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A CONCAVE PROPERTY OF THE HYPERGEOMETRIC FUNCTION WITH RESPECT TO A PARAMETER*

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Abstract. The hypergeometric function is shown to be logarithmically concave in integer values of one of its parameters. The methods used are probabilistic.

THEOREM. Let m, i and g be positive integers satisfying \(3 \leq i + 1 \leq g\), and let z be a negative real number. Then

\[
\left\{ 2F1[-m, i : g ; z] \right\}^2 > 2F1[-m, i + 1 ; g ; z] 2F1[-m, i - 1 ; g ; z].
\]

We first establish the following lemma concerning the evaluation of the generating function of the negative hypergeometric distribution.

LEMMA. \(\sum_{j=0}^{k} \binom{b + j - 1}{j} \binom{k + a - j - 1}{k - j} s^j / \binom{a + b + k - 1}{k} = 2F1[-k, b ; a + b ; 1 - s]\)

for all positive integers k, and all real s, and positive real values of a and b.

Proof of Lemma. Skellam [2] has shown that if \(X\) follows a binomial distribution with parameters \(p\) and \(k\), and if \(p\) is integrated with respect to the normalized beta function

\[
\frac{p^{b-1}(1-p)^{a-1} dp}{B(a, b)},
\]

then the unconditional distribution of \(X\) is negative hypergeometric, that is,

\[
\Pr\{X = j\} = \binom{b + j - 1}{j} \binom{k + a - j - 1}{k - j} / \binom{a + b + k - 1}{k}.
\]

The left-hand side of (1), denoted below by \(I\), is then the probability generating function of the negative hypergeometric distribution. Thus

\[
I = \mathcal{E}(s^X) = \mathcal{E}_p\{s^X|p\} = \mathcal{E}_p(1 - p + ps)^k
\]

\[
= \frac{1}{B(a, b)} \int_0^1 [1 - p(1 - s)]^k p^{b-1}(1 - p)^{a-1} dp
\]

\[
= 2F1[-k, b ; a + b ; 1 - s].
\]

See [3, p. 20]. This proves the lemma.

Proof of Theorem. The essence of the proof is to use two theorems proved elsewhere [1], one on the existence of a probability distribution with a certain property, the other giving an inequality relating to such a distribution.

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Let \( s = 1 + z \), \( h = g - 1 \) and \( n = m + g - 1 \). Theorem 3 of [1] states that there is a probability distribution \( F \) such that \( a_{i,n} \), the expected value of the \( i \)th largest of a sample of size \( n \) drawn independently from \( F \), satisfies
\[
a_{i,n} = s^{i-1} \quad \text{for all } i, \ 1 \leq i \leq n.
\]
By use of a standard recurrence relation, quoted in [1, (4)], the expected value of the \( i \)th largest of some smaller sample of size \( h \) can be deduced as follows: For \( 1 \leq i \leq g \leq n \),
\[
a_{i,h} = \sum_{j=0}^{n-h} \binom{i+j-1}{j} \binom{n-j-i}{n-h-j} \frac{a_{i+j,n}}{n}\binom{n}{h}
\]
(2)
\[
= \sum_{j=0}^{n-h} \binom{n-j-i}{n-h-j} \binom{i+j-1}{j} s^{i+j-1} \frac{a_{i+j,n}}{n}\binom{n}{h}
= s^{i-1} \, {}_2F_1[h-n, i; h+1; 1-s]
\]
on using the lemma with \( k = n-h, b = i \) and \( a = h - i + 1 \).

Theorem 4 of [1] states that if \( a_{i,n} = s^{i-1} \) for all \( i = 1, \ldots, n \), then
\[
a_{i,h}^2 > a_{i-1,h}a_{i+1,h} \quad \text{for } i = 2, \ldots, h-1 \quad \text{and } h \leq n-1.
\]
Applying (2), we obtain
\[
s^{2i-2} \left\{ {}_2F_1[h-n, i; h+1; 1-s] \right\}^2
> s^{i-2} \left\{ {}_2F_1[h-n, i-1; h+1; 1-s] \right\} s^i \left\{ {}_2F_1[h-n, i+1; h+1; 1-s] \right\}.
\]
The theorem now follows by substituting for \( h, n \) and \( s \).

Remark. An analytic proof of the theorem has been shown to the author by Dr. Tyson of the Center for Naval Analyses.

REFERENCES