Shape recognition in three dimensions

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Abstract. The subshape recognition problem for three-dimensional shapes under linear transformations is considered. The problem is analysed in a series of cases, some that provide a determinate number of solutions and others that have indeterminately many solutions. Procedures for its solution for general shapes are developed. Difficulties posed by strict adherence to rational transformations are examined. As a corollary, an outline of a procedure for determining the symmetries of a shape is presented.

Introduction

A central computational problem associated with shape grammar theory is discussed, namely, under which transformations is one shape a subshape of another? Further, the question is examined whether these transformations—if they exist—between shapes that can be rationally described also can be described by rational coefficients. In other words, can the subshape recognition problem for rational shapes always be resolved using exact arithmetic?

The importance of the subshape problem can be gauged from the fact that its resolution is a prerequisite for the composition of shapes by the application of spatial rules. A spatial relation \((\alpha, \beta)\) between shapes \(\alpha\) and \(\beta\) can be considered as a spatial rule that applies to a shape \(\gamma\) if we can find a similarity (in general, a linear transformation) \(\tau\) such that \(\tau(\alpha)\) occurs as a shape in \(\gamma\) in which case \(\tau(\beta)\) replaces \(\tau(\alpha)\) in \(\gamma\) under rule application.

The subshape recognition problem in the form considered here has direct application to shape grammars (Stiny, 1980a; 1990), and to the ways in which spatial relations between shapes can be used to specify shape rules (Earl, 1986; Earl and Krishnamurti, 1984; Stiny, 1980b). Although acquaintance with shape grammar concepts would be an advantage, it is not crucial. The properties of shapes and definitions of spatial terms necessary for the arguments in this paper will be developed as needed.

The treatment for two-dimensional shapes given in Krishnamurti (1980; 1981) is the basis for implementations of shape generation systems (Chase, 1989; Krishnamurti, 1982; Krishnamurti and Giraud, 1986). A discussion of three-dimensional shapes is presented in Earl (1986), and an independent approach both for two-dimensional and for three-dimensional shapes is given in Stiny (forthcoming). This paper completes the details of the three-dimensional case.

The following ideas are basic.

A shape is a finite set of maximal straight lines of finite, nonzero length, where each line is specified by the coordinates of its endpoints. A shape is rational if its lines have endpoints given by rational coordinates, and real otherwise. The lines in a shape are maximal in the sense that colinear lines are separated by a gap, but
otherwise lines may touch or intersect. A subshape of a shape is a shape the lines of which each have endpoints that are coincident with a line in the original shape. The verification of the subshape relation for any two shapes \( s \) and \( s' \) can be approached as follows.

Let \( s \) denote a shape and the set of maximal lines that describes it. Shape \( s \) may be structured by an equivalence relation that partitions the set into equivalence classes of colinear maximal lines. That is, each line \( l \) is associated with an unambiguous descriptor, say \( \text{co}(l) \), such that colinear lines have identical descriptors.

An equivalence class of colinear lines can be organised by the order relation, \(<\), extended to \( n \)-tuples as follows. Let \( A \) and \( B \) denote two arbitrary \( n \)-tuples of numbers, \( A = (a_1, a_2, \ldots, a_n) \) and \( B = (b_1, b_2, \ldots, b_n) \). Then \( A < B \), if for some \( j \in \{1, 2, \ldots, n\}, a_k = b_k, k \neq j, \) and \( a_j < b_j \). Equality between \( A \) and \( B \) holds whenever the components of \( A \) and \( B \) are identical.

The endpoints of a line segment \( l \) can be ordered such that one is designated as the \( \text{tail}(l) \) and the other as the \( \text{head}(l) \) where \( \text{tail}(l) < \text{head}(l) \). A line \( l \) may be thus described by the triple \( \langle \text{co}(l), \text{tail}(l), \text{head}(l) \rangle \). A set of colinear maximal lines can be arranged as a sequence of lines in (increasing) order of head or tail values.

A further structuring of a shape can be carried out. Suppose that the descriptor function \( \text{co} \) is expressed as a tuple of numbers, then the equivalence classes of colinear lines can be arranged as a sequence in order of their descriptor values.

Two shapes can then be checked for equality, denoted \( s = s' \), by comparing the sequences in their description term by term. Observe that by arranging the description of a shape in the manner suggested above, shape equality can be tested for in a time linear in the number of lines in shapes \( s \) or \( s' \).

In a similar fashion, the subshape relation \( s \subseteq s' \) can be verified. Here the description of \( s \) must be wholly contained within the description of \( s' \). That is, for any two colinear lines, \( l = \langle \text{co}(l), \text{tail}(l), \text{head}(l) \rangle \) in \( s \), and \( l' = \langle \text{co}(l'), \text{tail}(l'), \text{head}(l') \rangle \) in \( s' \), \( l \) is contained in \( l' \) if and only if:

\[
\text{co}(l) = \text{co}(l'), \quad \text{tail}(l) \preceq \text{tail}(l'), \quad \text{and} \quad \text{head}(l) \preceq \text{head}(l').
\]

The subshape relation can be decided in a time linear in the number of lines in \( s \) or \( s' \).

### Line geometry in three dimensions

Points are described in homogeneous coordinates \( (x, y, z, w) \) in order to facilitate the description of points with rational coordinates as 4-tuples of integers. For real shapes we take \( w = 1 \). For rational shapes, any three-dimensional point \( (x, y, z) \) is expressed as a 4-tuple of relatively prime integers \( (xw, yw, zw, w) \) for some integer \( w \) chosen such that the integers are relatively prime. A simple way to do this is to set \( w = \frac{1}{\gcd(xw', yw', zw', w')} \), where \( w' \) is the product of the denominators of the nonzero \( x, y, \) and \( z, \) and \( a = \gcd(xw', yw', zw', w') \), the greatest common divisor. We will assume that all coordinates are given in their (in the case of rational shapes, to their reduced relatively prime) homogeneous coordinate form.

Points can be defined by vectors (which are indicated by boldface). A point \( p \) with coordinates \( (x, y, z, w) \) can be represented as \( p = (x, y, z, w) \). The operations of vector arithmetic, scalar, and cross products are defined for any two vectors \( p = (x_p, y_p, z_p, w_p) \) and \( q = (x_q, y_q, z_q, w_q) \) as:

\[
\lambda p = (\lambda x_p, \lambda y_p, \lambda z_p, w_p),
\]

\[
p \pm q = (x_p, y_p, z_p, w_p) \pm (x_q, y_q, z_q, w_q),
\]

where \( \lambda \) is a scalar.

1. \[
\lambda p = (\lambda x_p, \lambda y_p, \lambda z_p, w_p),
\]
2. \[
p \pm q = (x_p, y_p, z_p, w_p) \pm (x_q, y_q, z_q, w_q),
\]
and

\[
p \cdot q = \frac{1}{w_p w_q} \left( x_p x_q + y_p y_q + z_p z_q \right),
\]

(3)

\[
p \times q = \left( y_p z_q - y_q z_p, z_p x_q - z_q x_p, y_q x_p - y_p x_q, w_p w_q \right).
\]

(4)

**Line descriptor**

The descriptor function for a line is now established. Let \( p \) and \( p' \) be endpoints of a line segment \( l, p = (x, y, z, w) \) and \( p' = (x', y', z', w') \). The line \( l \) can be described by a pair of vectors \( (L, L_0) \) representing the direction of the line and its moment about the origin respectively (figure 1). The vectors are given by the following expressions:

\[
L = p - p' = (xw' - x'w, yw' - y'w, zw' - z'w, ww')
\]

(5)

\[
L_0 = p \times p' = (yz' - y'z, zx' - z'x, xy' - x'y, ww')
\]

(6)

The components of \( L \) are denoted by \( L = (L, M, N, W) \) and those of \( L_0 \) by \( L_0 = (L_0, M_0, N_0, W) \). For a line segment \( (q, q') \) colinear with \( (p, p') \), the positions of \( q \) and \( q' \) can be expressed as \( q = \lambda p + \mu p' \) and \( q' = \rho p + \upsilon p' \), where \( \lambda + \mu = \rho + \upsilon = 1 \).

Equations (5) and (6) reduce to

\[
q - q' = (\lambda \rho - \mu \upsilon) (p - p') = (\lambda \rho - \mu \upsilon) L
\]

\[
q \times q' = (\lambda \rho - \mu \upsilon) (p \times p') = (\lambda \rho - \mu \upsilon) L_0
\]

Thus colinear line segments have descriptions \((kL, kL_0), k \neq 0\).

Suppose the descriptors are **normalised** according to the schemes: (a) for real line segments, set \( 1/k^2 = L^2 + M^2 + N^2 \), and \( W = 1 \); and (b) for rational line segments, divide \( L \) and \( L_0 \) by their greatest common denominator (gcd) by setting \( 1/k = \gcd(L, M, N, L_0, M_0, N_0) \), and \( W = 1 \). Further, the first nonzero entry of \( L \) is assumed positive by setting the sign of \( k \) the same as the sign of the first nonzero entry of \( L \). This, in effect, assigns an orientation to the line segment.

Then, colinear line segments have the same descriptor provided all lines belong to the same class (real or rational), and thus have a consistent method of description. Note that parallel lines share the same normalised direction descriptor \( L \) to within a scalar factor.

The definition of the line descriptor means that vectors \( L \) and \( L_0 \) are perpendicular and their scalar product \( L \cdot L_0 \) is zero:

\[
LL_0 + MM_0 + NN_0 = 0.
\]

(7)

Conversely, a vector pair \( (a, b), a, b \neq 0 \), is a line descriptor if \( a \cdot b = 0 \). This may be shown as follows. Consider vectors \( x \) satisfying \( b = x \times a \). These have the

![Figure 1](image_url)

**Figure 1.** The direction and moment vectors of a line.
general vector form:
\[ x = \lambda a + \frac{a \times b}{a \cdot a}, \]
where \( \lambda \) is an arbitrary scalar. A line segment with endpoints \( x \) and \( x' \) of this form has descriptor \((ka, kb)\) where \( k \) is a scalar. The descriptor \((L, L_0)\) is also referred to as the Plücker (Brand, 1947) coordinates of the infinite line on which the line segment lies. Note that a point \( p = (x, y, z, w) \) lies on the line \((L, L_0)\) if and only if \( L_0 = p \times L \).

Figure 2 illustrates examples of spatial relations between pairs of line segments and *distinguished* points that result from the relations. These points are functions of the relations between the lines. They remain unaltered by Euclidean and scale transformations of the shape in the sense that the image of a point of a given kind under a Euclidean transformation is another point of the same kind. For instance, figure 2(a) shows lines that intersect at points of intersection which are distinguished points; figure 2(b) shows a line, a designated point, and the perpendicular from this point to the line the foot of which is a distinguished point; and figure 2(c) illustrates skew lines and the feet of their common perpendicular which are distinguished points. In determining the transformations under which the subshape relation holds it is often necessary to construct distinguished points. In the next section we examine some of these distinguished points which are used subsequently in determining the transformations. The distinguished points considered all have the property that they are rational if constructed on rational shapes.

![Figure 2](image)

**Figure 2.** Spatial relations between pairs of lines and distinguished points: (a) point of intersection of two lines, (b) the foot of the perpendicular from a point to a line, (c) the feet of the common perpendicular between two skew lines.

**Intersecting lines**
The point of intersection of two noncollinear lines is the point that is coincident with both lines or their extensions. Parallel line segments may be considered to intersect on a plane at infinity (according to the homogeneous coordinate description of points).

Suppose \((L, L_0)\) and \((L', L'_0)\) are two noncollinear line segments. The segments intersect if and only if they are coplanar, and they are coplanar if and only if the volume of the tetrahedron with the segments as opposite edges is zero. This geometric condition can be expressed in a number of ways. If the endpoints of the segments are \((p, q)\) and \((p', q')\), respectively, then the condition for intersection is...
expressed by the vanishing determinant $\Delta$, where

$$
\Delta = \begin{vmatrix}
  x_p & y_p & z_p & w_p \\
  x_q & y_q & z_q & w_q \\
  x_p' & y_p' & z_p' & w_p' \\
  x_q' & y_q' & z_q' & w_q'
\end{vmatrix}.
$$

(8)

The condition $\Delta = 0$ expresses the linear dependence or equivalently the coplanarity of the four given points. This condition can be expressed succinctly in vector form as:

$$
L \cdot L_0 + L' \cdot L_0' = 0.
$$

(9)

The point of intersection $c$, $c = (x_c, y_c, z_c, w_c)$, of the two lines is the common solution of the two equations

$$
c \times L = L_0, \quad c \times L' = L_0'.
$$

The vector product of these two equations gives

$$
c = \frac{L_0 \times L_0'}{L_0 \cdot L} = \frac{L_0' \times L_0}{L_0' \cdot L'}.
$$

(10)

In coordinates this corresponds to

$$
x_c = \frac{M_0 N_0' - M_0' N_0}{L_0 L + M_0 M' + N_0 N'}, \quad y_c = \frac{N_0 L_0' - N_0' L_0}{L_0 L' + M_0 M' + N_0 N'},
$$

$$
z_c = \frac{L_0 M_0' - L_0' M_0}{L_0 L + M_0 M' + N_0 N'}, \quad w_c = \frac{M_0 N_0 - M_0' N_0}{L_0 L' + M_0 M' + N_0 N'}.
$$

(11)

Thus the intersection of two intersecting lines with rational descriptors is a rational point.

**Perpendiculars to lines**

Let $p, p = (x_p, y_p, z_p, w_p)$, be a point not coincident with line $l(L, L_0)$. Let $f(p, l)$ be the foot of the perpendicular from $p$ to the line $l$, $f(p, l) = (x_f, y_f, z_f, w_f)$. Perpendicularity and the coincidence of $f$ and $l(L, L_0)$ implies that

$$
f \times L = L_0, \quad (f - p) \cdot L = 0.
$$

Taking the vector product of the first equation with $L$ gives

$$
f = \frac{(p \cdot L) L - L_0 \times L}{L \cdot L}.
$$

(12)

This can be expressed in coordinate form. The $x$-coordinate is given by

$$
x_f = \frac{(x_p L + y_p M + z_p N) L - (M_0 N_0 - MN_0) w_p}{(L^2 + M^2 + N^2) w_p}.
$$

(13)

The other coordinates are expressed similarly. Thus for line $l$ and point $p$ both rational, the foot of the perpendicular from $p$ to $l$ is a rational point.

**Skew lines**

The arguments above can be extended to apply to the case of two skew lines $l(L, L_0)$ and $l'(L', L_0')$. Skew lines are neither coplanar nor parallel. There is a unique common perpendicular which meets the line segments or their extensions at $f = (x_f, y_f, z_f, w_f)$ and $f' = (x_{f'}, y_{f'}, z_{f'}, w_{f'})$, respectively. The feet of the common
perpendiculars are given by
\[
    f = \frac{L_0 \times (L \times L_0 - L' \times L_0)}{L_0 \cdot (L \times L')}, \quad f' = \frac{L_0 \times (L \times L_0 - L' \times L_0)}{L_0 \cdot (L \times L')},
\]
and the descriptor of the common perpendicular line is \((L \times L', L \times L_0 - L' \times L_0)\).
The common perpendicular of two skew rational lines is a rational line.

Transformations

A transformation, \(\tau\), of a shape \(s\) is the shape \(\tau(s)\). The transformation \(\tau\) acts on the lines in \(s\) so that a line with endpoints \(p\) and \(q\) in \(s\) is transformed to a line with endpoints \(\tau(p)\) and \(\tau(q)\) in \(\tau(s)\). The transformation \(\tau: p \to p'\) acting on general points \(p = (x, y, z, w)\) to give points \(p' = (x', y', z', w')\) can be represented by a matrix \(T\) of point coordinates:
\[
    \begin{bmatrix}
        x' \\
        y' \\
        z' \\
        w'
    \end{bmatrix} = \begin{bmatrix}
        a & b & c & d \\
        a & b & c & d \\
        a & b & c & d \\
        0 & 0 & 0 & E
    \end{bmatrix}
    \begin{bmatrix}
        x \\
        y \\
        z \\
        w
    \end{bmatrix} = T(x, y, z, w)^T.
\]

Note that \(E \neq 0\). A transformation is determined if the correspondence between two sets of four noncoplanar points is given. The treatment will be restricted to Euclidean transformations augmented by scale. The appropriate correspondences are thus the similarity transformations of the tetrahedra formed by the sets of points.

Consider the correspondence between two sets of noncoplanar points \((p_i, p_2, p_3, p_4)\) and \((p'_i, p'_2, p'_3, p'_4)\) in which \(p_i\) and \(p'_i\) correspond for \(i = 1, 2, 3, 4\). The points are expressed in homogeneous coordinates of the forms \((x_i, y_i, z_i, w_i)\) and \((x'_i, y'_i, z'_i, w'_i)\). It is assumed that the two sets of noncoplanar points are similar tetrahedra under the correspondence. If the tetrahedra have symmetry, then there are further correspondences involving the two sets of points which give rise to distinct transformations related by the symmetry.

In order to determine the transformation it is necessary to find the coefficients of the matrix \(T\) which satisfy equation (15) for the points \((x_i, y_i, z_i, w_i), (x'_i, y'_i, z'_i, w'_i), i = 1, 2, 3, 4\). The matrix equation (15) provides three groups of four equations in four unknowns: \([a_i, b_i, c_i, d_i], [a_i, b_i, c_i, d_i], [a_i, b_i, c_i, d_i]\). Each group can be solved provided the equations are nonsingular. This is guaranteed by the requirement that the points are noncoplanar.

However, we note that the solution is not quite as straightforward in the formulation proposed for the matrix representation. The points are represented in homogeneous coordinates which means that points \((x, y, z, w)\) and \((kx, ky, kz, kw)\) are the same. The matrix will transform these onto the same point. The difficulty arises because the four pairs of points in the correspondence will not in general be expressed in a way which allows direct use in a system of linear equations. Essentially, the value of \(E\) for each pair of points will be different. This is dealt with by using the scalar freedom in the homogeneous point representation. We notice that if each of the matrix elements is multiplied by the same scalar factor, then the transformation remains unchanged. The freedom to express points and transformations in homogeneous coordinates can be incorporated in the equations derived from the matrix representation of the transformation as follows.
The matrix gives the following sets of equations
\[
\begin{align*}
\begin{aligned}
  a_i x_i + b_i y_i + c_i z_i + d_i w_i &= k_i x'_i, \\
  a_i x_i + b_i y_i + c_i z_i + d_i w_i &= k_i y'_i, \\
  a_i x_i + b_i y_i + c_i z_i + d_i w_i &= k_i z'_i, \\
  E w_i &= k_i w'_i,
\end{aligned}
\end{align*}
\]
for \(i = 1, 2, 3, 4\). The value of \(E\) can be chosen as convenient, giving a value for \(k_i\) and thus three sets of four equations in four unknowns. Essentially what is happening here is that we are choosing a scale factor for all the matrix entries. This does not affect the transformation because homogeneous coordinates are being used.

Define the determinant
\[
\Delta = \begin{vmatrix}
  x_1 & y_1 & z_1 & w_1 \\
  x_2 & y_2 & z_2 & w_2 \\
  x_3 & y_3 & z_3 & w_3 \\
  x_4 & y_4 & z_4 & w_4
\end{vmatrix}.
\]
The equations (16) have a solution provided they are nonsingular. This is guaranteed by the condition \(\Delta \neq 0\), which is equivalent to the condition that the four points are noncoplanar. Assign \(E = \Delta\), which sets values for \(k_i\), \(i = 1, 2, 3, 4\). With this assignment, the solution of the equations are found by standard results to be
\[
\begin{align*}
  a_i &= \Delta_{x'_i|zw}, \\
  b_i &= \Delta_{y'_i|zw}, \\
  c_i &= \Delta_{z'_i|w}, \\
  d_i &= \Delta_{z'_i|w},
\end{align*}
\]
\[
\begin{align*}
  a_i &= \Delta_{y'_i|xw}, \\
  b_i &= \Delta_{x'_i|xw}, \\
  c_i &= \Delta_{x'_i|w}, \\
  d_i &= \Delta_{x'_i|w},
\end{align*}
\]
\[
\begin{align*}
  a_i &= \Delta_{z'_i|xyw}, \\
  b_i &= \Delta_{z'_i|xyw}, \\
  c_i &= \Delta_{y'_i|zw}, \\
  d_i &= \Delta_{y'_i|zw}.
\end{align*}
\]
\(\Delta_{x'_i|zw}\) is the determinant derived from \(\Delta\) by replacing the first column by the corresponding \(x'_i\) entries. The other determinants are defined similarly. For example, the determinant \(c_i = \Delta_{y'_i|w}\) is the determinant derived from \(\Delta\) by replacing the third column by \(y'_i\) entries.

For real shapes, the transformation matrix is most conveniently given by dividing each element by \(\Delta\) and setting \(E = 1\). For rational shapes, each coefficient of the transformation can be calculated by using integer arithmetic. The points are expressed as integer coordinates and thus determinants are calculated by integer arithmetic.

The transformation can be considered graphically in a number of ways. One of these is described below (figure 3). There are five steps in all.

1. Translation \(\tau_1\) taking \(p_i\) to the origin, that is, \(\tau_1(p_i) = 0\).
2. Translation \(\tau_2\) taking \(p'_i\) to the origin, that is, \(\tau_2(p'_i) = 0\).
3. Rotation and scale \(\tau_3\) about the common normal of the lines \(\tau_1(p_i, p'_i)\) and \(\tau_2(p_i, p'_i)\) such that the lines \(\tau_1(p_i, p'_i)\) and \(\tau_2(p_i, p'_i)\) coincide and the points \(\tau_3(p_i)\) and \(\tau_3(p'_i)\) coincide, where \(p_i\) and \(p'_i\) denote the centroids of the triangles formed by the points \(p_2, p_3, p_4\) and \(p'_2, p'_3, p'_4\), respectively.
4. Rotation and scale \(\tau_4\) about the line through \(\tau_3(p_i)\) normal to the plane formed by \(\tau_1(p_2, p_3, p_4)\) and \(\tau_3(p_2, p_3, p_4)\) such that \(\tau_4(p_i)\) and \(\tau_4(p'_i)\) coincide, \(\tau_4(p_2, p_3, p_4)\) and \(\tau_4(p'_2, p'_3, p'_4)\) coincide, and \(\tau_3(p_i)\) and \(\tau_3(p'_i)\) coincide.
[For similarity (isometric) transformations, these latter correspondences will always hold provided the two tetrahedra are initially similar (congruent).]
5. Translation \((-\tau_5)\) taking the origin to \(p_i\).

The composition \(\tau = (-\tau_5)_1 \tau_3 \tau_2 \tau_1\) gives the desired transformation.
Figure 3. Graphical construction of the transformation between a pair of similar tetrahedra:
(a) correspondence $p_i \rightarrow p_i'$ between similar tetrahedra; (b) step 1: translation $\tau_1(p_i)$ to the origin;
(c) step 2: translation $\tau_2(p_i)$ to the origin; (d) step 3: rotation and scale $\tau_3$ about the common normal to position vectors $\tau_1(p_i)$ and $\tau_1(p_i')$;
(e) step 4: rotation and scale $\tau_4$ about the common normal through $\tau_3(p_i)$; (f) step 5: translation $-\tau_2$ back to $p_i'$. 

axis of rotation and scale normal to the position vectors to $\tau_1(p_i)$ and $\tau_2(p_i)$; the centroids of the two triangles

axis of rotation and scale normal to the triangles through $\tau_3(p_i') = \tau_3(p_i)$
Subshape recognition under similarity transformations

When composing shapes by the recursive application of shape rules, it is imperative to know when two shapes are equal under a linear transformation. Of particular interest are the similarity transformations that are made up from finite compositions of the Euclidean transformations of translation, rotation, and reflection, augmented by a scale transformation. In the sequel, we restrict the discussion to the similarity transformations.

Under similarity, transformed shapes maintain the same spatial relation between the lines composing the shape. That is, angles and separation among lines are preserved.

When considering the problem of determining when a shape is a subshape of another, difficulties arise because there are general cases where there are an indeterminate number of valid transformations under which the subshape relation will be satisfied. This difficulty is compounded by the fact that endpoints of lines cannot be used as distinguished points because endpoints are not necessarily preserved in a subshape relation. Nevertheless, it is possible to demonstrate that, in such cases where there are an indeterminate number of valid transformations, one can isolate a ‘base’ transformation from which the other transformations can be generated.

Before this problem is addressed, the augmentation of shapes by labelled points is considered. In the generation of shapes, labelled points are used to guide both the choice of shape rules and the locations where they are applied. A labelled point is a point \( p \) with an associated label set \( A \), and is denoted by \( p : A \). The convention is adopted that no two distinct labelled points in a shape share the same coordinates. That is, if there are two labelled points \( p : A_1 \) and \( p : A_2 \) with \( A_1 \neq A_2 \), they may be replaced by a single maximally labelled point \( p : A_1 \cup A_2 \), where \( \cup \) denotes set union. The label set \( A \) does not carry any geometrical import in the sense that under a linear transformation \( \tau \), the labelled point \( p : A \) maps onto the labelled point \( \tau(p) : A \).

Labelled points are distinguished in the sense that the spatial relations of the lines and labelled points in a labelled shape are preserved under a similarity transformation. A labelled point \( p : A \) is a subshape of \( p' : A \) whenever \( p = p' \) and \( A \leq A' \).

The procedures to determine the set, \( T \), of possible similarity transformations \( \tau \) of a shape \( s \) augmented with a set of labelled points \( P \), denoted by \( \sigma = (s, P) \), such that \( \tau(\sigma) = (\tau(s), \tau(P)) \) is a subshape of the second shape \( \sigma' = (s', P') \) are enumerated in a series of cases which depend on certain conditions being satisfied in \( \sigma \). Precisely, one of these cases must hold for \( \sigma \). The subshape expression \( \tau(\sigma) \leq \sigma' \) is used as a shorthand for the pair of subshape expressions \( \tau(s) \leq s' \) and \( \tau(P) \leq P' \).

To determine whether or not the subshape relation between \( \sigma \) and \( \sigma' \) holds, it is more than sufficient to examine the mapping between corresponding sets of four noncolinear distinguished points in the two shapes. We start by selecting labelled points as candidates for distinguished points. If there are an insufficient number of these, other points are selected as possible candidates. If this is still not possible, we have a situation where there are potentially indeterminately many ways of satisfying the subshape relation between \( \sigma \) and \( \sigma' \).

In the next two sections we exhaustively enumerate the determinate and indeterminate cases, respectively. The cases are enumerated in decreasing order of the number of labelled points in \( \sigma \). That is, case 1 prefixes the case when there are four labelled points, case 2 corresponds to cases when there are three labelled points and so on. The case numbering is preserved over the two sections.
In some of the cases, we separate the discussion into shapes when they are rational and into shapes when they are real. In all other cases, the constructions apply equally both to rational and to real shapes.

The determinate cases

Case 1: There are at least four noncoplanar labelled points in $P$. Let the labelled points be $p_i : A_i, p_i = (x_i, y_i, z_i, w_i), 1 \leq i \leq 4$. The four points are noncoplanar if and only if $\Delta \neq 0$ (as defined in the previous section). For each set of labelled points $p_i : A_i$ in $P'$, satisfying $A_i \preceq A_i'$ and with $|p_i|$ similar to $|p_i'|$, $1 \leq i \leq 4$, a transformation $\tau$ can be generated by the procedure described in the previous section. If $\tau (o) \preceq o'$ is satisfied, then $\tau$ is added to the set of valid transformations. Distinct correspondences of labelled points generate distinct transformations. Distinct correspondences on the same sets of points are possible if the labelled tetrahedra possess symmetries.

If all possible correspondences are examined, then the procedure is exhaustive.

If the labelled points in both sets are rational, the valid transformations will have rational coefficients.

Case 2: There are three labelled points in $P$ and they are not all colinear. Let the three labelled points be $p_i : A_i, p_i = (x_i, y_i, z_i, w_i), i = 1, 2, 3$. If there is a line which is not in the plane of the points and not perpendicular to the plane, then it is possible to construct distinguished points as feet of the perpendiculars from $p_i$ to the line which is not in the plane of the points. The procedure in case 1 is then used.

If there is no such line, then define the points $c^+ (p_1, p_2, p_3)$ and $c^- (p_1, p_2, p_3)$ as illustrated in figure 4 and given by the expression,

$$c^\pm (p_1, p_2, p_3) = c(p_1, p_2, p_3) \pm k[(p_1 - p_2) \times (p_1 - p_3)],$$

where

$$c(p_1, p_2, p_3) = \frac{1}{3} (p_1 + p_2 + p_3)$$

is the centroid of the points $p_1, p_2,$ and $p_3$, and $k$ is a scale factor equal to the inverse of the magnitude of $(p_1 - p_2)$. The vector product $(p_1 - p_2) \times (p_1 - p_3)$ represents a line normal to the plane of the three labelled points. Fix $p_4 = c^+ (p_1, p_2, p_3)$.

Figure 4. Constructing possible correspondences from similar triangles.
For each set of three labelled points \( p'_i : A' \), \( i = 1, 2, 3 \), in \( P' \), satisfying \( A_i \leq A' \), and with triangles \([p_i]\) and \([p'_i]\) similar under the correspondence \( p_i \rightarrow p'_i \), \( i = 1, 2, 3 \), construct the points \( c^{-}(p'_i, p'_j, p'_k) \) and \( c^+(p'_i, p'_j, p'_k) \). Potential transformations \( \tau \) can be generated by setting \( p'_i \) to each of these points in turn. The construction of the fourth point guarantees that the tetrahedra are similar if the original triangles are similar. If \( \tau(a) \leq a' \) is satisfied, then \( \tau \) is added to the set of valid transformations. Distinct correspondences of labelled points generate distinct transformations. Distinct correspondences on the same sets of points are possible if the labelled triangles possess symmetries.

If all possible correspondences are examined, then the procedure is exhaustive.

The above argument demonstrates that the correspondence between two sets of three noncolinear distinguished points is sufficient to generate possible transformations.

This above procedure will not always succeed for rational shapes in the sense that the constructed points may not be rational and thus the transformation may not be rational. However, if the constructed points in \( a \) and \( a' \) are both irrational, then the transformation may be rational. This can only occur if the values of \( k \) and \( k' \) used to construct \( c^+(p_i, p_j, p_k) \) and \( c^-(p_i, p_j, p_k) \), respectively, have \( k/k' \) rational. Clever choice of the constructed points is irrelevant and one construction is as good as another for the purpose of determining the transformation. If only one of the constructed points in \( a \) or \( a' \) is rational, then the transformation is irrational as the following examples indicate.

**Examples**

1. Let \( p_1 = (0, 0, 0), p_2 = (0, 2, 0), \) and \( p_3 = (2, 0, 0) \) be the \((x, y, z)\)-coordinates of the three labelled points. Let \( p'_i = f[p_i, (p_2, p_3)] = (1, 1, 0) \). Two possible choices for \( p'_i \) based on \( p_i \) are \((1, 1, \pm 2)^{(1/2)} \). That is, \( p'_i \) is irrational. Note that \( p'_i \) is not constructed according to the above procedure but is given by an equivalent construction corresponding to a right triangle \([p_1, p_2, p_3]\) orthogonal to the triangle \([p_1, p_2, p_3]\). Let \( p'_1 = (0, 0, 0), p'_2 = (-1, 1, 0), \) and \( p'_3 = (1, 1, 0) \) be the corresponding labelled points. Let \( p'_i = f[p_i, (p'_2, p'_3)] = (0, 1, 0) \). The corresponding choices for \( p'_i \) are \((0, 1, \pm 1) \). That is, \( p'_i \) is rational. The similarity transformation that maps between the two tetrahedra has a scale factor of \( 2^{-1/2} \) and, hence, must have irrational coefficients.

2. Let \( p_1, p_2, \) and \( p_3 \) be the same points as above. Let \( p'_1, p'_2, \) and \( p'_3 \) be their mirror images in the \( yz \)-plane. Then, \( p'_i \) and \( p'_i \) are not rational but the similarity transformation between the two tetrahedra has rational coefficients.

3. Let \( p_1 = (0, 0, 0), p_2 = (0, 1, 0), \) and \( p_3 = (2, 0, 0) \). Then, a possible choice for \( p'_i \) is \((2/5, 4/5, 1/5)^{(1/2)} \) which is clearly not rational. Let \( p'_1 = (0, 0, 0), p'_2 = (1, -1, 0), \) and \( p'_3 = (-2, -2, 0) \). The corresponding choices for \( p'_i \) are \((2/5, -6/5, \pm 2^{1/2}/5^{1/2}) \) which are clearly not rational. The two tetrahedra are similar with a scale factor of \( 2^{1/2} \).

Thus, it is not always possible to find valid transformations with rational coefficients that map a rational shape as a subshape of another rational shape.

**Case 3:** There are two labelled points in \( P \). Let the two labelled points be \( p_1 : A_1 \) and \( p_2 : A_2 \). There are a number of subcases which are conveniently enumerated by considering the lines in the shape \( a \) if they are present. The maximum number of lines considered is limited to one. In general, it may be possible to use a number of lines in \( a \) for the construction of distinguished points, thus allowing the application of cases already considered. However, this is not always a sensible strategy to be considered in checking subshape because the number of possible constructions in \( a' \) increases exponentially with the number of extra lines. As will
be demonstrated below, the use of a single extra line, if it exists, is sufficient to determine the possible transformations. This guideline is augmented by the use of distinguished points if they are available within the configuration selected in $\sigma$. This is relevant to case 3 because it is possible to specify a general construction for two labelled points and a noncoincident line, but determining which of the resulting transformations are valid requires checking the subshape relation in all cases. However, with the use of distinguished points, an initial similarity check can be done before transformation generation. This reduces the operations required in subshape recognition. The result of these observations is that the minimum of extra lines are used in constructing the distinguished points, and all the distinguished points available are used. In all there are three determinate subcases to be considered.

**Subcase 3(a):** There is a line $l$ not coplanar and not perpendicular to $(p_1, p_2)$. The condition on the line descriptor and the point for this condition is straightforward. The feet of the perpendiculars from $p_1$ and $p_2$ to the line (figure 5) are distinct and can be used as two additional points to construct possible transformations as in the procedure for case 1. All possible constructions of the additional points must be made in $\sigma'$ to enumerate the transformations exhaustively. However, repetitions of valid transformations will be generated if there are two or more colinear lines in $\sigma'$ used in the construction of the additional points. This is avoided by using only one line from each colinear class of lines in $\sigma'$ during the construction.

![Figure 5](image)

*Figure 5.* Constructing distinguished points on a line not coplanar and not perpendicular to the line specified by two labelled points.

**Subcase 3(b):** There is a line $l$: (1) perpendicular and not coplanar to $(p_1, p_2)$, or (2) coplanar, not perpendicular, and not coincident with $(p_1, p_2)$. In this subcase the foot of the perpendicular from $p_1$ or $p_2$ defines a third point $p_3$, noncolinear with $p_1$ and $p_2$ (figure 6). Using this additional point the procedure for constructing the transformations is as case 2 above. All possible constructions of the additional points must be made in $\sigma'$ based upon pairs of labelled points $(p_1', p_2')$, $p_3': A'_1$, $A_1' \leq A_1$. Associated correspondences are then used as the basis for generating possible transformations. Further, the constructions in $\sigma'$ should follow the choice of $p_1$ or $p_2$ in the construction of the third point in $\sigma$.

![Figure 6](image)

*(1)*

(2)

*Figure 6.* Constructing a distinguished point on a line which is exclusively (1) perpendicular, or (2) coplanar with (and distinct from) the line specified by two labelled points.
Subcase 3(c): There is a line \( l \) coplanar and perpendicular to \((p_1, p_2)\). There are several ways of considering this case (figure 7). One of these will be examined. Let the point of intersection of \( l \) and \((p_1, p_2)\) be \( c \). Construct points \( a^\pm \) on \( l \) at a distance from \( c \) equal to the length of \((p_1, p_2)\), where \( a^- \) lies along the positive direction of \( l \) as determined by the descriptor. Further, construct the line \( m \) through \( c \) and perpendicular to \( l \) and \((p_1, p_2)\) and points \( b^\pm \) on \( m \) at a distance from \( c \) equal to the length of \((p_1, p_2)\), where \( b^- \) lies along the positive direction of \( m \) as determined by its descriptor. Assign \( p_1 = a^- \) and \( p_4 = b^- \).

For each set of two labelled points \( p_i': A', i = 1, 2, \) in \( P' \), satisfying \( A_i \leq A_i' \) and lines \( l' \) in \( s' \) perpendicular and coplanar with \((p_1', p_2')\) such that the intersection point \( c' \) on \( l' \) and \((p_1', p_2')\) divides \((p_1', p_2')\) in the same ratio as \( c \) divides \((p_1, p_2)\), construct points \( a'^\pm \) and \( b'^\pm \). For each pairwise selection of the constructed points, assigned to \( p_1', p_2' \), generate the transformation for the associated correspondence between \((p_1, p_2, p_3, p_4)\) and \((p_1', p_2', p_3', p_4')\). Transformations with \( \tau(\sigma) \leq \sigma' \) represent valid transformations. Add these to the list of transformations. All possible constructions of the additional points must be made for the exhaustive enumeration of potential transformations. The procedure can generate repetitions of the transformations if there are two or more colinear lines in \( \sigma' \) which are used in the generation. To avoid this only one line from each colinear class of lines in \( \sigma' \) is considered in the construction of the additional points.

![Figure 7](image_url)

**Figure 7.** Distinguished points from a configuration consisting of a line which is coplanar and perpendicular to the line specified by two labelled points.

Case 4: There is one labelled point in \( P \).

Subcase 4(a): There is a line \( l \) in \( s \) not coincident with the labelled point \( p_1 \). Let \( p_2 = f(p_1, l) \) be the foot of the perpendicular from \( p_1 \) to \( l \) (figure 8). The case is similar to case 3(c) with all lines in \( \sigma' \) not coincident with a corresponding labelled point \( p_1' \), satisfying \( A_i \leq A_i' \). Note that the constructed transformations may produce transformations that do not have rational coefficients.

![Figure 8](image_url)

**Figure 8.** Distinguished points constructed from a line not coincident with a given labelled point.
We illustrate by a simple example a situation where rational shapes are mapped onto irrational shapes. Figure 9(a) illustrates a shape rule with the left-hand shape consisting of a single rational line and a single rational labelled point. Figure 9(b) is a subshape of the 'current' shape to which it applies. The appropriate transformation that maps the left-hand shape of the rule to the current shape has irrational coefficients and when applied to the right-hand side of the rule will produce an irrational shape.

\[ (1, 0, 1) \rightarrow (0, 0, 2) \]
\[ (-1, 0, -1) \rightarrow (0, 0, -1) \]

(a)  

\[ (1, 0, 1) \rightarrow (0, 0, 2) \]
\[ (-1, 0, -1) \rightarrow (0, 0, -1) \]

(b)  

**Figure 9.** A simple example to show that a rational shape rule (a) applied to a rational shape (b) may produce irrational shapes.

**Case 5:** There are no labelled points in \( P \). That is, \( P = \emptyset \). There are two determinate cases depending on the configuration of lines in \( \sigma \).

**Case a:** There are two skew lines in \( s \). Skew lines are not parallel by definition. Let the lines be \( l_1 \) and \( l_2 \). Construct their common perpendicular and denote the corresponding intersections with \( l_1 \) and \( l_2 \) by \( p_1 \) and \( p_2 \) (Figure 10). This case may be considered similarly to case 3(c) using just one of those lines. However, this ignores some of the geometrical information in the shape. A variant of the treatment is considered. Construct points \( a_i^\pm, i = 1, 2 \) on \( l_1 \) at a distance from \( p_i \) equal to the length of \( (p_1, p_2) \), where \( a_i^+ \) lies along the positive direction of \( l_i \) as determined by the descriptor. Assign \( p_3 = a_1^+ \) and \( p_4 = a_2^+ \).

For each pair of skew lines \( l_1 \) and \( l_2 \) in \( \sigma \) construct the common perpendicular and denote the feet by \( p_1 \) and \( p_2 \). Further construct the points \( a_i^\pm \) and \( a_i^\mp \). For each pairwise selection from \( \{a_i^+, a_i^-\} \) assigned to \( p_1, p_2 \), generate the transformation from the associated correspondence between \( \{p_1, p_2, p_3, p_4\} \) and \( \{p_1, p_2, p_3, p_4\} \). A preliminary check is made to ensure similarity of the correspondence. This effectively checks that the skew lines have equal twist angle. All possible constructions of the additional points must be made for the exhaustive enumeration of potential transformations. Colinear lines can give rise to repetitions of the constructions and the corresponding transformations. As with previous cases only one line from each colinear class in \( \sigma \) is considered.

**Figure 10.** Distinguished points from two skew lines.
Case 5(b): There are three coplanar lines in s not all parallel and not all concurrent at a common point. The situation considered here is that of three lines forming the sides of a triangle [figure 11(a)] or of parallel lines with transversal [figure 11(b)]. These two situations are similar to cases 2 and 3(b), respectively. However, if the transversal is perpendicular to the parallel lines [figure 11(c)], then a situation similar to case 3(c) arises.

Figure 11. The determinate cases when there are no labelled points and three coplanar lines in \( \sigma \).

The indeterminate cases

The cases in which there are an indeterminate number of transformations are now considered. In these cases the number of valid transformations effecting subshape relation may not be finite, but the nature of the indeterminacy is specified. The first instance arises in case 3.

Case 3: There are two labelled points in \( P \).
Subcase 3(d): All labelled points in \( P \) and all lines in \( s \) are colinear. This case subsumes the case when \( s \) is empty. For each set of two labelled points \( p_i : A_i^1 \), \( i = 1, 2 \) in \( P' \), satisfying \( A_i \leq A_i^1 \), one possible transformation is generated by a combination of translations, rotations, and scale taking \( (p_1, p_2) \) to \( (p_1^1, p_2^1) \). The particular transformation is not critical. The complete set of transformations is generated by composing this with rotations about the axis \( (p_1^1, p_2^1) \) and with a reflection in a plane containing \( p_1^1 \) and \( p_2^1 \). The first transformation may be considered as a base transformation for the complete set.

The base transformation can be constructed as follows (figure 12). At \( p_i \) construct the lines \( l_i \) and \( l_i' \), where \( l_i \) has the descriptor \( (L_i, L_{i0}) \), \( L_1 = p_i \times p_2 \) and \( L_{i0} = p_i \times (p_i \times p_5) \), and \( l_i' \) has the descriptor \( (L_i', L_{i0}) \), \( L_2 = (p_1 - p_2) \times p_1 \times p_3 \) and \( L_{i0} = p_i \times [(p_1 - p_i) \times p_1 \times p_3] \). The expressions for \( L_{i0} \) and \( L_{i0} \) are not used. Construct the points \( a_i^\circ \) on lines \( l_i, i = 1, 2 \) at a distance from \( p_i \) equal to the length of \( (p_1, p_2) \). Select \( p_3 = a_i^\circ \) and \( p_4 = a_i^\circ \). For each pairwise selection \( (a_1^\circ, a_2^\circ) \)

Figure 12. The distinguished points generated from two labelled points.
assigned to \(p_5, p_4\) generate the transformation from the associated correspondence between \((p_1, p_2, p_3, p_4)\) and \((p_1', p_2', p_3', p_4')\). These base transformations will incorporate possible reflections. We can construct the complete set of transformations by taking the composition of these base transformations with rotations about the axis \((p_1', p_2')\). Note that there is a degree of redundancy in this specification, arising from the generation of four base transformations three of which incorporate reflections. Only one of these three is required when composing with the general rotation.

**Case 4:** There is exactly one labelled point, \(p_1 : A_1\), in \(p\). There are three indeterminate situations. These are illustrated in figure 13.

![Figure 13](image)

**Subcase 4(b):** \(p_1\) is coincident with all lines in \(s\) and there are two noncolinear lines in \(s\). Let \(p_2\) be an endpoint of a line in \(s\) distinct from \(p_1\). For each labelled point \(p_1' : A_1'\) in \(P\), satisfying \(A_1 \leq A_1'\), consider each endpoint \(p_2' \neq p_1'\) of lines through \(p_1'\). For each pair \((p_1', p_2')\) and each line through \(p_1'\) but not through \(p_2'\), generate the transformation as in cases 3(b) or 3(c) above. The full set of transformations is obtained by composing this transformation with scales keeping \(p_1\) as the fixed centre of the scale.

**Subcase 4(c):** \(p_1\) is coincident with all lines in \(s\) and all lines are colinear. Let \(p_2\) be an endpoint of a line in \(s\). For each labelled points \(p_1' : A_1'\) in \(P\), satisfying \(A_1 \leq A_1'\), consider each endpoint \(p_2' \neq p_1'\) of lines through \(p_1'\). For each pair \((p_1', p_2')\) generate the base transformation as in case 3(d) above. The full set of transformations is obtained by composing these transformations with rotations about \((p_1', p_2')\) and with scale having \(p_1\) as the fixed centre of the scale.

**Subcase 4(d):** There are no lines in \(s\). For each labelled point \(p_1' : A_1'\) in \(P\), satisfying \(A_1 \leq A_1'\), construct the translation that takes \(p_1\) into \(p_1'\). The full set of transformations consists of this base transformation composed with rotations and scales having \(p_1'\) as a fixed point and a reflection in a plane through \(p_1'\).

**Case 5:** There are no labelled points in \(p\). There are three possible indeterminate cases as shown in figure 14.

![Figure 14](image)

**Figure 14:** The three indeterminate subshape conditions with no labelled points in \(\sigma\).
Subcase 5(c): All lines in $s$ are coincident at a common point and are not all colinear. Let $p_1$ be the common point and construct the transformations as in case 4(b).

Subcase 5(d): All lines in $s$ are parallel and are not all colinear. Choose two lines $l_1$ and $l_2$. Let $p_1$ be an endpoint of $l_1$, and $p_2$ the foot of the perpendicular from $p_1$ to $l_2$. For each pair of parallel lines $(l'_1, l'_2)$ in $s'$ construct in a similar way the two pairs of points $(p'_1, p'_2)$ corresponding to the choice of endpoint of $l'_1$. The base transformations are then generated as for case 3(c). The full set of transformations is generated by composing these with translations along the direction of the lines $l'_1$ and $l'_2$ together with a reflection in a plane perpendicular to the parallel lines. This case may have either a determinate or an indeterminate number of valid transformations.

Subcase 5(e): All lines in $s$ are colinear. Select endpoints $(p_1, p_2)$ of a line in $s$. Find all pairs of endpoints $(p'_1, p'_2)$ of all lines in $s'$ and generate transformations for each pair $(p'_1, p'_2)$ as in the generation of the base transformations in case 3(d). The full set of possible transformations is found by composing the base set with rotations about $(p'_1, p'_2)$, appropriate scale reductions, translations along $(p'_1, p'_2)$, and reflections in a plane containing $p'_1$ and $p'_2$ and perpendicular to $(p'_1, p'_2)$.

This completes the enumeration of the determinate and indeterminate cases.

The symmetries of a shape
We have considered the problem of, given two shapes, determining all possible similarity transformations of one shape such that the transformed shape is a subshape of the second shape. In this section we consider the related problem of determining the possible similarity transformations which render two unlabelled shapes equal. Procedures will now be given to determine all possible similarity transformations which maintain shape equality. A corollary of these procedures is that, if we are given two equal shapes, then the possible similarity transformations that take a shape into itself (other than the identity transformation) may be determined. That is, each transformation defines a symmetry of the shape. The symmetries of a given shape may thus be determined by these transformations.

In general, the set of transformations that preserve equality of shape is finite. There are three exceptional cases to consider: namely, when two unlabelled shapes each consist of single sets of finite colinear lines, and when the shape consists of no more than two labelled points and has no lines. In all cases, the shapes are always equal and there are an indeterminate number of potential transformations that yield equality. For the unlabelled shapes with single sets of finite colinear lines and the labelled shapes with two labelled points and no lines, one particular transformation can be singled out, from which the other transformations can be derived. This is the screw transformation along the common perpendicular with a translation along the direction of one of the lines—the two labelled points may be treated as a line—and a scale transformation. Further, note that the heads and tails are preserved.

To determine the potential transformation for shape equality four noncoplanar distinguished points in each shape are mapped onto one another. The coordinates of these points are sufficient to calculate the coefficients of a linear transformation. For rational points that map onto rational points, the coefficients are also rational. If distinguished points of the shapes are arbitrarily chosen, say the endpoints of line segments, or their points of intersection or the feet of the common perpendicular of two skew lines, then many potential transformations may have to be considered. Thus, labelled points are used whenever possible.
In situations where there are insufficient labelled points, other suitable distinguished points are used. If there are still insufficient distinguished points, then transformations of point and line descriptors are invoked by constructing additional points. The only case where this is not possible are the three cases mentioned above.

The procedure for determining the symmetries of a shape is summarised as follows. In principle, four distinguished points in a shape are kept fixed. Then, each possible transformation corresponds to a permutation of four distinguished points in the shape such that corresponding points are identical in type or label. That is, a labelled point in the shape must map onto another labelled point with the same label set in the shape; an endpoint of a line must map onto an endpoint of another line in the shape; and so on. Moreover, the tetrahedra formed by the two sets of corresponding points must be congruent. Once a correspondence between distinguished points in the shape has been constructed, the shape equality procedure can be applied to compare the shape and its transformation.

Conclusion
We have shown the existence of procedures for solving the recognition problem for three-dimensional line shapes. We have also shown that valid transformations that map between rational shapes may not have rational coefficients; consequently, it is not always possible to resolve the shape recognition problem using exact arithmetic. This would inexorably lead to round-off and other numerical errors that would have to be taken care of in practice. However, if computations for shape recognition are restricted to integers, we would then have to take greater care in the way in which we specify shape rules. Principally, this means that the shape rules have to be sufficiently dense to ensure that there are always four noncoplanar labelled or other distinguished points.(1)

Nonetheless, we believe, we have established that a grammatical approach to the construction of form-led three-dimensional spatial designs is a viable proposition. In doing so, we have accomplished one half of a two-part objective, laid out and argued for by Earl (1986), namely, to develop a constructive computational paradigm for the creation of worlds of designs. To fulfill this ambition and to provide a practical computational framework we would need to implement the theory described here, taking into consideration the numerical problems that are introduced by arithmetic on reals, or by the constraints that would have to be imposed on the specification of shape rules as a result of limiting arithmetic to integers. That, however, remains the object of another endeavour, the actions of another incarnation, and the subject of another report.

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(1) There are other issues that surface when we restrict shapes to rational descriptors. More often than not, we are interested in shapes that have rational or integral dimensions rather than have rational endpoints. Rational shapes raise issues that are more of a theoretical nature, whereas real shapes raise practical issues that demand accuracy of numerical computation. One permits exact subshape testing and hence, accurate shape recognition; the other greater flexibility in shape representation, for example, the unit cube rotated through 45° about the z-axis inevitably has nonrational descriptors. A fuller discussion of the rational–real divide requires greater depth and is beyond the scope of this paper.
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