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Some Applications of Recursive Functionals to the Foundations of Mathematics and Physics

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December 2010

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Abstract

We consider two applications of recursive functionals. The first application concerns Gödel's theory $\mathcal{T}$, which provides a rudimentary foundation for the formalization of mathematics. $\mathcal{T}$ can be understood as a theory of the simply-typed lambda calculus that is extended to include the constant 0_{N}, the successor function S, and the operator R_τ for primitive recursion on objects of type τ. It is known that the functions from non-negative integers to non-negative integers that can be defined in this theory are exactly the $<\varepsilon_0$-recursive functions of non-negative integers. But it is not well-known which *functionals* of arbitrary type can be defined in $\mathcal{T}$. We show that when the domain and codomain are restricted to pure closed normal forms, the functionals of arbitrary type that are definable in $\mathcal{T}$ are exactly those functionals that can be encoded as $<\varepsilon_0$-recursive functions of non-negative integers. This result has many interesting consequences, including a new characterization of $\mathcal{T}$.

The second application is concerned with the question: "When can a model of a physical system be regarded as computable?" We provide the definition of a *computable physical model* to answer this question. The connection between our definition and Kreisel's notion of a mechanistic theory is discussed, and several examples of computable physical models are given, including models which feature discrete motion, a model which features non-discrete continuous motion, and non-deterministic models such as radioactive decay. We show how computable physical models on effective topological spaces can be formulated with recursive functionals in the theory of type-two effectivity (TTE). Various common operations on computable physical models are described, such as the operation of coarse-graining and the formation of statistical ensembles. The definition of a computable physical model also allows for a precise formalization of the *computable universe hypothesis*—the claim that all the laws of physics are computable.

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Preface

The word *functional*, when used as a noun, does not have a precise, universally agree-upon meaning. It has generally been used to describe a function whose domain is itself a set of functions, but the question of what sorts of functions are allowed in the domain, and the question of what sorts of operations may be performed on inputs taken from that domain, have varied from author-to-author and from context-to-context. A *recursive functional*, in the sense used by Rogers [33], is a partial recursive function that takes a function variable f as input. A recursive functional can then be thought of, informally, as a computer program that is allowed to give inputs to f and to observe the corresponding outputs of f, regardless of whether or not f is itself a computable function.

There is a long history of applications of functionals to the foundations of mathematics. Indeed, functionals played an important role in Russell's theory of types [36], in Church's lambda calculus [10], and in the combinatory logics of Schönfinkel [37] and Curry [11]. Functionals were also central to Gödel's *Dialectica* interpretation [20]. Gödel demonstrated that the theorems of intuitionistic arithmetic can be interpreted as equations between primitive recursive functionals of finite type in a theory that he denoted by the letter $\mathcal{T}$. We study the class of primitive recursive functionals that are definable in $\mathcal{T}$ in Chapter 1, using terminological and notational conventions similar to reference [2]. In particular, we use serif Latin letters to denote terms in the theory $\mathcal{T}$. Additionally, with only a few exceptions, we use sans-serif Latin letters as type constants and Greek letters as type variables.

Recursive functionals have also been used in applications to the foundations of physics—most notably in the context of Kreisel's definition of a mechanistic theory [25]. Kreisel's definition provides one way to interpret the claim that a physical model is *computable*. But in Chapter 2, we provide an alternate definition that utilizes recursive functions, rather than recursive functionals. Although the two definitions are quite different in character, we show that models which satisfy Kreisel's definition can be interpreted as models which satisfy our definition. Throughout Chapter 2, we use terminological and notational conventions similar to reference [33], supplemented by reference [31] for topological content. With only a few exceptions, we use lower-case Latin letters to denote numbers and points, we use upper-case Latin letters to denote sets, and we use Greek letters to denote functions.

Both chapters are self-contained and can be read independently.

Chapter 1

The Definability of Functionals in Gödel's Theory $\mathcal{T}$

1.1 Introduction

For the formalization of his *Dialectica* interpretation of intuitionistic arithmetic, Gödel [20] introduced the theory $\mathcal{T}$. It was later shown [15, 22, 23] that $\mathcal{T}$ can be formalized as an extension of the simply-typed lambda calculus.[1] In this formalization, the terms of the theory $\mathcal{T}$ are simply-typed lambda terms with ground type N, extended to include the constants $0_{\mathsf{N}}^{\mathsf{N}}$, $\mathrm{S}^{\mathsf{N}\to\mathsf{N}}$, and $\mathrm{R}_{\tau}^{\tau\to(\tau\to\mathsf{N}\to\tau)\to\mathsf{N}\to\tau}$ for each type τ. We use superscripts to denote the types of terms, freely omitting the superscript when the type of a term can be deduced from its context. The formulas of $\mathcal{T}$ are equations between terms, with formulas of the following forms taken as axioms for each type τ

$$\begin{aligned} \mathrm{R}_{\tau}\, A\, B\, 0_{\mathsf{N}} &= A \\ \mathrm{R}_{\tau}\, A\, B\, (\mathrm{S}\, C) &= B\, (\mathrm{R}_{\tau}\, A\, B\, C)\, C \end{aligned}$$

where A, B, and C are metavariables for terms of types τ, $\tau \to \mathsf{N} \to \tau$, and N, respectively. The rules of inference of $\mathcal{T}$ are the rules of $\beta\eta$-conversion and the rules of substitution of equality. For any terms A and B in the language of $\mathcal{T}$, we write $\mathcal{T} \vdash A = B$ to denote that the equation $A = B$ is provable in $\mathcal{T}$. We say that a term is a $\beta\eta\mathcal{T}$-normal form if and only if that term is a $\beta\eta$-normal form which, for each type τ, contains no subterms of the form $\mathrm{R}_{\tau}\, A\, B\, C$. Since $\mathcal{T}$ is strongly normalizing and has the Church-Rosser property [2], $\mathcal{T} \vdash A = B$ if and only if A and B have the same $\beta\eta\mathcal{T}$-normal forms.[2]

The closed terms of type N in the language of $\mathcal{T}$ are called *numerals*. Each numeral has a

[1] Readers unfamiliar with the simply-typed lambda calculus should consult reference [2].

[2] Gödel did not clearly define equality between higher-type terms in $\mathcal{T}$. He only required that equality "be understood as intensional or definitional equality" [21]. Most formalizations of $\mathcal{T}$ in the simply-typed lambda calculus take equality to mean $\beta\mathcal{T}$-equality, omitting η-conversion as a rule of inference. But we require $\beta\eta\mathcal{T}$-equality for Curry's pairing function in Section 1.3 and for the Statman Reducibility Theorem in Section 1.6. This formalization of $\mathcal{T}$ is not uncommon (see reference [2], for example). A survey of several different commonly-used formalizations of equality in $\mathcal{T}$ is contained in reference [1].

$\beta\eta\mathcal{T}$-normal form

$$\overbrace{\mathrm{S}\,(\mathrm{S}\,(\cdots\,(\mathrm{S}}^{n}\,0_{\mathsf{N}})\cdots))$$

where the successor function S is applied n many times to 0_{N}, and we abbreviate any such term as $\overline{n}$.

A function f from $\beta\eta\mathcal{T}$-normal forms of types $\sigma_1, \sigma_2, \ldots, \sigma_n$ to $\beta\eta\mathcal{T}$-normal forms of type τ is said to be *definable* in $\mathcal{T}$ if and only if there is a closed term $F^{\sigma_1\to\sigma_2\to\cdots\to\sigma_n\to\tau}$ in the language of $\mathcal{T}$ such that $\mathcal{T} \vdash F\,A_1^{\sigma_1}\,A_2^{\sigma_2}\,\ldots\,A_n^{\sigma_n} = B^{\tau}$ whenever $f\,A_1\,A_2\,\ldots\,A_n = B$ is true. For example, because the constant R_{N} denotes the operation of primitive recursion, every primitive recursive function of non-negative integers is definable in $\mathcal{T}$, using the numerals $\overline{n}$ to represent the non-negative integers n. Indeed, it has been shown [23, 26, 39, 41] that the closed terms of type $\mathsf{N} \to \mathsf{N} \to \cdots \to \mathsf{N}$ in the language of $\mathcal{T}$ define exactly the $<\varepsilon_0$-recursive functions of non-negative integers. We will demonstrate that when the $\beta\eta\mathcal{T}$-normal forms of types σ and τ are restricted to be *pure* closed $\beta\eta$-normal forms (that is, closed normal forms that do not contain any of the constants), then the functionals of type $\sigma \to \tau$ that are definable in Gödel's theory $\mathcal{T}$ are precisely those functionals of one argument whose graphs are $<\varepsilon_0$-recursive sets. This result can naturally be extended to functionals of more than one argument.

1.2 Examples of Primitive Recursion in $\mathcal{T}$

Every primitive recursive function is definable in $\mathcal{T}$. For example, addition, multiplication, and predecessor are defined as follows.

$$\begin{aligned}
\mathrm{Add}^{\mathsf{N}\to\mathsf{N}\to\mathsf{N}} &= \lambda x^{N}.\,\mathrm{R}_{\mathsf{N}}\,x\,(\lambda a^{\mathsf{N}} b^{\mathsf{N}}.\,\mathrm{S}\,a)\\
\mathrm{Mult}^{\mathsf{N}\to\mathsf{N}\to\mathsf{N}} &= \lambda x^{\mathsf{N}}.\,\mathrm{R}_{\mathsf{N}}\,0_{\mathsf{N}}\,(\lambda a^{\mathsf{N}} b^{\mathsf{N}}.\,\mathrm{Add}\,a\,x)\\
\mathrm{Pred}^{\mathsf{N}\to\mathsf{N}} &= \mathrm{R}_{\mathsf{N}}\,0_{\mathsf{N}}\,(\lambda a^{\mathsf{N}} b^{\mathsf{N}}.\,b)
\end{aligned}$$

We write $A + B$ and $A \times B$ as abbreviations for $\mathrm{Add}\,A\,B$ and $\mathrm{Mult}\,A\,B$, respectively. We define

$$\mathrm{Monus}^{\mathsf{N}\to\mathsf{N}\to\mathsf{N}} = \lambda x^{\mathsf{N}}.\,\mathrm{R}_{\mathsf{N}}\,x\,(\lambda a^{\mathsf{N}} b^{\mathsf{N}}.\,\mathrm{Pred}\,a)$$

and we write $A \dot{-} B$ as an abbreviation for $\mathrm{Monus}\,A\,B$. Note that for all non-negative integers m and n, if $m \geq n$ then

$$\mathcal{T} \vdash \overline{m} \dot{-} \overline{n} = \overline{m-n}$$

Otherwise, if $m < n$ then

$$\mathcal{T} \vdash \overline{m} \dot{-} \overline{n} = \overline{0}$$

We write $|A - B|$ as an abbreviation for $(A \dot{-} B) + (B \dot{-} A)$.

The conditional function is defined as

$$\mathrm{Cond}^{\mathsf{N}\to\mathsf{N}\to\mathsf{N}\to\mathsf{N}} = \lambda x^{\mathsf{N}} y^{\mathsf{N}}.\,\mathrm{R}_{\mathsf{N}}\,x\,(\lambda a^{\mathsf{N}} b^{\mathsf{N}}.\,y)$$

For each non-negative integer n, if $n = 0$ then

$$\mathcal{T} \vdash \mathrm{Cond}\, x^{\mathsf{N}}\, y^{\mathsf{N}}\, \overline{n} = x$$

Alternatively, if $n \neq 0$ then

$$\mathcal{T} \vdash \mathrm{Cond}\, x^{\mathsf{N}}\, y^{\mathsf{N}}\, \overline{n} = y$$

Functionals can also be defined by primitive recursion. For example, the summation functional is given by

$$\mathrm{Sum}^{\mathsf{N}\to(\mathsf{N}\to\mathsf{N})\to\mathsf{N}} = \lambda x^{\mathsf{N}} f^{\mathsf{N}\to\mathsf{N}}.\, \mathrm{R}_{\mathsf{N}}\, 0_{\mathsf{N}}\, \big(\lambda a^{\mathsf{N}} b^{\mathsf{N}}.\, a + (f\, b)\big)\, (\mathrm{S}\, x)$$

It is common practice to write $\sum_{i=0}^{n} F\, \overline{i}$ as an abbreviation for $\mathrm{Sum}\, \overline{n}\, F$, where i is a dummy variable. Similarly, a functional for bounded maximization is given by

$$\mathrm{Max}_{\leq}^{\mathsf{N}\to(\mathsf{N}\to\mathsf{N})\to\mathsf{N}} = \lambda x^{\mathsf{N}} f^{\mathsf{N}\to\mathsf{N}}.\, \mathrm{R}_{\mathsf{N}}\, 0_{\mathsf{N}}\, \big(\lambda a^{\mathsf{N}} b^{\mathsf{N}}.\, \mathrm{Cond}\, b\, a\, (f\, b)\big)\, (\mathrm{S}\, x)$$

Note that for each closed term $F^{\mathsf{N}\to\mathsf{N}}$ in the language of $\mathcal{T}$ and for each non-negative integer n, if m is the largest non-negative integer less than or equal to n such that $\mathcal{T} \vdash F\, \overline{m} = \overline{0}$, then

$$\mathcal{T} \vdash \mathrm{Max}_{\leq}\, \overline{n}\, F = \overline{m}$$

Otherwise, if no such m exists, then

$$\mathcal{T} \vdash \mathrm{Max}_{\leq}\, \overline{n}\, F = \overline{0}$$

Division can be defined in terms of bounded maximization.

$$\mathrm{Div}^{\mathsf{N}\to\mathsf{N}\to\mathsf{N}} = \lambda x^{\mathsf{N}} y^{\mathsf{N}}.\, \mathrm{Max}_{\leq}\, x\, \big(\lambda a^{\mathsf{N}}.\, (a \times y) \dot{-} x\big)$$

We write $\lfloor A/B \rfloor$ as an abbreviation for $\mathrm{Div}\, A\, B$.

1.3 Examples of Pairing Functions in $\mathcal{T}$

A variant of Cantor's pairing function [7] can be defined as follows

$$\mathrm{P}_0^{\mathsf{N}\to\mathsf{N}\to\mathsf{N}} = \lambda x^{\mathsf{N}} y^{\mathsf{N}}.\, \big\lfloor \big(x \times \big(x + \overline{3}\big) + y \times \big(y + \overline{1}\big) + \overline{2} \times x \times y\big)/\overline{2} \big\rfloor$$

We write $\langle A, B \rangle$ as an abbreviation for $\mathrm{P}_0\, A\, B$. For each non-negative integer n there is exactly one pair of non-negative integers m_1 and m_2 such that

$$\mathcal{T} \vdash \langle \overline{m_1}, \overline{m_2} \rangle = \overline{n}$$

And since $2m_1 \leq m_1(m_1 + 3)$ and $2m_2 \leq m_2(m_2 + 1)$ for all non-negative integers m_1 and m_2, it follows from the definition of P_0 that $m_1 \leq n$ and $m_2 \leq n$. (In fact, if $m_1 \neq 0$ then $m_1 < n$ and $m_2 < n$.) Therefore, if we define

$$\mathrm{P}_1^{\mathsf{N}\to\mathsf{N}} = \lambda z^{\mathsf{N}}.\, \mathrm{Sum}\, z\, \big(\lambda y^{\mathsf{N}}.\, \mathrm{Max}_{\leq}\, z\, \big(\lambda x^{\mathsf{N}}.\, \big|z - \langle x, y \rangle\big|\big)\big)$$
$$\mathrm{P}_2^{\mathsf{N}\to\mathsf{N}} = \lambda z^{\mathsf{N}}.\, \mathrm{Sum}\, z\, \big(\lambda x^{\mathsf{N}}.\, \mathrm{Max}_{\leq}\, z\, \big(\lambda y^{\mathsf{N}}.\, \big|z - \langle x, y \rangle\big|\big)\big)$$

then

$$\mathcal{T} \vdash \mathrm{P}_1 \langle \overline{m_1}, \overline{m_2} \rangle = \overline{m_1}$$
$$\mathcal{T} \vdash \mathrm{P}_2 \langle \overline{m_1}, \overline{m_2} \rangle = \overline{m_2}$$

for each pair of non-negative integers m_1 and m_2.

Now, note that for each type τ there is a non-negative integer n and there are types τ_1, τ_2, ..., τ_n such that $\tau = \tau_1 \to \tau_2 \to \cdots \to \tau_n \to \mathsf{N}$. Given any two types

$$\sigma = \sigma_1 \to \sigma_2 \to \cdots \to \sigma_m \to \mathsf{N}$$
$$\tau = \tau_1 \to \tau_2 \to \cdots \to \tau_n \to \mathsf{N}$$

we define

$$\sigma \times \tau = \sigma_1 \to \sigma_2 \to \cdots \to \sigma_m \to \tau_1 \to \tau_2 \to \cdots \to \tau_n \to (\mathsf{N} \to \mathsf{N} \to \mathsf{N}) \to \mathsf{N}$$

Moreover, for each pair of types σ and τ, there are pure closed terms $\mathrm{D}_{0,\sigma,\tau}^{\sigma \to \tau \to \sigma \times \tau}$, $\mathrm{D}_{1,\sigma,\tau}^{\sigma \times \tau \to \sigma}$, and $\mathrm{D}_{2,\sigma,\tau}^{\sigma \times \tau \to \tau}$ such that

$$\mathcal{T} \vdash \mathrm{D}_{1,\sigma,\tau} \, (\mathrm{D}_{0,\sigma,\tau} \, x^\sigma \, y^\tau) = x$$
$$\mathcal{T} \vdash \mathrm{D}_{2,\sigma,\tau} \, (\mathrm{D}_{0,\sigma,\tau} \, x^\sigma \, y^\tau) = y$$

Reference [3] provides explicit definitions for these terms. $\mathrm{D}_{0,\sigma,\tau}$ is commonly known as Curry's pairing function. We write $\{A_1^{\tau_1}, A_2^{\tau_2}\}$ as an abbreviation for $\mathrm{D}_{0,\tau_1,\tau_2} \, A_1^{\tau_1} \, A_2^{\tau_2}$, and we write $\{A_1^{\tau_1}, A_2^{\tau_2}, \ldots, A_n^{\tau_n}\}$ as an abbreviation for the term

$$\{A_1^{\tau_1}, \{A_2^{\tau_2}, \{\ldots, \{A_{n-1}^{\tau_{n-1}}, A_n^{\tau_n}\} \ldots\}\}\}$$

of type $\tau_1 \times \tau_2 \times \cdots \times \tau_n$. Note that for each type $\tau = \tau_1 \times \tau_2 \times \cdots \times \tau_n$ such that $n > 1$, and for each positive integer $i \leq n$, there is a pure closed term $\mathrm{D}_{i,\tau_1,\tau_2,\ldots,\tau_n}^{\tau \to \tau_i}$ such that

$$\mathcal{T} \vdash \mathrm{D}_{i,\tau_1,\tau_2,\ldots,\tau_n} \, \{x_1^{\tau_1}, x_2^{\tau_2}, \ldots, x_n^{\tau_n}\} = x_i$$

Now, for each type $\sigma = \sigma_1 \to \sigma_2 \to \cdots \to \sigma_m \to \mathsf{N}$ such that $m > 0$, define

$$0_\sigma^\sigma = \lambda x_1^{\sigma_1} x_2^{\sigma_2} \ldots x_m^{\sigma_m} . \, 0_{\mathsf{N}}$$

and for each type τ define

$$\mathrm{Cons}_\tau^{\tau \to (\mathsf{N} \to \tau) \to \mathsf{N} \to \tau} = \lambda x^\tau y^{\mathsf{N} \to \tau} . \, \mathrm{R}_\tau \, x \, (\lambda a^\tau . \, y)$$

We write $[A_0^\tau, A_1^\tau, \ldots, A_n^\tau]$ as an abbreviation for the term

$$\mathrm{Cons}_\tau \, A_0 \, (\mathrm{Cons}_\tau \, A_1 \, (\cdots \, (\mathrm{Cons}_\tau \, A_n \, 0_{\mathsf{N} \to \tau}) \cdots))$$

of type $\mathsf{N} \to \tau$, which we use to represent a *list* of terms of type τ. Note that for each non-negative integer $i \leq n$,

$$\mathcal{T} \vdash [x_0^\tau, x_1^\tau, \ldots, x_n^\tau] \, \bar{i} = x_i$$

1.4 Enumerating Pure Closed $\beta\eta$-Normal Forms

Let $\mathrm{FV}(A)$ denote the set of free variables in the term A.

Lemma 1.4.1. *If A^τ is a pure $\beta\eta$-normal form, then one of the following three conditions must hold.*

1. *A^τ is a variable.*
2. *A^τ is of the form $\lambda B^{\sigma_1}.\, C^{\sigma_2}$, where $\tau = \sigma_1 \to \sigma_2$ and C is a pure $\beta\eta$-normal form with free variables in the set $\{B\} \cup \mathrm{FV}(A)$.*
3. *A^τ is of the form*

$$B^{\sigma_n \to \sigma_{n-1} \to \cdots \to \sigma_1 \to \tau}\, C_n^{\sigma_n}\, C_{n-1}^{\sigma_{n-1}}\, \cdots\, C_1^{\sigma_1}$$

where B is a member of $\mathrm{FV}(A)$ and C_n, C_{n-1}, $\ldots$, C_1 are pure $\beta\eta$-normal forms with free variables in the set $\mathrm{FV}(A)$.

Proof. Any term A^τ must either be a variable, be of the form $\lambda B^{\sigma_1}.\, C^{\sigma_2}$ with $\tau = \sigma_1 \to \sigma_2$, or be of the form $D_1^{\sigma_1 \to \tau}\, C_1^{\sigma_1}$. Condition 1 and condition 2 follow immediately from the first two cases. In the third case, if $A^\tau = D_1^{\sigma_1 \to \tau}\, C_1^{\sigma_1}$ and A^τ is a $\beta\eta$-normal form, then $D_1^{\sigma_1 \to \tau}$ must either be a variable or of the form $D_2^{\sigma_2 \to \sigma_1 \to \tau}\, C_2^{\sigma_2}$. Therefore, by induction, A^τ must be of the form

$$B^{\sigma_n \to \sigma_{n-1} \to \cdots \to \sigma_1 \to \tau}\, C_n^{\sigma_n}\, C_{n-1}^{\sigma_{n-1}}\, \cdots\, C_1^{\sigma_1}$$

for some positive integer n, where B is a variable. Condition 3 immediately follows. □

If $\tau = \mathsf{N}$, then we define σ to be a *subtype* of τ if and only if $\sigma = \mathsf{N}$. Otherwise, if $\tau = \tau_1 \to \tau_2$, then we define σ to be a subtype of τ if and only if σ is a subtype of τ_1, σ is a subtype of τ_2, or $\sigma = \tau$.

Lemma 1.4.2. *If B^σ is a subterm of a pure closed $\beta\eta$-normal form A^τ, then σ is a subtype of τ and the type of each free variable in B is a subtype of τ.*

Proof. The proof is by induction on the depth of each subterm B in A. Suppose that A^τ is a pure closed $\beta\eta$-normal form. As the base case, note that if $B^\sigma = A^\tau$ then $\sigma = \tau$ is a subtype of τ and B has no free variables because A is closed. As the inductive hypothesis, suppose that B^σ is a subterm of A^τ with σ a subtype of τ and with the type of each member of $\mathrm{FV}(B)$ a subtype of τ. Since A is a pure $\beta\eta$-normal form, B is a pure $\beta\eta$-normal form and B satisfies one of the three conditions in Lemma 1.4.1. In particular, if B satisfies condition 1, then B has no subterms except for itself. Alternatively, if B satisfies condition 2, then $B^\sigma = \lambda C^{\sigma_1}.\, D^{\sigma_2}$ with $\sigma = \sigma_1 \to \sigma_2$, and the free variables of D are members of $\{C^{\sigma_1}\} \cup \mathrm{FV}(B)$. But because $\sigma = \sigma_1 \to \sigma_2$ is a subtype of τ, σ_1 and σ_2 are subtypes of τ. So, the type of D is a subtype of τ and the type of each free variable in D is a subtype of τ. Finally, if B satisfies condition 3, then

$$B^\sigma = C^{\sigma_n \to \sigma_{n-1} \to \cdots \to \sigma_1 \to \sigma}\, D_n^{\sigma_n}\, D_{n-1}^{\sigma_{n-1}}\, \cdots\, D_1^{\sigma_1}$$

Where $C^{\sigma_n \to \sigma_{n-1} \to \cdots \to \sigma_1 \to \sigma}$ is a variable. But because C is a free variable of B, it follows from the inductive hypothesis that $\sigma_n \to \sigma_{n-1} \to \cdots \to \sigma_1 \to \sigma$ is a subtype of τ, as are σ_1 and $\sigma_1 \to \sigma$.

Hence, the type of D_1 is a subtype of τ, as is the type of the subterm $C\, D_n\, D_{n-1} \cdots D_2$. Of course, the free variables of these subterms are members of $\mathrm{FV}(B)$, so the types of the free variables in these subterms are subtypes of τ. $\square$

For each type τ there is an algorithm [4] for enumerating the pure closed $\beta\eta$-normal forms of type τ. In fact, the algorithm can be implemented within $\mathcal{T}$. In other words, for each type τ there is a closed term $\mathrm{E}_\tau^{\mathsf{N}\to\tau}$ in the language of $\mathcal{T}$ such that for each pure closed term A^τ,

$$\mathcal{T} \vdash \mathrm{E}_\tau \ulcorner A^\tau \urcorner = A^\tau$$

where $\ulcorner A^\tau \urcorner$ is a numeral that encodes the $\beta\eta$-normal form of A^τ. We rely on Lemma 1.4.2 to encode pure closed $\beta\eta$-normal forms as follows.

Let A^τ be a pure closed $\beta\eta$-normal form where each occurrence of λ in A binds a distinct variable. Let $\tau_1, \tau_2, \ldots, \tau_n$ be all the subtypes of τ, and let $\tau_1 = \tau$. We define $\ulcorner A^\tau \urcorner$ recursively in terms of an encoding function Enc_τ. In particular,

$$\ulcorner A^\tau \urcorner = \mathrm{Enc}_\tau(0_{\mathsf{N}\to\tau_1}, 0_{\mathsf{N}\to\tau_2}, \ldots, 0_{\mathsf{N}\to\tau_n}, A)$$

For each positive integer $i \leq n$, let X_i be the empty list $0_{\mathsf{N}\to\tau_i}$ or let

$$X_i = [V_{0,\tau_i}^{\tau_i}, V_{1,\tau_i}^{\tau_i}, \ldots, V_{m_i,\tau_i}^{\tau_i}]$$

be a list of variables, where m_i is some non-negative integer. We use $\vec{X}$ to denote the sequence

$$X_1, X_2, \ldots, X_n$$

and we use $\vec{X}_{V,j}$ to denote the sequence

$$X_1, X_2, \ldots, \mathrm{Cons}_{\tau_j}\, V\, X_j, \ldots, X_n$$

that is obtained from $\vec{X}$ by appending the variable V of type τ_j to the list X_j. Define

$$\mathrm{Enc}_\tau(\vec{X}, B^{\tau_i}) = \begin{cases} \langle \overline{0}, \overline{j} \rangle & \text{if } B = V_{j,\tau_i} \\ \langle \overline{j}, \langle \mathrm{Enc}_\tau(\vec{X}, C), \mathrm{Enc}_\tau(\vec{X}, D) \rangle \rangle & \text{if } B = C^{\tau_j \to \tau_i}\, D^{\tau_j} \\ \langle \overline{n+1}, \mathrm{Enc}_\tau(\vec{X}_{C,j}, D) \rangle & \text{if } B = \lambda C^{\tau_j}.\, D^{\tau_k} \end{cases}$$

where $X_1, X_2, \ldots, X_n$ contain the free variables of types $\tau_1, \tau_2, \ldots, \tau_n$ that occur in B. Note that Enc_τ is defined in the metalanguage, not within the theory $\mathcal{T}$ itself.[3]

Constructing a procedure to enumerate pure closed $\beta\eta$-normal forms of type τ is now a matter of inverting the encoding. That is, given a numeral $\langle \overline{j}, \overline{k} \rangle$ together with the lists $X_1, X_2, \ldots, X_n$, we wish to find a pure term B^{τ_i} such that $\mathcal{T} \vdash \mathrm{Enc}_\tau(\vec{X}, B^{\tau_i}) = \langle \overline{j}, \overline{k} \rangle$. Since $\overline{k}$ might itself be an encoding for a subterm or a pair of subterms, this procedure depends recursively on the enumeration of the subterms of B^{τ_i}. But Cantor's pairing function possesses the property that

[3] It is also worth mentioning that if B^{τ_i} is a variable, then its encoding $\langle \overline{0}, \overline{j} \rangle$ is a sort of de Bruijn index [12] for the variable. In particular, j is the number of λ-abstractions of type τ_i that contain B^{τ_i} within their scopes, but that are themselves within the scope of the λ-abstraction which binds B^{τ_i}.

if $\mathcal{T} \vdash \langle \overline{j}, \overline{k} \rangle = \overline{l}$, then $k < l$ whenever $j \neq 0$. That is, it suffices to use a course-of-values recursion, where the computation of the term encoded as $\langle \overline{j}, \overline{k} \rangle$ depends on the list of all those potential subterms whose encodings are strictly less than $\langle \overline{j}, \overline{k} \rangle$. This approach also requires a simultaneous recursion over all of the types $\tau_1, \tau_2, \ldots, \tau_n$, because a subterm of B^{τ_i} might not itself be of type τ_i.

Hence, for each type τ, a decoding function Dec_τ can be defined as follows

$$\mathrm{Dec}_\tau^{\mathsf{N}\to\mathsf{N}\to\upsilon} = \lambda x^{\mathsf{N}}.\, \mathrm{R}_{\mathsf{N}\to\upsilon}\, 0_{\mathsf{N}\to\upsilon} \left(\lambda a^{\mathsf{N}\to\upsilon} b^{\mathsf{N}}.\, \mathrm{Cons}_\upsilon \left(\lambda x_1^{\mathsf{N}\to\tau_1} x_2^{\mathsf{N}\to\tau_2} \ldots x_n^{\mathsf{N}\to\tau_n}.\, \{\mathrm{B}_{1,\tau}, \mathrm{B}_{2,\tau}, \ldots, \mathrm{B}_{n,\tau}\}\right) a\right) (\mathrm{S}\, x)$$

where

$$\upsilon = (\mathsf{N} \to \tau_1) \to (\mathsf{N} \to \tau_2) \to \cdots \to (\mathsf{N} \to \tau_n) \to \tau_1 \times \tau_2 \times \cdots \times \tau_n$$

In this case, $\mathrm{B}_{i,\tau}$ denotes the term of type τ_i encoded by the numeral b, with free variables chosen from the lists $x_1, x_2, \ldots, x_n$. Specifically, we define

$$\mathrm{B}_{i,\tau}^{\tau_i} = \left(\lambda p_1^{\mathsf{N}} p_2^{\mathsf{N}}.\, [\mathrm{J}_{0,i,\tau}, \mathrm{J}_{1,i,\tau}, \ldots, \mathrm{J}_{n+1,i,\tau}]\, p_1\right) (\mathrm{P}_1\, b)\, (\mathrm{P}_2\, b)$$

for each positive integer $i \leq n$, where $\mathrm{J}_{0,i,\tau}, \mathrm{J}_{1,i,\tau}, \ldots, \mathrm{J}_{n+1,i,\tau}$ compute the inverses for each of the separate cases in the definition of Enc_τ. In particular, we define $\mathrm{J}_{0,i,\tau}^{\tau_i} = x_i\, p_2$. And for each positive integer $j \leq n$, if there exists a positive integer $k \leq n$ such that $\tau_k = \tau_j \to \tau_i$, then we define

$$\mathrm{J}_{j,i,\tau}^{\tau_i} = \left(\mathrm{D}_{k,\tau_1,\tau_2,\ldots,\tau_n} \left(a\, (b \dot{-} \mathrm{S}\, (\mathrm{P}_1\, p_2))\, x_1\, x_2\, \ldots\, x_n\right)\right) \left(\mathrm{D}_{j,\tau_1,\tau_2,\ldots,\tau_n} \left(a\, (b \dot{-} \mathrm{S}\, (\mathrm{P}_2\, p_2))\, x_1\, x_2\, \ldots\, x_n\right)\right)$$

Otherwise, if no such k exists, then we define $\mathrm{J}_{j,i,\tau} = 0_{\tau_i}$. Similarly, if there exist positive integers $j \leq n$ and $k \leq n$ such that $\tau_i = \tau_j \to \tau_k$, then we define

$$\mathrm{J}_{n+1,i,\tau}^{\tau_i} = \lambda y^{\tau_j}.\, \left(\mathrm{D}_{k,\tau_1,\tau_2,\ldots,\tau_n} \left(a\, (b \dot{-} \mathrm{S}\, p_2)\, x_1\, x_2\, \cdots\, (\mathrm{Cons}_{\tau_j}\, y\, x_j)\, \cdots\, x_n\right)\right)$$

Otherwise, if no such j and k exist, then we define $\mathrm{J}_{n+1,i,\tau} = 0_{\tau_i}$.

Finally, for each type τ we define

$$\mathrm{E}_\tau^{\mathsf{N}\to\tau} = \lambda x^{\mathsf{N}}.\, \mathrm{D}_{1,\tau_1,\tau_2,\ldots,\tau_n} \left(\mathrm{Dec}_\tau\, x\, \overline{0}\, 0_{\mathsf{N}\to\tau_1}\, 0_{\mathsf{N}\to\tau_2}\, \cdots\, 0_{\mathsf{N}\to\tau_n}\right)$$

and it immediately follows that

$$\mathcal{T} \vdash \mathrm{E}_\tau \ulcorner A^\tau \urcorner = A^\tau$$

for each pure closed term A^τ. Of course,[4] there are numerals $\overline{l}$ that do not encode any pure closed $\beta\eta$-normal form of type τ, and in that case,

$$\mathcal{T} \vdash \mathrm{E}_\tau\, \overline{l} = 0_\tau$$

[4]Since we have chosen an encoding where no two distinct numerals encode the same pure closed $\beta\eta$-normal form, there must be some numerals that do not encode any pure closed $\beta\eta$-normal form of type τ if τ has only finitely many pure closed $\beta\eta$-normal forms. For example, $\lambda x^{\mathsf{N}}.\, x$ is the only pure closed $\beta\eta$-normal form of type $\tau = \mathsf{N} \to \mathsf{N}$.

1.5 Consistent Sets of Functional Equations

Consider any $<\varepsilon_0$-recursive set of equations of the form

$$\begin{aligned} F\,A_1^\sigma &= B_1^\tau \\ F\,A_2^\sigma &= B_2^\tau \\ F\,A_3^\sigma &= B_3^\tau \\ &\vdots \end{aligned} \tag{1.5.1}$$

where F is unknown and where, for each positive integer i, A_i and B_i are closed terms. We will say that such a set of equations is *consistent* if and only if the set contains no two equations $F\,A = B$ and $F\,C = D$ where A and C have the same $\beta\eta\mathcal{T}$-normal forms and where B and D have distinct $\beta\eta\mathcal{T}$-normal forms. We say that a set of equations of the form (1.5.1) is *satisfied* by some closed term F in the theory $\mathcal{T}$ if and only if there is a closed term in the language of $\mathcal{T}$ such that each of the equations in (1.5.1) is provable in $\mathcal{T}$ when that term is substituted for F.

Lemma 1.5.1. *Every consistent $<\varepsilon_0$-recursive set of equations of the form (1.5.1) is satisfied by a closed term F in the theory $\mathcal{T}$ if $\sigma = \mathsf{N}$ and $\tau = \mathsf{N}$.*

Proof. F is a $<\varepsilon_0$-recursive function of non-negative integers, and therefore is satisfied by a closed term in $\mathcal{T}$. See references [4, 23]. □

Lemma 1.5.2. *Every consistent $<\varepsilon_0$-recursive set of equations of the form (1.5.1) is satisfied by a closed term F in the theory $\mathcal{T}$ if $\sigma = \mathsf{N}$ and if, for each positive integer i, B_i is a pure closed term.*

Proof. Given a consistent $<\varepsilon_0$-recursive set of equations

$$\begin{aligned} F\,A_1^{\mathsf{N}} &= B_1^\tau \\ F\,A_2^{\mathsf{N}} &= B_2^\tau \\ F\,A_3^{\mathsf{N}} &= B_3^\tau \\ &\vdots \end{aligned}$$

where B_i is a pure closed term for each positive integer i, consider the set of equations

$$\begin{aligned} G\,A_1^{\mathsf{N}} &= \ulcorner B_1^\tau \urcorner \\ G\,A_2^{\mathsf{N}} &= \ulcorner B_2^\tau \urcorner \\ G\,A_3^{\mathsf{N}} &= \ulcorner B_3^\tau \urcorner \\ &\vdots \end{aligned}$$

Note that this is a consistent $<\varepsilon_0$-recursive set of equations of the form (1.5.1) where A_i and $\ulcorner B_i \urcorner$ are of type N for each positive integer i. By Lemma 1.5.1 there is a closed term G in $\mathcal{T}$ which satisfies this set of equations. But then F is satisfied by the closed term $\lambda x^{\mathsf{N}}.\, E_\tau\,(G\,x)$ in $\mathcal{T}$. □

1.6 The Statman Reducibility Theorem

The following theorem asserts that each type τ is $\beta\eta$-reducible to the type $(\mathsf{N} \to \mathsf{N} \to \mathsf{N}) \to \mathsf{N} \to \mathsf{N}$.

Statman Reducibility Theorem. *For each type τ there exists a pure closed term M_τ of type $\tau \to (\mathsf{N} \to \mathsf{N} \to \mathsf{N}) \to \mathsf{N} \to \mathsf{N}$ such that for all pure closed terms A^τ and B^τ*

$$\mathcal{T} \vdash \mathrm{M}_\tau\, A = \mathrm{M}_\tau\, B$$

if and only if

$$\mathcal{T} \vdash A = B$$

Proof. See references [4, 40]. □

In fact, in the context of the theory $\mathcal{T}$ we can prove a somewhat stronger statement.

Lemma 1.6.1. *For each type τ there exists a closed term $\mathrm{N}_\tau^{\tau\to\mathsf{N}}$ in the language of $\mathcal{T}$ such that for all pure closed terms A^τ and B^τ*

$$\mathrm{N}_\tau\, A = \mathrm{N}_\tau\, B$$

if and only if

$$A = B$$

Proof. The type $(\mathsf{N} \to \mathsf{N} \to \mathsf{N}) \to \mathsf{N} \to \mathsf{N}$ is the type of binary trees [4]. That is, it can be shown by Lemma 1.4.1 that every pure closed $\beta\eta$-normal form of type $(\mathsf{N} \to \mathsf{N} \to \mathsf{N}) \to \mathsf{N} \to \mathsf{N}$ is of the form $\lambda x^{\mathsf{N}\to\mathsf{N}\to\mathsf{N}} y^{\mathsf{N}}.\, A^{\mathsf{N}}$, where the free variables of A^{N} are members of the the set $\{x^{\mathsf{N}\to\mathsf{N}\to\mathsf{N}}, y^{\mathsf{N}}\}$. But again by Lemma 1.4.1, it must be the case that either $A^{\mathsf{N}} = y^{\mathsf{N}}$ or A^{N} is of the form $x^{\mathsf{N}\to\mathsf{N}\to\mathsf{N}}\, C^{\mathsf{N}}\, D^{\mathsf{N}}$, where the free variables of C^{N} and D^{N} are members of the the set $\{x^{\mathsf{N}\to\mathsf{N}\to\mathsf{N}}, y^{\mathsf{N}}\}$. But this same argument applies to the subterms C and D themselves. Hence, A is a binary tree with leaves y and branching nodes x. Furthermore, each tree A can be encoded as a distinct numeral $\|A\|$ by letting $\|y\| = \overline{0}$, and by letting $\|x\, C\, D\| = \mathrm{S}\left\langle \|C\|, \|D\| \right\rangle$.

Now, for each type τ define

$$\mathrm{N}_\tau^{\tau\to\mathsf{N}} = \lambda x^\tau.\, \mathrm{M}_\tau^{\tau\to(\mathsf{N}\to\mathsf{N}\to\mathsf{N})\to\mathsf{N}\to\mathsf{N}}\, x \left(\lambda c^{\mathsf{N}} d^{\mathsf{N}}.\, \mathrm{S}\left\langle c, d\right\rangle\right) \overline{0}$$

By the Statman Reducibility Theorem, for any two distinct $\beta\eta$-normal forms A^τ and B^τ, $\mathrm{M}_\tau\, A$ and $\mathrm{M}_\tau\, B$ must have distinct $\beta\eta$-normal forms of type $(\mathsf{N} \to \mathsf{N} \to \mathsf{N}) \to \mathsf{N} \to \mathsf{N}$. Then,

$$\mathcal{T} \vdash \mathrm{N}_\tau\, A = \mathrm{M}_\tau\, A \left(\lambda c^{\mathsf{N}} d^{\mathsf{N}}.\, \mathrm{S}\left\langle c, d\right\rangle\right) \overline{0}$$
$$\mathcal{T} \vdash \mathrm{N}_\tau\, B = \mathrm{M}_\tau\, B \left(\lambda c^{\mathsf{N}} d^{\mathsf{N}}.\, \mathrm{S}\left\langle c, d\right\rangle\right) \overline{0}$$

Therefore, $\mathrm{N}_\tau\, A$ and $\mathrm{N}_\tau\, B$ have distinct $\beta\eta\mathcal{T}$-normal forms. Namely, the $\beta\eta\mathcal{T}$-normal form of $\mathrm{N}_\tau\, A$ is the numeral assigned to the tree $\mathrm{M}_\tau\, A$, and the $\beta\eta\mathcal{T}$-normal form of $\mathrm{N}_\tau\, B$ is the numeral assigned to the tree $\mathrm{M}_\tau\, B$. □

This result can be immediately applied to solve sets of equations of the form (1.5.1).

Theorem 1.6.2. *Every consistent $<\varepsilon_0$-recursive set of equations of the form (1.5.1) is satisfied by a closed term F in the theory $\mathcal{T}$ if, for each positive integer i, A_i and B_i are a pure closed terms.*

Proof. Given a consistent $<\varepsilon_0$-recursive set of equations of the form (1.5.1), where A_i and B_i are pure closed terms for each positive integer i, substitute $\lambda x^\sigma.\, G^{\mathsf{N}\to\tau}\,(\mathrm{N}_\sigma\, x)$ for F. Then, by β-reduction we obtain the following consistent $<\varepsilon_0$-recursive set of equations which is also of the form (1.5.1).

$$\begin{gathered} G\,(\mathrm{N}_\sigma\, A_1^\sigma) = B_1^\tau \\ G\,(\mathrm{N}_\sigma\, A_2^\sigma) = B_2^\tau \\ G\,(\mathrm{N}_\sigma\, A_3^\sigma) = B_3^\tau \\ \vdots \end{gathered}$$

It follows from Lemma 1.5.2 that this system has a solution G in the theory $\mathcal{T}$. Hence, the original system is solved by the closed term

$$F^{\sigma\to\tau} = \lambda x^\sigma.\, G\,(\mathrm{N}_\sigma\, x)$$

in the theory $\mathcal{T}$. □

1.7 Some Consequences of Theorem 1.6.2

Theorem 1.6.2 has several consequences.

Corollary 1.7.1. *Any $<\varepsilon_0$-recursive function from pure closed $\beta\eta$-normal forms of type σ to pure closed $\beta\eta$-normal forms of type τ is definable in Gödel's theory $\mathcal{T}$.*

Proof. Given any $<\varepsilon_0$-recursive function from pure closed $\beta\eta$-normal forms of type σ to pure closed $\beta\eta$-normal forms of type τ, the function is determined uniquely by a consistent $<\varepsilon_0$-recursive set of equations of the form (1.5.1). Hence, by Theorem 1.6.2, the function is definable in $\mathcal{T}$. □

A *numeration* of a set is a surjective function ν from the non-negative integers to that set. Let ν_τ denote an *effective* numeration of the pure closed $\beta\eta$-normal forms of type τ. For example, if A^τ is any one pure closed $\beta\eta$-normal form, then for all non-negative integers n and all pure closed $\beta\eta$-normal forms B we could define

$$\nu_\tau(n) = \begin{cases} B & \text{if } \mathcal{T} \vdash \mathrm{E}_\tau\, \overline{n} = B \\ A & \text{if } \mathcal{T} \vdash \mathrm{E}_\tau\, \overline{n} = 0_\tau \end{cases}$$

An *Eršov morphism* (see references [16, 17, 45]) from ν_σ to ν_τ is a function f from pure closed $\beta\eta$-normal forms of type σ to pure closed $\beta\eta$-normal forms of type τ such that there is a recursive

function g with $f \circ \nu_\sigma = \nu_\tau \circ g$. We say that the Eršov morphism f from ν_σ to ν_τ is *given* by g. Theorem 1.6.2 can be phrased as a theorem about Eršov morphisms, rather than a theorem about consistent sets of equations.

Corollary 1.7.2. *Every Eršov morphism from ν_σ to ν_τ that is given by a $<\varepsilon_0$-recursive function is definable in Gödel's theory $\mathcal{T}$.*

Consider the theory of simply-typed lambda calculus with ground type o and with $\beta\eta$-equality, extended to include all primitive recursive consistent sets of equations of the form

$$\begin{aligned} \mathrm{F}_i\, A^\sigma_{1,i} &= B^\tau_{1,i} \\ \mathrm{F}_i\, A^\sigma_{2,i} &= B^\tau_{2,i} \\ \mathrm{F}_i\, A^\sigma_{3,i} &= B^\tau_{3,i} \\ &\vdots \end{aligned}$$

where $A_{j,i}$ and $B_{j,i}$ are pure closed terms for all positive integers i and j, and $\mathrm{F}_i^{\sigma\to\tau}$ is a new constant that is unique to the ith such set of equations. We call this theory $\mathcal{F}$.

Corollary 1.7.3. *The functions of pure closed $\beta\eta$-normal forms that are definable in $\mathcal{T}$ are exactly those functions of pure closed $\beta\eta$-normal forms that are definable in $\mathcal{F}$.*

Proof. For each non-negative integer n, let

$$\tilde{n}^{(\tau\to\tau)\to\tau\to\tau}$$

stand for the term

$$\lambda f^{\tau\to\tau} x^\tau.\, \overbrace{f\,(f\,(\cdots\,(f}^{n}\; x)\cdots))$$

with f applied n many times to x. Note that if we identify N with the type $(o \to o) \to o \to o$, then $\tilde{n}^{\mathsf{N}}$ is the Church numeral for n, and $\tilde{n}^{(\tau\to\tau)\to\tau\to\tau}$ is a higher-type analog of the numeral for each type τ higher than N. The successor function for Church numerals is

$$\mathrm{S}_{\mathcal{F}}^{\mathsf{N}\to\mathsf{N}} = \lambda c^{\mathsf{N}} f^{o\to o} x^o.\, f\,(c\, f\, x)$$

Now, for each type τ the theory $\mathcal{F}$ includes the consistent primitive-recursive set of axioms

$$\begin{aligned} \mathrm{F}_{i_\tau}^{\mathsf{N}\to(\tau\to\tau)\to\tau\to\tau}\, \tilde{1}^{\mathsf{N}} &= \tilde{1}^{(\tau\to\tau)\to\tau\to\tau} \\ \mathrm{F}_{i_\tau}^{\mathsf{N}\to(\tau\to\tau)\to\tau\to\tau}\, \tilde{2}^{\mathsf{N}} &= \tilde{2}^{(\tau\to\tau)\to\tau\to\tau} \\ \mathrm{F}_{i_\tau}^{\mathsf{N}\to(\tau\to\tau)\to\tau\to\tau}\, \tilde{3}^{\mathsf{N}} &= \tilde{3}^{(\tau\to\tau)\to\tau\to\tau} \\ &\vdots \end{aligned}$$

where F_{i_τ} is a constant unique to this set of axioms. Define

$$\mathrm{H}_\tau^{(\tau\to\mathsf{N}\to\tau)\to\tau\times\mathsf{N}\to\tau\times\mathsf{N}} = \lambda x^{\tau\to\mathsf{N}\to\tau} y^{\tau\times\mathsf{N}}.\, \left\{x\,(\mathrm{D}_{1,\tau,\mathsf{N}}\, y)\,(\mathrm{D}_{2,\tau,\mathsf{N}}\, y), \mathrm{S}_{\mathcal{F}}\,(\mathrm{D}_{2,\tau,\mathsf{N}}\, y)\right\}$$

for each type τ. It immediately follows that the term

$$\lambda a^{\tau} b^{\tau \to \mathsf{N} \to \tau} c^{\mathsf{N}}.\, \mathrm{D}_{1,\tau,\mathsf{N}} \left(\mathrm{F}_{i_{\tau \times \mathsf{N}}}\, c\, (\mathrm{H}_{\tau}\, b)\, \{a, \tilde{0}^{\mathsf{N}}\} \right)$$

in the language of $\mathcal{F}$ computes $\mathrm{R}_{\tau}^{\tau \to (\tau \to \mathsf{N} \to \tau) \to \mathsf{N} \to \tau}$. Hence, any function that is definable in the theory $\mathcal{T}$ is also definable in the theory $\mathcal{F}$.

Conversely, any function that is definable in $\mathcal{F}$ is also definable in $\mathcal{T}$, since the functions F_i are definable in $\mathcal{T}$ by Theorem 1.6.2. $\square$

Chapter 2

The Computable Universe Hypothesis

2.1 Introduction

A common way to formalize the concept of a *physical model* is to identify the states of the system being modeled with the members of some set S, and to identify each observable quantity of the system with a function from S to the real numbers.[1] For example, a simple model of planetary motion, with the Earth moving in a circular orbit and traveling at a uniform speed, is the following.

Model 2.1.1 (Simple Planetary Motion). *Let S be the set of all pairs of real numbers (t, a) such that $a = 360\big(t - \lfloor t \rfloor\big)$, where $\lfloor t \rfloor$ denotes the largest integer less than or equal to t. The angular position of the Earth, measured in degrees, is given by the function $\alpha(t, a) = a$. The time, measured in years, is given by the function $\tau(t, a) = t$.*

If we wish, for example, to compute the position of the Earth after 2.25 years, we ask: "For which states (t, a) does $\tau(t, a) = 2.25$?" There is only one such state, namely $(2.25, 90)$. Therefore, the position of the Earth after 2.25 years is $\alpha(2.25, 90) = 90$ degrees. We say that the model is *faithful* if and only if the values of the observable quantities in the model match the values that are physically observed.

Church and Turing hypothesized that the functions which are effectively computable by humans are exactly the recursive functions.[2] There have been several attempts [19, 34, 42, 50, 51] to extend the Church-Turing thesis to physics, hypothesizing that the laws of physics are, in some sense, computable. But given an arbitrary physical model, it has not been clear exactly how one determines whether or not that model is to be regarded as computable. To date, the best attempt at providing such a definition has been Kreisel's notion of a mechanistic theory [25]. Kreisel suggested the following.

Kreisel's Criterion. *The predictions of a physical model are to be regarded as computable if and only if every real number which is observable according to the model is recursive relative to*

[1] A more detailed account of this formalism is available in reference [35].

[2] Readers unfamiliar with the definition of a recursive function or related terminology, such as uniformity, should consult reference [33]. The original justifications for identifying the effectively computable functions with the recursive functions can be found in references [9, 43, 44].

the data uniformly.

But many seemingly innocuous models have failed to satisfy Kreisel's criterion. For example, the simple model of planetary motion (Model 2.1.1) fails because given a real number representing the time t, there is no effectively computable procedure which determines the corresponding angle a when a is near the discontinuity at 360 degrees. Models which intuitively seem to have computable predictions often fail to satisfy Kreisel's criterion because discontinuities in their formalisms prevent the models' predictions from being regarded as computable, despite the fact that there are no discontinuities in the actual physical phenomena being modeled [32].

Rather than using Kreisel's criterion to prove that the predictions of established models are computable, an alternate approach is to supply a restrictive formalism which guarantees that the predictions of models expressible in that formalism are computable. This has been the approach taken in references [19, 34, 50, 51]. But difficulties have been encountered expressing important established models in these formalisms. For example, Rosen [34] was unable to describe radioactive decay in the formalism that he had proposed, and work is ongoing to describe established physical models in other computable formalisms.

It is the goal of this chapter to provide a general formalism for describing physical models whose predictions are computable, and to show that the computable formalisms studied by previous authors are special cases of our general formalism. In particular, we show in Section 2.15 that among the members of a large class of physical models, each physical model satisfying Kreisel's criterion has a corresponding model in our formalism. We also avoid some of the difficulties which, for example, prevented the simple model of planetary motion (Model 2.1.1) from being regarded as computable, as will be seen in Section 2.4. Our approach also avoids the difficulty that Rosen encountered with radioactive decay, as will be seen in Section 2.6.

2.2 Computable Physical Models

The central problem is that physical models use real numbers to represent the values of observable quantities, but that recursive functions are functions of non-negative integers, not functions of real numbers. To show that a model is computable, the model must somehow be expressed using recursive functions. Careful consideration of this problem, however, reveals that the real numbers are not actually necessary in physical models. Non-negative integers suffice for the representation of observable quantities because numbers measured in laboratory experiments necessarily have only finitely many digits of precision. For example, measurements of distances with a measuring stick will always be non-negative integer multiples of the smallest division on the measuring stick. So, we suffer no loss of generality by restricting the values of all observable quantities to be expressed as non-negative integers—the restriction only forces us to make the methods of error analysis, which were tacitly assumed when dealing with real numbers, an explicit part of each model.

Non-negative integers are not only sufficient for the description of direct physical measurements, but are also sufficient for encoding more complex data structures—allowing us to define recursive functions on those data structures. For example, a pair of two non-negative integers x

and y can be encoded as a single non-negative integer $\langle x, y\rangle$ using Cantor's pairing function

$$\langle x, y\rangle = \frac{1}{2}(x^2 + 2xy + y^2 + 3x + y)$$

A pair $\langle x, y\rangle$ of non-negative integers will also be called a *length two sequence* of non-negative integers. A triple (or equivalently, length three sequence) of non-negative integers x, y, and z can be encoded as $\langle\langle x, y\rangle, z\rangle$, and so on. We write $\langle x, y, z\rangle$ as an abbreviation for $\langle\langle x, y\rangle, z\rangle$. An integer i can be encoded as a non-negative integer $\zeta(i)$ using the formula

$$\zeta(i) = \begin{cases} -2i - 1 & \text{if } i < 0 \\ 2i & \text{if } i \geq 0 \end{cases}$$

And a rational number $\frac{a}{b}$ in lowest-terms with $b > 0$ can be encoded as a non-negative integer $\rho(\frac{a}{b})$ using the formula

$$\rho\big(\frac{a}{b}\big) = \zeta\big((\operatorname{sgn} a)2^{\zeta(a_1-b_1)}3^{\zeta(a_2-b_2)}5^{\zeta(a_3-b_3)}7^{\zeta(a_4-b_4)}11^{\zeta(a_5-b_5)}\cdots\big)$$

where $a = (\operatorname{sgn} a)2^{a_1}3^{a_2}5^{a_3}7^{a_4}11^{a_5}\cdots$ is the prime factorization of the integer a, and similarly for b. We write $(q\,;r)$ as an abbreviation for the pair of rational numbers $\big\langle\rho(q), \rho(r)\big\rangle$.

Historically, authors who have wished to restrict themselves to physical models whose predictions are computable have chosen from among a handful of formalisms. For example, Zuse [51] and Fredkin [19] have formalized their models as cellular automata, with each cell of an automaton representing a discrete unit of space and each step of computation in the automaton representing a discrete unit of time. Wolfram [50] has formalized his models in a variety of computational systems, including cellular automata, but has favored network systems for a model of fundamental physics. In each of these cases, the states of a physical system are represented by the states of a computational system (for example, a cellular automaton or a network system) which can be encoded as non-negative integers using the techniques just described. The resulting set of non-negative integers is a recursive set, and the observable quantities of the system are recursive functions of the members of that set. This immediately suggests the following definition.

Definition 2.2.1. *A* computable physical model *of a system is a recursive set S of states with a total recursive function ϕ for each observable quantity of the system. $\phi(s)$ is the value of that observable quantity when the system is in state s.*

So, in a computable physical model the set S is a set of non-negative integers, and each observable quantity is a function from non-negative integers to non-negative integers. The models considered by Zuse, Fredkin, and Wolfram are necessarily special sorts of computable physical models, and the set of all computable physical models is a proper subset of all physical models. In order to avoid all ambiguity, we insist that observable quantities be defined operationally [6] in computable physical models, so that, for example, if there were an observable quantity corresponding to time, then that observable quantity would be the time as measured with a specific conventionally-chosen clock in a specific conventionally-chosen reference frame.

An immediate consequence of the definition of a computable physical model is that we can give a precise formal counterpart to the informal claim that all the laws of physics are computable.

Computable Universe Hypothesis. *The universe has a recursive set of states U. For each observable quantity, there is a total recursive function ϕ. $\phi(s)$ is the value of that observable quantity when the universe is in state s.*

By a *distinguishable system*, we mean any system for which there is an observable quantity ϕ such that $\phi(s) = 1$ when the system exists in the universe, and such that $\phi(s) = 0$ otherwise. For example, if the system being studied is the orbit of the Earth, then $\phi(s) = 0$ when state s corresponds to a time before the formation of the Earth, and $\phi(s) = 1$ when the Earth exists and is orbiting the Sun. Note that the set of states s in U for which $\phi(s) = 1$ is itself a recursive set whenever U and ϕ are recursive. So, the computable universe hypothesis implies that computable physical models are sufficient for modeling any distinguishable system in the universe—the set of states of that distinguishable system is the set of all members s of U for which $\phi(s) = 1$, and the observable quantities of the distinguishable system are necessarily a subset of the observable quantities of the universe.

2.3 Discrete Planetary Motion

As a first example of a computable physical model, consider the following model of planetary motion.

Model 2.3.1 (Discrete Planetary Motion). *Let S be the set of all pairs $\big\langle (r\,;s), (p\,;q) \big\rangle$ such that*

$$r = \frac{i}{10} - \frac{1}{100} \qquad\qquad p = 360\big(r - \lfloor r \rfloor\big)$$
$$s = \frac{i+1}{10} + \frac{1}{100} \qquad\qquad q = 360\big(s - \lfloor s \rfloor\big)$$

for some integer i between -20000 and 20000. The angular position of the Earth, represented as a range of angles measured in degrees, is given by the function $\alpha\big\langle (r\,;s), (p\,;q) \big\rangle = (p\,;q)$. The time interval, measured in years, is given by the function $\tau\big\langle (r\,;s), (p\,;q) \big\rangle = (r\,;s)$.

This is a discrete model. That is, the position of the Earth in its orbit is not an exact real number, such as 90 degrees, but is instead an interval, such as $(68.4\,;111.6)$ representing a range of angles between 68.4 degrees and 111.6 degrees.[3] Similarly, time is measured in discrete intervals of length 0.12 years. The earliest time interval in the model is near the year -2000 and the latest time interval is near the year 2000. Moreover, this model is faithful—it is in exact agreement with all observations.

There are ten possible measurements for the angular position of the Earth in Model 2.3.1:

$(32.4\,;75.6)$	$(68.4\,;111.6)$	$(104.4\,;147.6)$	$(140.4\,;183.6)$	$(176.4\,;219.6)$
$(212.4\,;255.6)$	$(248.4\,;291.6)$	$(284.4\,;327.6)$	$(320.4\,;3.6)$	$(356.4\,;39.6)$

These are the intervals obtained by dividing the 360 degrees of the circle into ten equal intervals of 36 degrees each, then extending each interval by exactly 3.6 degrees on both sides, bringing

[3] We use decimal numbers to represent exact rational numbers. For example, 68.4 is to be understood as an abbreviation for $\frac{684}{10}$.

the total length of each interval to 43.2 degrees. Therefore, consecutive intervals overlap by 7.2 degrees (there is also overlap in consecutive time intervals), and this serves an important purpose. The Earth's orbit is not, in reality, a perfect circle, and the Earth does not spend an equal amount of time in each of the intervals. But because the eccentricity of the Earth's orbit contributes to, at most, only about a 2 degree deviation [18] from the simple model of planetary motion (Model 2.1.1), the overlap of these intervals is more than adequate to conceal evidence of the eccentricity, ensuring that this discrete model is faithful. Also note that the overlap is a realistic feature of all known instruments which measure angles, since each such instrument has only a limited accuracy. If angles were measured with a protractor, for example, the accuracy might be limited by the thickness of the lines painted on the protractor, which divide one reading from another. For example, if the lines are 7.2 degrees thick, then it might not be possible to distinguish a reading of $(32.4\,;75.6)$ from a reading of $(68.4\,;111.6)$ if the quantity being measured is somewhere on that line (that is, if the quantity is somewhere between 68.4 and 75.6 degrees). The accuracy of measuring instruments is discussed in greater detail in Section 2.9.

2.4 Non-Discrete Continuous Planetary Motion

Many commonly-studied computable physical models are discrete, but non-discrete continuous models are also possible. For example, a non-discrete continuous computable physical model of planetary motion is the following.

Model 2.4.1 (Non-Discrete Continuous Planetary Motion). *Let S be the set of all pairs $\big\langle (r\,;s), (p\,;q)\big\rangle$ such that*

$$r = \frac{i}{10^n} - \frac{1}{10^{n+1}} \qquad\qquad p = 360\big(r - \lfloor r \rfloor\big)$$
$$s = \frac{i+1}{10^n} + \frac{1}{10^{n+1}} \qquad\qquad q = 360\big(s - \lfloor s \rfloor\big)$$

for some integer i and some positive integer n. The angular position of the Earth, represented as a range of angles measured in degrees, is given by the function $\alpha\big\langle (r\,;s), (p\,;q)\big\rangle = (p\,;q)$. The time interval, measured in years, is given by the function $\tau\big\langle (r\,;s), (p\,;q)\big\rangle = (r\,;s)$.

Like the discrete model, angular position and time are measured in intervals, but in this case the intervals are not all the same length. In particular, there are arbitrarily small intervals for the observable quantities of position and time, meaning that these quantities may be measured to arbitrary precision. This feature of Model 2.4.1 allows us to speak about real-valued positions and times, despite the fact that the values of observable quantities in the model are all non-negative integers, not real numbers.

This is because a real number is not the result of a single measurement, but is instead the limit of a potentially-infinite sequence of measurements. Suppose, for example, that we wish to measure the circumference of a circle whose diameter is exactly one meter. Measured with unmarked metersticks, we measure the circumference to be 3 meters. If the sticks are marked with millimeters, then we measure the circumference to be about 3.141 meters. And if they are marked with micrometers, then we measure a circumference of about 3.141592 meters. If we

continue this process indefinitely with increasingly precise measuring instruments, then in the infinite limit we approach the real number π.

More formally, for each real number x there is an infinite sequence of nested intervals $(a_0\,;b_0)$, $(a_1\,;b_1)$, $(a_2\,;b_2)$, ... that converges to x. Given such a sequence, the function ϕ such that $\phi(n) = (a_n\,;b_n)$ for each non-negative integer n is said to be an *oracle* for x. Note that there is more than one distinct sequence of nested intervals converging to x, and therefore more than one oracle for each x. Of particular importance is the *standard decimal oracle* o_x for the real number x. By definition, $o_x(n) = (a_n\,;b_n)$, where

$$a_n = \frac{\lfloor 10^{n+1}x \rfloor}{10^{n+1}} - \frac{c}{10^{n+1}} \qquad\qquad b_n = \frac{\lfloor 10^{n+1}x \rfloor + 1}{10^{n+1}} + \frac{c}{10^{n+1}}$$

for each non-negative integer n, and where the *accuracy factor* c is a positive rational number constant. We say that x is a *recursive real number* if and only if o_x is a recursive function. Note that not all real numbers are recursive [43].

Now, returning to Model 2.4.1, suppose that we are asked to find the position of the Earth at some real-valued time t. Suppose further that we are given the oracle o_t with accuracy factor $c = \frac{1}{10}$. Note that as we increase n, the values $o_t(n)$ are increasingly precise measurements of the time t in Model 2.4.1. Therefore, for each n there is some state $\big\langle o_t(n), (p_n\,;q_n)\big\rangle$ in the set S of Model 2.4.1. Because S is a recursive set, and because there is exactly one state corresponding to each time measurement, the function ϵ such that $\epsilon(n) = (p_n\,;q_n)$ is a recursive function relative to the oracle o_t. In fact, if $o_t(n) = (r_n\,;s_n)$, then

$$\epsilon(n) = \Big(360\big(r_n - \lfloor r_n \rfloor\big)\,;360\big(s_n - \lfloor s_n \rfloor\big)\Big)$$

for each non-negative integer n. And since the sequence of intervals $(r_0\,;s_0)$, $(r_1\,;s_1)$, $(r_2\,;s_2)$, ... converges to t, it immediately follows that the sequence of intervals $\epsilon(0)$, $\epsilon(1)$, $\epsilon(2)$, ... converges to $a = 360\big(t - \lfloor t \rfloor\big)$ whenever t is not an integer. In other words, ϵ is an oracle for the angular position a.

But in the case that t is an integer,

$$\epsilon(n) = \Big(360 - \frac{36}{10^{n+1}}\,;\frac{396}{10^{n+1}}\Big)$$

for all non-negative integers n, and $(356.4\,;39.6)$, $(359.64\,;3.96)$, $(359.964\,;0.396)$, ... is the resulting sequence. In the standard topology of the real numbers an interval $(x\,;y)$ should have $x < y$, so the question of whether or not this sequence converges to a point a in that standard topology cannot be meaningfully answered. But if we are willing to abandon the standard topology of the real numbers, then we may conventionally define this sequence to converge to $a = 0$. In fact, this definition is tantamount to establishing the topology of a circle of circumference 360 for all angles a.[4] Of course, this definition is justified since the readings after 360 on a measuring instrument for angles are identified with those readings after 0. In other words, angles really do lie in a circle.

[4] A basis for this topology is represented by the set of all possible angle measurements. In particular, if $x < y$ then $(x\,;y)$ represents the set of all real numbers a such that $x < a < y$, and if $x > y$ then $(x\,;y)$ represents the set of all real numbers a such that $0 \leq a < y$ or $x < a < 360$.

So, given the oracle o_t for a real-valued time t, Model 2.4.1 allows us to compute an oracle ϵ for the angular position a of the Earth at that time. These predictions are in complete agreement with the predictions of the simple model of planetary motion (Model 2.1.1). In fact, imposing the appropriate topology on the space of angles a, the mapping from t to a in Model 2.4.1 is continuous. The same mapping is discontinuous in the standard topology of the real numbers, which leads Kreisel's criterion to fail for Model 2.1.1. The formulation of computable physical models on effective topological spaces is discussed in greater detail in Sections 2.11 through 2.15.

2.5 Coarse-Graining

Observable quantities in computable physical models are defined operationally. This means that each observable quantity is defined so as to correspond to a specific physical operation, such as the operation of comparing a length to the markings on a meterstick (where the meterstick itself is constructed according to a prescribed operation). This is problematic for the non-discrete continuous model of planetary motion (Model 2.4.1) because, for example, arbitrary precision angle measurements are made with a single observable quantity in the model. That is, to assert that a model such as Model 2.4.1 is faithful, one must assert that there exists an operation which is capable of measuring angles to arbitrary precision. It is not known whether or not such an operation actually exists. And although the point is somewhat moot, since Model 2.4.1 is clearly not faithful, it raises the question of whether this is an accidental feature of Model 2.4.1, or whether it is a feature common to all non-discrete continuous computable physical models.

A more practical alternative to Model 2.4.1 might introduce an infinite sequence of observable quantities α_1, α_2, α_3, ... , each with finitely many digits of precision, and each more precise than its predecessor in the sequence. In this case, given a state s, the values $\alpha_1(s)$, $\alpha_2(s)$, $\alpha_3(s)$, ... would form a sequence of intervals converging to a real number representing the angular position of the Earth in that state. But a computable physical model has only countably many states, and there are uncountably many real numbers in the interval $(0\,;360)$. Therefore, there must be some real number position in the interval $(0\,;360)$ that the Earth never attains.[5] That is, a computable physical model of this alternative form is not continuous in the intended topology.

Rather than considering arbitrary precision measurements, let us introduce just one additional level of precision into the discrete model of planetary motion (Model 2.3.1).

[5] In particular, this is a real number constructed by diagonalizing over those real numbers which are associated with each of the countably many states.

Model 2.5.1. *Let S be the set of all quadruples $\big\langle (r_1\,;s_1),(r_2\,;s_2),(p_1\,;q_1),(p_2\,;q_2)\big\rangle$ such that*

$$\begin{aligned} r_1 &= \frac{i}{10} - \frac{1}{10^2} & p_1 &= 360\big(r_1 - \lfloor r_1 \rfloor\big) \\ s_1 &= \frac{i+1}{10} + \frac{1}{10^2} & q_1 &= 360\big(s_1 - \lfloor s_1 \rfloor\big) \\ r_2 &= \frac{j}{10^2} - \frac{1}{10^3} & p_2 &= 360\big(r_2 - \lfloor r_2 \rfloor\big) \\ s_2 &= \frac{j+1}{10^2} + \frac{1}{10^3} & q_2 &= 360\big(s_2 - \lfloor s_2 \rfloor\big) \end{aligned}$$

for some integers i and j with $10i \le j \le 10i+9$. The angular position of the Earth, represented as a range of angles measured in degrees with a low-precision measuring instrument, is given by the function

$$\alpha_1\big\langle (r_1\,;s_1),(r_2\,;s_2),(p_1\,;q_1),(p_2\,;q_2)\big\rangle = (p_1\,;q_1)$$

The angular position of the Earth, represented as a range of angles measured in degrees with a high-precision measuring instrument, is given by the function

$$\alpha_2\big\langle (r_1\,;s_1),(r_2\,;s_2),(p_1\,;q_1),(p_2\,;q_2)\big\rangle = (p_2\,;q_2)$$

The time interval, measured in years by a low-precision measuring instrument, is given by the function

$$\tau_1\big\langle (r_1\,;s_1),(r_2\,;s_2),(p_1\,;q_1),(p_2\,;q_2)\big\rangle = (r_1\,;s_1)$$

The time interval, measured in years by a high-precision measuring instrument, is given by the function

$$\tau_2\big\langle (r_1\,;s_1),(r_2\,;s_2),(p_1\,;q_1),(p_2\,;q_2)\big\rangle = (r_2\,;s_2)$$

Note that if the high-precision observable quantities α_2 and τ_2 are ignored, then the predictions of Model 2.5.1 agree exactly with the predictions of Model 2.3.1.[6] The process of removing observable quantities from a model to obtain a new model with fewer observable quantities is called *coarse-graining*. But while Model 2.3.1 is faithful, Model 2.5.1 is not faithful—physical measurements do not agree with the values of the observable quantities α_2 and τ_2 because the orbit of the Earth is not a perfect circle.

A traditional conception of science regards all physical models as inexact approximations of reality, and holds that the goal of science is to produce progressively more accurate models whose predictions more closely match observations than the predictions of previous models. That conception of science is reasonable when the values of observable quantities are real numbers, since the real numbers predicted by physical models are never exactly the same as the real numbers 'measured' in the laboratory. But when non-negative integers are used for the values of observable quantities, then an alternate conception of science is possible.

In this alternate conception there exist faithful models that are in exact agreement with reality, but perhaps only for a small subset of all physically observable quantities. For example,

[6] But it should be noted that the model obtained by omitting α_2 and τ_2 from Model 2.5.1 is not identical to Model 2.3.1. In particular, for each state in Model 2.3.1, there are ten indistinguishable states in the model obtained by omitting α_2 and τ_2 from Model 2.5.1. That is, these models are not isomorphic. See Section 2.10.

Model 2.3.1 is faithful, but only predicts the angular position of the Earth to within 43.2 degrees, and only for a limited range of times. The goal of science is then to produce *more refined* models. That is, the goal of science is to discover faithful models which have larger sets of observable quantities, and are therefore capable of predicting increasing numbers of facts.

2.6 Radioactive Decay

Given non-negative integers x and y, let $\beta(x, y)$ be the length y sequence composed of the first y bits in the binary expansion of x. For example $\beta(13, 6) = \langle 0, 0, 1, 1, 0, 1\rangle$. Now suppose that a single atom of a radioactive isotope, such as nitrogen-13, is placed inside a detector at time $t = 0$. We say that the detector has status 1 if it has detected the decay of the isotope, and has status 0 otherwise. The *history* of the detector at time t is the length t sequence of bits corresponding to the status of the detector at times 1 through t. For example, if the isotope decays sometime between $t = 2$ and $t = 3$, then the history of the detector at time $t = 5$ is $\langle 0, 0, 1, 1, 1\rangle$. The following computable physical model models the status of the detector as a function of time.

Model 2.6.1 (Radioactive Decay). *Let S be the set of all pairs $\big\langle t, \beta(2^n - 1, t)\big\rangle$ where n and t are non-negative integers such that $n \leq t$ and $t \neq 0$. The history of the detector is given by the function $\eta\langle t, h\rangle = h$, and the time, measured in units of the half-life of the isotope, is given by the function $\tau\langle t, h\rangle = t$.*

This is a model of the many-worlds interpretation [13] of radioactive decay. Suppose that one asks, "What will the status of the detector be at time $t = 3$?" There are four states $\langle t, h\rangle$ such that $\tau\langle t, h\rangle = 3$, namely

$$\big\langle 3, \langle 0, 0, 0\rangle\big\rangle \qquad \big\langle 3, \langle 0, 0, 1\rangle\big\rangle \qquad \big\langle 3, \langle 0, 1, 1\rangle\big\rangle \qquad \big\langle 3, \langle 1, 1, 1\rangle\big\rangle$$

In three of these states, the detector has status 1, and in one state it has status 0. If we assume that each state of the system is equally likely, then there is a $\frac{3}{4}$ probability that the detector will have status 1 at time $t = 3$. But if we ask, "If the detector has status 1 at time $t = 1$, then what will its status be at time $t = 3$?" The answer is "1", since $\langle t, h\rangle = \big\langle 3, \langle 1, 1, 1\rangle\big\rangle$ is the only state such that $\tau\langle t, h\rangle = 3$ and such that the status of the detector at time 1 is 1. These results are in agreement with conventional theory.

2.7 Ensembles of Physical Models

Suppose that a planet orbits a distant star and that we are uncertain of the planet's orbital period. In particular, suppose that we believe its motion is faithfully described by either the discrete model of planetary motion (Model 2.3.1) or by the following computable physical model.

Model 2.7.1. *Let S be the set of all pairs $\big\langle (r\,;s),(p\,;q)\big\rangle$ such that*

$$r = \frac{i}{10} - \frac{1}{100} \qquad\qquad p = 360\left(\frac{n}{10} - \frac{1}{100} - \left\lfloor \frac{n}{10} - \frac{1}{100} \right\rfloor\right)$$
$$s = \frac{i+1}{10} + \frac{1}{100} \qquad\qquad q = 360\left(\frac{n+1}{10} + \frac{1}{100} - \left\lfloor \frac{n+1}{10} + \frac{1}{100} \right\rfloor\right)$$

for some integer i between -20000 and 20000, and such that $n = \lfloor i/4 \rfloor$. The angular position of the planet, represented as a range of angles measured in degrees, is given by the function $\alpha\big\langle (r\,;s),(p\,;q)\big\rangle = (p\,;q)$. The time interval, measured in Earth years, is given by the function $\tau\big\langle (r\,;s),(p\,;q)\big\rangle = (r\,;s)$.

Note that this model is similar to Model 2.3.1, except that the orbital period of the planet is 4 Earth years, rather than 1 Earth year.

If for each of the two models we are given a rational number expressing the probability that that model is faithful, then a *statistical ensemble* of the models may be constructed. For example, if Model 2.3.1 is twice as likely as Model 2.7.1, then a corresponding statistical ensemble is the following. Note that this statistical ensemble is itself a computable physical model.

Model 2.7.2 (Ensemble of Models). *Let S be the set of all triples $\big\langle (r\,;s),(p\,;q),j\big\rangle$ such that*

$$r = \frac{i}{10} - \frac{1}{100} \qquad\qquad p = 360\left(\frac{n}{10} - \frac{1}{100} - \left\lfloor \frac{n}{10} - \frac{1}{100} \right\rfloor\right)$$
$$s = \frac{i+1}{10} + \frac{1}{100} \qquad\qquad q = 360\left(\frac{n+1}{10} + \frac{1}{100} - \left\lfloor \frac{n+1}{10} + \frac{1}{100} \right\rfloor\right)$$

for some integer i between -20000 and 20000, where $j = 0$, 1, or 2, and where

$$n = \begin{cases} i & \text{if } j = 0 \text{ or } 1 \\ \lfloor i/4 \rfloor & \text{if } j = 2 \end{cases}$$

The angular position of the planet, represented as a range of angles measured in degrees, is given by the function $\alpha\big\langle (r\,;s),(p\,;q),j\big\rangle = (p\,;q)$. The time interval, measured in Earth years, is given by the function $\tau\big\langle (r\,;s),(p\,;q),j\big\rangle = (r\,;s)$.

Since Model 2.3.1 is twice as likely as Model 2.7.1, there are two states, $\big\langle (r\,;s),(p\,;q),0\big\rangle$ and $\big\langle (r\,;s),(p\,;q),1\big\rangle$ in the ensemble for each state $\big\langle (r\,;s),(p\,;q)\big\rangle$ in Model 2.3.1, and there is one state $\big\langle (r\,;s),(p\,;q),2\big\rangle$ in the ensemble for each state $\big\langle (r\,;s),(p\,;q)\big\rangle$ in Model 2.7.1. Note that the index j in each state $\big\langle (r\,;s),(p\,;q),j\big\rangle$ of the ensemble is not observable.

Now, if we ask for the position of the planet during the time interval $(0.29\,;0.41)$, for example, there are three possible states $\big\langle (r\,;s),(p\,;q),j\big\rangle$ in the ensemble such that

$$\tau\big\langle (r\,;s),(p\,;q),j\big\rangle = (0.29\,;0.41)$$

namely

$$\big\langle (0.29\,;0.41),(104.4\,;147.6),0\big\rangle$$
$$\big\langle (0.29\,;0.41),(104.4\,;147.6),1\big\rangle$$
$$\big\langle (0.29\,;0.41),(356.4\,;39.6),2\big\rangle$$

Since the planet's angular position is $(104.4\,;147.6)$ for two of these three states, the position measurement $(104.4\,;147.6)$ has a probability of $\frac{2}{3}$. Similarly, because the planet's angular position is $(356.4\,;39.6)$ for one of the three states, the position measurement $(356.4\,;39.6)$ has a probability of $\frac{1}{3}$. These probabilities are a direct reflection of our uncertainty about which of the two underlying physical models, Model 2.3.1 or Model 2.7.1, is the true faithful model. In particular, because Model 2.3.1 has been deemed twice as likely as Model 2.7.1, the position of the planet in Model 2.3.1, namely $(104.4\,;147.6)$, has twice the probability of the position predicted by Model 2.7.1, namely $(356.4\,;39.6)$.

It is important to note that there is no observable quantity corresponding to probability in Model 2.7.2. Instead, probability is a mathematical tool used to interpret the model's predictions. This sort of interpretation of an ensemble of models is appropriate whenever the ensemble is composed from all possible models which could describe a particular system, with the number of copies of states of the individual models reflecting our confidence in the predictions of those models. See reference [24] for a more detailed account of this subjectivist interpretation of probability in physics.

Ensembles may be constructed in other circumstances as well, and we may refer to such ensembles as *non-statistical ensembles* of physical models. Non-statistical ensembles of physical models are commonplace in the sciences. For example, they result whenever a constant, such as an initial position, is left unspecified in the statement of a model. That model can then be used to describe any member of a family of systems, each of which may have a different value for the constant. But most importantly, when a non-statistical ensemble of physical models is constructed, no claims as to the likelihood of one value of the constant, as compared to some other value of the constant, are being made. In fact, this is the defining characteristic of a non-statistical ensemble of models. Non-statistical ensembles can be useful because they provide a convenient way to collect together sets of closely-related models.

2.8 Incompatible Measurements

A pair of measurements is said to be *simultaneous* if and only if they are both performed while the system is in a single state. An essential feature of quantum mechanical systems is that there may be quantities which are not simultaneously measurable. For example, the measurement of one quantity, such as the position of a particle, might affect the subsequent measurement of another quantity, such as the particle's momentum. Such measurements are said to be *incompatible*. It is natural to ask whether computable physical models can be used to describe systems which feature incompatible measurements.

Discrete quantum mechanical systems are often formalized as follows [14, 46]. The *quantum mechanical state* of a system is a normalized vector v in some normed complex vector space V. Typically, V is a Hilbert space and v is a *wave function*. For each *quantum mechanical measurement* there is a corresponding set $B = \{v_1, v_2, v_3, \ldots\}$ of normalized basis vectors for V. Each member of B corresponds to a possible value of the measurement. Because B is a basis for V, $v = a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots$ for some complex numbers a_1, a_2, a_3, ... If the system is in quantum mechanical state v and no two members of B correspond to the same measurement

value,[7] then the probability that the measurement will have the value corresponding to v_n is $|a_n|^2$. In this case, if the actual value which is measured is the value corresponding to v_n, then the quantum mechanical state of the system immediately after that measurement is v_n. The state v is said to have *collapsed* to v_n. During the time between measurements, the quantum mechanical state of a system may evolve according to a rule such as Schrödinger's equation.

Consider, for example, the problem of measuring the components of the spin of an isolated electron. In this case, V is the set of all vectors (a, b) such that a and b are complex numbers, where the norm $\|(a, b)\|$ is defined to be $\sqrt{|a|^2 + |b|^2}$. A quantum mechanical measurement of the z component of the electron's spin has two possible values, $-\frac{1}{2}\hbar$ and $+\frac{1}{2}\hbar$. The basis vectors corresponding to these values are $(0, 1)$ and $(1, 0)$, respectively. The quantum mechanical measurement of another component of the electron's spin, lying in the xz plane at an angle of 60 degrees to the z axis, also has two possible values, $-\frac{1}{2}\hbar$ and $+\frac{1}{2}\hbar$. The basis vectors corresponding to these values are $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, respectively. So, for example, if the spin component in the z direction is measured to have a value of $+\frac{1}{2}\hbar$ at time $t = 0$, then since

$$(1, 0) = -\frac{1}{2}\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

there is a $\left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4}$ probability that if the 60-degree electron spin component is measured at time $t = 1$, then that component will also have a value of $+\frac{1}{2}\hbar$.

Supposing that the 60-degree electron spin component is measured to have a value of $+\frac{1}{2}\hbar$ at time $t = 1$, a similar line of reasoning implies that if the spin's z component is measured at time $t = 2$, then there is a $\left|\frac{1}{2}\right|^2 = \frac{1}{4}$ probability that the value of that measurement will be $-\frac{1}{2}\hbar$, since

$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{1}{2}(0, 1) + \frac{\sqrt{3}}{2}(1, 0)$$

Therefore, if the z component of the electron's spin is measured at time $t = 0$, followed by a measurement of the 60-degree spin component at time $t = 1$, and followed by another measurement the z component at time $t = 2$, then the values of the two measurements of the z component need not be the same. Indeed, the quantum mechanical state of the system does not change between times $t = 0$ and $t = 1$, or between times $t = 1$ and $t = 2$, but the measurement of the 60-degree spin component at time $t = 1$ disturbs the system and can potentially change the value of any subsequent measurement of the z component.[8] That is, measurement of the electron's 60-degree spin component is incompatible with measurement of its z component.

Let us formalize this system as a computable physical model. The system is composed of the electron, the apparatus used to make the quantum mechanical measurements, and the researcher

[7] Alternatively, if $v_{n_1}, v_{n_2}, v_{n_3}, \ldots$ are distinct basis vectors corresponding to the same measurement value, then the probability of measuring the value is $|a_{n_1}|^2 + |a_{n_2}|^2 + |a_{n_3}|^2 + \cdots$. If that value is actually measured, then the state of the system immediately after the measurement is the normalization of $a_{n_1}v_{n_1} + a_{n_2}v_{n_2} + a_{n_3}v_{n_3} + \cdots$. See reference [28].

[8] In this case, the quantum mechanical state of the system does not change between measurements because the electron is isolated. For example, the electron is free from external electromagnetic fields or other influences that might cause its spin to precess.

who chooses which components to measure.[9] We assume that the quantum mechanical state of the electron is $(1, 0)$ at time $t = 0$, and that the researcher makes subsequent quantum mechanical measurements of the electron's spin components at times 1 and 2. When a quantum mechanical measurement is performed, a record is made (perhaps in the researcher's notebook) of the value of this measurement and of the component that was measured. We construct the computable physical model of this system from the point of view of an agent who observes only this recorded history and the time.

Model 2.8.1 (Electron Spin Measurement). *Let S be the set of all triples $\langle t, h, j\rangle$ such that*

$$\begin{array}{lcl} t = 1 & & t = 1 \\ h = (0\,;+1) & \textit{or} & h = (60\,;-1) \\ j = 0 & & j = 1 \end{array} \quad \textit{or} \quad \begin{array}{l} t = 1 \\ h = (60\,;+1) \\ j = 2 + m \end{array}$$

or

$$\begin{array}{l} t = 2 \\ h = \big\langle (0\,;+1), (0\,;+1) \big\rangle \\ j = 5 \end{array} \quad \textit{or} \quad \begin{array}{l} t = 2 \\ h = \big\langle (0\,;+1), (60\,;-1) \big\rangle \\ j = 6 \end{array}$$

or

$$\begin{array}{l} t = 2 \\ h = \big\langle (0\,;+1), (60\,;+1) \big\rangle \\ j = 7 + m \end{array} \quad \textit{or} \quad \begin{array}{l} t = 2 \\ h = \big\langle (60\,;-1), (0\,;-1) \big\rangle \\ j = 10 \end{array}$$

or

$$\begin{array}{l} t = 2 \\ h = \big\langle (60\,;-1), (0\,;+1) \big\rangle \\ j = 11 + m \end{array} \quad \textit{or} \quad \begin{array}{l} t = 2 \\ h = \big\langle (60\,;-1), (60\,;-1) \big\rangle \\ j = 14 \end{array}$$

or

$$\begin{array}{l} t = 2 \\ h = \big\langle (60\,;+1), (0\,;-1) \big\rangle \\ j = 15 + n \end{array} \quad \textit{or} \quad \begin{array}{l} t = 2 \\ h = \big\langle (60\,;+1), (0\,;+1) \big\rangle \\ j = 24 + m \end{array}$$

[9] We refrain from asking questions about the probability with which the researcher chooses which components to measure. That is, this model describes a non-statistical ensemble of researchers.

or

$$\begin{aligned} t &= 2 \\ h &= \big\langle (60\,;+1), (60\,;+1) \big\rangle \\ j &= 27 + m \end{aligned}$$

for some integers m and n with $0 \leq m \leq 2$ and $0 \leq n \leq 8$. The time is given by the function $\tau\langle t, h, j\rangle = t$. The history is given by the function $\eta\langle t, h, j\rangle = h$. A history is a chronological sequence of records, with the leftmost record being the oldest. Each record is a pair $(a\,;b)$ of rational numbers, where a is the angle from the z axis, measured in degrees, of a component of the electron's spin, and where b is the value of that component, measured in units of $\frac{1}{2}\hbar$.

Note that each state $\langle t, h, j\rangle$ has a distinct index j, which we will use to identify that particular state.

Model 2.8.1 corresponds to the quantum mechanical system in the following sense. First, the quantum mechanical state of the system at time t corresponds to a set of states in the computable physical model. For example, if the researcher decides to measure the 60-degree component of the electron's spin at time $t = 1$, then the quantum mechanical state of the system is represented by the set of states with indices 1 through 4. Assuming that the states in the set are equally likely, there is a $\frac{3}{4}$ probability that this component will have a measured value of $+\frac{1}{2}\hbar$, for example. Immediately after the measurement is made, the quantum mechanical state collapses, becoming either the set of states with indices 2 through 4, or the singleton set containing only the state with index 1. The collapse occurs because the information provided by the quantum mechanical measurement allows us to identify the state of the system more precisely, eliminating those states which disagree with the measurement result.[10] The quantum mechanical state then evolves to a new set of states at time $t = 2$. For example, if the measured value of the 60-degree electron spin component is $+\frac{1}{2}\hbar$ at time $t = 1$, and if the researcher plans to measure the 0-degree electron spin component (that is, the z component) at time $t = 2$, then the quantum mechanical state immediately before that measurement at time $t = 2$ is the set of states with indices 15 through 26.

Computable physical models similar to Model 2.8.1 can be constructed for quantum mechanical systems which satisfy the following criteria.

1. There is a set of possible measurements $\{m_0, m_1, m_2, \ldots, m_i, \ldots\}$ indexed by non-negative integers i.
2. Every discrete time step, one measurement from this set is performed.
3. The possible values of each measurement m_i are identified with non-negative integers.
4. If $\phi(i, n, t, h)$ is the probability that the measurement with index i has the value n, given that the measurement is performed at time step t and that $h = \big\langle (i_1\,;n_1), \ldots, (i_{t-1}\,;n_{t-1}) \big\rangle$ is the history of past measurements and their values, then $\phi(i, n, t, h)$ is a rational number.
5. If there is no measurement with index i or if the non-negative integer n does not correspond to a value of the measurement with index i, then $\phi(i, n, t, h) = 0$.

[10] For a more detailed discussion of this ensemble interpretation of the collapse of a quantum mechanical state, see reference [5].

6. For each choice of non-negative integers i, t, and h, there are only finitely many non-negative integers n such that $\phi(i, n, t, h) > 0$.
7. ϕ is a recursive function.

If a quantum mechanical system satisfies these criteria, then we can determine whether or not

$$s = \big\langle t, \big\langle (i_1\,; n_1), (i_2\,; n_2), \ldots, (i_t\,; n_t) \big\rangle, j \big\rangle$$

is in the set S of states of the corresponding computable physical model as follows. First, if $t = 0$, then s is not in S. Next, let $h_1 = 0$ and for each positive integer k with $1 < k \leq t$, let

$$h_k = \big\langle (i_1\,; n_1), (i_2\,; n_2), \ldots, (i_{k-1}\,; n_{k-1}) \big\rangle$$

Now we perform the following calculations for each positive integer $k \leq t$. If $\phi(i_k, n_k, k, h_k) = 0$, then s is not in S. Otherwise, there must be finitely many non-negative integers n such that the probability $\phi(i_k, n, k, h_k)$ is greater than zero. Since probabilities must sum to 1, those values for n may be found exhaustively by calculating $\phi(i_k, 0, k, h_k)$, $\phi(i_k, 1, k, h_k)$, $\phi(i_k, 2, k, h_k)$, and so on, until the the sum of these probabilities reaches 1. Let d_k be the least common denominator of these rational probabilities, and let a_k be the unique positive integer such that

$$\phi(i_k, n_k, k, h_k) = \frac{a_k}{d_k}$$

If $j < a_1 a_2 \cdots a_t$, then s is in S. Otherwise, s is not in S.

2.9 The Accuracy of Measuring Instruments

An important feature of the discrete model of planetary motion (Model 2.3.1) is that the intervals representing time and angle measurements overlap. The amount of overlap between adjacent intervals is determined by the accuracy of the corresponding measuring instrument. The introduction of overlapping intervals is motivated by an argument such as the following.

If Model 2.3.1 were constructed using disjoint, non-overlapping intervals, then the states $\big\langle (r\,; s), (p\,; q) \big\rangle$ of that model would be given by

$$r = i/10 \qquad\qquad p = 360\big(r - \lfloor r \rfloor\big)$$
$$s = (i+1)/10 \qquad\qquad q = 360\big(s - \lfloor s \rfloor\big)$$

where i is an integer. In particular, $\big\langle (0.2\,; 0.3), (72\,; 108) \big\rangle$ and $\big\langle (0.3\,; 0.4), (108\,; 144) \big\rangle$ would be two such states, with $(r\,; s)$ representing the state's time interval, measured in years, and with $(p\,; q)$ representing the corresponding interval of angular positions for the Earth, measured in degrees. According to this model, if the position of the Earth is measured at time $t = 0.298$ years, then t is within the interval $(0.2\,; 0.3)$, and the state of the system is $\big\langle (0.2\,; 0.3), (72\,; 108) \big\rangle$. Therefore, according to this model, the position of the Earth should be between 72 and 108 degrees. Indeed, the simple model of planetary motion (Model 2.1.1) predicts that the angular position of the Earth at time $t = 0.298$ years should be $360\big(0.298 - \lfloor 0.298 \rfloor\big) \approx 107$ degrees. But the true position of the Earth in its orbit deviates from Model 2.1.1. In this case, the true

position of the Earth at time $t = 0.298$ years is about 109 degrees,[11] which is outside the interval $(72\,;108)$. Therefore, if the discrete model were constructed using disjoint, non-overlapping intervals, then the model would fail when $t = 0.298$ years.

But the discrete model of planetary motion (Model 2.3.1) was constructed using overlapping intervals. In particular,

$$\big\langle (0.19\,;0.31), (68.4\,;111.6) \big\rangle \qquad\qquad \big\langle (0.29\,;0.41), (104.4\,;147.6) \big\rangle$$

are two states in Model 2.3.1. Note that at time $t = 0.298$ years, Model 2.3.1 could be in either of these two states. Furthermore, any pair of real-valued time t and angle a measurements which satisfy

$$\big| a - 360\big(t - \lfloor t \rfloor\big) \big| < 7.2$$

fall within the time and angle intervals of some common state of Model 2.3.1. Since $|a - 360\big(t - \lfloor t \rfloor\big)|$ is at most 2 degrees [18] for all physically observed angles a measured at times t, Model 2.3.1 is faithful.

It is important to point out, though, that Model 2.3.1 is faithful only if the results of measurements are uncertain when they occur within the region of overlap. For example, at time $t = 0.306$ years, two results of a time measurement are possible, $(0.19\,;0.31)$ and $(0.29\,;0.41)$, and an observer cannot be certain which of these intervals is the value of the measurement. The actual angular position of the Earth at time $t = 0.306$ years is about 112 degrees, so $(104.4\,;147.6)$ is the only possible result of a position measurement. Since

$$\big\langle (0.19\,;0.31), (104.4\,;147.6) \big\rangle$$

is not one of the states of Model 2.3.1, the observer is expected to realize, in retrospect, after measuring the angular position, that the true time measurement must have been $(0.29\,;0.41)$. After providing a model for the phenomenon of accuracy, we will be able to reformulate Model 2.3.1 so that the results of measurements no longer possess this sort of ambiguity.

But first, note that the accuracy of a measuring instrument, by definition, can only be quantified relative to some other, more precise quantity. For example, the argument above, concerning accuracy in Model 2.3.1, makes frequent reference to exact real-valued angles and times. Indeed, even when we express an angle measurement as an interval, such as $(68.4\,;111.6)$, we are implying that it is possible to distinguish an angle of 68.4 degrees from an angle of 111.6 degrees, and that other angles lie between those two values. In principle, though, it is possible to describe the accuracy of a measuring instrument in a purely discrete manner, without any mention of real numbers. For example, let us consider an instrument for measuring distances in meters, with the value of a measurement represented as an integer number of meters. The accuracy of this measuring instrument can be quantified relative to a second instrument which measures distances in decimeters.

Presumably, the phenomenon of accuracy results from our inability to properly calibrate measuring instruments. Although there are many different underlying causes of calibration error, it

[11] This is assuming that time is measured in anomalistic years, with each year beginning at perihelion passage. During the course of a year, the position of the Earth is the true anomaly, measured relative to that perihelion passage.

suffices to consider only one such cause for a simple model of this phenomenon. We will suppose that when we measure a distance in meters, that we have difficulty aligning the measuring instrument with the origin, so that sometimes the instrument is aligned a decimeter too far in the negative direction, and at other times a decimeter too far in the positive direction. Hence, there are two different physical models for the measurement. In one model the instrument is misaligned in the negative direction, and in the other model it is misaligned in the positive direction. Since we do not know which of these two models describes any one particular measurement, it is appropriate to combine them in the following statistical ensemble.

Model 2.9.1. *Let S be the set of all triples $\big\langle \zeta(m), \zeta(d), \zeta(i) \big\rangle$ such that*

$$m = \left\lfloor \frac{d+i}{10} \right\rfloor$$

where d is an integer, and where $i = -1$ or $+1$. The distance, measured in meters, is given by the function

$$\mu\big\langle \zeta(m), \zeta(d), \zeta(i) \big\rangle = \zeta(m)$$

The same distance, measured in decimeters, is given by the function

$$\delta\big\langle \zeta(m), \zeta(d), \zeta(i) \big\rangle = \zeta(d)$$

Note that the function ζ was defined in Section 2.2. Also note that the index i in each state $\big\langle \zeta(m), \zeta(d), \zeta(i) \big\rangle$ represents the calibration error, which is either -1 decimeter or $+1$ decimeter. Model 2.9.1 is a computable physical model.

A measurement of d decimeters in Model 2.9.1 can be interpreted as corresponding to an interval of $\big(\frac{d}{10}\,;\frac{d+1}{10}\big)$ meters. Note that a measurement of 9 decimeters (corresponding to an interval of $(0.9\,;1.0)$ meters), for example, is possible in two distinct states of the model:

$$\big\langle \zeta(0), \zeta(9), \zeta(-1) \big\rangle \qquad\qquad \big\langle \zeta(1), \zeta(9), \zeta(+1) \big\rangle$$

Similarly, a measurement of 10 decimeters (corresponding to an interval of $(1.0\,;1.1)$ meters) is possible in the states

$$\big\langle \zeta(0), \zeta(10), \zeta(-1) \big\rangle \qquad\qquad \big\langle \zeta(1), \zeta(10), \zeta(+1) \big\rangle$$

Hence, a measurement of 0 meters overlaps with a measurement of 1 meter on the intervals $(0.9\,;1.0)$ and $(1.0\,;1.1)$. And in general, a measurement of m meters overlaps with a measurement of $m+1$ meters on the intervals $(m+0.9\,;m+1.0)$ and $(m+1.0\,;m+1.1)$. Therefore, a measurement of m meters in Model 2.9.1 can be understood as corresponding to an interval of $(m-0.1\,;m+1.1)$ meters, with adjacent intervals overlapping by 0.2 meters.

Of course, this interpretation of Model 2.9.1 presumes that decimeters can be measured with perfect accuracy. A more realistic computable physical model can be constructed by supposing that decimeter measurements can also be misaligned, for example, by -1 centimeter or $+1$ centimeter. Note that centimeters are treated as unobserved, purely theoretical constructions in this model—there is no observable quantity for centimeter measurements.

Model 2.9.2. *Let S be the set of all quintuples $\big\langle \zeta(m), \zeta(d), \zeta(c), \zeta(i), \zeta(j) \big\rangle$ such that*

$$m = \left\lfloor \frac{c + 10i}{100} \right\rfloor \qquad\qquad d = \left\lfloor \frac{c + j}{10} \right\rfloor$$

where c is an integer, $i = -1$ or $+1$, and $j = -1$ or $+1$. The distance, measured in meters, is given by the function

$$\mu\big\langle \zeta(m), \zeta(d), \zeta(c), \zeta(i), \zeta(j) \big\rangle = \zeta(m)$$

The same distance, measured in decimeters, is given by the function

$$\delta\big\langle \zeta(m), \zeta(d), \zeta(c), \zeta(i), \zeta(j) \big\rangle = \zeta(d)$$

As an important application, the model of accuracy described in this section can be used to reformulate the discrete model of planetary motion (Model 2.3.1).

Model 2.9.3. *Let S be the set of all quintuples $\big\langle (r\,;s), (p\,;q), \zeta(i), \zeta(j), \zeta(k) \big\rangle$ such that*

$$\begin{aligned} m &= \left\lfloor \frac{k+i}{10} \right\rfloor & n &= \left\lfloor \frac{k+j}{10} \right\rfloor \\ r &= \frac{m}{10} - \frac{1}{100} & p &= 360\left(\frac{n}{10} - \frac{1}{100} - \left\lfloor \frac{n}{10} - \frac{1}{100} \right\rfloor \right) \\ s &= \frac{m+1}{10} + \frac{1}{100} & q &= 360\left(\frac{n+1}{10} + \frac{1}{100} - \left\lfloor \frac{n+1}{10} + \frac{1}{100} \right\rfloor \right) \end{aligned}$$

for some integers m and n, for some integer k between -200000 and 200000, and where $i = -1$ or $+1$, and $j = -1$ or $+1$. The angular position of the Earth, represented as a range of angles measured in degrees, is given by the function

$$\alpha\big\langle (r\,;s), (p\,;q), \zeta(i), \zeta(j), \zeta(k) \big\rangle = (p\,;q)$$

The time interval, measured in years, is given by the function

$$\tau\big\langle (r\,;s), (p\,;q), \zeta(i), \zeta(j), \zeta(k) \big\rangle = (r\,;s)$$

Note that like Model 2.3.1, this model is faithful. But the faithfulness, in this case, no longer requires that the results of some measurements be uncertain. Instead, given any particular measurement, the state of the system is uncertain. For example, there are 40 distinct states s such that $\tau(s) = (0.29\,;0.41)$.

In contrast to Model 2.3.1, consider what happens if Model 2.9.3 is used to explain measurements taken at time $t = 0.306$ years. Associated with each state

$$\big\langle (r\,;s), (p\,;q), \zeta(i), \zeta(j), \zeta(k) \big\rangle$$

in Model 2.9.3 is an integer k, intended to represent the time interval $(\frac{k}{100}\,;\frac{k+1}{100})$ during which the system is in that state. At time $t = 0.306$ years, $k = 30$, and the system could be in one of

the following four states:

$$\big\langle (0.19\,;0.31), (68.4\,;111.6), \zeta(-1), \zeta(-1), \zeta(30) \big\rangle$$
$$\big\langle (0.29\,;0.41), (68.4\,;111.6), \zeta(+1), \zeta(-1), \zeta(30) \big\rangle$$
$$\big\langle (0.19\,;0.31), (104.4\,;147.6), \zeta(-1), \zeta(+1), \zeta(30) \big\rangle$$
$$\big\langle (0.29\,;0.41), (104.4\,;147.6), \zeta(+1), \zeta(+1), \zeta(30) \big\rangle$$

Like Model 2.3.1, two time measurements are possible, $(0.19\,;0.31)$ or $(0.29\,;0.41)$. And since the actual angular position of the Earth at time $t = 0.306$ years is about 112 degrees, the measured position of the Earth is $(104.4\,;147.6)$ degrees at that time. Unlike Model 2.3.1, this position measurement is compatible with either time measurement, since

$$\big\langle (0.19\,;0.31), (104.4\,;147.6), \zeta(-1), \zeta(+1), \zeta(30) \big\rangle$$
$$\big\langle (0.29\,;0.41), (104.4\,;147.6), \zeta(+1), \zeta(+1), \zeta(30) \big\rangle$$

are both states of Model 2.9.3.

2.10 Isomorphism Theorems

Given a physical model with a set S of states and a set $A = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ of observable quantities, we write (S, A) as an abbreviation for that model.[12]

Definition 2.10.1. *Two physical models (S, A) and (T, B) are* isomorphic *if and only if there exist bijections $\phi : S \to T$ and $\psi : A \to B$ such that $\alpha(s) = \psi(\alpha)\big(\phi(s)\big)$ for all $s \in S$ and all $\alpha \in A$.*

Intuitively, isomorphic physical models can be thought of as providing identical descriptions of the same system.[13]

Given any particular computable physical model (S, A), there are many different models which are isomorphic to (S, A). The following two theorems provide some convenient forms for the representation of computable physical models. Let π_i^n be the *projection function* that takes a length n sequence of non-negative integers and outputs the ith element of the sequence. That is, $\pi_i^n\langle x_1, x_2, \ldots, x_n\rangle = x_i$ for any positive integer $i \leq n$.

Theorem 2.10.2. *If A is a finite set, then the computable physical model (S, A) is isomorphic to some computable physical model whose observable quantities are all projection functions.*

Proof. Given a computable physical model (S, A) with $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, let (T, B) be the computable physical model such that

$$T = \big\{\, \big\langle \alpha_1(s), \alpha_2(s), \ldots, \alpha_n(s), s \big\rangle \;\big|\; s \in S \,\big\}$$

[12] In the interest of generality, the definition of a computable physical model places no restrictions on the set A except that its members must be total recursive functions. Some authors prefer to restrict their attention to finite sets A. For example, see reference [8]. Other authors may prefer to restrict their attention to observable quantities computed by programs which belong to a recursively enumerable set.

[13] Rosen [35] defined a weaker notion of isomorphism. Physical models that are isomorphic in Rosen's sense are not necessarily isomorphic in the sense described here.

and let B be the set of projection functions $\{\pi_1^{n+1}, \pi_2^{n+1}, \ldots, \pi_n^{n+1}\}$. By construction, (T, B) is a computable physical model isomorphic to (S, A). □

Theorem 2.10.3. *If S is an infinite set, then the computable physical model (S, A) is isomorphic to some computable physical model whose set of states is the set of all non-negative integers. If S has n elements, then the computable physical model (S, A) is isomorphic to some computable physical model whose set of states is $\{0, 1, \ldots, n-1\}$.*

Proof. By definition, if (S, A) is a computable physical model, then S is a recursive set. It immediately follows that S is recursively enumerable. In particular, if S is infinite, then let T be the set of non-negative integers and there is a bijective recursive function ψ from T to S. If S has n elements, then there is a bijective recursive function ψ from $T = \{0, 1, \ldots, n-1\}$ to S. Now, given $A = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$, let $B = \{\alpha_1 \circ \psi, \alpha_2 \circ \psi, \alpha_3 \circ \psi, \ldots\}$, where $\alpha \circ \psi$ denotes the composition of the functions α and ψ. By construction, (S, A) is isomorphic to (T, B). □

Although Theorem 2.10.3 implies that any computable physical model (S, A) is isomorphic to a computable physical model (T, B) where T is a set of consecutive non-negative integers beginning with zero, there is no effective procedure for constructing a program that computes the characteristic function of T, given a program for computing the characteristic function of S when S is a finite set. That is, Theorem 2.10.3 does not hold uniformly.

Definition 2.10.4. *A* non-negative integer physical model *is a pair (S, A) where S is a set of non-negative integers and where each member of A is a partial function from the non-negative integers to the non-negative integers. S is the set of states of the model, and A is the set of observable quantities of the model.*

Note that if α is an observable quantity of a non-negative integer physical model (S, A), then α might be undefined for some inputs. The non-negative integer physical models form a more general class of objects than the computable physical models. In particular, a computable physical model is a non-negative integer physical model whose set of states is a recursive set and whose observable quantities are total recursive functions.

Theorem 2.10.5. *A non-negative integer physical model (S, A) is isomorphic to some computable physical model if S is a recursively enumerable set and if each member of A is a partial recursive function whose domain includes all the members of S.*

Proof. Given a non-negative integer physical model (S, A), note that the construction of the computable physical model (T, B) in the proof of Theorem 2.10.3 only requires that S be a recursively enumerable set and that each member of A be a partial recursive function whose domain includes all the members of S. Therefore, any such non-negative integer physical model (S, A) is isomorphic to the computable physical model (T, B). □

Let ϕ be any partial recursive function and let S be the largest set of consecutive non-negative integers beginning with zero such that $\phi(s)$ is defined for each $s \in S$. Let A be the set $\{\pi_1^m \circ \phi, \pi_2^m \circ \phi, \ldots, \pi_m^m \circ \phi\}$ for some non-negative integer m. By Theorem 2.10.5, (S, A) is isomorphic to a computable physical model. We say that any such computable physical model is *determined* by ϕ.

Theorem 2.10.6. *Every computable physical model with finitely many observable quantities is isomorphic to a computable physical model determined by a partial recursive function.*

Proof. Let (S, A) be a computable physical model with $A = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, and let T, B, and ψ be defined as in the proof of Theorem 2.10.3. We will show that (T, B) is determined by a partial recursive function ϕ. Now, if S has only n states, then let $\phi(i)$ be undefined for all non-negative integers $i \geq n$. Otherwise, define

$$\phi(i) = \big\langle \alpha_1(\psi(i)), \alpha_2(\psi(i)), \ldots, \alpha_m(\psi(i)) \big\rangle$$

By construction, (T, B) is determined by ϕ. □

A physical model (S, A) is said to be *reduced* if and only if for each pair of distinct states s_1 and s_2 in S, there exists an $\alpha \in A$ with $\alpha(s_1) \neq \alpha(s_2)$.

Theorem 2.10.7. *If (S, A) and (T, B) are isomorphic physical models and (S, A) is reduced, then (T, B) is also a reduced physical model.*

Proof. Let (S, A) and (T, B) be isomorphic physical models and let (S, A) be reduced. Since (S, A) and (T, B) are isomorphic, there exist bijections $\phi : S \to T$ and $\psi : A \to B$ such that $\alpha(s) = \psi(\alpha)\big(\phi(s)\big)$ for all $s \in S$ and all $\alpha \in A$. Now suppose that t_1 and t_2 are distinct states in T. Because ϕ is a bijection, $\phi^{-1}(t_1)$ and $\phi^{-1}(t_2)$ are distinct states in S. But S is reduced, so there must exist an $\alpha \in A$ such that $\alpha\big(\phi^{-1}(t_1)\big) \neq \alpha\big(\phi^{-1}(t_2)\big)$. Furthermore,

$$\alpha\big(\phi^{-1}(t_1)\big) = \psi(\alpha)\big(\phi\big(\phi^{-1}(t_1)\big)\big) = \psi(\alpha)(t_1)$$

and

$$\alpha\big(\phi^{-1}(t_2)\big) = \psi(\alpha)\big(\phi\big(\phi^{-1}(t_2)\big)\big) = \psi(\alpha)(t_2)$$

Hence, there exists a $\beta \in B$ such that $\beta(t_1) \neq \beta(t_2)$, namely $\beta = \psi(\alpha)$. We may conclude that the physical model (T, B) is reduced. □

An *epimorphism* from a physical model (S, A) to a physical model (T, B) is a pair of functions (ϕ, ψ) such that ϕ is a surjection from S to T and ψ is a bijection from A to B, where $\alpha(s) = \psi(\alpha)\big(\phi(s)\big)$ for all $s \in S$ and all $\alpha \in A$. Two physical models (S_1, A_1) and (S_2, A_2) are said to be *observationally equivalent* if and only if there are epimorphisms from (S_1, A_1) to (T, B) and from (S_2, A_2) to (T, B), where (T, B) is some reduced physical model.

Theorem 2.10.8. *If (S_1, A_1) and (S_2, A_2) are isomorphic physical models, then (S_1, A_1) and (S_2, A_2) are observationally equivalent.*

Proof. Define an equivalence relation on S_2 so that $r \in S_2$ is related to $s \in S_2$ if and only if $\alpha(r) = \alpha(s)$ for all $\alpha \in A_2$. Let T be the corresponding set of equivalence classes of S_2. For each $\alpha \in A_2$, define a function α' so that if $s \in t \in T$, then $\alpha'(t) = \alpha(s)$. Let $B = \{\, \alpha' \mid \alpha \in A_2 \,\}$. By construction, (T, B) is a reduced physical model. Also note that there is an epimorphism (ϕ, ψ) from (S_2, A_2) to (T, B). Namely, ϕ is the function that maps each member of S_2 to its corresponding equivalence class in T, and ψ is the function that maps each $\alpha \in A_2$ to $\alpha' \in B$.

Now suppose that (S_1, A_1) and (S_2, A_2) are isomorphic. By definition, there are bijections ϕ' from S_1 to S_2 and ψ' from A_1 to A_2 such that $\alpha(s) = \psi'(\alpha)\big(\phi'(s)\big)$ for all $s \in S_1$ and all $\alpha \in A_1$. Since (ϕ, ψ) is an epimorphism from (S_2, A_2) to (T, B), it immediately follows that $(\phi \circ \phi', \psi \circ \psi')$ is an epimorphism from (S_1, A_1) to (T, B). We may conclude, by definition, that (S_1, A_1) and (S_2, A_2) are observationally equivalent. $\square$

Intuitively, two physical models are observationally equivalent when they both make the same observable predictions. For example, as was discussed in Section 2.5, the model obtained by omitting the observable quantities α_2 and τ_2 from Model 2.5.1 is observationally equivalent to the discrete model of planetary motion (Model 2.3.1). Moreover, if a physical model (S_1, A_1) is faithful, and if (S_1, A_1) is observationally equivalent to (S_2, A_2), then (S_2, A_2) is also faithful.

It is important to note that the converse of Theorem 2.10.8 does not hold. That is, observationally equivalent models are not necessarily isomorphic. Consider, for example, the computable physical models $\big(\{0, 1\}, \{\alpha\}\big)$ and $\big(\{0, 1, 2\}, \{\beta\}\big)$ where $\alpha(s) = s$ for all $s \in \{0, 1\}$ and where $\beta(s) = \lfloor s/2 \rfloor$ for all $s \in \{0, 1, 2\}$. These models are not isomorphic because $\{0, 1\}$ and $\{0, 1, 2\}$ have different cardinalities. Yet, they are observationally equivalent, since both models have a single observable quantity whose only possible values are 0 and 1. Physical models that are isomorphic must not only make the same observable predictions, but must also have the same *structure*. The models $\big(\{0, 1\}, \{\alpha\}\big)$ and $\big(\{0, 1, 2\}, \{\beta\}\big)$ have different structures because, assuming that the states are equally likely, they both give different answers to the question, "What is the probability that the observable quantity has value 0?"

2.11 Oracles and Effective Topologies

Given any set X, we can impose a topology on X. Let $\mathcal{B}$ be a basis for this topology. The members of $\mathcal{B}$ are said to be *basis elements*. We say that a set $\mathcal{L}_x \subseteq \mathcal{B}$ is a *local basis* for a point $x \in X$ if and only if the following two conditions hold.

1. For each $L \in \mathcal{L}_x$, x is a member of L.
2. For each $B \in \mathcal{B}$ with $x \in B$, there exists an $L \in \mathcal{L}_x$ with $L \subseteq B$.

Note that every point $x \in X$ has a local basis. For example, the set

$$\mathcal{L}_x = \{\, B \in \mathcal{B} \mid x \in B \,\}$$

of all basis elements that contain x is a local basis for x.

If the basis $\mathcal{B}$ is countable, then each basis element can be encoded as a non-negative integer. In that case, choose some encoding and let $\nu(n)$ be the basis element encoded by n. We allow for the possibility that a basis element may be encoded by more than one non-negative integer. (That is, ν is not necessarily an injection.) The *domain* of ν, denoted $\operatorname{dom}_{\mathcal{B}} \nu$, is the set of all non-negative integers n such that $\nu(n) \in \mathcal{B}$. For any function $\phi : A \to B$ and any set $C \subseteq A$, let $\phi(C) = \{\phi(c) \mid c \in C\}$ denote the *image* of C under ϕ. We let $\mathbb{N}$ denote the set of non-negative integers.

Definition 2.11.1. *A function $\phi : \mathbb{N} \to \operatorname{dom}_{\mathcal{B}} \nu$ is said to be an* oracle *for a point x, with basis $\mathcal{B}$ and coding ν, if and only if $\nu\big(\phi(\mathbb{N})\big)$ is a local basis for x.*

An oracle ϕ for x is *complete* if and only if every $n \in \operatorname{dom}_{\mathcal{B}} \nu$ such that $x \in \nu(n)$ is a member of $\phi(\mathbb{N})$. An oracle is said to be *nested* if and only if $\nu\big(\phi(n+1)\big) \subseteq \nu\big(\phi(n)\big)$ for all $n \in \mathbb{N}$.

A pair $(\mathcal{B}, \nu)$ is said to be an *effective topology* if and only if $\mathcal{B}$ is a countable basis for a T_0 topology and ν is a coding for $\mathcal{B}$. Effective topologies were first introduced in the theory of type-two effectivity [27, 48]. In accordance with that theory, we use an oracle for x, with a basis $\mathcal{B}$ and coding ν, as a representation of the point x in an effective topology $(\mathcal{B}, \nu)$. Because effective topologies are T_0, no two distinct points are ever represented by the same oracle.

Of special interest are effective topologies where the subset relation

$$\{\, \langle b_1, b_2 \rangle \mid \nu(b_1) \subseteq \nu(b_2) \;\&\; b_1 \in \operatorname{dom}_{\mathcal{B}} \nu \;\&\; b_2 \in \operatorname{dom}_{\mathcal{B}} \nu \,\}$$

is a recursively enumerable set.[14] In particular, if $(\mathcal{B}, \nu)$ has a recursively enumerable subset relation, then

$$\operatorname{dom}_{\mathcal{B}} \nu = \{\, b \mid \nu(b) \subseteq \nu(b) \;\&\; b \in \operatorname{dom}_{\mathcal{B}} \nu \,\}$$

is also a recursively enumerable set.

Theorem 2.11.2. *Let ϕ be an oracle for x in an effective topology $(\mathcal{B}, \nu)$ with a recursively enumerable subset relation. Then there exists a complete oracle ψ for x in $(\mathcal{B}, \nu)$ that is recursive relative to ϕ uniformly.*

Proof. Suppose that ϕ is an oracle for x in an effective topology $(\mathcal{B}, \nu)$ with a recursively enumerable subset relation. Since $\nu\big(\phi(\mathbb{N})\big)$ is a local basis for x, it follows that for each $b \in \operatorname{dom}_{\mathcal{B}} \nu$, $x \in \nu(b)$ if and only if there exists an $n \in \mathbb{N}$ such that $\nu\big(\phi(n)\big) \subseteq \nu(b)$. Hence,

$$\big\{\, b \in \operatorname{dom}_{\mathcal{B}} \nu \;\big|\; (\exists\, n \in \mathbb{N}) \big[\nu\big(\phi(n)\big) \subseteq \nu(b)\big] \,\big\}$$

is the set of encodings of all basis elements that contain x. But this set is recursively enumerable relative to ϕ because $(\mathcal{B}, \nu)$ has a recursively enumerable subset relation. Therefore, there exists a function $\psi : \mathbb{N} \to \operatorname{dom}_{\mathcal{B}} \nu$, recursive relative to ϕ, such that $\nu\big(\psi(\mathbb{N})\big)$ is this set. By definition, ψ is a complete oracle for x. $\square$

Theorem 2.11.3. *Let ϕ be an oracle for x in an effective topology $(\mathcal{B}, \nu)$ with a recursively enumerable subset relation. Then there exists a nested oracle ψ for x in $(\mathcal{B}, \nu)$ that is recursive relative to ϕ uniformly.*

Proof. Suppose that ϕ is an oracle for x in an effective topology $(\mathcal{B}, \nu)$ with a recursively enumerable subset relation. Note that for each pair of basis elements B_1 and B_2 such that $x \in B_1 \cap B_2$, there exists a basis element B_3 with $x \in B_3 \subseteq B_1 \cap B_2$, by the definition of a basis. Therefore, since $\nu\big(\phi(\mathbb{N})\big)$ is a local basis for x, there must exist an $m \in \mathbb{N}$ such that

$$x \in \nu\big(\phi(m)\big) \subseteq B_3 \subseteq B_1 \cap B_2$$

Now define $\psi : \mathbb{N} \to \operatorname{dom}_{\mathcal{B}} \nu$ recursively, relative to ϕ, as follows. Let $\psi(0) = \phi(0)$ and for each $n \in \mathbb{N}$ let $\psi(n+1) = \phi(m)$ for some $m \in \mathbb{N}$ such that

$$\nu\big(\phi(m)\big) \subseteq \nu\big(\psi(n)\big) \cap \nu\big(\phi(n+1)\big)$$

[14] An effective topology with a recursively enumerable subset relation is an example of a *computable topology*, as defined in reference [48]. Not all computable topologies have recursively enumerable subset relations.

We can find m recursively given $\psi(n)$ and ϕ because the subset relation for $(\mathcal{B}, \nu)$ is recursively enumerable, and the set of all $m \in \mathbb{N}$ such that

$$\nu\big(\phi(m)\big) \subseteq \nu\big(\psi(n)\big) \;\&\; \nu\big(\phi(m)\big) \subseteq \nu\big(\phi(n+1)\big)$$

is therefore recursively enumerable relative to ϕ. We may conclude that ψ is nested because

$$\nu\big(\psi(n+1)\big) \subseteq \nu\big(\psi(n)\big) \cap \nu\big(\phi(n+1)\big) \subseteq \nu\big(\psi(n)\big)$$

for all $n \in \mathbb{N}$, and that ψ is an oracle for x because $x \in \nu\big(\psi(0)\big) = \nu\big(\phi(0)\big)$ and

$$x \in \nu\big(\psi(n+1)\big) \subseteq \nu\big(\psi(n)\big) \cap \nu\big(\phi(n+1)\big) \subseteq \nu\big(\phi(n+1)\big)$$

for all $n \in \mathbb{N}$. □

Define $\iota(a\,;b)$ to be the set of all real numbers x such that $a < x < b$, and let $\mathcal{I}$ be the set of all $\iota(a\,;b)$ such that a and b are rational numbers with $a < b$. The members of $\mathcal{I}$ are said to be *rational intervals*. Note that $\mathcal{I}$ is a basis for the standard topology of the real numbers. Indeed, the oracles for real numbers that were introduced in Section 2.4 were nested oracles with basis $\mathcal{I}$ and coding ι. Another commonly-used basis for the standard topology of the real numbers is the set $\mathcal{I}_{10,c}$ of *decimal intervals* with accuracy factor c, where c is a positive rational number, and where $\mathcal{I}_{10,c}$ is defined to be the set of all $\iota(a\,;b)$ such that

$$a = \frac{m}{10^n} - \frac{c}{10^n} \qquad\qquad b = \frac{m+1}{10^n} + \frac{c}{10^n}$$

for some integer m and some positive integer n. We call n the *number of digits of precision* of $(a\,;b)$.

Note that both $(\mathcal{I}, \iota)$ and $(\mathcal{I}_{10,c}\,, \iota)$ have recursively enumerable subset relations. The following theorem asserts that if ϕ is an oracle for a real number x with basis $\mathcal{I}$ and coding ι, then there exists an oracle ψ for x with basis $\mathcal{I}_{10,c}$ and coding ι that is recursive relative to ϕ uniformly.

Theorem 2.11.4. *Let $(\mathcal{A}, \nu)$ and $(\mathcal{B}, \nu)$ be effective topologies with recursively enumerable subset relations such that $\mathcal{B} \subseteq \mathcal{A}$, and such that $\mathcal{A}$ and $\mathcal{B}$ are bases for the same topology. If ϕ is an oracle for x in $(\mathcal{A}, \nu)$, then there exists an oracle ψ for x in $(\mathcal{B}, \nu)$ that is recursive relative to ϕ uniformly.*

Proof. Suppose that $(\mathcal{A}, \nu)$ and $(\mathcal{B}, \nu)$ are effective topologies as described in the statement of the theorem, and that ϕ is an oracle for a point x in $(\mathcal{A}, \nu)$. By definition,

$$x \in \nu\big(\phi(n)\big)$$

for every $n \in \mathbb{N}$. And because $\nu\big(\phi(n)\big)$ is an open set, it is a union of basis elements from $\mathcal{B}$. Hence, there must exist a $B \in \mathcal{B}$ such that

$$x \in B \subseteq \nu\big(\phi(n)\big)$$

But $\nu\big(\phi(\mathbb{N})\big)$ is a local basis for x, and B is a basis element in $\mathcal{A}$, so there exists an $m \in \mathbb{N}$ such that $x \in \nu\big(\phi(m)\big) \subseteq B$. Therefore, we have that for each $n \in \mathbb{N}$ there exist $B \in \mathcal{B}$ and $m \in \mathbb{N}$ such that

$$x \in \nu\big(\phi(m)\big) \subseteq B \subseteq \nu\big(\phi(n)\big)$$

Now, since $(\mathcal{B}, \nu)$ has a recursively enumerable subset relation, $\operatorname{dom}_{\mathcal{B}} \nu$ is a recursively enumerable set. Then, because $(\mathcal{A}, \nu)$ also has a recursively enumerable subset relation, the set

$$\big\{\, \langle m, b\rangle \;\big|\; \nu\big(\phi(m)\big) \subseteq \nu(b) \subseteq \nu\big(\phi(n)\big) \;\&\; m \in \mathbb{N} \;\&\; b \in \operatorname{dom}_{\mathcal{B}} \nu \,\big\}$$

is recursively enumerable relative to ϕ, for any $n \in \mathbb{N}$. Therefore, there is a function $\psi : \mathbb{N} \to \operatorname{dom}_{\mathcal{B}} \nu$, recursive relative to ϕ, such that $\psi(n) = b$ for all $n \in \mathbb{N}$, where

$$\nu\big(\phi(m)\big) \subseteq \nu(b) \subseteq \nu\big(\phi(n)\big)$$

for some $m \in \mathbb{N}$. But this function ψ is an oracle for x in $(\mathcal{B}, \nu)$, because

$$x \in \nu\big(\phi(m)\big) \subseteq \nu\big(\psi(n)\big) \subseteq \nu\big(\phi(n)\big)$$

for all $n \in \mathbb{N}$. $\square$

2.12 Basic Representations of Sets

Definition 2.12.1. *Let $(\mathcal{B}, \nu)$ be an effective topology on a set X, and let A be any subset of X. We say that a set R of non-negative integers is a* basic representation *of A in the effective topology $(\mathcal{B}, \nu)$ if and only if the following two conditions hold.*

1. $R \subseteq \operatorname{dom}_{\mathcal{B}} \nu$
2. *$x \in A$ if and only if there exists a local basis $\mathcal{L}_x$ for x with $\mathcal{L}_x \subseteq \nu(R)$.*

Note that condition 2 of the definition ensures that no two distinct sets in $(\mathcal{B}, \nu)$ have the same basic representation. Note further that if R is a basic representation of A, then $\{\, A \cap \nu(r) \mid r \in R \,\}$ is a basis for the subspace topology on A, and this is an effective topology with coding $\lambda r\big[A \cap \nu(r)\big]$.

In an effective topology we use basic representations to represent sets of points, but not all sets of points have basic representations. For example, there are $2^{2^{\aleph_0}}$ many sets of real numbers, but since a basic representation is a set of non-negative integers, there are at most $2^{\aleph_0}$ many basic representations. Nevertheless, many commonly-studied sets have basic representations.[15]

[15] In the effective topology $(\mathcal{I}, \iota)$, the set of rational numbers does not have a basic representation, but the set of irrational numbers has the basic representation

$$R = \Big\{\, \Big(\frac{m}{n!}\,;\frac{m+1}{n!}\Big) \;\Big|\; m \in \mathbb{Z} \;\&\; n \in \mathbb{N} \,\Big\}$$

where $\mathbb{Z}$ denotes the set of integers. It is tempting to conjecture that the sets with basic representations in an effective topology $(\mathcal{B}, \nu)$ are exactly the G_δ sets, but there is a trivial counterexample to this conjecture if the effective topology is not T_1.

Theorem 2.12.2. *Let A be a set in an effective topology $(\mathcal{B}, \nu)$.*

1. *If A is an open set, then A has a basic representation.*
2. *If A is a closed set, then A has a basic representation.*

Proof. Suppose that A is an open set in the effective topology $(\mathcal{B}, \nu)$ and let

$$R = \{\, r \in \operatorname{dom}_{\mathcal{B}} \nu \mid \nu(r) \subseteq A \,\}$$

Clearly, $R \subseteq \operatorname{dom}_{\mathcal{B}} \nu$ and if $x \notin A$ then there does not exist a local basis $\mathcal{L}_x$ for x with $\mathcal{L}_x \subseteq \nu(R)$, since no member of $\nu(R)$ contains x. Alternatively, if $x \in A$ then, by the definition of a basis, for each basis element B_1 that contains x there exists some basis element B_2 such that $x \in B_2 \subseteq B_1 \cap A$. That is, if $x \in A$ then for each basis element B_1 with $x \in B_1$, there exists a basis element $B_2 \in \{\, B \in \mathcal{B} \mid x \in B \subseteq A \,\}$ with $B_2 \subseteq B_1$. It immediately follows that $\mathcal{L}_x = \{\, B \in \mathcal{B} \mid x \in B \subseteq A \,\}$ is a local basis for x and $\mathcal{L}_x \subseteq \nu(R)$. By definition, R is a basic representation of A.

Now, if A is a closed set in $(\mathcal{B}, \nu)$, then let

$$R = \{\, r \in \operatorname{dom}_{\mathcal{B}} \nu \mid A \cap \nu(r) \neq \varnothing \,\}$$

Clearly, $R \subseteq \operatorname{dom}_{\mathcal{B}} \nu$ and if $x \in A$ then there exists a local basis $\mathcal{L}_x$ for x with $\mathcal{L}_x \subseteq \nu(R)$. Namely, $\mathcal{L}_x$ is the set of all basis elements that contain x. Alternatively, if $x \notin A$, then since A is closed, every local basis $\mathcal{L}_x$ for x contains a basis element that does not intersect A. Therefore, $\mathcal{L}_x \nsubseteq \nu(R)$. We may conclude, by definition, that R is a basic representation of A. □

Although we use a basic representation R to represent a set of points in an effective topology, the following theorem demonstrates that there is, in general, no effective procedure (relative to R) for finding oracles for those points. Nevertheless, if we restrict our attention to certain special classes of basic representations R, then effective procedures do exist. See Section 2.15.

Theorem 2.12.3. *Let $\mathcal{B}$ be a countable basis for the standard topology of $\mathbb{R}^n$ and let ν be a coding for the basis. Then there does not exist a partial recursive function ϕ satisfying the condition that for every singleton set $\{x\} \subseteq \mathbb{R}^n$ and for every basic representation R_x of $\{x\}$ in the effective topology $(\mathcal{B}, \nu)$, the function $\lambda m\big[\phi(R_x, m)\big]$ is an oracle for x in $(\mathcal{B}, \nu)$.*

Proof. Let $(\mathcal{B}, \nu)$ be an effective topology as in the statement of the theorem and suppose, as an assumption to be shown contradictory, that there exists a partial recursive function ϕ satisfying the condition that for every singleton set $\{x\} \subseteq \mathbb{R}^n$ and for every basic representation R_x of $\{x\}$ in the effective topology $(\mathcal{B}, \nu)$, the function $\lambda m\big[\phi(R_x, m)\big]$ is an oracle for x in $(\mathcal{B}, \nu)$. Now consider any two distinct points $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, and let R_x be a basic representation of $\{x\}$ in $(\mathcal{B}, \nu)$. Since the standard topology of $\mathbb{R}^n$ is T_1, there must exist a non-negative integer k such that $\phi(R_x, k)$ is defined and

$$y \notin \nu\big(\phi(R_x, k)\big)$$

Next, choose a program for computing ϕ. Note that since the computation for $\phi(R_x, k)$ has only finitely many steps, only finitely many non-negative integers are tested for membership in R_x during the course of the computation. Let C be the collection of all $i \in \mathbb{N}$ such that $i \in R_x$ and such that i is tested for membership in R_x during the course of the computation of $\phi(R_x, k)$.

Similarly, let D be the collection of all $i \in \mathbb{N}$ such that $i \notin R_x$ and such that i is tested for membership in R_x during the course of the computation of $\phi(R_x, k)$.

Now, choose any oracle ψ for y in $(\mathcal{B}, \nu)$. Note that $\psi(\mathbb{N})$ is a basic representation for $\{y\}$ in $(\mathcal{B}, \nu)$. And because C and D are finite sets,

$$R_y = \big(\psi(\mathbb{N}) \cup C\big) - D$$

is also a basic representation for $\{y\}$ in $(\mathcal{B}, \nu)$. It follows that $\phi(R_x, k) = \phi(R_y, k)$, because whenever i is tested for membership in R_x during the course of the computation of $\phi(R_x, k)$, $i \in R_x$ if and only if $i \in R_y$. Therefore,

$$y \notin \nu\big(\phi(R_x, k)\big) = \nu\big(\phi(R_y, k)\big)$$

But by the definition of ϕ, $\lambda m\big[\phi(R_y, m)\big]$ is an oracle for y. Hence,

$$y \in \nu\big(\phi(R_y, k)\big)$$

This is a contradiction, so the assumption must be false. The partial recursive function ϕ does not exist. $\square$

2.13 Basic Representations of Physical Models

Let $\mathbb{R}$ be the set of all real numbers. For any two sets A and B, let $A \times B = \{\, (a, b) \mid a \in A \;\&\; b \in B \,\}$ be the Cartesian product of A with B. We write A^k to denote the set formed by taking the Cartesian product of A with itself k many times. For example, $A^3 = (A \times A) \times A$. As with Cantor's pairing function, (a, b, c) is an abbreviation for $((a, b), c)$, and so on. Similarly, we define the *Cartesian projection function* ϖ_i^n so that $\varpi_i^n(x_1, x_2, \ldots, x_n) = x_i$ for each positive integer $i \leq n$.

A physical model (S, A) with finitely many observable quantities is said to be in *normal form* if and only if $S \subseteq \mathbb{R}^n$ and $A = \{\varpi_1^n, \varpi_2^n, \ldots, \varpi_n^n\}$.

Theorem 2.13.1. *The following two conditions hold for any physical model (S, A) with finitely many observable quantities.*

1. *(S, A) is observationally equivalent to a physical model in normal form.*
2. *(S, A) is isomorphic to a physical model in normal form if and only if (S, A) is a reduced physical model.*

Proof. Begin by noting that if (T, B) is a physical model in normal form, and if $t_1 \neq t_2$ for any $t_1 \in T$ and $t_2 \in T$, then $\varpi_i^n(t_1) \neq \varpi_i^n(t_2)$ for some positive integer $i \leq n$. Therefore, by definition, every physical model in normal form is a reduced physical model. It immediately follows from Theorem 2.10.7 that if a physical model (S, A) is isomorphic to a physical model in normal form, then (S, A) is a reduced physical model.

To prove condition 1, suppose that (S, A) is a physical model such that $A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Define

$$T = \big\{\, \big(\alpha_1(s), \alpha_2(s), \ldots, \alpha_n(s)\big) \;\big|\; s \in S \,\big\}$$

and let $B = \{\varpi_1^n, \varpi_2^n, \ldots, \varpi_n^n\}$. Note that (T, B) is a physical model in normal form. Also note that the function $\phi : S \to T$ given by

$$\phi(s) = \big(\alpha_1(s), \alpha_2(s), \ldots, \alpha_n(s)\big)$$

is a surjection, and that $\alpha_i(s) = \varpi_i^n\big(\phi(s)\big)$ for all $s \in S$ and all positive integers $i \leq n$. Therefore, there is an epimorphism from (S, A) to the reduced physical model (T, B). Trivially, there is also an epimorphism from (T, B) to itself. We may conclude that (S, A) is observationally equivalent to (T, B).

To prove condition 2, consider the special case where (S, A) is a reduced physical model. Because (S, A) is reduced, we have that if $s_1 \neq s_2$ for any $s_1 \in S$ and $s_2 \in S$, then there exists a positive integer $i \leq n$ such that $\alpha_i(s_1) \neq \alpha_i(s_2)$. This implies that if $s_1 \neq s_2$ then $\phi(s_1) \neq \phi(s_2)$. Hence, ϕ is an injection. Since ϕ is also a surjection, ϕ is a bijection. Therefore, if (S, A) is a reduced physical model, then (S, A) and (T, B) are isomorphic. We have already proved the converse, that if (S, A) is isomorphic to a physical model in normal form, then (S, A) is a reduced physical model. Hence, condition 2 holds. □

The notion of a basic representation of a set can be generalized so that we may speak of basic representations of physical models in normal form. Given a physical model (S, A) in normal form with $A = \{\varpi_1^n, \varpi_2^n, \ldots, \varpi_n^n\}$, we may choose sets X_1, X_2, … , X_n such that $\varpi_i^n(S) \subseteq X_i \subseteq \mathbb{R}$ for each positive integer $i \leq n$, and we may impose effective topologies $(\mathcal{B}_1, \nu_1)$, $(\mathcal{B}_2, \nu_2)$, … , $(\mathcal{B}_n, \nu_n)$ on these sets.[16] Define

$$(\mathcal{B}_1, \nu_1) \otimes (\mathcal{B}_2, \nu_2) \otimes \cdots \otimes (\mathcal{B}_n, \nu_n)$$

to be the effective topology with basis $\mathcal{B}$ such that

$$\mathcal{B} = \{\, B_1 \times B_2 \times \cdots \times B_n \mid B_1 \in \mathcal{B}_1 \;\&\; B_2 \in \mathcal{B}_2 \;\&\; \cdots \;\&\; B_n \in \mathcal{B}_n \,\}$$

and with coding ν such that

$$\nu\langle a_1, a_2, \ldots, a_n\rangle = \nu_1(a_1) \times \nu_2(a_2) \times \cdots \times \nu_n(a_n)$$

We call $(\mathcal{B}, \nu)$ the *effective product* of $(\mathcal{B}_1, \nu_1)$, $(\mathcal{B}_2, \nu_2)$, … , $(\mathcal{B}_n, \nu_n)$. Note that S is a set of points in the effective topology $(\mathcal{B}, \nu)$. A physical model (R, H) is said to be a *basic representation* of the physical model (S, A) if R is a basic representation of S in the effective topology $(\mathcal{B}, \nu)$ and if H is the set $\{\pi_1^n, \pi_2^n, \ldots, \pi_n^n\}$ of projection functions.

For example, the non-discrete continuous computable physical model of planetary motion (Model 2.4.1) is a basic representation of the simple model of planetary motion (Model 2.1.1). In particular, Model 2.4.1 is obtained by imposing the effective topology $(\mathcal{I}_{10,c}, \iota)$ on the time in Model 2.1.1, where $c = \frac{1}{10}$, and by imposing the effective topology described in Footnote 4 on the angular position in Model 2.1.1. The product of these topologies is the topology for the surface of a cylinder. The states of Model 2.1.1 are a spiral path on the surface of that cylinder, and the set of states of Model 2.4.1 is a basic representation of the path.

[16] If (S, A) is faithful, then the bases for these topologies are uniquely determined by the physical operations used to measure each of the observable quantities. For example, if an observable quantity is an angle measurement, then the corresponding topology is the topology of a circle, and each basis element corresponds to a particular reading on the instrument that is used to measure angles. The idea that basis elements correspond to the values of measurements appears to have originated with reference [49].

2.14 Data and Predictions

In order to make predictions, we are often interested in finding the set of all states of a physical model which could account for a given collection of simultaneous measurements. That is, given a physical model (S, A) with $A = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$, and given real numbers x_1, x_2, ... , x_k, we are interested in the set

$$P = \{\, s \in S \mid \alpha_1(s) = x_1 \;\&\; \alpha_2(s) = x_2 \;\&\; \cdots \;\&\; \alpha_k(s) = x_k \,\}$$

In this context, the real numbers $x_1, x_2, \ldots, x_k$ are said to be the *data*, and P is the corresponding set of states *predicted* by the model.

The following theorem shows that if we are given a basic representation of a physical model (S, A) in normal form, together with complete oracles for the real numbers x_1, x_2, ... , x_k, then there is an effective procedure for finding a basic representation of the set P, provided that the underlying topology is T_1. (This is rather weak requirement, since almost all topologies with practical applications in the sciences are T_1.)

Theorem 2.14.1. *Let (S, A) be a physical model in normal form with $A = \{\varpi_1^n, \varpi_2^n, \ldots, \varpi_n^n\}$ and let (R, H) be a basic representation of (S, A) in a T_1 effective topology*

$$(\mathcal{B}, \nu) = (\mathcal{B}_1, \nu_1) \otimes (\mathcal{B}_2, \nu_2) \otimes \cdots \otimes (\mathcal{B}_n, \nu_n)$$

If $k \leq n$ and ϕ_1, ϕ_2, ... , ϕ_k are complete oracles for x_1, x_2, ... , x_k in the effective topologies $(\mathcal{B}_1, \nu_1)$, $(\mathcal{B}_2, \nu_2)$, ... , $(\mathcal{B}_k, \nu_k)$, then there is a basic representation of

$$P = \{\, s \in S \mid \varpi_1^n(s) = x_1 \;\&\; \varpi_2^n(s) = x_2 \;\&\; \cdots \;\&\; \varpi_k^n(s) = x_k \,\}$$

in $(\mathcal{B}, \nu)$ that is recursively enumerable relative to R, ϕ_1, ϕ_2, ... , ϕ_k uniformly.

Proof. Let the variables be defined as in the statement of the theorem and note that the set

$$Q = \big\{\, r \in R \;\big|\; \big(\forall\, i \in \{1, 2, \ldots, k\}\big)\big(\exists\, m \in \mathbb{N}\big)\big[\pi_i^n(r) = \phi_i(m)\big] \,\big\}$$

is recursively enumerable relative to R, ϕ_1, ϕ_2, ... , ϕ_k uniformly. (In fact, Q is recursively enumerable relative to ξ, ϕ_1, ϕ_2, ... , ϕ_k uniformly, where ξ is a function that merely enumerates the members of R.) We claim that Q is a basic representation of P in $(\mathcal{B}, \nu)$. Since $Q \subseteq R \subseteq \mathrm{dom}_{\mathcal{B}}\, \nu$, it suffices to prove that $s \in P$ if and only if there exists a local basis $\mathcal{L}_s$ for s with $\mathcal{L}_s \subseteq \nu(Q)$. Or equivalently, it suffices to prove that $s \in P$ if and only if there exists an oracle ψ for s with $\psi(\mathbb{N}) \subseteq Q$.

Suppose $s \in P$. Because R is a basic representation of S in $(\mathcal{B}, \nu)$, there is an oracle ψ for s in $(\mathcal{B}, \nu)$ such that $\psi(\mathbb{N}) \subseteq R$. Moreover, for each positive integer $i \leq k$, the set $\nu_i\big(\pi_i^n\big(\psi(\mathbb{N})\big)\big)$ is a local basis for $\varpi_i^n(s) = x_i$. And since ϕ_i is a complete oracle for x_i, we have that $\pi_i^n\big(\psi(\mathbb{N})\big) \subseteq \phi_i(\mathbb{N})$. Hence, if $r = \psi(l)$ for some $l \in \mathbb{N}$, then there exists an $m \in \mathbb{N}$ such that $\pi_i^n(r) = \phi_i(m)$. Therefore, by the definition of Q, $\psi(\mathbb{N}) \subseteq Q$.

Conversely, suppose that ψ is an oracle for some point s in $(\mathcal{B}, \nu)$, and that $\psi(\mathbb{N}) \subseteq Q$. Then, for each positive integer $i \leq k$, we have that $\pi_i^n\big(\psi(\mathbb{N})\big) \subseteq \phi_i(\mathbb{N})$. Of course, $\nu_i\big(\pi_i^n\big(\psi(\mathbb{N})\big)\big)$ is a local basis for $\varpi_i^n(s)$ because $\nu\big(\psi(\mathbb{N})\big)$ is a local basis for s. And by the definition of ϕ_i,

$\nu_i\big(\phi_i(\mathbb{N})\big)$ is a local basis for x_i. Hence, a local basis for $\varpi_i^n(s)$ is a subset of a local basis for x_i in the effective topology $(\mathcal{B}_i, \nu_i)$. But because $(\mathcal{B}, \nu)$ is a T_1 effective topology, $(\mathcal{B}_i, \nu_i)$ is also T_1. In a T_1 topology, local bases for any two distinct points z_1 and z_2 must contain basis elements B_1 and B_2, respectively, such that $z_2 \notin B_1$ and $z_1 \notin B_2$. Therefore, since a local basis for $\varpi_i^n(s)$ is a subset of a local basis for x_i, it must be the case that $\varpi_i^n(s) = x_i$. We may conclude, by the definition of P, that $s \in P$. $\square$

A set $S \subseteq \mathbb{R}^n$ is said to be the *graph* of a function $\psi : \mathbb{R}^k \to \mathbb{R}^{n-k}$, if and only if

$$S = \{\, (x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) \in \mathbb{R}^n \mid \psi(x_1, \ldots, x_k) = (x_{k+1}, \ldots, x_n) \,\}$$

And we say that a physical model (S, A) is *induced* by a function $\psi : \mathbb{R}^k \to \mathbb{R}^{n-k}$ if and only if (S, A) is in normal form and S is the graph of ψ. Therefore, if (S, A) is induced by $\psi : \mathbb{R}^k \to \mathbb{R}^{n-k}$ and we are given k real numbers x_1, x_2, ... , x_k as data, then the corresponding set of states predicted by (S, A) is a singleton set $P = \{s\}$. Namely,

$$s = (x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n)$$

where x_{k+1}, x_{k+2}, ... , x_n are the real numbers uniquely determined by the equation

$$\psi(x_1, x_2, \ldots, x_k) = (x_{k+1}, x_{k+2}, \ldots, x_n)$$

It then follows from Theorem 2.14.1 that given complete oracles for x_1, x_2, ... , x_k in the standard topology of $\mathbb{R}$, and given a basic representation R of (S, A) in the standard topology of $\mathbb{R}^n$, there is an effective procedure (relative to the given oracles and R) for finding a basic representation of $\{s\}$. But by Theorem 2.12.3 there is, in general, no effective procedure for finding an *oracle* for s. In the next section we describe a special class of basic representations for which such an effective procedure does exist.

2.15 Kreisel's Criterion

A common way to interpret Kreisel's criterion is to say that a physical model (S, A) satisfies Kreisel's criterion on $\mathbb{R}^k$ if and only if (S, A) is induced by a function $\psi : \mathbb{R}^k \to \mathbb{R}^{n-k}$ for some positive integer $n > k$, and for each positive integer $j \leq n - k$ there is a partial recursive function κ_j such that if ϕ_1, ϕ_2, ... , ϕ_k are nested oracles for real numbers x_1, x_2, ... , x_k in the effective topology $(\mathcal{I}, \iota)$, then $\kappa_j(\phi_1, \phi_2, \ldots, \phi_k, m)$ is defined for all $m \in \mathbb{N}$ and $\lambda m\big[\kappa_j(\phi_1, \phi_2, \ldots, \phi_k, m)\big]$ is a nested oracle for $\varpi_j^{n-k}\big(\psi(x_1, x_2, \ldots, x_k)\big)$ in $(\mathcal{I}, \iota)$. Note by Theorem 2.11.4 that the effective topology $(\mathcal{I}, \iota)$ in this statement can be replaced, without loss of generality, with any effective topology $(\mathcal{B}, \iota)$ that has a recursively enumerable subset relation and such that $\mathcal{B} \subseteq \mathcal{I}$ is a basis for the standard topology of the real numbers.

Practical computer models that use multiple-precision interval arithmetic [30] provide examples of physical models satisfying Kreisel's criterion. Typically, such models are induced by a function $\psi : \mathbb{R}^k \to \mathbb{R}^{n-k}$ where, for each positive integer $j \leq n - k$, there is a recursive function ξ_j such that if the data x_1, x_2, ... , x_k lie within the intervals $(a_1\,; b_1)$, $(a_2\,; b_2)$, ... , $(a_k\,; b_k)$ respectively, then $\varpi_j^{n-k}\big(\psi(x_1, x_2, \ldots, x_k)\big)$ lies within the interval

$$\xi_j\big((a_1\,; b_1), (a_2\,; b_2), \ldots, (a_k\,; b_k)\big)$$

In such a case, the partial recursive function κ_j in Kreisel's criterion is given by

$$\kappa_j(\phi_1, \phi_2, \ldots, \phi_k, m) = \xi_j\big(\phi_1(m), \phi_2(m), \ldots, \phi_k(m)\big)$$

We are now prepared to state the following theorem, which holds uniformly.

Theorem 2.15.1. *If a physical model satisfies Kreisel's criterion on $\mathbb{R}^k$, then the model has a basic representation that is isomorphic to a computable physical model.*

Proof. Suppose that (S, A) is a physical model satisfying Kreisel's criterion on $\mathbb{R}^k$. In particular, suppose that (S, A) is induced by a function $\psi : \mathbb{R}^k \to \mathbb{R}^{n-k}$, and for each positive integer $j \leq n - k$ suppose there is a partial recursive function κ_j such that if ϕ_1, ϕ_2, ... , ϕ_k are nested oracles for real numbers x_1, x_2, ... , x_k in the effective topology $(\mathcal{I}_{10,c}\,, \iota)$, then $\kappa_j(\phi_1, \phi_2, \ldots, \phi_k, m)$ is defined for all $m \in \mathbb{N}$ and $\lambda m\big[\kappa_j(\phi_1, \phi_2, \ldots, \phi_k, m)\big]$ is a nested oracle for $\varpi_j^{n-k}\big(\psi(x_1, x_2, \ldots, x_k)\big)$ in $(\mathcal{I}_{10,c}\,, \iota)$, where c is a positive rational number.

Let $I = \mathrm{dom}_{\mathcal{I}_{10,c}}\, \iota$. Then, for each interval $u \in I$ with midpoint p and with d digits of precision, define the partial recursive function σ_u so that

$$\sigma_u(l) = \begin{cases} o_p(l) & \text{if } l < d \\ \text{undefined} & \text{if } l \geq d \end{cases}$$

for each $l \in \mathbb{N}$, where o_p is the standard decimal oracle described in Section 2.4. Note that for each $x \in \mathbb{R}$ and each $m \in \mathbb{N}$, $\sigma_{o_x(m)}(l) = o_x(l)$ for all non-negative integers $l \leq m$. Now, for each positive integer $j \leq n-k$, choose a program to compute κ_j and let $\kappa_j(\sigma_{u_1}, \sigma_{u_2}, \ldots, \sigma_{u_k}, m)$ be undefined if for some positive integer $i \leq k$ and some $l \in \mathbb{N}$ the program calls $\sigma_{u_i}(l)$ in the course of the computation and $\sigma_{u_i}(l)$ is undefined. Note that given u_1, u_2, ... , u_k, the set of all $m \in \mathbb{N}$ such that $\kappa_j(\sigma_{u_1}, \sigma_{u_2}, \ldots, \sigma_{u_k}, m)$ is defined is a recursively enumerable set, since for each m we can follow the computation and test whether or not $\sigma_{u_i}(l)$ is defined whenever $\sigma_{u_i}(l)$ is called by the program, for any i and l. Let R be the set of all $\langle u_1, u_2, \ldots, u_n\rangle$ such that $u_i \in I$ for each positive integer $i \leq k$, and such that $u_{k+j} = \kappa_j(\sigma_{u_1}, \sigma_{u_2}, \ldots, \sigma_{u_k}, m)$ for some $m \in \mathbb{N}$ if $j \leq n - k$ is a positive integer. Note that R is also recursively enumerable. Let $H = \{\pi_1^n, \pi_2^n, \ldots, \pi_n^n\}$. We claim that (R, H) is a basic representation of (S, A).

As a brief digression from the proof, suppose that $\langle u_1, u_2, \ldots, u_n\rangle \in R$ and note that for each positive integer $i \leq k$, if $x_i \in \iota(u_i)$ then there exists a nested oracle ϕ_i for x_i in $(\mathcal{I}_{10,c}\,, \iota)$ such that $\phi_i(m) = \sigma_{u_i}(m)$ for all $m \in \mathbb{N}$ where $\sigma_{u_i}(m)$ is defined. And since for each positive integer $j \leq n - k$ we have that $\lambda m\big[\kappa_j(\phi_1, \phi_2, \ldots, \phi_k, m)\big]$ is a nested oracle for $x_{k+j} = \varpi_j^{n-k}\big(\psi(x_1, x_2, \ldots, x_k)\big)$, it follows that

$$x_{k+j} \in \iota\big(\kappa_j(\phi_1, \phi_2, \ldots, \phi_k, m)\big) = \iota\big(\kappa_j(\sigma_{u_1}, \sigma_{u_2}, \ldots, \sigma_{u_k}, m)\big) = \iota(u_{k+j})$$

for some $m \in \mathbb{N}$. Hence, given any $\langle u_1, u_2, \ldots, u_n\rangle \in R$, if $x_i \in \iota(u_i)$ for each positive integer $i \leq k$, then $x_{k+j} = \varpi_j^{n-k}\big(\psi(x_1, x_2, \ldots, x_k)\big) \in \iota(u_{k+j})$ for each positive integer $j \leq n - k$.

Now, returning to the proof of Theorem 2.15.1, let

$$(\mathcal{B}, \nu) = \overbrace{(\mathcal{I}_{10,c}\,, \iota) \otimes (\mathcal{I}_{10,c}\,, \iota) \otimes \cdots \otimes (\mathcal{I}_{10,c}\,, \iota)}^{n \text{ factors}}$$

and note that the basis of $(\mathcal{B}, \nu)$ is a basis for the standard topology of $\mathbb{R}^n$. Note that because (S, A) satisfies Kreisel's criterion, the function ψ which induces (S, A) is continuous. And since continuous real functions have closed graphs, the set S is closed in the standard topology of $\mathbb{R}^n$. Because S is closed, to prove that R is a basic representation of S in $(\mathcal{B}, \nu)$, it suffices to show that for each $r \in R$ there exists an $x \in S$ with $x \in \nu(r)$, and that for each $x \in S$ there is a local basis $\mathcal{L}_x$ for x with $\mathcal{L}_x \subseteq \nu(R)$.

By the definition of R, if $r = \langle u_1, u_2, \ldots, u_n \rangle \in R$ then for each positive integer $j \leq n - k$,

$$u_{k+j} = \kappa_j(\sigma_{u_1}, \sigma_{u_2}, \ldots, \sigma_{u_k}, m)$$

for some $m \in \mathbb{N}$. So, if p_i is the midpoint of the interval u_i for each positive integer $i \leq k$, then

$$u_{k+j} = \kappa_j(o_{p_1}, o_{p_2}, \ldots, o_{p_k}, m)$$

And by Kreisel's criterion $\lambda m\big[\kappa_j(o_{p_1}, o_{p_2}, \ldots, o_{p_k}, m)\big]$ is an oracle for $\varpi_j^{n-k}\big(\psi(p_1, p_2, \ldots, p_k)\big)$. Therefore, for each $r \in R$ there exists an $(x_1, x_2, \ldots, x_n) \in S$ with $(x_1, x_2, \ldots, x_n) \in \nu(r)$. Namely, $x_i = p_i$ for each positive integer $i \leq k$ and

$$x_{k+j} = \varpi_j^{n-k}\big(\psi(p_1, p_2, \ldots, p_k)\big)$$

for each positive integer $j \leq n - k$.

Now suppose that $(x_1, x_2, \ldots, x_n)$ is an arbitrary member of S. Again, by Kreisel's criterion, for each positive integer $j \leq n - k$, the function $\lambda m\big[\kappa_j(o_{x_1}, o_{x_2}, \ldots, o_{x_k}, m)\big]$ is an oracle for x_{k+j}. But for each $m \in \mathbb{N}$ and each positive integer $i \leq k$, the computation for $\kappa_j(o_{x_1}, o_{x_2}, \ldots, o_{x_k}, m)$ has only finitely many steps, and so the oracle o_{x_i} can only be called finitely many times during the course of the computation. Hence, for each $m \in \mathbb{N}$ there exists a non-negative integer l_i for each $i \leq k$, such that for any non-negative integer $l_i' \geq l_i$, if $u_i = o_{x_i}(l_i')$ then $\kappa_j(\sigma_{u_1}, \sigma_{u_2}, \ldots, \sigma_{u_k}, m)$ is defined and

$$\kappa_j(\sigma_{u_1}, \sigma_{u_2}, \ldots, \sigma_{u_k}, m) = \kappa_j(o_{x_1}, o_{x_2}, \ldots, o_{x_k}, m)$$

Of course, for each positive integer $j \leq n - k$ the interval

$$u_{k+j} = \kappa_j(o_{x_1}, o_{x_2}, \ldots, o_{x_k}, m)$$

can be made arbitrarily small by choosing a suitably large value of m, and for each positive integer $i \leq k$ the interval u_i can be made arbitrarily small by choosing a suitably large value of l_i'. Furthermore, by definition, $\langle u_1, u_2, \ldots, u_n \rangle \in R$. It immediately follows that for each $x = (x_1, x_2, \ldots, x_n) \in S$ there is a local basis $\mathcal{L}_x$ for x such that $\mathcal{L}_x \subseteq \nu(R)$. We may conclude that (R, H) is a basic representation of (S, A). And since R is recursively enumerable, it follows from Theorem 2.10.5 that (R, H) is isomorphic to a computable physical model. $\square$

A physical model (S, A) that satisfies Kreisel's criterion on $\mathbb{R}^k$ is uniquely determined by the functions $\kappa_1, \kappa_2, \ldots, \kappa_{n-k}$. Moreover, the proof of Theorem 2.15.1 describes an effective procedure for finding a basic representation of (S, A), given programs for computing $\kappa_1, \kappa_2, \ldots, \kappa_{n-k}$. Let $\mathcal{K}_{k,n,c}$ be the collection of all basic representations of physical models that are constructed

from physical models satisfying Kreisel's criterion according to the procedure in the proof of Theorem 2.15.1, where c is the positive rational number which appears in that proof. An immediate question is whether there exists an effective procedure for the inverse operation. That is, given a basic representation in $\mathcal{K}_{k,n,c}$, is there an effective procedure for constructing partial recursive functions $\kappa_1, \kappa_2, \ldots, \kappa_{n-k}$? In the proof of the following theorem, we show that the answer is "Yes." Therefore, for every physical model satisfying Kreisel's criterion on $\mathbb{R}^k$, there is a computable physical model that may be used in its place, to predict the values of observable quantities given the data.

Theorem 2.15.2. *If $(R, H) \in \mathcal{K}_{k,n,c}$ and if $\psi : \mathbb{R}^k \to \mathbb{R}^{n-k}$ is the function whose graph has basic representation R, then there exist partial recursive functions $\kappa_1, \kappa_2, \ldots, \kappa_{n-k}$ such that if $\phi_1, \phi_2, \ldots, \phi_k$ are nested oracles for real numbers $x_1, x_2, \ldots, x_k$ in the effective topology $(\mathcal{I}_{10,c}, \iota)$, then for each positive integer $j \leq n-k$, $\kappa_j(\phi_1, \phi_2, \ldots, \phi_k, m)$ is defined for all $m \in \mathbb{N}$ and $\lambda m\big[\kappa_j(\phi_1, \phi_2, \ldots, \phi_k, m)\big]$ is a nested oracle for $\varpi_j^{n-k}\big(\psi(x_1, x_2, \ldots, x_k)\big)$ in $(\mathcal{I}_{10,c}, \iota)$.*

Proof. Suppose that $(R, H) \in \mathcal{K}_{k,n,c}$ and that $\psi : \mathbb{R}^k \to \mathbb{R}^{n-k}$ is the function whose graph has basic representation R. Note by the proof of Theorem 2.15.1 that R is recursively enumerable. Now, given any oracles $\phi_1, \phi_2, \ldots, \phi_k$ for real numbers $x_1, x_2, \ldots, x_k$ in $(\mathcal{I}_{10,c}, \iota)$, it follows from Theorem 2.11.2 that there are complete oracles $\phi_1', \phi_2', \ldots, \phi_k'$ for $x_1, x_2, \ldots, x_k$ in $(\mathcal{I}_{10,c}, \iota)$, such that $\phi_1', \phi_2', \ldots, \phi_k'$ are recursive relative to $\phi_1, \phi_2, \ldots, \phi_k$ uniformly. Let x_{k+1}, $x_{k+2}, \ldots, x_n$ be the real numbers uniquely determined by the equation

$$\psi(x_1, x_2, \ldots, x_k) = (x_{k+1}, x_{k+2}, \ldots, x_n)$$

Then by the proof of Theorem 2.14.1,

$$Q = \big\{\, \langle u_1, u_2, \ldots, u_n\rangle \in R \;\big|\; \big(\forall\, i \in \{1, 2, \ldots, k\}\big)\big(\exists\, m \in \mathbb{N}\big)\big[u_i = \phi_i'(m)\big] \,\big\}$$

is a basic representation of $\big\{(x_1, x_2, \ldots, x_n)\big\}$ in

$$(\mathcal{B}, \nu) = \overbrace{(\mathcal{I}_{10,c}, \iota) \otimes (\mathcal{I}_{10,c}, \iota) \otimes \cdots \otimes (\mathcal{I}_{10,c}, \iota)}^{n \text{ factors}}$$

And since R is recursively enumerable, the set Q is recursively enumerable relative $\phi_1, \phi_2, \ldots, \phi_k$ uniformly.

Now, since $x_i \in \iota\big(\phi_i(m)\big)$ for each positive integer $i \leq k$, it follows from the definition of Q that $x_i \in \iota(u_i)$ for each $\langle u_1, u_2, \ldots, u_n\rangle \in Q$. But recall from the proof of Theorem 2.15.1 that R has the property that if $x_i \in \iota(u_i)$ for each positive integer $i \leq k$, then $x_j \in \iota(u_{k+j})$ for each positive integer $j \leq n-k$. Hence,

$$(x_1, x_2, \ldots, x_n) \in \nu\langle u_1, u_2, \ldots, u_n\rangle$$

for each $\langle u_1, u_2, \ldots, u_n\rangle \in Q$. It immediately follows from the definition of a basic representation that $\nu(Q)$ is a local basis for the point $(x_1, x_2, \ldots, x_n)$. Therefore, for any function $\kappa : \mathbb{N} \to R$ such that $\kappa(\mathbb{N}) = Q$, the function κ is an oracle for $(x_1, x_2, \ldots, x_n)$ in $(\mathcal{B}, \nu)$. And

since Q is recursively enumerable relative to ϕ_1, ϕ_2, $\ldots$, ϕ_k uniformly, κ is recursive relative to ϕ_1, ϕ_2, $\ldots$, ϕ_k uniformly. So, if we define

$$\kappa_j(\phi_1, \phi_2, \ldots, \phi_k, m) = \pi^n_{k+j}\big(\kappa(m)\big)$$

for each positive integer $j \leq n-k$ and for each $m \in \mathbb{N}$, then κ_j is partial recursive, $\kappa_j(\phi_1, \phi_2, \ldots, \phi_k, m)$ is defined for all $m \in \mathbb{N}$, and $\lambda m\big[\kappa_j(\phi_1, \phi_2, \ldots, \phi_k, m)\big]$ is a nested oracle for

$$\varpi^{n-k}_j\big(\psi(x_1, x_2, \ldots, x_k)\big) = x_{k+j}$$

in $(\mathcal{I}_{10,c}\,, \iota)$. $\square$

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