Measuring Uncertainty Using Survey-Based Diffusion Indices

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Learning About Consumer Uncertainty from Qualitative Surveys: As Uncertain As Ever

Santiago Pinto, Pierre-Daniel Sarte, Robert Sharp
Federal Reserve Bank of Richmond

July 2015
Introduction

- Information compiled by statistical agencies (e.g. BLS, BEA) on state of economic activity,
  - is not comprehensive, e.g. regional information on certain series are not compiled
  - involves lags, i.e. published with at least a one-month lag, and subject to 3-month and 1-year revisions
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- A growing number of institutions and government agencies produce diffusion indices constructed from qualitative survey data
  - Michigan Survey of Consumers indices of consumer sentiment, Institute of Supply Management index of manufacturing production, Federal Reserve Banks regional indices, etc.
Introduction

- Summarizing qualitative survey data in the form of a diffusion index:

\[ \mu \left( \frac{n^u}{n} - \frac{n^d}{n} \right) + \kappa = \mu D + \kappa \]

ISM: \( \mu = 1/2, \kappa = 1/2 \)
Richmond: \( \mu = 1, \kappa = 0 \)
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- Aggregate growth may arise in different ways: e.g. IP, a few sectors doing well while others muddle through, all sectors doing moderately well, etc.

- Diffusion indices summarize the direction of change in a set of disaggregated series: the breadth of change
Actual change and the breadth of change

\[
\Delta x_t = \frac{1}{N} \sum_{i=1}^{n^u_t} \Delta x_{i,t}^u - \frac{1}{N} \sum_{i=1}^{n^d_t} \Delta x_{i,t}^d,
\]

where \( \Delta x_{i,t}^u = \Delta x_{i,t} \) if \( \Delta x_{i,t} \geq 0 \), and \( \Delta x_{i,t}^d = -\Delta x_{i,t} \) if \( \Delta x_{i,t} < 0 \).
Actual change and the breadth of change

\[ \Delta x_t = \frac{1}{N} \sum_{i=1}^{n_t^u} \Delta x_{i,t}^u - \frac{1}{N} \sum_{i=1}^{n_t^d} \Delta x_{i,t}^d, \]

where \( \Delta x_{i,t}^u = \Delta x_{i,t} \) if \( \Delta x_{i,t} \geq 0 \), and \( \Delta x_{i,t}^d = -\Delta x_{i,t} \) if \( \Delta x_{i,t} < 0 \)

\[ \Delta x_t = \frac{n_t^u}{N} \mu_t^u - \frac{n_t^d}{N} \mu_t^d \]
Actual change and the breadth of change

- Define $\mu^u = T^{-1} \sum_{t=1}^{T} \mu_t^u$, $\phi^u = T^{-1} \sum_{t=1}^{T} n_t^u / N$,

- Then ...

$$\Delta x_t^u = \left( \frac{n_t^u}{N} - \phi^u \right) \mu^u + \phi^u (\mu_t^u - \mu^u) + \left( \frac{n_t^u}{N} - \phi^u \right) (\mu_t^u - \mu^u),$$
Actual change and the breadth of change

Define $\mu^u = T^{-1} \sum_{t=1}^{T} \mu_u^t$, $\varphi^u = T^{-1} \sum_{t=1}^{T} n_u^t / N$,

Then ...

$$\Delta x_t^u = \left( \frac{n_t^u}{N} - \varphi^u \right) \mu^u + \varphi^u (\mu_u^t - \mu^u) + \left( \frac{n_t^u}{N} - \varphi^u \right) (\mu_u^t - \mu^u),$$

Similarly, let $\mu^d = T^{-1} \sum_{t=1}^{T} \mu_d^t$, $\varphi^d = T^{-1} \sum_{t=1}^{T} n_d^t / N$,

Then ...

$$\Delta x_t^d = + \left( \frac{n_t^d}{N} - \varphi^d \right) \mu^d + \varphi^d (\mu_d^t - \mu^d) + \left( \frac{n_t^d}{N} - \varphi^d \right) (\mu_d^t - \mu^d),$$
Actual change and the breadth of change

- Decomposing an expansion/contraction,

\[
\Delta x_t \approx \varphi^u(\mu^u_t - \mu^u) - \varphi^d(\mu^d_t - \mu^d)
\]

Change in “how much” or intensive margin

\[
+ \mu^u D_t,
\]

Change in “how many” or extensive margin
Actual change and the breadth of change

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Change in “how much” or intensive margin

\[ + \mu^u D_t , \]

Change in “how many” or extensive margin

- Overall growth arises from:

  - the difference between how fast ”up” sectors grew and how badly ”down” sectors declined,

  - the difference between the proportion of sectors that expanded versus those that declined, the breadth of the expansion
Decomposition of BLS $\Delta x_t$

Annualized month/month % change

Month

Ext. margin: $D_t \mu^u$

Int. margin: $\varphi^u (\mu^u_t - \mu^u) - \varphi^d (\mu^d_t - \mu^d)$

Interaction: $\varepsilon \varphi^d_t + (\varphi^u_t - \varphi^u) (\mu^u_t - \mu^u) - (\varphi^d_t - \varphi^d) (\mu^d_t - \mu^d)$

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Measuring the breadth of change using qualitative surveys

A sample of $n$ survey participants, drawn randomly from a population at a point in time, is surveyed – e.g. overall business conditions?
Measuring the breadth of change using qualitative surveys

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- Possible answers: \( \mathcal{A} = \{1, 2, \ldots, r\} \). Answers are indexed by \( a \in \mathcal{A} \), e.g. \( a \in \mathcal{A} = \{u, d, s\} \)
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- Number of respondents associated with answer \( a \in \mathcal{A} \), is \( n^a \), where \( \sum_{a=1}^{r} n^a = n \). Answers of type \( a \) are assigned a value of \( \omega^a \in \mathcal{R} \), e.g. \( \omega^u = 1 \), \( \omega^s = 0 \), and \( \omega^d = -1 \).
Measuring the breadth of change using qualitative surveys

- A sample of $n$ survey participants, drawn randomly from a population at a point in time, is surveyed – e.g. overall business conditions?

- Possible answers: $\mathcal{A} = \{1, 2, \ldots, r\}$. Answers are indexed by $a \in \mathcal{A}$, e.g. $a \in \mathcal{A} = \{u, d, s\}$

- Number of respondents associated with answer $a \in \mathcal{A}$, is $n^a$, where $\sum_{a=1}^{r} n^a = n$. Answers of type $a$ are assigned a value of $\omega^a \in \mathcal{R}$, e.g. $\omega^u = 1$, $\omega^s = 0$, and $\omega^d = -1$.

- The answers are summarized in a diffusion index

$$\hat{D} = \sum_{a=1}^{r} \omega^a \frac{n^a}{n}.$$
Measuring the breadth of change using qualitative surveys

- $p^a$ is the probability that a participant’s answer is $a \in A = \{1, 2, \ldots, r\}$, with $\sum_{a=1}^{r} p^a = 1$
Measuring the breadth of change using qualitative surveys

- $p^a$ is the probability that a participant’s answer is $a \in A = \{1, 2, \ldots, r\}$, with $\sum_{a=1}^{r} p^a = 1$

- $\hat{p}^a = n^a / n$, the proportion of answers of type $a \in A$, where $\hat{p}^a = \frac{1}{n} \sum_{i=1}^{n} x_i^a$,

  where $x_i^a$ takes on the value 1 when survey participant $i$ answers $a$, zero otherwise
Measuring the breadth of change using qualitative surveys

- $p^a$ is the probability that a participant’s answer is $a \in \mathcal{A} = \{1, 2, \ldots, r\}$, with $\sum_{a=1}^{r} p^a = 1$

- $\hat{p}^a = n^a / n$, the proportion of answers of type $a \in \mathcal{A}$,

$$\hat{p}^a = \frac{1}{n} \sum_{i=1}^{n} x_i^a,$$

where $x_i^a$ takes on the value 1 when survey participant $i$ answers $a$, zero otherwise

- $\hat{p}^a$ then has the interpretation of a sample Bernoulli mean
The multivariate Central Limit Theorem immediately gives

\[
\sqrt{n} \left( \begin{array}{c}
\hat{p}^1 - p^1 \\
\hat{p}^2 - p^2 \\
\vdots \\
\hat{p}^r - p^r 
\end{array} \right) \xrightarrow{D} \mathcal{N} \left( \begin{array}{c}
0 \\
0 \\
\vdots \\
0
\end{array} \right), \quad \begin{array}{ccc}
p^1(1 - p^1) & -p^1 p^2 & \cdots & -p^1 p^r \\
-p^2 p^1 & p^2(1 - p^2) & \cdots & -p^2 p^r \\
\vdots & \vdots & \ddots & \vdots \\
-p^1 p^r & -p^2 p^r & \cdots & p^r(1 - p^r)
\end{array}
\]

where \( D = \mathbb{E}(\hat{D}) = \sum_{a=1}^{r} \omega\hat{p}^a = \sum_{a=1}^{r} \omega a \hat{p} \).
Uncertainty in the measure of direction of change

- The multivariate Central Limit Theorem immediately gives

$$\sqrt{n} \begin{pmatrix} \hat{p}^1 - p^1 \\ \hat{p}^2 - p^2 \\ \vdots \\ \hat{p}^r - p^r \end{pmatrix} \xrightarrow{D} \mathcal{N} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} p^1(1 - p^1) & -p^1 p^2 & \cdots & -p^1 p^r \\ -p^2 p^1 & p^2(1 - p^2) & \cdots & -p^2 p^r \\ \vdots & \vdots & \ddots & \vdots \\ -p^r p^1 & -p^r p^2 & \cdots & p^r(1 - p^r) \end{pmatrix}$$

- $\hat{D}$, is a linear combination of sample Bernoulli means, $\sum_{a=1}^{r} \omega^a \hat{p}^a$
Uncertainty in the measure of direction of change

- The multivariate Central Limit Theorem immediately gives

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\sqrt{n} \left( \begin{array}{c}
\hat{p}^1 - p^1 \\
\hat{p}^2 - p^2 \\
\vdots \\
\hat{p}^r - p^r
\end{array} \right) \overset{d}{\longrightarrow} \mathcal{N} \left( \begin{array}{c}
0 \\
0 \\
\vdots \\
0
\end{array}, \begin{pmatrix}
p^1(1 - p^1) & -p^1 p^2 & \cdots & -p^1 p^r \\
-p^2 p^1 & p^2(1 - p^2) & \cdots & -p^2 p^r \\
\vdots & \vdots & \ddots & \vdots \\
-p^r p^1 & -p^r p^2 & \cdots & p^r(1 - p^r)
\end{pmatrix} \right)
\]

- \( \hat{D} \), is a linear combination of sample Bernoulli means, \( \sum_{a=1}^r \omega^a \hat{p}^a \)

- Then

\[
\sqrt{n} \left( \hat{D} - D \right) \sim^a \mathcal{N} \left( 0, \left( \sum_{a=1}^r (\omega^a)^2 p^a \right) - D^2 \right),
\]

where \( D = E(\hat{D}) = \sum_{a=1}^r \omega^a p^a. \)
Uncertainty in the survey-measured direction of change

- Richmond indices: $\mathcal{A} = \{u, d, s\}$, and $\omega_u = 1$, $\omega_s = 0$, $\omega_d = -1 \Rightarrow \hat{D} = (\hat{\rho}_u - \hat{\rho}_d)$ and

$$\sqrt{n} \left( \hat{D} - D \right) \xrightarrow{D} \mathcal{N} \left( 0, (1 - p_s) - D^2 \right).$$
Uncertainty in the survey-measured direction of change

- Richmond indices: \( A = \{u, d, s\} \), and \( \omega_u = 1, \omega_s = 0, \omega_d = -1 \) \( \Rightarrow \)
  \( \hat{D} = (\hat{p}_u - \hat{p}_d) \) and

  \[
  \sqrt{n} \left( \hat{D} - D \right) \overset{D}{\longrightarrow} \mathcal{N} \left( 0, (1 - p_s) - D^2 \right).
  \]

- Uncertainty in the measured direction of change ... 
  - decreases with the square root of the sample size
  - decreases with the diffusion index itself, \( D \)
  - decreases with the degree of polarization, \( 1 - p_s \)
Distinguishing between different categories of participants

\[
\hat{D} = \sum_{j=1}^{J} \sum_{a=1}^{r} \omega^a \frac{n^a_j}{n} = \hat{D} = \sum_{j=1}^{J} \frac{n_j}{n} \sum_{a=1}^{r} \omega^a \frac{n^a_j}{n_j} \]

where \(\sum_{j=1}^{J} n^a_j = n^a\), and \(\sum_{j=1}^{J} n_j = n\)
Distinguishing between different categories of participants

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\text{where } \sum_{j=1}^{J} n_j^a = n^a, \text{ and } \sum_{j=1}^{J} n_j = n
\]

One may choose to rescale the weights, say by \( \gamma_j \),

\[
\hat{D} = \sum_{j=1}^{J} \gamma_j \frac{n_j}{n} \sum_{a=1}^{r} \omega^a \frac{n_j^a}{n_j}
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One may choose to rescale the weights, say by \( \gamma_j \),

\[ \hat{D} = \sum_{j=1}^{J} \gamma_j \frac{n_j}{n} \sum_{a=1}^{r} \omega^a \frac{n_j^a}{n_j} \]

\[ \sqrt{n} \left( \hat{D} - D \right) \sim^a \mathcal{N} \left( 0, \left( \sum_{a=1}^{r} \sum_{j=1}^{J} (\omega^a \gamma_j)^2 p_j^a \right) - D^2 \right) \]
The Distribution of Composite Indices

- \( n \) survey participants responding to questions concerning \( k \) economic conditions - e.g. household conditions, overall business conditions, spending on big ticket items
The Distribution of Composite Indices

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- Answers $a_k$, confined to a set $A_k$, each comprising $r$ possible types of responses, $\{1, 2, \ldots, r\}$.
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- $n$ survey participants responding to questions concerning $k$ economic conditions - e.g. household conditions, overall business conditions, spending on big ticket items

- Answers $a_k$, confined to a set $\mathcal{A}_k$, each comprising $r$ possible types of responses, $\{1, 2, ..., r\}$.

- Participants’ answers across all components, $k = 1, ..., k$, are collected in a $k$-tuple $\mathbf{a} = (a_1, ..., a_k)$ that lives in the set $\mathcal{A} = \Pi_{k=1}^{k} \mathcal{A}_k$. 
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- $n$ survey participants responding to questions concerning $k$ economic conditions - e.g. household conditions, overall business conditions, spending on big ticket items.

- Answers $a_k$, confined to a set $\mathcal{A}_k$, each comprising $r$ possible types of responses, $\{1, 2, \ldots, r\}$.

- Participants’ answers across all components, $k = 1, \ldots, \bar{k}$, are collected in a $k$-tuple $\mathbf{a} = (a_1, \ldots, a_{\bar{k}})$ that lives in the set $\mathcal{A} = \prod_{k=1}^{\bar{k}} \mathcal{A}_k$.

Consider an example with 3 components, each comprising 3 possible responses, $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_2$

$= \{u, d, s\} \times \{u, d, s\} \times \{u, d, s\} = \{uus, uud, uus, duu, ddu, dsu, suu, sdu, ssu, \ldots\}$ has 27 elements, in general $r^{\bar{k}}$. 

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The Distribution of Composite Indices

The Composite Index

\[ \hat{D} = \sum_{k=1}^{\bar{k}} \delta_k \hat{D}_k, \]

where \( 0 < \delta_k < 1. \)
The Distribution of Composite Indices

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\[ \hat{D} = \sum_{k=1}^{k} \delta_k \hat{D}_k, \]

where \(0 < \delta_k < 1\).

\[ \hat{D}_k = \sum_{a=1}^{r} \omega^a \sum_{a_k \in A/A_k} \frac{n(a_k=a,a_{-k})}{n} = \sum_{a=1}^{r} \omega^a \frac{n_k^a}{n}, \]
The Distribution of Composite Indices

- The Composite Index

\[ \hat{D} = \sum_{k=1}^{\bar{k}} \delta_k \hat{D}_k, \]

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Uncertainty in \( \hat{D} \) will need to take account of pairwise covariances between individual indices, say \( D_k \) and \( D_\ell \) - i.e. how participants’ answers compare/comove across different categories.
The Distribution of Composite Indices

- $n_{k\ell}^{aa'}$ - the number of survey participants answering $a_k = a$ and $a_\ell = a'$ for the components $k$ and $\ell$,

\[
n_{k\ell}^{aa'} = \sum_{a_{\{k,\ell\}} \in A/A_k \times A_\ell} n^{(a_k=a,a_\ell=a',a_{\{k,\ell\}})}, \ k \neq \ell,
\]
The Distribution of Composite Indices

- $n_{k\ell}^{aa'}$ - the number of survey participants answering $a_k = a$ and $a_\ell = a'$ for the components $k$ and $\ell$,

$$n_{k\ell}^{aa'} = \sum_{a_{\{-k,\ell\}} \in \mathcal{A}/\mathcal{A}_k \times \mathcal{A}_\ell} n(a_k = a, a_\ell = a', a_{\{-k,\ell\}}), \ k \neq \ell,$$

- The number of participants answering a given response $a$ for component $k$ satisfies $n_k^a = \sum_{a' \in \mathcal{A}_\ell} n_{k\ell}^{aa'}$.
The Distribution of Composite Indices

- $n_{k\ell}^{aa'}$ - the number of survey participants answering $a_k = a$ and $a_\ell = a'$ for the components $k$ and $\ell$,

$$n_{k\ell}^{aa'} = \sum_{a_{-\{k,\ell\}} \in \mathcal{A}/\mathcal{A}_k \times \mathcal{A}_\ell} n^{(a_k=a, a_\ell=a', a_{-\{k,\ell\}})}$$

- The number of participants answering a given response $a$ for component $k$ satisfies

$$n_k^a = \sum_{a' \in \mathcal{A}_\ell} n_{k\ell}^{aa'}$$

- Let $\hat{p}_k^a = n_k^a / n$, and $\hat{p}_{k\ell}^{aa'} = n_{k\ell}^{aa'} / n$,
The Distribution of Composite Indices

- \( n_{k\ell}^{aa'} \) - the number of survey participants answering \( a_k = a \) and \( a_\ell = a' \) for the components \( k \) and \( \ell \),

\[
n_{k\ell}^{aa'} = \sum_{a-\{k,\ell\} \in \mathcal{A}/\mathcal{A}_k \times \mathcal{A}_\ell} n(a_k=a, a_\ell=a', a-\{k,\ell\}) \quad k \neq \ell,
\]

The number of participants answering a given response \( a \) for component \( k \) satisfies \( n_k^a = \sum_{a' \in \mathcal{A}_\ell} n_{k\ell}^{aa'} \)

Let \( \hat{p}_k^a = n_k^a / n \), and \( \hat{p}_{k\ell}^{aa'} = n_{k\ell}^{aa'} / n \),

Individual indices are given by

\[
\hat{D}_k = \sum_{a=1}^{r} \omega^a \hat{p}_k^a = \sum_{a=1}^{r} \omega^a \sum_{a' \in \mathcal{A}_\ell} \hat{p}_{k\ell}^{aa'},
\]
The Distribution of Composite Indices

- $p_{k\ell}^{aa'}$ - the joint probability of observing $a_k = a$ and $a_\ell = a'$ for the components $k$ and $\ell$,

$$p_{k\ell}^{aa'} = \sum_{a_{-\{k,\ell\}} \in \mathcal{A}/\mathcal{A}_k \times \mathcal{A}_\ell} p(a_k=a,a_\ell=a',a_{-\{k,\ell\}}), \ k \neq \ell,$$

where $(a_k = a, a_\ell = a', a_{-\{k,\ell\}})$ distinguishes between answers for component $k$, component $\ell$, and all other components, $-\{k,\ell\}$.
The Distribution of Composite Indices

- $p_{k\ell}^{aa'}$ - the joint probability of observing $a_k = a$ and $a_\ell = a'$ for the components $k$ and $\ell$,

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where $(a_k = a, a_\ell = a', a_{-\{k,\ell\}})$ distinguishes between answers for component $k$, component $\ell$, and all other components, $-\{k,\ell\}$

- $p_{k\ell}$ - the vector comprising all pairwise joint probabilities, $p_{k\ell}^{aa'}$ for given components $k$ and $\ell$, where the dimension of $p_{k\ell}$ is $r^2$. 

Each element $\hat{p}_{k\ell}^{aa'}$ in the vector $\hat{p}_{k\ell}$ is the sample mean of $\sqrt{n}(\hat{p}_{k\ell}^{aa'} - p_{k\ell}) \rightarrow D \mathcal{N}(0, \Sigma_{p_{k\ell}})$. 
The Distribution of Composite Indices

- $p_{k\ell}^{aa'}$ - the joint probability of observing $a_k = a$ and $a_\ell = a'$ for the components $k$ and $\ell$,

$$p_{k\ell}^{aa'} = \sum_{a_{-{k,\ell}} \in \mathcal{A}/\mathcal{A}_k \times \mathcal{A}_\ell} p(a_k = a, a_\ell = a', a_{-{k,\ell}}), \ k \neq \ell,$$

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- $p_{k\ell}$ - the vector comprising all pairwise joint probabilities, $p_{k\ell}^{aa'}$ for given components $k$ and $\ell$, where the dimension of $p_{k\ell}$ is $r^2$.

Each element $\hat{p}_{k\ell}^{aa'}$, in the vector $\hat{p}_{k\ell}$, is the sample mean of Bernoulli random variables

$$\sqrt{n}(\hat{p}_{k\ell} - p_{k\ell}) \xrightarrow{D} \mathcal{N}(0, \Sigma_{p_{k\ell}}).$$
The Distribution of Composite Indices

\[ \sqrt{n}(\hat{D} - D) \sim a \mathcal{N}\left(0, \sum_{k=1}^{\bar{k}} \delta_k^2 \text{Var}(\hat{D}_k) + 2 \sum_{1 \leq k < \ell \leq \bar{k}} \delta_k \delta_\ell \text{Cov}(\hat{D}_k, \hat{D}_\ell)\right), \]

where

\[ D = \sum_{k=1}^{\bar{k}} \delta_k \sum_{a=1}^{r} \omega^a p_k^a, \]

\[ \text{Var}(\hat{D}_k) = \frac{1}{n} \left\{ \left( \sum_{a=1}^{r} (\omega^a)^2 p_k^a \right)^2 - (D_k)^2 \right\}, \]

\[ \text{Cov}(\hat{D}_k, \hat{D}_\ell) = \frac{1}{n} \left\{ \sum_{(a, a') \in A_k \times A_\ell} \omega^a \omega^{a'} \left[ p_{k\ell}^{aa'} - \sum_{(b, b') \in A_k \times A_\ell} p_{k\ell}^{ab} p_{k\ell}^{b'a'} \right] \right\}. \]
An Application: The Michigan Survey of Consumers

- Monthly survey of Consumers (about 500 interviews) conducted by the Survey Research Center, University of Michigan
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- Index of Current Conditions
  - $D_1$: Would you say that you (and your family living there) are better off or worse off financially than you were a year ago?
  - $D_5$: Generally speaking, do you think now is a good or bad time for people to buy major household items?

$$ICC = \frac{D_1 + D_5}{2.6424}$$
An Application: The Michigan Survey of Consumers

Index of Consumer Expectations

- $D_2$: Do you think that a year from now, you will be better off financially, or worse off, or just about the same as now?

- $D_3$: In the country as a whole—do you think that during the next twelve months we’ll have good times financially, or bad times, or what?

- $D_4$: In the country as a whole, will we have continuous good times during the next five years or so, will we have periods of widespread unemployment or depression, or what?

\[ ICE = \frac{D_2 + D_3 + D_4}{4.1134} \]
An Application: The Michigan Survey of Consumers

- **Index of Consumer Expectations**
  - $D_2$: Do you think that a year from now, you will be better off financially, or worse off, or just about the same as now?
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  $$ICE = \frac{D_2 + D_3 + D_4}{4.1134}$$

- **Index of Consumer Sentiment (headline number)**

  $$ICS = \frac{D_1 + D_2 + D_3 + D_4 + D_5}{6.7558}$$
Uncertainty and Qualitative Surveys

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