and (23.25), we see that

$$(\mathbf{Gt})^{\chi \circ \mathrm{le}_x}(y) = \nabla_y(\chi \circ \mathrm{le}_x)\mathbf{G}(y)\mathbf{t} = (\nabla_{xy}\chi)\mathbf{G}(xy)\mathbf{t}$$

for all $y \in \mathcal{G}$ and hence

$$(\mathbf{Gt})^{\chi \circ \mathrm{le}_x} = (\mathbf{Gt})^{\chi} \circ \mathrm{le}_x. \tag{48.12}$$

Using the chain rule again, we find

$$\nabla_u(\mathbf{Gt})^{\chi \circ \mathrm{le}_x} = \nabla_x(\mathbf{Gt})^{\chi}\mathbf{G}(x) \qquad \text{for all} \qquad \mathbf{t} \in \mathbf{T}_u \tag{48.13}$$

Now let $\mathbf{s}, \mathbf{t} \in \mathrm{T}_u\mathcal{M}$ be given and put $\mathbf{h} := \mathbf{Gt}$, $\mathbf{k} := \mathbf{Gs}$. Using (43.17) with x replaced by u and χ by $\chi \circ \mathrm{le}_x$ we conclude from (48.28) that

$$[\![\, \mathbf{h}\, ,\ \mathbf{k} \,]\!]_u = \nabla_u(\chi \circ \mathrm{le}_u)^{-1}\big((\nabla_x \mathbf{k}^{\chi})\mathbf{h}(x) - (\nabla_x \mathbf{h}^{\chi})\mathbf{k}(x)\big).$$

Using (48.26) with $y := u$ and observing (48.23), we obtain

$$[\![\, \mathbf{h}\ ,\ \mathbf{k}\,]\!]_u = \mathbf{G}(x)^{-1}\nabla_x\chi^{-1}\big((\nabla_x\mathbf{k}^\chi)\mathbf{h}(x) - (\nabla_x\mathbf{h}^\chi)\mathbf{k}(x)\big).$$

Since $x \in \mathcal{G}$ was arbitrary, we obtain (48.25) by applying (43.17) again. ∎

Proposition 6: *Define*

$$\big((\mathbf{t},\mathbf{s}) \mapsto [\mathbf{t},\mathbf{s}]\big) : \mathrm{T}_u\mathcal{M}^2 \to \mathrm{T}_u\mathcal{M} \tag{48.14}$$

by

$$[\mathbf{t},\mathbf{s}] := [\![\, \mathbf{Gt}\ ,\ \mathbf{Gs}\,]\!]_u, \tag{48.15}$$

where $\mathbf{G}$ *is defined by* (48.21)*. Then* (48.21) *endows* $\mathrm{T}_u\mathcal{M}$ *with the structure of a Lie-algebra, i.e. it is bilinear, skew, and satisfies the "Jacobi-identity"*

$$\big[[\mathbf{t}_1,\mathbf{t}_2],\mathbf{t}_3\big] + \big[[\mathbf{t}_2,\mathbf{t}_3],\mathbf{t}_1\big] + \big[[\mathbf{t}_3,\mathbf{t}_1],\mathbf{t}_2\big] = \mathbf{0} \tag{48.16}$$

for all $\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3 \in \mathrm{T}_u\mathcal{M}$*. We use the notation* $\mathrm{La}\,\mathcal{G} := \mathrm{T}_u\mathcal{M}$ *for this Lie-algebra and call it the* **Lie-algebra** *of* $\mathcal{G}$*.*

Proof: It is clear from the definition (48.30) and from (43.13) that $(\mathbf{t},\mathbf{s}) \mapsto [\mathbf{t},\mathbf{s}]$ is bilinear and skew. The Jacobi-indendity (48.31) follows from Prop. 7 of Sect. 43, applied to $\mathbf{h}_i := \mathbf{Gt}_i\ ,\ i \in 3^]$, and Prop. 5. ∎

For each $y \in \mathcal{G}$, define $\mathbf{C}(y) \in \mathrm{Lin}(\mathrm{T}_y\mathcal{M}, \mathrm{S}_y\mathrm{T}\,\mathcal{G})$ by

$$\mathbf{C}(y) := \nabla_y\big(z \mapsto \mathbf{G}(z)\mathbf{G}(y)^{-1}\big). \tag{48.17}$$

Then (48.32) defines, as described in (48.9), a natural connection $\mathbf{C} : \mathcal{G} \to \mathrm{Con}\,\mathcal{G}$ on $\mathcal{G}$. This connection is analytic.

Let a vector fuield $\mathbf{h} \in \mathfrak{X}^1(\mathrm{T}\,\mathcal{G})$ be given and let the lineon-field $\nabla_{\mathbf{C}}\mathbf{h}$ be defined according to (41.3). Then it follows from Prop.2 that $\nabla_{\mathbf{C}}\mathbf{h} = \mathbf{0}$ if $\mathbf{h} = \mathbf{Gt}$ for some $\mathbf{t} \in \mathrm{T}_u\mathcal{M}$, where $\mathbf{G}$ is defined by (48.21). Conversely, if $\nabla_{\mathbf{C}}\mathbf{h} = \mathbf{0}$ and if $\mathcal{G}$ is connected, then $\mathbf{h} = \mathbf{Gt}$ for some $\mathbf{t} \in \mathrm{T}_u\mathcal{M}$.

Proposition 7: *The Lie-algebra-operation of* $\mathrm{T}_u\mathcal{M}$ *is the opposite of the torsion* $\mathbf{T}_u(\mathbf{C}(u))$*, i.e.*

$$[\mathbf{t},\mathbf{s}] = \mathbf{T}_u(\mathbf{C}(u))(\mathbf{t},\mathbf{s}) \qquad \text{for all} \qquad \mathbf{t},\mathbf{s} \in \mathbf{T}_u. \tag{48.18}$$

Proof: Let $\mathbf{t},\mathbf{s} \in \mathbf{T}_u$ be given. Application of (43.18) to $\mathbf{h} := \mathbf{Gt}$, $\mathbf{k} := \mathbf{Gs}$, $x := u$ gives (48.33) if (48.30) is observed and $\nabla_{\mathbf{C}}\mathbf{h} = \mathbf{0} = \nabla_{\mathbf{C}}\mathbf{k}$, as described in above, is applied. ∎

Remark : The curvature field $\mathbf{R}(\mathbf{C}) = 0$???

Proposition 8: *Let $d \in {}^{\times}$ and $p \in [0,d] \to \mathcal{G}$, of class C^1 and with $p(0) = u$, be given. Then $\mathbf{G} \circ p : [0,d] \to \mathrm{Tlis}_u\mathcal{G}$ is the parallelism along p for $\mathbf{C}$.*

Proof: Put $\mathbf{T} := \mathbf{G} \circ p$. Then $\mathbf{T}(s)\mathbf{T}(t)^{-1} = \mathbf{G}(p(s))\mathbf{G}(p(t))^{-1}$ for all $s, t \in [0,d]$. Hence, by (48.32), (35.10), and the chain rule,

$$\mathrm{sd}_t\mathbf{T} = \mathbf{C}(p(t))p^{\cdot}(t) \qquad \text{for all} \qquad t \in [0,d],$$

i.e. $\mathrm{sd}\mathbf{T} = (\mathbf{C} \circ p)p^{\cdot}$. In view of (35.23) the assertion follows. ∎

An non-constant homomorphism $q : \ \to \mathcal{G}$ from the additive group of to $\mathcal{G}$ is called a **one-parameter subgroup** of $\mathcal{G}$ if it is of class C^1.

Proposition 9: *Let $d \in {}^{\times}$ and $p \in [0,d] \to \mathcal{G}$, of class C^1 and with $p(0) = u$, be given. Then p is geodesic if and only if $p = q|_{[0,d]}$ for some one-parameter subgroup q of $\mathcal{G}$.*

Proof: By Prop. 6 and (35.28), p is geodesic if and only if $p^{\cdot}(0) \neq \mathbf{0}$ and

$$\mathbf{G}(p(t))p^{\cdot}(0) = p^{\cdot}(t) \qquad \text{for all} \qquad t \in [0,d]. \tag{48.19}$$

Let q be a one-parameter subgroup of $\mathcal{G}$ and $p = q|_{[0,d]}$. Let $t \in [0,d[$ be given. Then

$$\mathrm{le}_{p(t)}p(s) = q(t)q(s) = q(t+s) = p(t+s)$$
$$\text{for all} \quad s \in [0,d] \cap ([0,d] - t) = [0, d-t[.$$

Differentiating with respect to s at 0 and using (48.21), we get

$$\mathbf{G}(p(t))p^{\cdot}(0) = p^{\cdot}(t).$$

Since $t \in [0.d[$ was arbitrary and since $p^{\cdot}$ is continuous at d, (48.34) follows.

Assume now that p is geodesic, *i.e.* that (48.34) holds. Let $q : I \to \mathcal{G}$ be the (unique) solution of the differential equation

$$? \quad q \in \mathrm{C}^1(I, \mathcal{G}) \qquad , \qquad (\mathbf{G} \circ q)p^{\cdot}(0) = q^{\cdot} \tag{48.20}$$

whose domain I is the maximal interval that contains $0 \in$. Then I is an open interval, $[0,d] \subset I$, and $p = q|_{[0,d]}$ by the standard uniqueness theorem for differential equations. Let $t \in I$ be given and define $u : I \to \mathcal{G}$ and $v : (I - t) \to \mathcal{G}$ by

$$u(s) := q(t)q(s) = \mathrm{le}_{q(t)}(q(s)) \qquad \text{for all} \qquad s \in I \tag{48.21}$$

and

$$v(s) := q(t+s) \qquad \text{for all} \qquad s \in I - t \tag{48.22}$$

Using the chain rule and (48.24), it follows from (48.36) that

$$u^{\cdot}(s) = (\nabla_{q(s)}\mathrm{le}_{q(t)})q^{\cdot}(s) = \mathbf{G}(q(t)q(s))\mathbf{G}(q(s))^{-1}q^{\cdot}(s)$$

for all $s \in I$ and hence, by (71.23) and (71.24), that

$$u^{\cdot} = (\mathbf{G} \circ u)p^{\cdot}(0) \quad , \quad u(0) = q(t). \tag{48.23}$$

On the other hand, it follows (48.35) and (48.36) that

$$v^{\cdot}(s) = q^{\cdot}(t+s) = \mathbf{G}(q(t+s))p^{\cdot}(0)$$

for all $s \in I - t$ and hence that

$$v^{\cdot} = (\mathbf{G} \circ v)p^{\cdot}(0) \quad , \quad v(0) = q(t). \tag{48.24}$$

Comparing (48.38) and (48.39), we see that u and v satisfiy the same differential equation and initial condition. Since the domain of q is the maximal interval containng 0, it is clear that the domains of u and v must both be the maximal interval containing 0. It follows that $I - t = I$, which can be valid for all $t \in I$ only if $I = $. The standard uniqueness theorem for differential equations shows that $u = v$ and hence, by (48.36) and (48.37), that $q(t+s) = q(t)q(s)$ for all $s \in$. Since $t \in$ was arbitrary, it follows that q must be a one-parameter subgroup of $\mathcal{G}$. ∎

Chapter 5

Geometric Structures.

We assume in this chapter that numbers $r, s \in \tilde{\ }$, with $r \geq 3$ and $s \in 0..r$, a C^r manifold $\mathcal{M}$ and a C^s linear-space bundle $\mathcal{B}$ over the manifold $\mathcal{M}$ are given. We also assume that both $\mathcal{M}$ and $\mathcal{B}$ have constant dimensions, and put $n := \dim \mathcal{M}$ and $m := \dim \mathcal{B} - \dim \mathcal{M}$. Then we have $n = \dim \mathrm{T}_x\mathcal{M}$ and $m = \dim \mathcal{B}_x$ for all $x \in \mathcal{M}$.

51. Compatible Connections

Let $x \in \mathcal{M}$ be fixed. Let $\mathbf{\Phi}$ be an analytic tensor functor and let $\mathbf{E} \in \mathbf{\Phi}(\mathcal{B}_x)$ be given.

Notation: *We define the mapping*

$$\mathbf{E}^{\diamond} : \mathrm{Tlis}_x\mathcal{B} \to \mathbf{\Phi}(\mathcal{B}) \tag{51.1}$$

by

$$\mathbf{E}^{\diamond}(\mathbf{T}) := \mathbf{\Phi}(\mathbf{T})\mathbf{E} \quad \text{for all} \quad \mathbf{T} \in \mathrm{Tlis}_x\mathcal{B}. \tag{51.2}$$

Since $\mathbf{\Phi}$ is analytic, it is clear that $\mathbf{E}^{\diamond}$ is differentiable at $\mathbf{1}_{\mathcal{B}_x}$.

Proposition 1: *We have* $\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^{\diamond} \in \mathrm{Lin}\,(\mathrm{S}_x\mathcal{B}, \mathrm{T}_{\mathbf{E}}\mathbf{\Phi}(\mathcal{B}))$ *and, for every bundle chart* $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$,

$$(\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^{\diamond})\mathbf{s} = \mathbf{A}_{\mathbf{E}}^{\mathbf{\Phi}(\phi)}\mathbf{P}_x\mathbf{s} + \mathbf{I}_{\mathbf{E}}\mathbf{\Phi}^{\bullet}_x\big(\mathbf{\Lambda}(\mathbf{A}_x^{\phi})\mathbf{s}\big)\mathbf{E} \tag{51.3}$$

for all $\mathbf{s} \in \mathrm{S}_x\mathcal{B}$.

Proof: By using (51.2) and the definition (23.21) of gradient, we obtain the desired result. ▮

Taking the gradient of $\mathbf{E}^{\diamond}\big|_{\mathrm{Lis}\mathcal{B}_x}^{\mathbf{\Phi}(\mathcal{B}_x)}$ at $\mathbf{1}_{\mathcal{B}_x}$, we have

$$\Big(\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^{\diamond}\big|_{\mathrm{Lis}\mathcal{B}_x}^{\mathbf{\Phi}(\mathcal{B}_x)}\Big)\mathbf{L} = \big(\mathbf{\Phi}^{\bullet}_x(\mathbf{L})\big)\mathbf{E} \tag{51.4}$$

for all $\mathbf{L} \in \mathrm{Lin}\mathcal{B}_x$. For the sake of simplicity, we use the following notation

$$\mathbf{E}^{\circ} := \nabla_{\mathbf{1}_{\mathcal{B}_x}}\Big(\mathbf{E}^{\diamond}\big|_{\mathrm{Lis}\mathcal{B}_x}^{\mathbf{\Phi}(\mathcal{B}_x)}\Big). \tag{51.5}$$

Given $r \in \backslash\{0\}$, we observe from (51.5) that $(r\mathbf{E})^\circ = r\mathbf{E}^\circ$ and hence

$$\mathrm{Null}\,\mathbf{E}^\circ = \mathrm{Null}\,(r\mathbf{E})^\circ. \tag{51.6}$$

It is follows from (51.3) and (51.4) that

$$\mathbf{P}_x = \mathbf{P}_\mathbf{E}(\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^\diamond) \quad \text{and} \quad (\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^\diamond)\mathbf{I}_x = \mathbf{I}_\mathbf{E}\mathbf{E}^\circ,$$

i.e. the diagram

$$\begin{array}{ccccc}
\mathrm{Lin}\,\mathcal{B}_x & \xrightarrow{\mathbf{I}_x} & \mathrm{S}_x\mathcal{B} & \xrightarrow{\mathbf{P}_x} & \mathrm{T}_x\mathcal{M} \\
\mathbf{E}^\circ \big\downarrow & & \nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^\diamond \big\downarrow & & \big\| \\
\mathbf{\Phi}(\mathcal{B}_x) & \xrightarrow{\mathbf{I}_\mathbf{E}} & \mathrm{T}_\mathbf{E}\mathbf{\Phi}(\mathcal{B}) & \xrightarrow{\mathbf{P}_\mathbf{E}} & \mathrm{T}_x\mathcal{M}
\end{array} \tag{51.7}$$

commutes. And it also clear from (51.3) that

$$\mathbf{A}_\mathbf{E}^{\mathbf{\Phi}(\phi)} = (\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^\diamond)\mathbf{A}_x^\phi \in \mathrm{Rcon}_\mathbf{E}\mathbf{\Phi}(\mathcal{B}) \tag{51.8}$$

for all bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$. More generally, we have

$$(\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^\diamond)\mathbf{K} \in \mathrm{Rcon}_\mathbf{E}\mathbf{\Phi}(\mathcal{B}) \qquad \text{for all} \quad \mathbf{K} \in \mathrm{Con}_x\mathcal{B}. \tag{51.9}$$

In view of (51.9), the mapping $\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^\diamond$ induces the following mapping.

<u>Definition</u>: *We define the mapping*

$$\mathbf{J}_\mathbf{E} : \mathrm{Con}_x\mathcal{B} \to \mathrm{Rcon}_\mathbf{E}\mathbf{\Phi}(\mathcal{B})$$

by

$$\mathbf{J}_\mathbf{E}(\mathbf{K}) := (\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{E}^\diamond)\mathbf{K} \quad \text{for all} \quad \mathbf{K} \in \mathrm{Con}_x\mathcal{B}. \tag{51.10}$$

Proposition 2: *The mapping $\mathbf{J}_\mathbf{E}$, defined in* (51.10)*, is flat. Hence, for every $\mathbf{D} \in \mathrm{Rng}\,\mathbf{J}_\mathbf{E}$, $\mathbf{J}_\mathbf{E}^<(\{\mathbf{D}\})$ is a flat in $\mathrm{Con}_x\mathcal{B}$ with*

$$\dim \mathbf{J}_\mathbf{E}^<(\{\mathbf{D}\}) = ????.$$

Let a cross section $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathcal{B})$, that is differentiable at $x \in \mathcal{M}$, be given. The gradient of $\mathbf{H}$ at x is a tangent connector of $\mathbf{\Phi}(\mathcal{B})$; i.e. $\nabla_x\mathbf{H} \in \mathrm{Rcon}_{\mathbf{H}(x)}\mathbf{\Phi}(\mathcal{B})$.

Proposition 3: *We have*

$$\nabla_{\mathbf{K}}\mathbf{H} = \mathbf{\Lambda}\big((\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{H}(x)^{\diamond})\mathbf{K}\big)\nabla_x\mathbf{H} \tag{51.11}$$

for all $\mathbf{K} \in \mathrm{Con}_x\mathcal{B}$ *and hence* $\nabla_{\mathbf{K}}\mathbf{H} = \mathbf{0}$ *if and only if* $\mathbf{J}_{\mathbf{H}(x)}(\mathbf{K}) = \nabla_x\mathbf{H}$, *i.e.* $\mathbf{K} \in \mathbf{J}^{<}_{\mathbf{H}(x)}(\{\nabla_x\mathbf{H}\})$.

Proof: The desired result (51.11) follows from (51.8), (41.11), (42.1) and Remark 1 of Sect. 32.

If $\mathbf{K} \in \mathrm{Con}_x\mathcal{B}$ be such that $\nabla_{\mathbf{K}}\mathbf{H} = \mathbf{0}$, then it follows from (51.11) that $\mathbf{\Lambda}\big((\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{H}(x)^{\diamond})\mathbf{K}\big)\nabla_x\mathbf{H} = \mathbf{0}$. Applyiny Prop.1 of Sect.14, we see that this can happen if and only if $(\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathbf{H}(x)^{\diamond})\mathbf{K} = \nabla_x\mathbf{H}$. Since $\mathbf{K} \in \mathrm{Con}_x\mathcal{B}$ was arbitrary, the assertion follows. ∎

Now, let a differentiable cross section $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathcal{B})$ be given.

__**Definition**__: *A connection* $\mathbf{C}\mathcal{M} \to \mathrm{Con}\mathcal{B}$ *is called a* **H-compatible connection** *if* $\nabla_{\mathbf{C}(x)}\mathbf{H} = \mathbf{0}$ *for all* $x \in \mathcal{M}$, *i.e.*

$$\nabla_{\mathbf{C}}\mathbf{H} = \mathbf{0}. \tag{51.12}$$

It clear from Prop.3 that a connection $\mathbf{C}$ is $\mathbf{H}$-compatiable if and only if

$$\mathbf{J}_{\mathbf{H}(x)}(\mathbf{C}(x)) = \nabla_x\mathbf{H} \qquad \text{for all} \quad x \in \mathcal{M}. \tag{51.13}$$

Proposition 4: *Let connectors* $\mathbf{K}_1, \mathbf{K}_2 \in \mathbf{J}^{<}_{\mathbf{H}(x)}(\{\nabla_x\mathbf{H}\})$ *be given and determine* $\mathbf{L} \in \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\,\mathcal{B}_x)$ *such that* $\mathbf{K}_1 - \mathbf{K}_2 = \mathbf{I}_x\mathbf{L}$; *then we have*

$$\mathbf{H}(x)^{\circ}(\mathbf{L}\mathbf{t}) = \mathbf{0} \qquad \text{for all} \quad \mathbf{t} \in \mathrm{T}_x\mathcal{M}. \tag{51.14}$$

52. Riemannian and Symplectic Bundles

We apply Sect.51 to the case when $\mathbf{\Phi} = \mathrm{Smf}_2$ or Skf_2 (see example (4) of Sect.13).

Let $x \in \mathcal{M}$ be fixed and $\mathbf{E} \in \mathbf{\Phi}(\mathcal{B}_x)$, $\mathbf{\Phi} = \mathrm{Smf}_2$ or Skf_2, be given. We have

$$\mathbf{E}^\circ(\mathbf{M}) = \mathbf{E} \circ (\mathbf{M} \times \mathbf{1}_{\mathcal{B}_x}) + \mathbf{E} \circ (\mathbf{1}_{\mathcal{B}_x} \times \mathbf{M}), \tag{52.1}$$

where $\mathbf{E}^\circ$ is given in (51.5), for every $\mathbf{M} \in \mathrm{Lin}\mathcal{B}_x$.

Proposition 1: *If* $\mathbf{E}$ *is invertiable, then* $\mathbf{E}^\circ$ *is surjective; i.e.*

$$\mathrm{Rng}\,\mathbf{E}^\circ = \mathrm{Sym}_2(\mathcal{B}_x^2,) \quad \text{when} \quad \mathbf{\Phi} = \mathrm{Smf}_2 \tag{52.2}$$

i.e., $\mathbf{E} \in \mathrm{Sym}_2(\mathcal{B}_x^2,)$ *and*

$$\mathrm{Rng}\,\mathbf{E}^\circ = \mathrm{Skw}_2(\mathcal{B}_x^2,) \quad \text{when} \quad \mathbf{\Phi} = \mathrm{Skf}_2 \tag{52.3}$$

i.e., $\mathbf{E} \in \mathrm{Skw}_2(\mathcal{B}_x^2,)$.

Proof: By using (52.1). ∎

Proposition 2: *If* $\mathbf{E}$ *is invertiable, then the flat mapping* $\mathbf{J}_\mathbf{E}$ *defined in* (51.10) *is surjective.*

Proof: The surjectivity follows directly from (51.3), (51.4), (51.5) and the surjectivity of $\mathbf{E}^\circ$. ∎

In view of Prop.2 we see taht, for every $\mathbf{D} \in \mathrm{Rcon}_\mathbf{E}\mathbf{\Phi}(\mathcal{B})$, the preimage $\mathbf{J}_\mathbf{E}^{<}(\{\mathbf{D}\})$ is a flat in $\mathrm{Con}_x\mathcal{B}$. Let $\mathbf{K}_1, \mathbf{K}_2 \in \mathbf{J}_\mathbf{E}^{<}(\{\mathbf{D}\})$ be given and determine $\mathbf{L} \in \mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x)$ such that $\mathbf{K}_2 - \mathbf{K}_2 = \mathbf{I}_x\mathbf{L}$. Applying (51.3), we have $\mathbf{0} = \mathbf{J}_\mathbf{E}(\mathbf{K}_2) - \mathbf{J}_\mathbf{E}(\mathbf{K}_1) = \mathbf{E}^\circ(\mathbf{L})$, that is $\mathbf{L} \in \mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathrm{Null}\,\mathbf{E}^\circ)$. Since $\mathbf{K}_1, \mathbf{K}_2 \in \mathbf{J}_\mathbf{E}^{<}(\{\mathbf{D}\})$ were arbitrary, we conclude that

$$\dim \mathbf{J}_\mathbf{E}^{<}(\{\mathbf{D}\}) = \dim \mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathrm{Null}\,\mathbf{E}^\circ). \tag{52.4}$$

Definition: *A cross section* $\mathbf{G} : \mathcal{M} \to \mathrm{Smf}_2(\mathcal{B})$ *is called a* **Riemannian field** *if, for every* $x \in \mathcal{M}$, $\mathbf{G}(x)$ *is invertiable when regard as element of* $\mathrm{Sym}(\mathcal{B}_x, \mathcal{B}_x^*)$.

A cross section $\mathbf{S} : \mathcal{M} \to \mathrm{Skf}_2(\mathcal{B})$ *is called a* **symplectic field of** $\mathcal{B}$ *if, for every* $x \in \mathcal{M}$, $\mathbf{S}(x)$ *is invertiable when regard as element of* $\mathrm{Skw}(\mathcal{B}_x, \mathcal{B}_x^*)$.

We say that $\mathcal{B}$ *is a* C^s **Riemannian linear space bundle** *if it is endowed with additional structure by the prescription of a* C^s *Riemannian field.*

We say that $\mathcal{B}$ *is a* C^s **symplectic linear space bundle** *if it is endowed with additional structure by the prescription of a* C^s *symplectic field.*

Remark 1: A symplectic field of $\mathcal{B}$ exist if and only if, for every $x \in \mathcal{M}$, $m := \dim \mathcal{B}_x$ is even (see Sect.11). If m is odd, then

$$\mathrm{Skw}(\mathcal{B}_x, \mathcal{B}_x^*) \cap \mathrm{Lis}(\mathcal{B}_x, \mathcal{B}_x^*) = \emptyset.$$

∎

Proposition 3: *If* $\mathbf{G} : \mathcal{M} \to \mathrm{Smf}_2(\mathcal{B})$ *is a Riemannian field, then*

$$\dim \mathbf{J}^{<}_{\mathbf{G}(x)}(\{\nabla_x \mathbf{G}\}) = n\binom{m}{2} \quad \text{for all} \quad x \in \mathcal{M}. \tag{52.5}$$

If $\mathbf{S} : \mathcal{M} \to \mathrm{Skf}_2(\mathcal{B})$ *is a symplectic field, then*

$$\dim \mathbf{J}^{<}_{\mathbf{S}(x)}(\{\nabla_x \mathbf{S}\}) = n\binom{m+1}{2} \quad \text{for all} \quad x \in \mathcal{M}. \tag{52.6}$$

Proof: It following easily from (52.4), (52.2) and (52.3). ∎

Remark 2: Let $\mathbf{G}$ be a Riemannian field and $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\mathcal{B}$ be a $\mathbf{G}$-compatible connection. Let $\mathbf{L} : \mathcal{M} \to \mathrm{Lis}\,\mathcal{B}$ be a cross section with $\nabla_{\mathbf{C}}\mathbf{L} = \mathbf{0}$ be given. Then, it follows from $\nabla_{\mathbf{C}}\mathbf{G} = \mathbf{0}$ and $\nabla_{\mathbf{C}}\mathbf{L} = \mathbf{0}$ that $\nabla_{\mathbf{C}}(\mathbf{G} \circ (\mathbf{L} \times \mathbf{L})) = \mathbf{0}$. Hence, the Riemannian field $\mathbf{H} := \mathbf{G} \circ (\mathbf{L} \times \mathbf{L})$ satisfies $\nabla_{\mathbf{C}}\mathbf{H} = \mathbf{0}$. ∎

53. Riemannian and Symplectic Manifolds.

Definition: *We say that $\mathcal{M}$ is a* **Riemannian manifold** *if the tangent bundle $\mathrm{T}\mathcal{M}$ is endowed with additional structure by the prescription of a C^{r-1} Riemannian field.*

We say that $\mathcal{M}$ is a **symplectic manifold** *if the tangent bundle $\mathrm{T}\mathcal{M}$ is endowed with additional structure by the prescription of a C^{r-1} symplectic field.*

Let a Riemannian field $\mathbf{G} : \mathcal{M} \to \mathrm{Sym}^{\mathrm{inv}}(\mathrm{T}\mathcal{M}, \mathrm{T}\mathcal{M}^*)$ of class C^{r-1} be given.

Proposition 1: *For every $x \in \mathcal{M}$, the restriction*

$$\mathbf{T}_x\big|_{\mathbf{J}^{<}_{\mathbf{G}(x)}(\{\nabla_x\mathbf{G}\})} : \mathbf{J}^{<}_{\mathbf{G}(x)}(\{\nabla_x\mathbf{G}\}) \to \mathrm{Skw}_2(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M}) \tag{53.1}$$

of the torsion mapping $\mathbf{T}_x$ is bijective.

Proof: Given $x \in \mathcal{M}$. If $\mathbf{K}_1, \mathbf{K}_2 \in \mathrm{Con}_x(\mathrm{T}\mathcal{M}, \mathcal{M})$, then we have $\mathbf{T}_x(\mathbf{K}_1) = \mathbf{T}_x(\mathbf{K}_2)$ if and only if $\mathbf{K}_1 - \mathbf{K}_2 = \mathbf{I}_x\mathbf{L}$ for some $\mathbf{L} \in \mathrm{Sym}_2((\mathrm{T}_x\mathcal{M})^2, \mathrm{T}_x\mathcal{M})$ and hence

$$(\mathbf{G}(x)\mathbf{L})(\mathbf{t}, \mathbf{b}, \mathbf{d}) = (\mathbf{G}(x)\mathbf{L})(\mathbf{b}, \mathbf{t}, \mathbf{d}) \tag{53.2}$$

for all $\mathbf{t}, \mathbf{b}, \mathbf{d} \in \mathrm{T}_x\mathcal{M}$.

Let $\mathbf{K}_1, \mathbf{K}_2 \in \mathbf{J}^{<}_{\mathbf{G}(x)}(\{\nabla_x\mathbf{G}\})$ with $\mathrm{T}_x(\mathbf{K}_1) = \mathrm{T}_x(\mathbf{K}_2)$ be given and determining $\mathbf{L} \in \mathrm{Lin}_2((\mathrm{T}_x\mathcal{M})^2, \mathrm{T}_x\mathcal{M})$ such that $\mathbf{K}_1 - \mathbf{K}_2 = \mathbf{I}_x\mathbf{L}$. Applying (52.1), (51.14) and (53.2), we have

$$\begin{aligned}(\mathbf{G}(x)\mathbf{L})(\mathbf{t}, \mathbf{b}, \mathbf{d}) &= -(\mathbf{G}(x)\mathbf{L})(\mathbf{t}, \mathbf{d}, \mathbf{b}) = -(\mathbf{G}(x)\mathbf{L})(\mathbf{d}, \mathbf{t}, \mathbf{b}) = \\ &= (\mathbf{G}(x)\mathbf{L})(\mathbf{d}, \mathbf{b}, \mathbf{t}) = (\mathbf{G}(x)\mathbf{L})(\mathbf{b}, \mathbf{d}, \mathbf{t}) = \\ &= -(\mathbf{G}(x)\mathbf{L})(\mathbf{b}, \mathbf{t}, \mathbf{d}) = -(\mathbf{G}(x)\mathbf{L})(\mathbf{t}, \mathbf{b}, \mathbf{d})\end{aligned}$$

for all $\mathbf{t}, \mathbf{b}, \mathbf{d} \in \mathrm{T}_x\mathcal{M}$. This shown that $\mathbf{G}(x)\mathbf{L} = \mathbf{0}$. Since $\mathbf{G}(x)$ is invertible, we observe that $\mathbf{L} = \mathbf{0}$ and hence $\mathbf{K}_1 = \mathbf{K}_2$. In other words, the restriction

$$\mathbf{T}_x\big|_{\mathbf{J}^{<}_{\mathbf{G}(x)}(\{\nabla_x\mathbf{G}\})} : \mathbf{J}^{<}_{\mathbf{G}(x)}(\{\nabla_x\mathbf{G}\}) \to \mathrm{Skw}_2(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M}) \tag{53.3}$$

of the flat mapping $\mathbf{T}_x$ is injective and hence bijective. Since $x \in \mathcal{M}$ was arbitrary, the assertion follows. ∎

Proposition 2: *For every $x \in \mathcal{M}$, we have*

$$\mathbf{J}^{<}_{\mathbf{G}(x)}(\{\nabla_x \mathbf{G}\}) = \Big\{ \mathbf{K} - \frac{1}{2}\mathbf{I}_x \mathbf{G}(x)^{-1}\big(\mathrm{S}\,(\nabla_{\mathbf{K}}\mathbf{G})\big) \,\Big|\, \mathbf{K} \in \mathrm{Con}_x(\mathcal{TM},\mathcal{M}) \Big\} \tag{53.4}$$

where

$$\big(\mathrm{S}\,(\nabla_{\mathbf{K}}\mathbf{G})\big) = \nabla_{\mathbf{K}}\mathbf{G} + \nabla_{\mathbf{K}}\mathbf{G}^{\sim(1,2)} - \nabla_{\mathbf{K}}\mathbf{G}^{\sim(1,3)}.$$

Moreover, if $\mathbf{K}_1, \mathbf{K}_2 \in \mathrm{Con}_x(\mathcal{TM},\mathcal{M})$ *with* $\mathbf{T}_x(\mathbf{K}_1) = \mathbf{T}_x(\mathbf{K}_2)$, *i.e.*

$$\mathbf{K}_1 - \mathbf{K}_2 \in \{\mathbf{I}_x\}\mathrm{Sym}_2(\mathcal{T}_x\mathcal{M}^2, \mathcal{T}_x\mathcal{M})),$$

then we have

$$\begin{aligned} &\mathbf{K}_1 - \frac{1}{2}\mathbf{I}_x\mathbf{G}(x)^{-1}\big(\nabla_{\mathbf{K}_1}\mathbf{G} + \nabla_{\mathbf{K}_1}\mathbf{G}^{\sim(1,2)} - \nabla_{\mathbf{K}_1}\mathbf{G}^{\sim(1,3)}\big) \\ &\qquad = \mathbf{K}_2 - \frac{1}{2}\mathbf{I}_x\mathbf{G}(x)^{-1}\big(\nabla_{\mathbf{K}_2}\mathbf{G} + \nabla_{\mathbf{K}_2}\mathbf{G}^{\sim(1,2)} - \nabla_{\mathbf{K}_2}\mathbf{G}^{\sim(1,3)}\big). \end{aligned} \tag{53.5}$$

Proof: By (41.8), we have

$$\begin{aligned} \big((\Box_x\mathbf{G})\mathbf{I}_x\mathbf{G}(x)^{-1}\nabla_{\mathbf{K}}\mathbf{G}\big)(\mathbf{s},\mathbf{t},\mathbf{u}) &= \nabla_{\mathbf{K}}\mathbf{G}(\mathbf{s},\mathbf{t},\mathbf{u}) + \nabla_{\mathbf{K}}\mathbf{G}(\mathbf{s},\mathbf{u},\mathbf{t}), \\ \big((\Box_x\mathbf{G})\mathbf{I}_x\mathbf{G}(x)^{-1}\nabla_{\mathbf{K}}\mathbf{G}^{\sim(1,2)}\big)(\mathbf{s},\mathbf{t},\mathbf{u}) &= \nabla_{\mathbf{K}}\mathbf{G}(\mathbf{t},\mathbf{s},\mathbf{u}) + \nabla_{\mathbf{K}}\mathbf{G}(\mathbf{u},\mathbf{s},\mathbf{t}), \\ \big((\Box_x\mathbf{G})\mathbf{I}_x\mathbf{G}(x)^{-1}\nabla_{\mathbf{K}}\mathbf{G}^{\sim(1,3)}\big)(\mathbf{s},\mathbf{t},\mathbf{u}) &= \nabla_{\mathbf{K}}\mathbf{G}(\mathbf{t},\mathbf{u},\mathbf{s}) + \nabla_{\mathbf{K}}\mathbf{G}(\mathbf{u},\mathbf{t},\mathbf{s}); \end{aligned} \tag{53.6}$$

for all $\mathbf{s},\mathbf{t},\mathbf{u} \in \mathcal{T}_x\mathcal{M}$. Observing $\nabla_{\mathbf{K}}\mathbf{G} \in \mathrm{Lin}\,\big(\mathcal{T}_x\mathcal{M}, \mathrm{Sym}_2(\mathcal{T}_x\mathcal{M}^2,)\big)$, we see that (53.4)) follows easily from (53.6). ∎

The more general version of "the fundamental theorem of Riemannian geometry" follows immediately from Prop. 1:

Fundamental Theorem of Riemannian Geometry (with torsion):
For every prescribed torsion field $\mathbf{L} : \mathcal{M} \to \mathrm{Skw}_2(\mathrm{T}\mathcal{M}^2, \mathrm{T}\mathcal{M})$ *of class* C^s, $s \in 0..r-2$, *there is exactly one* $\mathbf{G}$*-compatible connection* $\mathbf{C}$, *i.e. one satisfying* $\nabla_{\mathbf{C}}\mathbf{G} = \mathbf{0}$, *such that* $\mathbf{T}(\mathbf{C}) = \mathbf{L}$. $\mathbf{C}$ *is of class* C^s.

Remark 1: When $\mathbf{L} = \mathbf{0}$, the corresponding connection is called the **Levi-Cività connection**. ∎

Remark 2: It follows from Theorem 3 that for every connection $\mathbf{C}' : \mathcal{M} \to \mathrm{Con}\,\mathcal{TM}$ of class C^s, $s \in 0..r-2$, there is exactly one connection $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\,\mathcal{TM}$ such that $\mathbf{T}(\mathbf{C}) = \mathbf{T}(\mathbf{C}')$ and $\nabla_{\mathbf{C}}\mathbf{G} = \mathbf{0}$. Moreover, in view of Prop. 2, we have

$$\mathbf{C} = \mathbf{C}' - \frac{1}{2}\mathbf{I}\mathbf{G}^{-1}\big(\nabla_{\mathbf{C}'}\mathbf{G} - \nabla_{\mathbf{C}'}\mathbf{G}^{\sim(1,2)} + \nabla_{\mathbf{C}'}\mathbf{G}^{\sim(1,3)}\big). \tag{53.7}$$

Now let a connection $\mathbf{C} : \ \to \mathrm{Con}\mathrm{T}\mathcal{M}$ be given. We may define, for each $x \in \mathcal{M}$, a mapping

$$\mathbf{A}_x^{\mathbf{C}} : \mathrm{Con}_x\mathrm{T}\mathcal{M} \to \mathrm{Sym}_2(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M}) \tag{53.8}$$

by

$$\mathbf{A}_x^{\mathbf{C}}(\mathbf{K}) := \mathbf{\Lambda}(\mathbf{C}(x))\mathbf{K} + \big(\mathbf{\Lambda}(\mathbf{C}(x))\mathbf{K}\big)^{\sim} \quad \text{for all} \quad \mathbf{K} \in \mathrm{Con}_x\mathrm{T}\mathcal{M}. \tag{53.9}$$

Let a symplectic field $\mathbf{S} : \mathcal{M} \to \mathrm{Skw}^{\mathrm{inv}}(\mathrm{T}\mathcal{M}, \mathrm{T}^*\mathcal{M})$ of class C^{r-1} be given.

Proposition 3: *For every $x \in \mathcal{M}$, the restriction*

$$\mathbf{A}_x^{\mathbf{C}}\big|_{\mathbf{J}^{<}_{\mathbf{S}(x)}(\{\nabla_x\mathbf{S}\})} : \mathbf{J}^{<}_{\mathbf{S}(x)}(\{\nabla_x\mathbf{S}\}) \to \mathrm{Sym}_2(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M}) \tag{53.10}$$

of the mapping $\mathbf{A}_x^{\mathbf{C}}$ is bijective.

Proof: Similar to the proof of Prop. 1. ▮

Proposition 4: *For every connection $\mathbf{C}$ and each prescribed symmetric field $\mathbf{L} : \mathcal{M} \to \mathrm{Sym}_2(\mathrm{T}\mathcal{M}^2, \mathrm{T}\mathcal{M})$ of class C^s, $s \in 0..r-2$, there is exactly one $\mathbf{S}$-compatible connection $\mathbf{K}$, i.e. one satisfying $\nabla_{\mathbf{K}}\mathbf{S} = \mathbf{0}$, such that $\mathbf{A}^{\mathbf{C}}(\mathbf{K}) = \mathbf{L}$. $\mathbf{K}$ is of class C^s.*

Proof: It follows immediately from Prop.3. ▮

Notes 53

(1) The proof of the Fundamental Theorem of Riemannian Geometry given here is modelled on the proof given by Noll in [N1].

(2) In [Sp], Spivak, M. stated: “Perhaps its only defect [of the fundamental theorem of Riemannian geometry] is the restriction to symmetric connections.” We show that this restriction is not needed.

54. Identities

Let a C^r, $r \geq 2$, Riemannian manifold $\mathcal{M}$ with the Riemannian-field $\mathbf{G}$ be given. Assume that $\dim \mathcal{M} \geq 2$.

For every $A, B \in \mathfrak{X}(\mathrm{T}\mathcal{M})$ and a connection $\mathbf{C} : \mathcal{M} \to \mathrm{Con}(\mathrm{T}\mathcal{M})$, we use the following notations

$$\langle A, B\rangle := \mathbf{G}(A, B) \quad \text{and} \quad \nabla_A B := (\nabla_{\mathbf{C}} B)A.$$

Proposition 1: *A connection* $\mathbf{C}$ *on a Riemannian manifold* $\mathcal{M}$ *is compatible with the Riemannian-field* $\mathbf{G}$ *if and only if*

$$A\langle B, D\rangle = \langle \nabla_A B, D\rangle + \langle B, \nabla_A D\rangle \tag{54.1}$$

for all $A, B, D \in \mathfrak{X}(\mathrm{T}\mathcal{M})$.

Proof: Taking the covariant gradient of $\mathbf{G} \circ (B, D)$ with respect to $\mathbf{C}$, we obtain

$$\begin{aligned}(\nabla_{\mathbf{C}}(\mathbf{G} \circ (B, D)))A = \mathbf{G}((\nabla_{\mathbf{C}} B)A, D) + \mathbf{G}(B, (\nabla_{\mathbf{C}} D)A).\\ + (\nabla_{\mathbf{C}}\mathbf{G})(A, B, D)\end{aligned}$$

The equation (I.1) holds if and only if $\nabla_{\mathbf{C}}\mathbf{G} = \mathbf{0}$. ▮

For the sake of simplification, we adapt the following notation

$$\langle\!\langle X, Y, Z, T\rangle\!\rangle := \langle \mathbf{R}(X, Y)Z, T\rangle \quad \text{for all} \quad X, Y, Z, T \in \mathfrak{X}(\mathrm{T}\mathcal{M}),$$

where $\mathbf{R} := \mathbf{R}(\mathbf{C})$ is the curvature field for a given connection $\mathbf{C}$. Also recall that

$$\mathbf{R}(X, Y, Z) = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

for all $X, Y, Z \in \mathfrak{X}(\mathrm{T}\mathcal{M})$.

Proposition 2: *Let* $\mathbf{C}$ *be a connection on a Riemannian manifold* $\mathcal{M}$ *which is compatible with the Riemannian-field* $\mathbf{G}$*, then we have*

$$\langle\!\langle X, Y, Z, T\rangle\!\rangle = -\langle\!\langle X, Y, T, Z\rangle\!\rangle \tag{54.2}$$

for all $X, Y, Z, T \in \mathfrak{X}(\mathrm{T}\mathcal{M})$.

Proof: To prove (I.2) is equivalent to show

$$0 = \langle\!\langle X, Y, Z, Z\rangle\!\rangle = \langle \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, Z\rangle.$$

Applying (I.1), we have

$$\langle \nabla_Y \nabla_X Z, Z\rangle = Y\langle \nabla_X Z, Z\rangle - \langle \nabla_X Z, \nabla_Y Z\rangle$$

and

$$\langle \nabla_X \nabla_Y Z, Z\rangle = X\langle \nabla_Y Z, Z\rangle - \langle \nabla_Y Z, \nabla_X Z\rangle.$$

Hence

$$\langle\!\langle X, Y, Z, Z\rangle\!\rangle \;= Y\langle \nabla_X Z, Z\rangle - X\langle \nabla_Y Z, Z\rangle + \langle \nabla_{[X,Y]} Z, Z\rangle.$$

It follows from (I.1) and the symmetry of the Riemannian-field $\mathbf{G}$ that

$$\frac{1}{2}A\langle D, D\rangle = \langle \nabla_A D, D\rangle \quad \text{for all} \quad A, D \in \mathfrak{X}(\mathrm{T}\mathcal{M}). \tag{54.3}$$

And hence

$$\begin{aligned}\langle\!\langle X, Y, Z, Z\rangle\!\rangle \;&= \frac{1}{2}Y(X\langle Z, Z\rangle) - \frac{1}{2}X(Y\langle Z, Z\rangle) + \frac{1}{2}[X, Y]\langle Z, Z\rangle \\ &= -\frac{1}{2}[X, Y]\langle Z, Z\rangle + \frac{1}{2}[X, Y]\langle Z, Z\rangle = 0.\end{aligned}$$

Since $X, Y, Z \in \mathfrak{X}(\mathrm{T}\mathcal{M})$ were arbitrary, the equation (I.2) follows. ∎

Let $\mathbf{C}$ be a compatible connection with the Riemannian-field $\mathbf{G}$.

Given $x \in \mathcal{M}$. Since $\mathbf{R}_x(\mathbf{C}) \in \mathrm{Skw}_2(\mathrm{T}_x\mathcal{M}^2, \mathrm{Lin}\,\mathrm{T}_x\mathcal{M})$, we observe form Prop. 2 that

$$\langle\!\langle \,\cdot\,, \,\cdot\,, \,\cdot\,, \,\cdot\,\rangle\!\rangle \in \mathrm{Skw}_2(\mathrm{T}_x\mathcal{M}^2, \mathrm{Skw}_2(\mathrm{T}_x\mathcal{M}^2,)).$$

Lemma : *Let an inner-product space $\mathcal{T}$, with $\dim \mathcal{T} \geq 2$, and a two-dimensional subspace $\mathcal{S}$ of $\mathcal{T}$ be given. If both $\{\mathbf{u}, \mathbf{v}\}$ and $\{\mathbf{s}, \mathbf{t}\}$ are bases for $\mathcal{S}$, then we have*

$$\frac{\mathbf{W}(\mathbf{u}, \mathbf{v}, \mathbf{u}, \mathbf{v})}{(\mathbf{u} \wedge \mathbf{v})(\mathbf{u}, \mathbf{v})} = \frac{\mathbf{W}(\mathbf{s}, \mathbf{t}, \mathbf{s}, \mathbf{t})}{(\mathbf{s} \wedge \mathbf{t})(\mathbf{s}, \mathbf{t})} \tag{54.4}$$

for all $\mathbf{W} \in \mathrm{Skw}_2(\mathcal{T}^2, \mathrm{Skw}_2(\mathcal{T}^2,))$.

Proof: By calculations. ∎

Applying the above Lemma, we arrive the following definition.

Definition : *Let $\mathcal{V} \subset \mathrm{T}_x\mathcal{M}$ be a two-dimensional subspace of $\mathrm{T}_x\mathcal{M}$. Let $\{\mathbf{u}, \mathbf{v}\}$ be a basis for $\mathcal{S}$. The* **sectional curvature of $\mathcal{S}$ at** *x is defined by*

$$\mathbf{K}_x(\mathcal{S}) := \frac{\langle\!\langle \mathbf{u}, \mathbf{v}, \mathbf{u}, \mathbf{v}\rangle\!\rangle}{(\mathbf{u} \wedge \mathbf{v})(\mathbf{u}, \mathbf{v})} \tag{54.5}$$

which does not depend on the choice of $\{\mathbf{u}, \mathbf{v}\}$.

Remark : The definition of sectional curvature "*does not*" require the assuption of the compatible connection $\mathbf{C}$ to be torsion-free. ▮

Proposition 4: *Let* $\mathbf{C}$ *be a connection on a Riemannian manifold* $\mathcal{M}$ *which is compatible with the Riemannian-field* $\mathbf{G}$*, then we have*

$$\langle\!\langle X, Y, Z, W \rangle\!\rangle - \langle\!\langle Z, W, X, Y \rangle\!\rangle = \mathbf{V}(X, Y, Z, W) \tag{54.6}$$

for all $X, Y, Z, W \in \mathfrak{X}(\mathrm{T}\mathcal{M})$.

Proof:

$$\begin{aligned}
&\mathbf{R}(X,Y)Z\cdot W + \mathbf{R}(Y,Z)X\cdot W + \mathbf{R}(Z,X)Y\cdot W \\
&\qquad + \mathbf{R}(Y,Z)W\cdot X + \mathbf{R}(Z,W)Y\cdot X + \mathbf{R}(W,Y)Z\cdot X \\
&\qquad + \mathbf{R}(Z,W)X\cdot Y + \mathbf{R}(W,X)Z\cdot Y + \mathbf{R}(X,Z)W\cdot Y \\
&\qquad + \mathbf{R}(W,X)Y\cdot Z + \mathbf{R}(X,Y)W\cdot Z + \mathbf{R}(Y,W)X\cdot Z \\
&= \nabla\mathbf{T}(X,Y,Z)\cdot W + \nabla\mathbf{T}(Y,Z,X)\cdot W + \nabla\mathbf{T}(Z,X,Y)\cdot W \\
&\qquad + \nabla\mathbf{T}(Y,Z,W)\cdot X + \nabla\mathbf{T}(Z,W,Y)\cdot X + \nabla\mathbf{T}(W,Y,Z)\cdot X \\
&\qquad + \nabla\mathbf{T}(Z,W,X)\cdot Y + \nabla\mathbf{T}(W,X,Z)\cdot Y + \nabla\mathbf{T}(X,W,Z)\cdot Y \\
&\qquad + \nabla\mathbf{T}(W,X,Y)\cdot Z + \nabla\mathbf{T}(X,Y,W)\cdot Z + \nabla\mathbf{T}(Y,W,X)\cdot Z \\
&\qquad + \mathbf{T}(\mathbf{T}(X,Y),Z)\cdot W + \mathbf{T}(\mathbf{T}(Y,Z),X)\cdot W + \mathbf{T}(\mathbf{T}(Z,X),Y)\cdot W \\
&\qquad + \mathbf{T}(\mathbf{T}(Y,Z),W)\cdot X + \mathbf{T}(\mathbf{T}(Z,W),Y)\cdot X + \mathbf{T}(\mathbf{T}(W,Y),Z)\cdot X \\
&\qquad + \mathbf{T}(\mathbf{T}(Z,W),X)\cdot Y + \mathbf{T}(\mathbf{T}(W,X),Z)\cdot Y + \mathbf{T}(\mathbf{T}(X,Z),W)\cdot Y \\
&\qquad + \mathbf{T}(\mathbf{T}(W,X),Y)\cdot Z + \mathbf{T}(\mathbf{T}(X,Y),W)\cdot Z + \mathbf{T}(\mathbf{T}(Y,W),X)\cdot Z
\end{aligned}$$

Proposition 5: *Let* $\mathbf{C}$ *be a connection on a Riemannian manifold* $\mathcal{M}$ *which is compatible with the Riemannian-field* $\mathbf{G}$*, then we have*

$$\mathrm{tr}\Big(\mathbf{R}(x)(\mathbf{s},\cdot)\,\mathbf{t} - \mathbf{R}(x)(\mathbf{t},\cdot)\,\mathbf{s} + \mathbf{R}(x)(\mathbf{t},\mathbf{s})\Big) = ???? \tag{54.7}$$

for all $\mathbf{s}, \mathbf{t} \in \mathrm{T}_x\mathcal{M}$.

Second Proof of Pro. 2:

In view of (I.1) we have, for all $X, Y, Z, T \in \mathfrak{X}(\mathrm{T}\mathcal{M})$,

$$\langle \nabla_Y \nabla_X Z, T\rangle = Y\langle \nabla_X Z, T\rangle - \langle \nabla_X Z, \nabla_Y T\rangle,$$

$$\langle \nabla_X \nabla_Y Z, T\rangle = X\langle \nabla_Y Z, T\rangle - \langle \nabla_Y Z, \nabla_X T\rangle$$

and

$$\langle \nabla_{[X,Y]} Z, T\rangle = [X,Y]\langle Z,T\rangle - \langle Z, \nabla_{[X,Y]} T\rangle.$$

Hence

$$\begin{aligned}
\langle\!\langle X,Y,Z,T\rangle\!\rangle &= \langle \nabla_Y \nabla_X Z, T\rangle - \langle \nabla_X \nabla_Y Z, T\rangle + \langle \nabla_{[X,Y]} Z, T\rangle \\
&= Y\langle \nabla_X Z, T\rangle - \langle \nabla_X Z, \nabla_Y T\rangle - X\langle \nabla_Y Z, T\rangle + \langle \nabla_Y Z, \nabla_X T\rangle \\
&\quad + [X,Y]\langle Z,T\rangle - \langle Z, \nabla_{[X,Y]} T\rangle \\
&= Y(X\langle Z,T\rangle) - Y\langle Z, \nabla_X T\rangle - X(Y\langle Z,T\rangle) + X\langle Z, \nabla_Y T\rangle \\
&\quad - \langle \nabla_X Z, \nabla_Y T\rangle + \langle \nabla_Y Z, \nabla_X T\rangle + [X,Y]\langle Z,T\rangle - \langle Z, \nabla_{[X,Y]} T\rangle \\
&= -Y\langle Z, \nabla_X T\rangle + X\langle Z, \nabla_Y T\rangle \\
&\quad - \langle \nabla_X Z, \nabla_Y T\rangle + \langle \nabla_Y Z, \nabla_X T\rangle - \langle Z, \nabla_{[X,Y]} T\rangle \\
&= -\langle \nabla_Y \nabla_X T, Z\rangle + \langle \nabla_X \nabla_Y T, Z\rangle - \langle \nabla_{[X,Y]} T, Z\rangle \\
&= -\langle\!\langle X,Y,T,Z\rangle\!\rangle\,.
\end{aligned}$$

Since $X,Y,Z,T \in \mathfrak{X}(\mathrm{T}\mathcal{M})$ was arbitrary, the assertion of Prop. 2 follows.

55. Einstein-tensor field

Let a C^r manifold $\mathcal{M}$, with $r \geq 2$ and $\dim \mathcal{M} \geq 2$, and a C^r connection $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\,(\mathrm{T}\mathcal{M})$ be given. Assume that $\mathbf{G} : \mathcal{M} \to \mathrm{Sym}_2(\mathrm{T}\mathcal{M}^2,)$ be a Riemannian-field compatiable with the connection $\mathbf{C}$.

Let $x \in \mathcal{M}$ be given and assume that the following condition hold

$$\mathrm{tr}\,\Big(\mathbf{R}(x)(\mathbf{s},\cdot)\,\mathbf{t} - \mathbf{R}(x)(\mathbf{t},\cdot)\,\mathbf{s} + \mathbf{R}(x)(\mathbf{t},\mathbf{s})\Big) = 0, \tag{55.1}$$

i.e. we have

$$\mathrm{tr}\,(\mathbf{R}(x)(\mathbf{s},\cdot)\,\mathbf{t}) - \mathrm{tr}\,(\mathbf{R}(x)(\mathbf{t},\cdot)\,\mathbf{s}) + \mathrm{tr}\,(\mathbf{R}(x)(\mathbf{t},\mathbf{s})) = 0.$$

Since $\mathbf{R}(x)(\mathbf{t},\mathbf{s})$ is skew-symmetric with respect to $\mathbf{G}$, we obtain that

$$\mathrm{tr}\,(\mathbf{R}(x)(\mathbf{s},\cdot)\,\mathbf{t}) = \mathrm{tr}\,(\mathbf{R}(x)(\mathbf{t},\cdot)\,\mathbf{s}) \qquad \text{for all} \quad \mathbf{s},\mathbf{t} \in \mathrm{T}_x\mathcal{M}.$$

<u>Definition</u> : *The* **Ricci-tensor field** $\mathrm{Ric} : \mathcal{M} \to \mathrm{Sym}_2(\mathrm{T}\mathcal{M}^2,)$ *is defined by*

$$\mathrm{Ric}(x)(\mathbf{s},\mathbf{t}) := \mathrm{tr}\,(\mathbf{R}(x)(\mathbf{s},\cdot)\,\mathbf{t}) \tag{55.2}$$

for all $x \in \mathcal{M}$ *and all* $\mathbf{s},\mathbf{t} \in \mathrm{T}_x\mathcal{M}$.

Definition : *The* **Einstein-tensor field** $\mathrm{Ein}:\mathcal{M}\to\mathrm{Sym}_2(\mathrm{T}\mathcal{M}^2,)$ *is defined by*

$$\mathrm{Ein}(x) := \mathrm{Ric}(x) - \frac{1}{2}\mathrm{tr}\,(\mathbf{G}^{-1}(x)\mathrm{Ric}(x))\,\mathbf{G}(x) \tag{55.3}$$

for all $x\in\mathcal{M}$. (The factor $1/2$ is determined by the assumption $\dim \mathrm{T}_x\mathcal{M} = 4$!)

It follows from the 2nd Bianchi Identity (this condition should be weaken) that

$$\mathrm{div}_{\mathbf{C}}\,\mathrm{Ein} = 0. \tag{55.4}$$

Remark: The **matter tensor field** $\mathrm{Mat}:\mathcal{M}\to\mathrm{Sym}_2(\mathrm{T}\mathcal{M}^2,)$ satisfying

$$\mathrm{Ein}(x) = \kappa\,\mathrm{Mat}(x) \tag{55.5}$$

where $\kappa\in$ is the **universal gravitational constant**. It follows from (Ein.4) and (Ein 5) that

$$\mathrm{div}_{\mathbf{C}}\,\mathrm{Mat} = 0 \tag{55.6}$$

(balance of world-momentum). ∎

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