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Joseph B. Kadane

Carnegie Mellon University, kadane@stat.cmu.edu

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A CONCAVE PROPERTY OF THE HYPERGEOMETRIC FUNCTION WITH RESPECT TO A PARAMETER*

JOSEPH B. KADANE†

Abstract. The hypergeometric function is shown to be logarithmically concave in integer values of one of its parameters. The methods used are probabilistic.

THEOREM. Let m, i and g be positive integers satisfying $3 \leq i + 1 \leq g$, and let z be a negative real number. Then

$$\{ {}_2F_1[-m, i : g : z] \}^2 > {}_2F_1[-m, i + 1 : g : z] {}_2F_1[-m, i - 1 : g : z].$$

We first establish the following lemma concerning the evaluation of the generating function of the negative hypergeometric distribution.

LEMMA.

$$(1) \quad \sum_{j=0}^k \binom{b+j-1}{j} \binom{k+a-j-1}{k-j} s^j / \binom{a+b+k-1}{k} = {}_2F_1[-k, b : a+b; 1-s]$$

for all positive integers k , and all real s , and positive real values of a and b .

Proof of Lemma. Skellam [2] has shown that if X follows a binomial distribution with parameters p and k , and if p is integrated with respect to the normalized beta function

$$\frac{p^{b-1}(1-p)^{a-1} dp}{B(a, b)},$$

then the unconditional distribution of X is negative hypergeometric, that is,

$$\Pr\{X = j\} = \binom{b+j-1}{j} \binom{k+a-j-1}{k-j} / \binom{a+b+k-1}{k}.$$

The left-hand side of (1), denoted below by I , is then the probability generating function of the negative hypergeometric distribution. Thus

$$\begin{aligned} I &= \mathcal{E}(s^X) = \mathcal{E}_p\{(s^X|p)\} = \mathcal{E}_p(1-p+ps)^k \\ &= \frac{1}{B(a, b)} \int_0^1 [1-p(1-s)]^k p^{b-1}(1-p)^{a-1} dp \\ &= {}_2F_1[-k, b : a+b; 1-s]. \end{aligned}$$

See [3, p. 20]. This proves the lemma.

Proof of Theorem. The essence of the proof is to use two theorems proved elsewhere [1], one on the existence of a probability distribution with a certain property, the other giving an inequality relating to such a distribution.

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† Department of Statistics, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213.

Let $s = 1 + z$, $h = g - 1$ and $n = m + g - 1$. Theorem 3 of [1] states that there is a probability distribution F such that $a_{i,n}$, the expected value of the i th largest of a sample of size n drawn independently from F , satisfies

$$a_{i,n} = s^{i-1} \quad \text{for all } i, \quad 1 \leq i \leq n.$$

By use of a standard recurrence relation, quoted in [1, (4)], the expected value of the i th largest of some smaller sample of size h can be deduced as follows: For $1 \leq i \leq g \leq n$,

$$\begin{aligned} a_{i,h} &= \sum_{j=0}^{n-h} \binom{i+j-1}{j} \binom{n-j-i}{n-h-j} a_{i+j,n} / \binom{n}{h} \\ (2) \quad &= \sum_{j=0}^{n-h} \binom{n-j-i}{n-h-j} \binom{i+j-1}{j} s^{i+j-1} / \binom{n}{h} \\ &= s^{i-1} {}_2F_1[h-n, i; h+1; 1-s] \end{aligned}$$

on using the lemma with $k = n - h$, $b = i$ and $a = h - i + 1$.

Theorem 4 of [1] states that if $a_{i,n} = s^{i-1}$ for all $i = 1, \dots, n$, then

$$a_{i,h}^2 > a_{i-1,h} a_{i+1,h} \quad \text{for } i = 2, \dots, h-1 \quad \text{and } h \leq n-1.$$

Applying (2), we obtain

$$\begin{aligned} & s^{2i-2} \{ {}_2F_1[h-n, i; h+1; 1-s] \}^2 \\ & > s^{i-2} \{ {}_2F_1[h-n, i-1; h+1; 1-s] \} s^i \{ {}_2F_1[h-n, i+1; h+1; 1-s] \}. \end{aligned}$$

The theorem now follows by substituting for h, n and s .

Remark. An analytic proof of the theorem has been shown to the author by Dr. Tyson of the Center for Naval Analyses.

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