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Isocategories and Tensor Functors

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Isocategories and Tensor Functors

by Walter Noll, 1992

ABSTRACT. In this paper, I show how the concepts of an *isocategory* (category all of whose morphisms are isomorphisms) and the corresponding concept of an *isofunctor* can be used to improve the conceptural infrastructure of many branches of mathematics. The crucial new idea is that of a *natural assignment*, a modification of the idea of a natural tansformation introduced by Eilenberg and Mac Lane. Isofunctors that involve the isocategory LIS of all linear isomorphisms of finite-dimensional linear spaces are called *tensor functors*, because they can be used to clarify most uses of the word "tensor" in the literature of mathematics and physics. Of particular importance are the "*analytic tensor functors*", which can serve to simplify and generalize the treatment of tensor fields given in future textbooks on differentiable manifolds.

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Introduction

The term "tensor" has been used in mathematics and in physics for nearly 150 years. We know about "strain tensors", "stress tensors", "elasticity tensors", "electromagnetic field tensors", "energy-momentum tensors", "Riemann curvature tensors", "Ricci tensors", etc. The meaning assigned to the term "tensor" in the literature, howerever, is variable and the definitions given are often muddled. The distinction between a "tensor" and a "tensor field" is often blurred, at least in terminology. At any rate, I do not know of any precise definition in the literature of a concept that could be used to cover all or at least most of the uses of the the word "tensor" in the mathematical and physical literature. One of the motivations of this paper was to introduce such a precisely defined concept, using some of the conceptual machinery of category theory. The concept in question is that of a **tensor functor**, first used in [N] in 1984, and defined here in Sect. 4. Recently, I found that the ideas leading to the concept of a tensor functor might have wider applications and might actually be used to improve the conceptual infrastructure of mathematics in general. The basic idea is to use categories whose morphisms are all isomorphisms, or **isocategories** for short. The definitions given in Sect. 1, 2 and 3 are consistent with the ones used in the literature on category theory, although this literature deals primarily with categories that are not isocategories. However, the concept of **natural assignment** introduced in Sect. 3 is a modification of the concept of a "natural transformation" originally introduced by Eilenberg and Mac Lane (see, for example, [M], Sect. 4 of Ch.I). These authors introduced the notion of a category precisely because they wanted to clarify what "natural" means in the context of mathematical constructions. (Eilenberg confirmed this to me in a conversation in Tbilisi in November 1989.) I believe that my concept of a natural assignment will lead to additional clarification. The concept of a **tensor product** is understood here as a natural assignment (see Sect.5, (7) and Remark 2, and Sect.7). I introduced the concept of an **analytic tensor functor**, defined in Sect. 5 here, in an introductory graduate course on abstract differential geometry given in 1984 [N]. The purpose was to simplify and generalize the treatment of tensor-fields given in standard texbooks such as [K-N] and [M-T-W] (see the Remark in Sect. 4 below). Here, in Sect. 7, I indicate on how one can do this and how one can apply the concept of an analytic tensor functor to the theory of linear-space bundles, as has been carried out in detail recently by my doctoral student Sea-Mean Chiou [CS]. (He use simply "tensor functor" for what we call "analytic tensor functor" here.) Sect. 7 also contains brief descriptions of three different modifications of the ideas presented earlier. The notation and terminology of [FDS] is used in this paper. In particular, $\mathbb{N}$ denotes the set of all natural numbers including zero. Given $n \in \mathbb{N}$, the set of all natural numbers between 1 and n (inclusive) is denoted by $n^{\rfloor}$. The set of all subsets (a.k.a. "power set") of a given set $\mathcal{S}$ is denoted by $\mathrm{Sub}(\mathcal{S})$. The domain and codomain of a given mapping ϕ are denoted by $\mathrm{Dom}\,\phi$ and $\mathrm{Cod}\,\phi$, respectively. The inverse of an invertible mapping ϕ is denoted by $\phi^{\leftarrow}$. Given any mappings ϕ and ψ, $\phi \times \psi$

denotes the mapping from $\operatorname{Dom}\phi \times \operatorname{Dom}\psi$ to $\operatorname{Cod}\phi \times \operatorname{Cod}\psi$ defined by

$$(\phi \times \psi)(x, y) := (\phi(x), \psi(y)) \quad \text{for all} \quad x \in \operatorname{Dom}\phi \text{ and } y \in \operatorname{Dom}\psi.$$

Let an index set I be given. For every set $\mathcal{S}$, we denote by $\mathcal{S}^I$ the set of all families with terms in $\mathcal{S}$ and indexed on I. Given $a \in \mathcal{S}^I$, we denote the term of a with index $i \in I$ by a_i, and we often write $a = (a_i \,|\, i \in I\,)$. For every mapping ϕ, we denote by $\phi^{\times I}$ the mapping from $(\operatorname{Dom}\phi)^I$ to $(\operatorname{Cod}\phi)^I$ defined by

$$\phi^{\times I}(a_i \,|\, i \in I\,) := (\,\phi(a_i) \,|\, i \in I\,) \quad \text{for all} \quad a \in (\operatorname{Dom}\phi)^I.$$

We abbreviate $\mathcal{S}^n := \mathcal{S}^{n^]}$ and $\phi^{\times n} := \phi^{\times n^]}$ when $n \in \mathbb{N}$; members of $\mathcal{S}^n$ are called *lists of length* n. For *linear* mappings, evaluation is generally understood without parentheses, composition is understood without $\circ$, and $^{-1}$ is used instead of $^{\leftarrow}$ to denote inverses. We use "lineon" as an abbreviation for "linear transformation" in order to be able to form the adjective "lineonic".

1. Isocategories

An **isocategory** is given by the specification of

(i) a class OBJ whose members are called **objects**,

(ii) a class ISO whose members are called **isomorphisms**,

(iii) a rule that associates with each $\phi \in \mathrm{ISO}$ a pair $(\operatorname{Dom}\phi, \operatorname{Cod}\phi)$ of objects, called the **domain** and **codomain** of ϕ,

(iv) a rule that associates with each pair (ϕ, ψ) in ISO such that $\operatorname{Cod}\phi = \operatorname{Dom}\psi$ a member of ISO denoted by $\psi \circ \phi$ and called the **composite** of ϕ and ψ, with $\operatorname{Dom}(\psi \circ \phi) = \operatorname{Dom}\phi$ and $\operatorname{Cod}(\psi \circ \phi) = \operatorname{Cod}\psi$,

(v) a rule that associates with each $\mathcal{A} \in OBJ$ a member of ISO denoted by $1_{\mathcal{A}}$ and called the **identity** of $\mathcal{A}$,

(vi) a rule that associates with each $\phi \in \mathrm{ISO}$ a member of ISO denoted by $\phi^{\leftarrow}$ and called the **inverse** of ϕ.

The ingredients of an isocategory described above are subject to the following three axioms.

(I1) We have

$$\chi \circ (\psi \circ \phi) = (\chi \circ \psi) \circ \phi$$

for all $\phi, \psi, \chi \in \mathrm{ISO}$ such that $\operatorname{Cod}\phi = \operatorname{Dom}\psi$ and $\operatorname{Cod}\psi = \operatorname{Dom}\chi$.

(I2) We have

$$\phi \circ 1_{\operatorname{Dom}\phi} = \phi = 1_{\operatorname{Cod}\phi} \circ \phi$$

for all $\phi \in \mathrm{ISO}$.

(I3) we have

$$\phi^{\leftarrow} \circ \phi = 1_{\mathrm{Dom}\,\phi} \quad \text{and} \quad \phi \circ \phi^{\leftarrow} = 1_{\mathrm{Cod}\,\phi}$$

for all $\phi \in \mathrm{ISO}$.

Given $\phi \in \mathrm{ISO}$, one writes $\phi : \mathcal{A} \to \mathcal{B}$ or $\mathcal{A} \xrightarrow{\phi} \mathcal{B}$ to indicate that $\mathrm{Dom}\,\phi = \mathcal{A}$ and $\mathrm{Cod}\,\phi = \mathcal{B}$.

The class OBJ of an isocategory is determined by the class ISO because every $\mathcal{A} \in OBJ$ is determined by the corresponding identity $1_{\mathcal{A}}$. For this reason, we will usually name an isocategory by giving the name of its class of isomorphisms.

Let isocategories ISO and ISO$'$ with object-classes OBJ and OBJ' be given. We can then form the **product-isocategory** $\mathrm{ISO} \times \mathrm{ISO}'$ with object-class $OBJ \times OBJ'$ as follows:

(a) $\mathrm{ISO} \times \mathrm{ISO}'$ consists of pairs (ϕ, ϕ') with $\phi \in \mathrm{ISO}$, $\phi' \in \mathrm{ISO}'$.

(b) $OBJ \times OBJ'$ consists of pairs $(\mathcal{A}, \mathcal{A}')$ with $\mathcal{A} \in OBJ$, $\mathcal{A}' \in OBJ'$.

(c) For every $(\phi, \phi') \in \mathrm{ISO} \times \mathrm{ISO}'$, we put

$$\mathrm{Dom}\,(\phi, \phi') := (\mathrm{Dom}\,\phi, \mathrm{Dom}\,\phi') \quad , \quad \mathrm{Cod}\,(\phi, \phi') := (\mathrm{Cod}\,\phi, \mathrm{Cod}\,\phi').$$

(d) Composition in $\mathrm{ISO} \times \mathrm{ISO}'$ is defined by termwise composition, i.e. by

$$(\psi, \psi') \circ (\phi, \phi') := (\,\psi \circ \phi\,,\ \psi' \circ \phi'\,)$$

for all $\phi, \psi \in \mathrm{ISO}$ and $\phi', \psi' \in \mathrm{ISO}'$ such that $\mathrm{Dom}\,(\psi, \psi') = \mathrm{Cod}\,(\phi, \phi')$.

(e) The identity of a given pair $(\mathcal{A}, \mathcal{A}') \in OBJ \times OBJ'$ is defined to be

$$1_{(\mathcal{A}, \mathcal{A}')} = (1_{\mathcal{A}}, 1_{\mathcal{A}'}).$$

(f) The inverse of a given pair $(\phi, \phi') \in \mathrm{ISO} \times \mathrm{ISO}'$ is defined to be

$$(\phi, \phi')^{\leftarrow} := (\phi^{\leftarrow}, \phi'^{\leftarrow}).$$

The product of an arbitary family of isocategories can be defined in a similar manner. In particular, if an isocategory ISO and an index set I are given, one can form the I-**power-isocategory** ISO^I of ISO; its isomorphism-class consists of all families in ISO indexed on I. In the case when I is of the form $I := n^{]}$, we write $\mathrm{ISO}^n := \mathrm{ISO}^{n^{]}}$ for short. For example, we write $\mathrm{ISO}^2 := \mathrm{ISO} \times \mathrm{ISO}$. We identify ISO^1 with ISO and ISO^{m+n} with $\mathrm{ISO}^m \times \mathrm{ISO}^n$ for all $m, n \in \mathbb{N}$ in the obvious manner. The isocategory ISO^0 is the trival one whose only object is $\emptyset$ and whose only isomorphism is $1_{\emptyset}$.

We say that an isocategory ISO is **concrete** if ISO consists of invertible mappings, the object-class OBJ consists of sets, and if domain and codomain, composition, identity and inverse have the meanning they are usually given for sets and mappings. (See, e.g. Sect. 01-04 of [FDS]). In this paper, we will assume that a basic concrete isocategory ISO is given, and we will deal only with it and the isocategories obtained from it by product formation, such as $\mathrm{ISO}^m \times \mathrm{ISO}^n$ when $m, n \in \mathbb{N}$.

2. Isofunctors

An **isofunctor** Φ is given by the specification of:

(i) a pair $(\mathrm{DOM}\,\Phi, \mathrm{COD}\,\Phi)$ of isocategories, called the **domain-category** and **codomain-category** of Φ,

(ii) a rule that associates with every $\phi \in \mathrm{DOM}\,\Phi$ a member of $\mathrm{COD}\,\Phi$ denoted by $\Phi(\phi)$,

subject to the following conditions:

(F1) We have

$$\mathrm{Cod}\,\Phi(\phi) = \mathrm{Dom}\,\Phi(\psi) \quad \text{and} \quad \Phi(\psi \circ \phi) = \Phi(\psi) \circ \Phi(\phi) \tag{2.1}$$

for all $\phi, \psi \in \mathrm{DOM}\,\Phi$ such that $\mathrm{Cod}\,\phi = \mathrm{Dom}\,\psi$.

(F2) For every identity $1_{\mathcal{A}}$ in $\mathrm{DOM}\,\Phi$, where $\mathcal{A}$ belongs to the object-class of $\mathrm{DOM}\,\Phi$, $\Phi(1_{\mathcal{A}})$ is an identity in $\mathrm{COD}\,\Phi$.

Let isocategories ISO and ISO' with object-classes OBJ and OBJ' be given. We say that Φ is an **isofunctor from** ISO **to** ISO' and we write $\Phi : \mathrm{ISO} \to \mathrm{ISO}'$ to indicate that $\mathrm{ISO} = \mathrm{DOM}\,\Phi$ and $\mathrm{ISO}' = \mathrm{COD}\,\Phi$. By (F2), we can associate with each $\mathcal{A} \in OBJ$ exactly one object in OBJ', denoted by $\Phi(\mathcal{A})$, such that

$$\Phi(1_{\mathcal{A}}) = 1_{\Phi(\mathcal{A})}. \tag{2.3}$$

It easily follows from (I3) of Sect. 1 and from (F1) and (F2) above that every isofunctor Φ satisfies

$$\Phi(\phi^{\leftarrow}) = \big(\Phi(\phi)\big)^{\leftarrow} \quad \text{for all} \quad \phi \in \mathrm{DOM}\,\Phi. \tag{2.4}$$

One can construct new isofunctors from given isofunctors in the same way as new mappings are constructed from given mappings. (See, for example, Sect. 03

and 04 of [FDS].) Thus, if Φ and Ψ are isofunctors such that $\mathrm{COD}\,\Phi = \mathrm{DOM}\,\Psi$, one can define the **composite isofunctor** $\Psi \circ \Phi : \mathrm{DOM}\,\Phi \to \mathrm{COD}\,\Psi$ by

$$(\Psi \circ \Phi)(\phi) := \Psi(\Phi(\phi)) \quad \text{for all} \quad \phi \in \mathrm{DOM}\,\Phi \tag{2.5}$$

Also, given isofunctors Φ and Ψ, one can define the **product-isofunctor**

$$\Phi \times \Psi : \ \mathrm{DOM}\,\Phi \times \mathrm{DOM}\,\Psi \longrightarrow \mathrm{COD}\,\Phi \times \mathrm{COD}\,\Psi$$

of Φ and Ψ by

$$(\Phi \times \Psi)(\phi, \psi) := (\Phi(\phi), \Psi(\psi)) \tag{2.6}$$

for all $\phi \in \mathrm{DOM}\,\Phi$ and all $\psi \in \mathrm{DOM}\,\Psi$. Product-isofunctors of arbitary families of isofunctors are defined in a similar way. In particular, if an isofunctor Φ and an index set I are given, we define the I-**power-isofunctor** $\Phi^{\times I} : (\mathrm{DOM}\,\Phi)^I \to (\mathrm{COD}\,\Phi)^I$ of Φ by

$$\Phi^{\times I}(\phi_i \,|\, i \in I\,) = (\Phi(\phi_i) \,|\, i \in I\,) \tag{2.7}$$

for all families $(\phi_i \,|\, i \in I\,)$ in $\mathrm{DOM}\,\Phi$. We write $\Phi^{\times n} := \Phi^{\times n^]}$ when $n \in \mathbb{N}$.

We now assume that an isocategory ISO with object-class OBJ is given. The **identity-functor** $\mathrm{Id} : \mathrm{ISO} \to \mathrm{ISO}$ of ISO is defined by

$$\mathrm{Id}(\phi) = \phi \quad \text{for all} \quad \phi \in \mathrm{ISO}. \tag{2.8}$$

We then have

$$\mathrm{Id}(\mathcal{A}) = \mathcal{A} \quad \text{for all} \quad \mathcal{A} \in OBJ. \tag{2.9}$$

If I is an index set, then the identity-functor of ISO^I is $\mathrm{Id}^{\times I}$. In particular, the identity-functor of $\mathrm{ISO} \times \mathrm{ISO}$ is $\mathrm{Id} \times \mathrm{Id}$.

Let a specific object $\mathcal{C} \in OBJ$ be given. The **trivial-functor** $\mathrm{Tr}_{\mathcal{C}} : \mathrm{ISO} \to \mathrm{ISO}$ **for** $\mathcal{C}$ is defined by

$$\mathrm{Tr}_{\mathcal{C}}(\phi) = 1_{\mathcal{C}} \quad \text{for all} \quad \phi \in \mathrm{ISO}. \tag{2.10}$$

We then have

$$\mathrm{Tr}_{\mathcal{C}}(\mathcal{A}) = \mathcal{C} \quad \text{for all} \quad \mathcal{A} \in OBJ. \tag{2.11}$$

One often needs to consider a variety of "accounting isofunctors" whose domain and codomain categories are obtained from ISO by product formation. For example, the **switch-functor** $\mathrm{Sw} : \mathrm{ISO}^2 \to \mathrm{ISO}^2$ is defined by

$$\mathrm{Sw}(\phi, \psi) := (\psi, \phi) \quad \text{for all} \quad \phi, \psi \in \mathrm{ISO}. \tag{2.12}$$

Given any index set I, the **equalization-functor** $\mathrm{Eq}_I : \mathrm{ISO} \to \mathrm{ISO}^I$ is defined by

$$\mathrm{Eq}_I(\phi) := (\phi \,|\, i \in I\,) \quad \text{for all} \quad \phi \in \mathrm{ISO}. \tag{2.13}$$

We write $\mathrm{Eq}_n := \mathrm{Eq}_{n]}$ when $n \in \mathbb{N}$.

Let isofunctors Φ and Ψ with $\mathrm{Dom}\,\Phi = \mathrm{ISO} = \mathrm{Dom}\,\Psi$ be given. We then identify the pair (Φ, Ψ) with the **pair-formation functor**

$$(\Phi, \Psi) : \mathrm{ISO} \to \mathrm{COD}\,\Phi \times \mathrm{COD}\,\Psi$$

defined by

$$(\Phi, \Psi) := (\Phi \times \Psi) \circ \mathrm{Eq}_2,$$

so that

$$(\Phi, \Psi)(\phi) = (\Phi(\phi), \Psi(\phi)) \quad \text{for all} \quad \phi \in \mathrm{ISO}. \tag{2.14}$$

3. Natural assignments, examples.

We now assume that a *concrete* isocategory ISO with object-class OBJ is given.

A **natural assignment** α of **degree** $n \in \mathbb{N}$ is given by the specification of:

(i) a pair $(\mathrm{Dmf}_\alpha, \mathrm{Cdf}_\alpha)$ of isofunctors from ISO^n to ISO, called the **domain-functor** and **codomain-functor** of ϕ,

(ii) a rule that associates with every list $\mathcal{F} \in OBJ^n$ a mapping

$$\alpha_{\mathcal{F}} : \mathrm{Dmf}_\alpha(\mathcal{F}) \to \mathrm{Cdf}_\alpha(\mathcal{F}),$$

subject to the condition that

$$\mathrm{Cdf}_\alpha(\chi) \circ \alpha_{\mathrm{Dom}\,\chi} = \alpha_{\mathrm{Cod}\,\chi} \circ \mathrm{Dmf}_\alpha(\chi) \quad \text{for all} \quad \chi \in ISO^n. \tag{3.1}$$

Let isofunctors Φ and Ψ, both from ISO^n to ISO, be given. We say that α is a **natural assignment from Φ to Ψ**, and we write $\alpha : \Phi \to \Psi$ to indicate that $\mathrm{Dmf}_\alpha = \Phi$ and $\mathrm{Cdf}_\alpha = \Psi$.

A natural assignment α of degree $n \in \mathbb{N}$ is called a **natural equivalence** if, for every $\mathcal{F} \in OBJ^n$, the mapping $\alpha_{\mathcal{F}}$ of (ii) above belongs to ISO.

One can construct new natural assignments from given ones in the same way as new mappings from given ones. Let natural assignments α and β be given. If α and β have the same degree $n \in \mathbb{N}$ and if $\mathrm{Dmf}_\beta = \mathrm{Cdf}_\alpha$ we can define the **composite assignment** $\beta \circ \alpha : \mathrm{Dmf}_\alpha \to \mathrm{Cdf}_\beta$, also of degree n, by

assigning to each $\mathcal{F} \in OBJ^n$ the mapping $(\beta \circ \alpha)_{\mathcal{F}} := \beta_{\mathcal{F}} \circ \alpha_{\mathcal{F}}$. If α has degree $k \in \mathbb{N}$ and β has degree $m \in \mathbb{N}$, one can define the **product-assignment**

$$\alpha \times \beta : \mathrm{Pr} \circ (\mathrm{Dmf}_\alpha \times \mathrm{Dmf}_\beta) \to \mathrm{Pr} \circ (\mathrm{Cdf}_\alpha \times \mathrm{Cdf}_\beta),$$

of degree $k+m$, by assigning to each pair $(\mathcal{F}, \mathcal{G}) \in OBJ^k \times OBJ^m = OBJ^{k+m}$ the mapping $(\alpha \times \beta)_{(\mathcal{F},\mathcal{G})} := \alpha_{\mathcal{F}} \times \beta_{\mathcal{G}}$. Given a natural assignment α of degree $n \in \mathbb{N}$ and an isofunctor $\Phi : \mathrm{ISO}^k \to \mathrm{ISO}^n$ with $k \in \mathbb{N}$, one can define the **composite assignment** $\alpha \circ \Phi : \mathrm{Dmf}_\alpha \circ \Phi \to \mathrm{Cdf}_\alpha \circ \Phi$, of degree k, by assigning to each $\mathcal{F} \in \mathrm{ISO}^k$ the mapping $(\alpha \circ \Phi)_{\mathcal{F}} := \alpha_{\Phi(\mathcal{F})}$. The **identity-assignment** $\mathrm{id} : \mathrm{Id} \to \mathrm{Id}$ is defined by

$$\mathrm{id}_{\mathcal{A}} := \mathbf{1}_{\mathcal{A}} \quad \text{for all} \quad \mathcal{A} \in OBJ. \tag{3.2}$$

We have $\alpha \circ (\mathrm{id} \circ \mathrm{Dmf}_\alpha) = (\mathrm{id} \circ \mathrm{Cdf}_\alpha) \circ \alpha = \alpha$ for all natural assignments α. An illustration of the the use of the operations involving assignments is given by (5.18) below.

Examples: We now consider the concrete isocategory INV consisting of *all* invertible mappings. The corresponding object-class SET consists of *all* sets. The **subset-functor** $\mathrm{Sub} : \mathrm{INV} \to \mathrm{INV}$ is defined by

$$\mathrm{Sub}(\phi) := \phi_> \quad \text{for all} \quad \phi \in \mathrm{INV}, \tag{3.3}$$

where $\phi_>$ is the image mapping of ϕ (see [FDS], (03.7)). For every $\mathcal{S} \in SET$, $\mathrm{Sub}(\mathcal{S})$ is the set of all subsets of $\mathcal{S}$. For every $\mathcal{S} \in SET$ we denote the set of all *finite* subsets of $\mathcal{S}$ by $\mathrm{Fin}(\mathcal{S})$. The **finite-subset-functor** $\mathrm{Fin} : \mathrm{INV} \to \mathrm{INV}$ is defined by

$$\mathrm{Fin}(\phi) := \phi_> \Big|_{\mathrm{Fin}(\mathrm{Dom}\,\phi)}^{\mathrm{Fin}(\mathrm{Cod}\,\phi)} \quad \text{for all} \quad \phi \in \mathrm{INV}. \tag{3.4}$$

(See the definition of "adjustment" of a mapping in [FDS], Sect. 03.) The (finite) **cardinality** $\#$ can be interpreted to be the natural assignment $\# : \mathrm{Fin} \to \mathrm{Tr}_{\mathbb{N}}$ which associates with each set $\mathcal{S}$ the mapping $\#_{\mathcal{S}} : \mathrm{Fin}(\mathcal{S}) \to \mathbb{N}$ defined by

$$\#_{\mathcal{S}}(\mathcal{A}) := \#\mathcal{A} \quad \text{for all} \quad \mathcal{A} \in \mathrm{Fin}(\mathcal{S}). \tag{3.5}$$

The **set-product-functor** $\mathrm{Pr} : \mathrm{INV}^2 \to \mathrm{INV}$ is defined by

$$\mathrm{Pr}(\phi, \psi) := \phi \times \psi \quad \text{for all} \quad (\phi, \psi) \in \mathrm{INV}^2. \tag{3.6}$$

We have $\mathrm{Pr}(\mathcal{S}, \mathcal{T}) = \mathcal{S} \times \mathcal{T}$ for all $\mathcal{S}, \mathcal{T} \in SET$. A natural equivalence of degree 2 is the **switch-equivalence** $\mathrm{sw} : \mathrm{Pr} \to \mathrm{Pr} \circ \mathrm{Sw}$ which associates with every pair $(\mathcal{S}, \mathcal{T}) \in SET^2$ the mapping defined by

$$\mathrm{sw}_{(\mathcal{S},\mathcal{T})}(s, t) := (t, s) \quad \text{for all} \quad s \in \mathcal{S}, t \in \mathcal{T}. \tag{3.7}$$

(Here, the switch -functor $\mathrm{Sw} : \mathrm{INV}^2 \to \mathrm{INV}^2$ is defined according to (2.12).) The **map-functor** $\mathrm{Map} : \mathrm{INV}^2 \to \mathrm{INV}$ assigns to each pair $(\mathcal{S}, \mathcal{T}) \in SET^2$

the set $\mathrm{Map}(\mathcal{S},\mathcal{T})$ of all mappings from $\mathcal{S}$ to $\mathcal{T}$ and to each pair $(\phi,\psi)\in\mathrm{INV}^2$ the invertible mapping $\mathrm{Map}(\phi,\psi):\mathrm{Map}(\mathrm{Dom}\,\phi,\mathrm{Dom}\,\psi)\to\mathrm{Map}(\mathrm{Cod}\,\phi,\mathrm{Cod}\,\psi)$ defined by

$$(\mathrm{Map}(\phi,\psi))(f):=\psi\circ f\circ\phi^{\leftarrow}\quad\text{for all}\quad f\in\mathrm{Map}(\mathrm{Dom}\,\phi,\mathrm{Dom}\,\psi). \tag{3.8}$$

We can define a natural equivalence

$$\alpha:\mathrm{Sub}\circ\mathrm{Pr}\to\mathrm{Map}\circ(\mathrm{Id}\times\mathrm{Sub}) \tag{3.9}$$

of degree 2 by assigning to each pair $(\mathcal{S},\mathcal{T})\in SET^2$ the mapping

$$\alpha_{(\mathcal{S},\mathcal{T})}:\mathrm{Sub}\,(\mathcal{S}\times\mathcal{T})\to\mathrm{Map}(\mathcal{S},\mathrm{Sub}\,\mathcal{T}) \tag{3.10}$$

defined by

$$(\alpha_{(\mathcal{S},\mathcal{T})}(\mathcal{A}))(s):=\{t\in\mathcal{T}\mid(s,t)\in\mathcal{A}\} \tag{3.11}$$

for all $\mathcal{A}\in\mathrm{Sub}\,(\mathcal{S}\times\mathcal{T})$ and all $s\in\mathcal{S}$.

Remark: Most of the about 100 symbols listed in the Index of Multiple-Letter Symbols in [FDS] can be interpreted as standing for either isofunctors or natural assignments. ▮

4. Tensor functors

We now fix a field $\mathbb{F}$ and we consider the isocategory whose object-class *FDLS* consists of all finite dimensional linear spaces over $\mathbb{F}$ and whose isomorphism-class LIS consists of all linear isomorphism from one such space onto another or itself. We use the term **tensor functor of degree** $n\in\mathbb{N}$ for isofunctors from LIS^n to LIS. Here is a list of important tensor functors:

(1) The **product-space functor** $\mathrm{Pr}:\mathrm{LIS}^2\to\mathrm{LIS}$. It is defined by

$$\mathrm{Pr}(\mathbf{A},\mathbf{B}):=\mathbf{A}\times\mathbf{B}\quad\text{for all}\quad(\mathbf{A},\mathbf{B})\in\mathrm{LIS}^2. \tag{4.1}$$

We have $\mathrm{Pr}(\mathcal{V},\mathcal{W}):=\mathcal{V}\times\mathcal{W}$ (the product-*space* of $\mathcal{V}$ and $\mathcal{W}$) for all $\mathcal{V},\mathcal{W}\in$ *FDLS*.

(2) The **lin-map-functor** $\mathrm{Lin}:\mathrm{LIS}^2\to\mathrm{LIS}$. It assigns to every pair $(\mathcal{V},\mathcal{W})\in FDLS^2$ the linear space $\mathrm{Lin}\,(\mathcal{V},\mathcal{W})$ of all linear mappings from $\mathcal{V}$ to $\mathcal{W}$ and to every pair $(\mathbf{A},\mathbf{B})\in\mathrm{LIS}^2$ the invertible linear mapping

$$\mathrm{Lin}\,(\mathbf{A},\mathbf{B})\in\mathrm{Lis}\,(\mathrm{Lin}\,(\mathrm{Dom}\,\mathbf{A},\mathrm{Dom}\,\mathbf{B}),\mathrm{Lin}\,(\mathrm{Cod}\,\mathbf{A},\mathrm{Cod}\,\mathbf{B})) \tag{4.2}$$

defined by

$$\mathrm{Lin}\,(\mathbf{A},\mathbf{B})\mathbf{T}:=\mathbf{B}\mathbf{T}\mathbf{A}^{-1} \tag{4.3}$$

for all $\mathbf{T} \in \mathrm{Lin}\,(\mathrm{Dom}\,\mathbf{A}, \mathrm{Dom}\,\mathbf{B})$.

(3) The **duality-functor** $\mathrm{Dl} : \mathrm{LIS} \to \mathrm{LIS}$. It is defined by

$$\mathrm{Dl} := \mathrm{Lin} \circ (\mathrm{Id}, \mathrm{Tr}_{\mathbb{F}}). \tag{4.4}$$

We have

$$\mathrm{Dl}(\mathcal{V}) := \mathcal{V}^* \quad \text{for all} \quad \mathcal{V} \in \mathit{FDLS} \tag{4.5}$$

and

$$\mathrm{Dl}(\mathbf{A}) := (\mathbf{A}^\top)^{-1} \quad \text{for all} \quad \mathbf{A} \in \mathrm{LIS}. \tag{4.6}$$

(4) The **lineon-functor** $\mathrm{Ln} : \mathrm{LIS} \to \mathrm{LIS}$. It is defined by

$$\mathrm{Ln} := \mathrm{Lin} \circ \mathrm{Eq}_2. \tag{4.7}$$

We have

$$\mathrm{Ln}(\mathcal{V}) := \mathrm{Lin}\,(\mathcal{V}, \mathcal{V}) \quad \text{for all} \quad \mathcal{V} \in \mathit{FDLS} \tag{4.8}$$

and

$$\mathrm{Ln}(\mathbf{A})\mathbf{T} := \mathbf{A}\mathbf{T}\mathbf{A}^{-1} \quad \text{for all} \quad \mathbf{A} \in \mathrm{LIS} \;\text{ and }\; \mathbf{T} \in \mathrm{Ln}(\mathrm{Dom}\,\mathbf{A}). \tag{4.9}$$

(5) Given $k \in \mathbb{N}$, the k-**lin-map-functor** $\mathrm{Lin}_k : \mathrm{LIS}^k \times \mathrm{LIS} \to \mathrm{LIS}$. It assigns to each list $(\mathcal{V}_i \,|\, i \in k^]\,)$ in *FDLS* and each $\mathcal{W} \in \mathit{FDLS}$ the linear space

$$\mathrm{Lin}_k((\mathcal{V}_i \,|\, i \in k^]\,) \times \mathcal{W}) := \mathrm{Lin}_k\big(\bigtimes_{i \in k^]} \mathcal{V}_i, \mathcal{W}\big) \tag{4.10}$$

of all k-multilinear mappings from $\bigtimes_{i \in k^]} \mathcal{V}_i$ to $\mathcal{W}$, and it assigns to every list $(\mathbf{A}_i \,|\, i \in k^]\,)$ in LIS and each $\mathbf{B} \in \mathrm{LIS}$ the linear mapping

$$\mathrm{Lin}_k((\mathbf{A}_i \,|\, i \in k^]\,), \mathbf{B}) \tag{4.11}$$

from $\mathrm{Lin}_k\big(\bigtimes_{i \in k^]} \mathrm{Dom}\,\mathbf{A}_i, \mathrm{Dom}\,\mathbf{B}\big)$ to $\mathrm{Lin}_k\big(\bigtimes_{i \in k^]} \mathrm{Cod}\,\mathbf{A}_i, \mathrm{Cod}\,\mathbf{B}\big)$ defined by

$$\mathrm{Lin}_k((\mathbf{A}_i \,|\, i \in k^]\,), \mathbf{B})\mathbf{T} := \mathbf{B}\mathbf{T} \circ \bigtimes_{i \in k^]} \mathbf{A}_i^{-1} \tag{4.12}$$

for all $\mathbf{T} \in \mathrm{Lin}\big(\bigtimes_{i \in k^]} \mathrm{Dom}\,\mathbf{A}_i, \mathrm{Dom}\,\mathbf{B}\big)$.

(6) Given $k \in \mathbb{N}$, the k-**multilin-functor** $\mathrm{Ln}_k : \mathrm{LIS}^2 \to \mathrm{LIS}$. It is defined by

$$\mathrm{Ln}_k := \mathrm{Lin}_k \circ (\mathrm{Eq}_k \times \mathrm{Id}). \tag{4.13}$$

We have

$$\mathrm{Ln}_k(\mathcal{V}, \mathcal{W}) := \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W}) \tag{4.14}$$

for all $\mathcal{V}, \mathcal{W} \in \mathit{FDLS}$ and

$$\mathrm{Ln}_k(\mathbf{A}, \mathbf{B})\mathbf{T} := \mathbf{B}\mathbf{T} \circ (\mathbf{A}^{-1})^{\times k} \tag{4.15}$$

for all $\mathbf{A}, \mathbf{B} \in \mathrm{LIS}$ and all $\mathbf{T} \in \mathrm{Lin}_k((\mathrm{Dom}\,\mathbf{A})^k, \mathrm{Dom}\,\mathbf{B})$.

(7) Given $k \in \mathbb{N}$, the **symmetric-k-multilin-functor** $\mathrm{Sm}_k : \mathrm{LIS}^2 \to \mathrm{LIS}$. It is assigns to every pair $(\mathcal{V}, \mathcal{W}) \in \mathit{FDLS}^2$ the linear sapce

$$\mathrm{Sm}_k(\mathcal{V}, \mathcal{W}) := \mathrm{Sym}_k(\mathcal{V}^k, \mathcal{W}) \tag{4.16}$$

of all *symmetric* k-multilinear mappings from $\mathcal{V}^k$ to $\mathcal{W}$. We have

$$\mathrm{Sm}_k(\mathbf{A}, \mathbf{B})\mathbf{T} := \mathbf{B}\mathbf{T} \circ (\mathbf{A}^{-1})^{\times k} \tag{4.17}$$

for all $\mathbf{A}, \mathbf{B} \in \mathrm{LIS}$ and all $\mathbf{T} \in \mathrm{Sym}_k((\mathrm{Dom}\,\mathbf{A})^k, \mathrm{Dom}\,\mathbf{B})$.

(8) Given $k \in \mathbb{N}$, the **skew-k-multilin-functor** $\mathrm{Sk}_k : \mathrm{LIS}^2 \to \mathrm{LIS}$. It is defined in the same manner as Sm_k, except that $\mathrm{Sym}_k(\mathcal{V}^k, \mathcal{W})$ in (4.16) is replaced by the linear space $\mathrm{Skew}_k(\mathcal{V}^k, \mathcal{W})$ of all *skew* k-multilinear mappings from $\mathcal{V}^k$ to $\mathcal{W}$.

(9) Given $n \in \mathbb{N}$, the **k-linform-functor** Lnf_k, the **k-symform-functor** Smf_k, the **k-skewform-functor** Skf_k, all from LIS to LIS. They are defined by

$$\mathrm{Lnf}_k := \mathrm{Ln}_k \circ (\mathrm{Id}, \mathrm{Tr}_{\mathbb{F}})\ ,\ \mathrm{Smf}_k := \mathrm{Sm}_k \circ (\mathrm{Id}, \mathrm{Tr}_{\mathbb{F}})\ ,\ \mathrm{Skf}_k := \mathrm{Sk}_k \circ (\mathrm{Id}, \mathrm{Tr}_{\mathbb{F}}). \tag{4.18}$$

Given $\mathcal{V} \in \mathit{FDLS}$, we have

$$\mathrm{Lnf}_k(\mathcal{V}) := \mathrm{Lin}_k(\mathcal{V}^k, \mathbb{F}), \tag{4.19}$$

the space of all k-multilinear forms on $\mathcal{V}^k$. We have

$$\mathrm{Lnf}_k(\mathbf{A})\boldsymbol{\omega} \circ (\mathbf{A}^{-1})^{\times k} \quad \text{for all} \quad \boldsymbol{\omega} \in \mathrm{Lin}_k((\mathrm{Dom}\,\mathbf{A})^k, \mathbb{F}) \tag{4.20}$$

and all $\mathbf{A} \in \mathrm{LIS}$. The formulas (4.19) and (4.20) remain valid if Lin is replaced by Sym or Skew and Lnf by Smf or Skf correspondingly.

Remark : In much of the literature (see [K-N], Sect. 2 of Ch.I or [M-T-W], §3.2) the use of the term "tensor" is limited to tensor functors of the form $\mathbf{T}^r_s := \mathrm{Lin} \circ (\mathrm{Lnf}_s, \mathrm{Lnf}_r) : \mathrm{LIS} \to \mathrm{LIS}$ with $r, s \in \mathbb{N}$, or to tensor functors that are naturally equivalent to one of this form. Given $\mathcal{V} \in \mathit{FDLS}$ a member of the linear space $\mathbf{T}^r_s(\mathcal{V})$ is called a "tensor of contravariant degree r and covariant degree s." ∎

5. Natural assignments in LIS, identifications

Again, we fix a field $\mathbb{F}$ and we consider the isocategory LIS described in Sect. 4. We say that a natural assignment α of degree $n \in$ is **linear** if, for every $\mathcal{F} \in FDLS^n$, the mapping $\alpha_{\mathcal{F}} : \mathrm{Dmf}_\alpha(\mathcal{F}) \to \mathrm{Cdf}_\alpha(\mathcal{F})$ is linear. We give a list of important natural assignments.

(1) The **trace** $\mathrm{tr} : \mathrm{Ln} \to \mathrm{Tr}_{\mathbb{F}}$ assigns to each $\mathcal{V}$ the linear mapping

$$\mathrm{tr}_{\mathcal{V}} : \mathrm{Ln}(\mathcal{V}) \to \mathbb{F} \tag{5.1}$$

described, for example, by the Characterization of the Trace in [FDS], Sect. 26. tr is linear.

(2) The **determinant** $\det : \mathrm{Ln} \to \mathrm{Tr}_{\mathbb{F}}$ assigns to each $\mathcal{V}$ the linear mapping

$$\det_{\mathcal{V}} : \mathrm{Ln}(\mathcal{V}) \to \mathbb{F} \tag{5.2}$$

whose values are determinants of lineons in the usual sense. Of course, det is not linear.

Remark 1: The term **lineonic invariant** would be appropriate for any natural assignment from Ln to $\mathrm{Tr}_{\mathbb{F}}$. The literature describes special lineonic invariants called "principal invariants", which include tr and det. (See, for example, [FDS], Vol 2, Ch.1.) A part of "Invariant Theory" deals with classifying all posible lineonic invariants. The term **lineonic covariant** would be appropriate for any natural assignment from Ln to Ln. Among the lineonic covariants are the "adjugate" and the "principal covariants" described in [FDS], Vol 2, Ch.1. ▮

(3) The **transposition** $\mathrm{tp} : \mathrm{Lin} \to \mathrm{Lin} \circ (\mathrm{Dl}, \mathrm{Dl}) \circ \mathrm{Sw}$ assigns to each pair $(\mathcal{V}, \mathcal{W}) \in FDLS^2$ the mapping

$$\mathrm{tp}_{(\mathcal{V},\mathcal{W})} : \mathrm{Lin}\,(\mathcal{V}, \mathcal{W}) \to \mathrm{Lin}\,(\mathcal{W}^*, \mathcal{V}^*) \tag{5.3}$$

defined by

$$\mathrm{tp}_{(\mathcal{V},\mathcal{W})}(\mathbf{L}) := \mathbf{L}^\top \quad \text{for all} \quad \mathbf{L} \in \mathrm{Lin}\,(\mathcal{V}, \mathcal{W}) \tag{5.4}$$

where $\mathbf{L}^\top$ is the transpose of $\mathbf{L}$ as defined, for example, in [FDS], Sect.21. Transposition is a natural equivalence.

It is useful and customary to employ certain natural equivalences as identifications in the following way: Let tensor functors Φ and Φ' of a given degree $n \in \mathbb{N}$ be given. We single out a certain natural equivalence from Φ to Φ' and use it to treat Φ and Φ' as if they were the same tensor functor. We write $\Phi(\mathcal{F}) \cong \Phi'(\mathcal{F})$ for all $\mathcal{F} \in FDLS^n$ and $\Phi(\mathbf{F}) = \Phi'(\mathbf{F})$ for all $\mathbf{F} \in \mathrm{LIS}^n$. One must be very cautious with introducing identifications because they can lead to unexpected ambiguities or clashes. The following three identifications are customary and useful.

(4) The **biduality identification** $\mathrm{bdi} : \mathrm{Id} \to \mathrm{Dl} \circ \mathrm{Dl}$ assigns to each $\mathcal{V} \in FDLS$ the mapping $\mathrm{bdi}_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}^{**}$ defined by

$$\mathrm{bdi}_{\mathcal{V}}(\mathbf{v}) := \boldsymbol{\lambda}\mathbf{v} \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}, \boldsymbol{\lambda} \in \mathcal{V}^*. \tag{5.5}$$

We write $\mathcal{V} \cong \mathcal{V}^{**}$ and $\mathbf{v} = \mathrm{bdi}_{\mathcal{V}}(\mathbf{v})$ for all $\mathbf{v} \in \mathcal{V}$, so that (5.5) reduces to $\mathbf{v}\boldsymbol{\lambda} := \boldsymbol{\lambda}\mathbf{v}$.

(5) The **bilinearity-identification** $\mathrm{bli} : \mathrm{Lin}_2 \to \mathrm{Lin} \circ (\mathrm{Id}, \mathrm{Lin})$ assigns to each triple $(\mathcal{V}_1, \mathcal{V}_2, \mathcal{W}) \in \mathit{FDLS}^3$ the mapping

$$\mathrm{bli}_{(\mathcal{V}_1,\mathcal{V}_2,\mathcal{W})} : \mathrm{Lin}_2(\mathcal{V}_1 \times \mathcal{V}_2, \mathcal{W}) \to \mathrm{Lin}(\mathcal{V}_1, \mathrm{Lin}(\mathcal{V}_2, \mathcal{W})) \tag{5.6}$$

defined by

$$(\mathrm{bli}_{(\mathcal{V}_1,\mathcal{V}_2,\mathcal{W})}(\mathbf{B})\mathbf{v}_1)\mathbf{v}_2 := \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2) \tag{5.7}$$

for all $\mathbf{B} \in \mathrm{Lin}_2(\mathcal{V}_1 \times \mathcal{V}_2, \mathcal{W})$ and all $\mathbf{v}_1 \in \mathcal{V}_1$, $\mathbf{v}_2 \in \mathcal{V}_2$. We write $\mathrm{Lin}_2(\mathcal{V}_1 \times \mathcal{V}_2, \mathcal{W}) \cong \mathrm{Lin}(\mathcal{V}_1, \mathrm{Lin}(\mathcal{V}_2, \mathcal{W}))$ and $\mathbf{B} = \mathrm{bli}_{(\mathcal{V}_1,\mathcal{V}_2,\mathcal{W})}(\mathbf{B})$ for all $\mathbf{B} \in \mathrm{Lin}_2(\mathcal{V}_1 \times \mathcal{V}_2, \mathcal{W})$, so that (5.7) reduces to $(\mathbf{B}\mathbf{v}_1)\mathbf{v}_2 = \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2)$.

(6) The **dual-linmap-identification** $\mathrm{dli} : \mathrm{Lin} \circ (\mathrm{Dl} \times \mathrm{Dl}) \to \mathrm{Dl} \circ \mathrm{Lin}$ assigns each pair $(\mathcal{V}, \mathcal{W}) \in \mathit{FDLS}^2$ the mapping

$$\mathrm{dli}_{(\mathcal{V},\mathcal{W})} : \mathrm{Lin}(\mathcal{V}^*, \mathcal{W}^*) \to (\mathrm{Lin}(\mathcal{V}, \mathcal{W}))^* \tag{5.8}$$

defined by

$$(\mathrm{dli}_{(\mathcal{V},\mathcal{W})}(\mathbf{M}))\mathbf{L} := \mathrm{tr}_{\mathcal{V}}(\mathbf{M}^\top \mathbf{L}) \tag{5.9}$$

for all $\mathbf{M} \in \mathrm{Lin}(\mathcal{V}^*, \mathcal{W}^*)$ and all $\mathbf{L} \in \mathrm{Lin}(\mathcal{V}, \mathcal{W})$. We write $\mathrm{Lin}(\mathcal{V}^*, \mathcal{W}^*) \cong \mathrm{Lin}(\mathcal{V}, \mathcal{W})^*$ and $\mathbf{M} = \mathrm{dli}_{(\mathcal{V},\mathcal{W})}(\mathbf{M})$ for all $\mathbf{M} \in \mathrm{Lin}(\mathcal{V}^*, \mathcal{W}^*)$, so that (5.9) reduces to $\mathbf{M}\mathbf{L} = \mathrm{tr}_{\mathcal{V}}(\mathbf{M}^\top \mathbf{L})$.

The implications of the first two of above identifications are described, in some detail, in [FDS], Ch.2.

Let tensor functors Φ_1, Φ_2, Ψ, all of degree $n \in \mathbb{N}$, be given. We say that a natural assignment $\beta : \mathrm{Pr} \circ (\Phi_1, \Phi_2) \to \Psi$ is a **bilinear assignment** if, for every $\mathcal{F} \in \mathit{LIS}^n$, the mapping

$$\beta_{\mathcal{F}} : \Phi_1(\mathcal{F}) \times \Phi_2(\mathcal{F}) \to \Psi(\mathcal{F}) \tag{5.10}$$

is bilinear. The following are examples.

(7) The **dual-evaluation** $\mathrm{de} : \mathrm{Pr} \circ (\mathrm{Dl}, \mathrm{Id}) \to \mathrm{Tr}_{\mathbb{F}}$ assigns to each $\mathcal{V} \in \mathit{FDLS}$ the mapping

$$\mathrm{de}_{\mathcal{V}} : \mathcal{V}^* \times \mathcal{V} \to \mathbb{F} \tag{5.11}$$

defined by

$$\mathrm{de}_{\mathcal{V}}(, \mathbf{v}) = \mathbf{v} \quad \text{for all} \quad \in \mathcal{V}^*, \mathbf{v} \in \mathcal{V}. \tag{5.12}$$

(8) The **lineonic composition** $\mathrm{lc} : \mathrm{Pr} \circ (\mathrm{Ln}, \mathrm{Ln}) \to \mathrm{Ln}$ assigns to each $\mathcal{V} \in \mathit{FDLS}$ the mapping

$$\mathrm{lc}_{\mathcal{V}} : \mathrm{Ln}(\mathcal{V}) \times \mathrm{Ln}(\mathcal{V}) \to \mathrm{Ln}(\mathcal{V}) \tag{5.13}$$

defined by

$$\mathrm{lc}_{\mathcal{V}}(\mathbf{L},\mathbf{M}) = \mathbf{LM} \quad \text{for all} \quad \mathbf{L},\mathbf{M} \in \mathrm{Ln}(\mathcal{V}). \tag{5.14}$$

(9) The **tensor product** $\mathrm{tpr} : \mathrm{Pr} \circ (\mathrm{Id} \times \mathrm{Id}) \to \mathrm{Lin} \circ (\mathrm{Dl} \times \mathrm{Id}) \circ \mathrm{Sw}$ assigns each pair $(\mathcal{V},\mathcal{W}) \in \mathit{FDLS}^2$ the mapping

$$\mathrm{tpr}_{(\mathcal{V},\mathcal{W})} : \mathcal{V} \times \mathcal{W} \to \mathrm{Lin}\,(\mathcal{W}^*,\mathcal{V}) \tag{5.15}$$

defined by

$$\mathrm{tpr}_{(\mathcal{V},\mathcal{W})}(\mathbf{v},\mathbf{w}) := \mathbf{v} \otimes \mathbf{w} \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}, \mathbf{w} \in \mathcal{W}, \tag{5.16}$$

where $\mathbf{v} \otimes \mathbf{w}$ is the tensor product defined according to [FDS], Def. 1 of Sect. 25, with the identification $\mathcal{W} \cong \mathcal{W}^{**}$ (see (4) above).

Remark 2: In accord with definitions common in the literature (see for example [CC], Ch.3, Sect. 8), one might use the term "tensor product" for any bilinear assignment $\tau : \mathrm{Pr} \circ (\mathrm{Id} \times \mathrm{Id}) \times \Phi$, where Φ is a tensor functor of degree 2, provided that the following condition is satisfied: For every pair $(\mathcal{V},\mathcal{W}) \in \mathit{FDLS}^2$ and every bilinear mapping $\mathbf{B}$ with $\mathrm{Dom}\,\mathbf{B} = \mathcal{V} \times \mathcal{W}$, there is exactly one linear mapping $\mathbf{B}^\tau$ with $\mathrm{Dom}\,\mathbf{B}^\tau = \Phi(\mathcal{V},\mathcal{W})$ such that $\mathbf{B}^\tau \circ \tau_{(\mathcal{V},\mathcal{W})} = \mathbf{B}$. The tensor product tpr defined in (9) satisfies this condition by [FDS], Prop.6 of Sect.26. ∎

A special case of the dual-linmap-identification (6) can be expressed by the formula

$$\mathbf{M}^\top \mathbf{L} = \mathrm{tr}_{\mathcal{V}}(\mathbf{ML}) \quad \text{for all} \quad \mathbf{M},\mathbf{L} \in \mathrm{Ln}(\mathcal{V}), \tag{5.17}$$

valid for all $\mathcal{V} \in \mathit{FDLS}$. In terms of the operations involving natural assignments and isofunctors described in Sect.3, the fact that (5.17) is valid for all $\mathcal{V} \in \mathit{FDLS}$ can be expressed by

$$(\mathrm{de} \circ \mathrm{Ln}) \circ \Big(\big(\mathrm{tp} \circ (\mathrm{Id} \times \mathrm{Id})\big) \times (\mathrm{id} \circ \mathrm{Ln}) \Big) = \mathrm{tr} \circ \mathrm{lc} \tag{5.18}$$

where tr, lc, de and tp are described in (1), (8), (7) and (3), and where id is the idntity-assignment defined by (3.2).

6. Analytic tensor functors

We now assume that the field relative to which *FDLS* and LIS are defined in Sect.4 is the field $\mathbf{R}$ of real number. Given $\mathcal{V},\mathcal{W} \in \mathit{FDLS}$, the set

$$\mathrm{Lis}\,(\mathcal{V},\mathcal{W}) := \big\{\, \mathbf{A} \in \mathrm{LIS} \;\big|\; \mathrm{Dom}\,\mathbf{A} = \mathcal{V}, \mathrm{Cod}\,\mathbf{A} = \mathcal{W} \,\big\} \tag{6.1}$$

is then an open subset of the linear space $\mathrm{Lin}\,(\mathcal{V},\mathcal{W})$. (See, for example, the Differentiation Theorem for Inversion Mappings in Sect.68 of [FDS].).

Let a tensor functor Φ of degree 1 be given. For every pair $(\mathcal{V},\mathcal{W}) \in FDLS^2$, we define the mapping

$$\Phi_{(\mathcal{V},\mathcal{W})} : \mathrm{Lis}\,(\mathcal{V},\mathcal{W}) \to \mathrm{Lis}\,(\Phi(\mathcal{V}),\Phi(\mathcal{W})) \tag{6.2}$$

by

$$\Phi_{(\mathcal{V},\mathcal{W})}(\mathbf{A}) := \Phi(\mathbf{A}) \quad \text{for all} \quad \mathbf{A} \in \mathrm{Lis}\,(\mathcal{V},\mathcal{W}). \tag{6.3}$$

We say that the tensor functor Φ is **analytic** if $\Phi_{(\mathcal{V},\mathcal{W})}$ is an analytic mapping for every pair $(\mathcal{V},\mathcal{W}) \in FDLS^2$. We say that a natural assignment α of degree $n \in \mathbb{N}$ is an **analytic** assignment if the mapping $\alpha_{\mathcal{F}}$ is an analytic assignment for every list $\mathcal{F} \in FDLS^n$. All the tensor functors of degree 1 and all natural assignments listed in Sects.4 and 5 are in fact analytic. (The fact that they are of class C^∞ can easily be inferred from the results of Ch.6 of [FDS]. Proofs that they are analytic can be inferred, for example, from the results that will be presented in Ch.2 of Vol.2 of [FDS].)

Theorem : *Let an analytic tensor functor Φ be given and associate with each $\mathcal{V} \in FDLS$ the mapping*

$$\Phi^{\bullet}_{\mathcal{V}} : \mathrm{Ln}(\mathcal{V}) \to \mathrm{Ln}(\Phi(\mathcal{V})) \tag{6.4}$$

defined by

$$\Phi^{\bullet}_{\mathcal{V}} := \nabla_{\mathbf{1}_{\mathcal{V}}} \Phi_{(\mathcal{V},\mathcal{V})}. \tag{6.5}$$

(The gradient-notation used here is explained in [FDS], Sect.63.) *Then $\Phi^{\bullet}$ is a linear assignment from* Ln *to* $\mathrm{Ln} \circ \Phi$. *We call $\Phi^{\bullet}$ the* **derivative** *of Φ.*

Proof: Let $(\mathcal{V},\mathcal{W}) \in FDLS^2$ and $\mathbf{A} \in \mathrm{Lis}\,(\mathcal{V},\mathcal{W})$ be given. It follows from (6.3), from axiom (F1), and from (2.4) that

$$\Phi_{(\mathcal{W},\mathcal{W})}(\mathbf{A}\mathbf{L}\mathbf{A}^{-1}) = \Phi(\mathbf{A})\Phi_{(\mathcal{V},\mathcal{V})}(\mathbf{L})\Phi(\mathbf{A})^{-1} \tag{6.6}$$

for all $\mathbf{L} \in \mathrm{Lis}\,(\mathcal{V},\mathcal{V})$. By (4.9) we may write (6.6) as

$$\big(\Phi_{(\mathcal{W},\mathcal{W})} \circ \mathrm{Ln}(\mathbf{A})\big)(\mathbf{L}) = \big(\mathrm{Ln}(\Phi(\mathbf{A})) \circ \Phi_{(\mathcal{V},\mathcal{V})}\big)(\mathbf{L}) \tag{6.7}$$

for all $\mathbf{L} \in \mathrm{Lis}\,(\mathcal{V},\mathcal{V})$. Taking the gradient of (6.7) with respect to $\mathbf{L}$ at $\mathbf{L} := \mathbf{1}_{\mathcal{V}}$ yields

$$\Phi^{\bullet}_{\mathcal{W}} \circ \mathrm{Ln}(\mathbf{A}) = (\mathrm{Ln} \circ \Phi)(\mathbf{A}) \circ \Phi^{\bullet}_{\mathcal{V}}. \tag{6.8}$$

In view of (3.1) it follows that $\Phi^{\bullet}$ is a natural assignment from Ln to $\mathrm{Ln} \circ \Phi$. The linearity of $\Phi^{\bullet}$ follows from the definition of gradient. ∎

We now list the derivatives of a few analytic tensor functors (see (3), (4), and (9) of Sect.4). The formulas given are valid for every $\mathcal{V} \in FDLS$.

(1) We have

$$\mathrm{Id}^{\bullet}_{\mathcal{V}} = \mathbf{1}_{\mathrm{Ln}(\mathcal{V})}. \tag{6.9}$$

(2) We have

$$\mathrm{Dl}^{\bullet}_{\mathcal{V}} = \mathrm{tp}_{(\mathcal{V},\mathcal{V})}, \tag{6.10}$$

where tp is the transposition described in (3) of Sect.5.

(3) $\mathrm{Ln}^{\bullet}_{\mathcal{V}} : \mathrm{Ln}(\mathcal{V}) \to \mathrm{Ln}(\mathrm{Ln}(\mathcal{V}))$ is given by

$$(\mathrm{Ln}^{\bullet}_{\mathcal{V}}\mathbf{L})\mathbf{M} = \mathbf{LM} - \mathbf{ML} \quad \text{for all} \quad \mathbf{L}, \mathbf{M} \in \mathrm{Ln}(\mathcal{V}) \tag{6.11}$$

(This formula is an easy consequence of (4.8) and, [FDS] (68.9).).

(4) Let $k \in$ be given. In order to describe

$$(\mathrm{Lnf}_k)^{\bullet}_{\mathcal{V}} : \mathrm{Ln}(\mathcal{V}) \to \mathrm{Ln}(\mathrm{Lin}_k(\mathcal{V}^k, \mathbf{R})), \tag{6.12}$$

we define, for every $\mathbf{L} \in \mathrm{Ln}(\mathcal{V})$ and every $j \in k^{]}$, $D_j(\mathbf{L}) \in (\mathrm{Ln}(\mathcal{V}))^k$ by

$$(D_j(\mathbf{L}))_i := \left\{ \begin{array}{lll} \mathbf{L} & if & i = j \\ \mathbf{1}_{\mathcal{V}} & if & i \neq j \end{array} \right\} \qquad \text{for all} \quad i \in k^{]}. \tag{6.13}$$

We then have

$$((\mathrm{Lnf}_k)^{\bullet}_{\mathcal{V}}\mathbf{L})\boldsymbol{\omega} = -\sum_{j \in k^{]}} \boldsymbol{\omega} \circ D_j(\mathbf{L}) \quad \text{for all} \quad \boldsymbol{\omega} \in \mathrm{Lin}_k(\mathcal{V}^k, \mathbf{R}) \tag{6.14}$$

and all $\mathbf{L} \in \mathrm{Ln}(\mathcal{V})$. The formula (6.14) remains valid if Lnf is replaced by Smf or Skf and Lin by Sym or Skew, correspondingly.

The General Chain Rule for gradients (see [FDS], Sect.63) and the definition (6.5) immediately lead to the following

Chain Rule for Analytic Tensor Functors

Let Φ and Ψ be analytic tensor functors. Then the composite functor $\Psi \circ \Phi$ is also an analytic tensor functor and we have

$$(\Psi \circ \Phi)^{\bullet} = (\Psi^{\bullet} \circ \Phi) \circ \Phi^{\bullet}, \tag{6.15}$$

where the composite assignments on the right are explained in Sect.3.

For example, (6.15) shows that, for each $\mathcal{V} \in FDLS$,

$$(\mathrm{Ln} \circ \mathrm{Ln})^{\bullet}_{\mathcal{V}} : \mathrm{Ln}(\mathcal{V}) \to \mathrm{Ln}(\mathrm{Ln}(\mathrm{Ln}(\mathcal{V})))$$

is given by

$$(\mathrm{Ln} \circ \mathrm{Ln})^{\bullet}_{\mathcal{V}} = \mathrm{Ln}^{\bullet}_{\mathrm{Ln}(\mathcal{V})}\mathrm{Ln}^{\bullet}_{\mathcal{V}}. \tag{6.16}$$

In view of (6.11) above, (6.16) gives

$$\begin{aligned}\big(\big((\mathrm{Ln}\circ\mathrm{Ln})^{\bullet}_{\mathcal{V}}\mathbf{L}\big)\mathbf{K}\big)\mathbf{M} &= \big((\mathrm{Ln}^{\bullet}_{\mathcal{V}}\mathbf{L})\mathbf{K}-\mathbf{K}(\mathrm{Ln}^{\bullet}_{\mathcal{V}}\mathbf{L})\big)\mathbf{M}\\ &= \mathbf{L}(\mathbf{KM})-(\mathbf{KM})\mathbf{L}-\mathbf{K}(\mathbf{LM}-\mathbf{ML})\end{aligned} \tag{6.17}$$

for all $\mathcal{V}\in\mathit{FDLS}$, all $\mathbf{K}\in\mathrm{Ln}(\mathrm{Ln}(\mathcal{V}))$, and all $\mathbf{L},\mathbf{M}\in\mathrm{Ln}(\mathcal{V})$.

If Φ and Ψ are analytic tensor functors so is $\mathrm{Pr}\circ(\Phi,\Psi)$ and we have

$$(\mathrm{Pr}\circ(\Phi,\Psi))^{\bullet}_{\mathcal{V}} = (\Phi^{\bullet}_{\mathcal{V}}\mathbf{L})\times\mathbf{1}_{\Psi(\mathcal{V})}+\mathbf{1}_{\Psi(\mathcal{V})}\times(\Phi^{\bullet}_{\mathcal{V}}\mathbf{L}) \tag{6.18}$$

for all $\mathcal{V}\in\mathit{FDLS}$ and all $\mathbf{L}\in\mathrm{Ln}(\mathcal{V})$.

In the present situation, we can add an important natural assignment to the list given in Sect.5, namely the **lineonic exponential** $\exp:\mathrm{Ln}\to\mathrm{Ln}$; it assigns to each $\mathcal{V}\in\mathit{FDLS}$ the lineonic exponential

$$\exp_{\mathcal{V}}:\mathrm{Ln}(\mathcal{V})\to\mathrm{Ln}(\mathcal{V}) \tag{6.19}$$

for $\mathcal{V}$ as defined in [FDS], Prop.2 of Sect.612. exp is an analytic assignment of degree 1.

Let α be an analytic assignment of degree $n\in\mathbb{N}$. If we associate with each $\mathcal{V}\in\mathit{FDLS}$ the mapping $(\nabla\alpha)_{\mathcal{V}}:=\nabla(\alpha_{\mathcal{V}})$, the gradient of the mapping $\alpha_{\mathcal{V}}$, then $\nabla\alpha$ is again an analytic assignment of degree n and we have $\mathrm{Dmf}_{\nabla\alpha}=\mathrm{Dmf}_{\alpha}$ and $\mathrm{Cdf}_{\nabla\alpha}=\mathrm{Lin}\circ(\mathrm{Dmf}_{\alpha},\mathrm{Cdf}_{\alpha})$. We call $\nabla\alpha$ the **gradient** of α.

Let tensor functors Φ_1, Φ_2, Ψ, all of degree $n\in$ but not necessarily analytic, be given. Each bilinear assignment $\beta:\mathrm{Pr}\circ(\Phi_1,\Phi_2)\to\Psi$ is then analytic and its gradient $\nabla\beta:\mathrm{Pr}\circ(\Phi_1,\Phi_2)\to\mathrm{Lin}\circ(\mathrm{Pr}\circ(\Phi_1,\Phi_2),\Psi)$ is given by

$$\big((\nabla\beta)_{\mathcal{V}}(\mathbf{v}_1,\mathbf{v}_2)\big)(\mathbf{u}_1,\mathbf{u}_2)=\beta_{\mathcal{V}}(\mathbf{v}_1,\mathbf{u}_2)+\beta_{\mathcal{V}}(\mathbf{u}_1,\mathbf{v}_2) \tag{6.20}$$

for all $\mathcal{V}\in\mathit{FDLS}$, all $\mathbf{v}_1,\mathbf{u}_1\in\Phi_1(\mathcal{V})$, and all $\mathbf{v}_2,\mathbf{u}_2\in\Phi_2(\mathcal{V})$.

If α is an analytic assignment of degree $n\in\mathbb{N}$ and if Φ is any isofunctor from LIS^k to LIS^n with $k\in\mathbb{N}$, then $\alpha\circ\Phi$ is an analytic assignment of degree k and we have $\nabla(\alpha\circ\Phi)=(\nabla\alpha)\circ\Phi$.

7. Applications and modifications

We put $\widetilde{\mathbb{N}}=\mathbb{N}\cup\{\infty,\omega\}$ and consider $\widetilde{\mathbb{N}}$ to be totally ordered in such a way that $n<\infty<\omega$ for all $n\in\mathbb{N}$. Let a manifold $\mathcal{M}$ of class C^r, with $r\in\widetilde{\mathbb{N}}$ and

$r > 1$, and a linear-space bundle $\mathcal{B}$ over $\mathcal{M}$ and of class C^s with $1 \leq s < r$ be given. The fiber-space of $\mathcal{B}$ at a given point $x \in \mathcal{M}$ is denoted by $\mathcal{B}_x$.

Let an analytic tensor functor Φ be given. The set

$$\Phi(\mathcal{B}) := \bigcup_{x \in \mathcal{M}} \Phi(\mathcal{B}_x) \tag{7.1}$$

has the natural structure of a linear-space bundle over $\mathcal{M}$ of class C^s (see [CS], Sect.34). If $\mathcal{B} := \mathcal{TM}$ is the tangent-bundle of $\mathcal{M}$, then $\Phi(\mathcal{TM})$ is called the **tensor bundle** of $\mathcal{M}$ associated with Φ. A cross section of $\Phi(\mathcal{TM})$ is called a **tensor-field** of type Φ. In particular, a tensor field of type Id or Dl is called a **vector-field** or **covector-field**, respectively. The derivative $\Phi^\bullet$ of Φ as defined in Sect.6 is needed when one considers gradients of tensor-fields of type Φ or, more generally, gradients of cross sections of $\Phi(\mathcal{B})$ when $\mathcal{B}$ is any linear-space bundle. (See [CS], Ch.5.)

All of the considerations of Sect.4 can be applied if one replaces the isocategory considered there by the isocategory whose object-class *LS* consisting of *all* linear spaces over $\mathbb{F}$, finite-dimensional or not, or even the class *MOD* of all modules over a given commutative ring. Some of the considerations of Sect.5 can also be applied if this modifications is made. However, the natural assignments (1), (2) and (6) lose their meaning and biduality assignment bdi of (4) is no longer a natural equivalence and cannot be used for identification. Moreover, the domain-functor of tpr in (9) must be replaced by $\mathrm{Pr} \circ (\mathrm{Id} \times \mathrm{Dl} \circ \mathrm{Dl})$ and tpr is no longer a "tensor-product" in the sense described in Remark 2. (However, suitable tensor-product assignments can be constructed by the method described, for example, in Sect.8 of Ch.2 of [CC].) The considerations of Sect.6 lose their meaning.

All of the considerations of Sects.4, 5 and 6 can be applied if one replaces the isocategory considered there by the isocategory whose object-class *IPS* consists of all finite-dimensional *inner product* spaces and whose isomorphism-class OIS consists of all *orthogonal* isomorphisms. In this case, there is also a natural equivalence from Id to Dl, which can be used to deal efficiently with tensor fields on Riemannian manifolds.

Many of the considerations of Sects.4, 5 and 6 can also be applied if one replaces the isocategory considered there by the isocategory whose object-class *BS* consists of all Banach spaces (in the sense of "Banachable" space as explained in [L], p.4) and if one interprets LIS appropriately. The necessary details can be destilled from [L], Ch.1. However, as in the case when *FDLS* is replaced by *LS* the natural assignments (1), (2) and (6) of Sect.5 lose their meaning, the biduality assignment bdi of (4) is no longer an equivalence, and (9) must be modified. The isocategory LIS interpreted with the object-class *BS* can be used to deal with infinite-dimensional manifolds in the way described in the beginning of this section.

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